Variation on Kolmogorov's theorem: KAM with knobs

Veronica Danesi

In collaboration with M. Sansottera

University of Milan-Dep. of Mathematics

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Starting point

Giorgilli, Locatelli, Sansottera, CM&DA (2014):

- proof of the existence of lower dimensional elliptic tori for planetary systems;
- Kolmogorov-like normal form + result in measure [Pöschel (1989)];
- classical expansions in powers of ε;
- internal and transversal frequencies vary at each step.

Christodoulidi, Efthymiopoulos, Bountis, Phys. Rev. E (2010) and Christodoulidi, Efthymiopoulos, Physica D (2013):

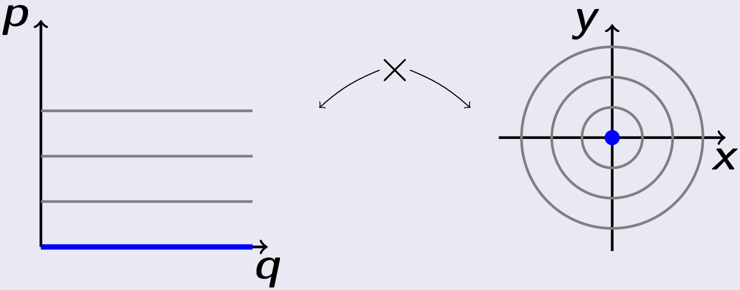
- location of lower dimensional elliptic tori in the FPU model;
- Poincaré-Lindstedt series;
- internal and transversal frequencies are fixed;
- numerical study of the convergence properties.

Lower dimensional elliptic tori

$$H(p,q,x,y) = \langle \omega, p \rangle + \sum_{j=1}^{n_2} \frac{\Omega_j(\omega)}{2}(x_j^2 + y_j^2) + \mathcal{O}(p^2) + \mathcal{O}(x^3, y^3) + \mathcal{O}(px, py)$$

$(p,q) \in \mathcal{U}(0) \times \mathbb{T}^{n_1}$ are action-angle variables; $(x,y) \in \mathbb{R}^{2n_2}$ are Cartesian variables.

$p = 0$ and $x = y = 0$ is a lower dimensional elliptic invariant torus



The Goal (we are working on it!)

Provide a constructive proof of the existence of **lower dimensional elliptic tori**, improving the control on the frequencies.

Since we cannot *fix* the frequencies along the normalization procedure, we aim to fix the *final* frequencies and determine *a posteriori* the corresponding initial ones.

The Talk (we are here!)

As a first result, we adapt the Kolmogorov's algorithm for **full dimensional tori**, so as to avoid the translation that keeps the frequencies fixed and introduce a *detuning* between the fixed *final* frequencies and the corresponding initial ones.

The detuning can be figured as the action of turning a control *knob*.

Kolmogorov's normal form (for a quadratic Hamiltonian)

$$H^{(0)}(p,q) = \langle \omega_0, p \rangle + f_2^{(0,0)} + \sum_{s \geq 1} \varepsilon^s \left(f_0^{(0,s)} + f_1^{(0,s)} + f_2^{(0,s)} \right) ,$$

$f_\ell^{(0,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K > 0$, in q.

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$$\begin{array}{c|cccccc}
f_2^{(0,0)} & f_2^{(0,1)} & f_2^{(0,2)} & f_2^{(0,3)} & \ldots & f_2^{(0,s)} & \ldots \\
\langle \omega_0, p\rangle & f_1^{(0,1)} & f_1^{(0,2)} & f_1^{(0,3)} & \ldots & f_1^{(0,s)} & \ldots \\
 & f_0^{(0,1)} & f_0^{(0,2)} & f_0^{(0,3)} & \ldots & f_0^{(0,s)} & \ldots
\end{array}$$

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 & f_0^{(0,1)} & f_0^{(0,2)} & f_0^{(0,3)} & \ldots & f_0^{(0,s)} & \ldots
\end{array}$$

Focus on the torus $p = 0$:

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f_2^{(0,0)} & f_2^{(0,1)} & f_2^{(0,2)} & f_2^{(0,3)} & \ldots & f_2^{(0,s)} & \ldots \\
\langle \omega_0, p \rangle & f_1^{(0,1)} & f_1^{(0,2)} & f_1^{(0,3)} & \ldots & f_1^{(0,s)} & \ldots \\
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\end{array}$$

Focus on the torus $p = 0$: $\qquad \dot{p} = -\left.\frac{\partial H}{\partial q}\right|_{p=0} = 0 \ ,$

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f_2^{(0,0)} & f_2^{(0,1)} & f_2^{(0,2)} & f_2^{(0,3)} & \ldots & f_2^{(0,s)} & \ldots \\
\langle \omega_0, p \rangle & f_1^{(0,1)} & f_1^{(0,2)} & f_1^{(0,3)} & \ldots & f_1^{(0,s)} & \ldots \\
 & f_0^{(0,1)} & f_0^{(0,2)} & f_0^{(0,3)} & \ldots & f_0^{(0,s)} & \ldots
\end{array}$$

Focus on the torus $p = 0$: $\dot{p} = -\left.\frac{\partial H}{\partial q}\right|_{p=0} = 0 \ , \qquad \dot{q} = \left.\frac{\partial H}{\partial p}\right|_{p=0} = \omega_0 \ .$

Adding the knobs

- ω_0 is the unperturbed initial frequency vector;
- pick a *strongly non resonant* frequency vector ω;
- introduce a sequence of detunings $\{\omega_s\}_{s\geq 1}$ between ω and ω_0 as

$$\omega = \omega_0 + \sum_{s\geq 1} \omega_s \ .$$

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$$\omega = \omega_0 + \sum_{s\geq 1} \omega_s \ .$$

Remarks

- the sequence of detunings $\{\omega_s\}_{s\geq 1}$ is unknown and will be determined step by step by the normalization procedure;
- at any given finite normal form order r we get an approximation of the initial frequency vector given by $\omega_0^{(r)} = \omega - \sum_{s=1}^{r} \omega_s$;
- this approach, in principle, allows to start from a *resonant* torus with frequencies ω_0 which falls into a strongly non resonant one *by construction*.

Kolmogorov's normal form with knobs (for a quadratic Hamiltonian)

$$H^{(0)}(p,q) = \langle \omega, p \rangle + f_2^{(0,0)} + \sum_{s \geq 1} \varepsilon^s \left(f_0^{(0,s)} + f_1^{(0,s)} - \langle \omega_s, p \rangle + f_2^{(0,s)} \right) \ ,$$

$f_\ell^{(0,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K > 0$, in q.

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$f_2^{(0,0)}$	$f_2^{(0,1)}$	$f_2^{(0,2)}$	$f_2^{(0,3)}$	$\ldots$
$\langle \omega, p\rangle$	$f_1^{(0,1)} - \langle \omega_1, p\rangle$	$f_1^{(0,2)} - \langle \omega_2, p\rangle$	$f_1^{(0,3)} - \langle \omega_3, p\rangle$	$\ldots$
	$f_0^{(0,1)}$	$f_0^{(0,2)}$	$f_0^{(0,3)}$	$\ldots$

Focus on the torus $p = 0$: $\qquad \dot{p} = -\left.\frac{\partial H}{\partial q}\right|_{p=0} = 0 \ ,$

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$f_\ell^{(0,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K>0$, in q.

$f_2^{(0,0)}$	$f_2^{(0,1)}$	$f_2^{(0,2)}$	$f_2^{(0,3)}$	$\dots$
$\langle\omega,p\rangle$	$f_1^{(0,1)} - \langle\omega_1,p\rangle$	$f_1^{(0,2)} - \langle\omega_2,p\rangle$	$f_1^{(0,3)} - \langle\omega_3,p\rangle$	$\dots$
	$f_0^{(0,1)}$	$f_0^{(0,2)}$	$f_0^{(0,3)}$	$\dots$

Focus on the torus $p=0$: $\quad \dot p = -\left.\frac{\partial H}{\partial q}\right|_{p=0} = 0 \ , \qquad \dot q = \left.\frac{\partial H}{\partial p}\right|_{p=0} = \omega \ .$

Kolmogorov's normal form with knobs (for a quadratic Hamiltonian)

$$H^{(0)}(p,q) = \langle \omega, p \rangle + f_2^{(0,0)} + \sum_{s\ge 1} \varepsilon^s \left(f_0^{(0,s)} + f_1^{(0,s)} - \langle \omega_s, p \rangle + f_2^{(0,s)} \right) \ ,$$

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$\langle \omega, p \rangle$	$f_1^{(0,1)} - \langle \omega_1, p \rangle$	$f_1^{(0,2)} - \langle \omega_2, p \rangle$	$f_1^{(0,3)} - \langle \omega_3, p \rangle$	$\ldots$
	$f_0^{(0,1)}$	$f_0^{(0,2)}$	$f_0^{(0,3)}$	$\ldots$

Generating functions:

Kolmogorov's normal form with knobs (for a quadratic Hamiltonian)

$$\hat{H}^{(1)}(p,q) = \langle \omega, p \rangle + \hat{f}_2^{(1,0)} + \sum_{s\geq 1} \varepsilon^s \left(\hat{f}_0^{(1,s)} + \hat{f}_1^{(1,s)} - \langle \omega_s, p \rangle + \hat{f}_2^{(1,s)} \right) \ ,$$

$\hat{f}_\ell^{(1,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K > 0$, in q.

$\hat{f}_2^{(1,0)}$	$\hat{f}_2^{(1,1)}$	$\hat{f}_2^{(1,2)}$	$\hat{f}_2^{(1,3)}$	$\ldots$
$\langle \omega, p \rangle$	$\hat{f}_1^{(1,1)} - \langle \omega_1, p \rangle$	$\hat{f}_1^{(1,2)} - \langle \omega_2, p \rangle$	$\hat{f}_1^{(1,3)} - \langle \omega_3, p \rangle$	$\ldots$
	0	$\hat{f}_0^{(1,2)}$	$\hat{f}_0^{(1,3)}$	$\ldots$

Generating functions: $\chi_0^{(1)}$,

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$f_\ell^{(1,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K > 0$, in q.

$f_2^{(1,0)}$	$f_2^{(1,1)}$	$f_2^{(1,2)}$	$f_2^{(1,3)}$	$\dots$
$\langle \omega, p\rangle$	0	$f_1^{(1,2)} - \langle \omega_2, p\rangle$	$f_1^{(1,3)} - \langle \omega_3, p\rangle$	$\dots$
	0	$f_0^{(1,2)}$	$f_0^{(1,3)}$	$\dots$

Generating functions: $\chi_0^{(1)}$, $\chi_1^{(1)}$,

Kolmogorov's normal form with knobs (for a quadratic Hamiltonian)

$$\hat{H}^{(2)}(p,q) = \langle \omega, p \rangle + \hat{f}_2^{(2,0)} + \sum_{s\ge 1} \varepsilon^s \left(\hat{f}_0^{(2,s)} + \hat{f}_1^{(2,s)} - \langle \omega_s, p \rangle + \hat{f}_2^{(2,s)} \right) ,$$

$\hat{f}_\ell^{(2,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K > 0$, in q.

$\hat{f}_2^{(2,0)}$	$\hat{f}_2^{(2,1)}$	$\hat{f}_2^{(2,2)}$	$\hat{f}_2^{(2,3)}$	$\dots$
$\langle \omega, p \rangle$	0	$\hat{f}_1^{(2,2)} - \langle \omega_2, p \rangle$	$\hat{f}_1^{(2,3)} - \langle \omega_3, p \rangle$	$\dots$
	0	0	$\hat{f}_0^{(2,3)}$	$\dots$

Generating functions: $\chi_0^{(1)}$, $\chi_1^{(1)}$, $\chi_0^{(2)}$,

Kolmogorov's normal form with knobs (for a quadratic Hamiltonian)

$$H^{(2)}(p,q) = \langle \omega, p\rangle + f_2^{(2,0)} + \sum_{s\ge 1} \varepsilon^s \left(f_0^{(2,s)} + f_1^{(2,s)} - \langle \omega_s, p\rangle + f_2^{(2,s)} \right) \ ,$$

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$f_2^{(2,0)}$	$f_2^{(2,1)}$	$f_2^{(2,2)}$	$f_2^{(2,3)}$	$\dots$
$\langle \omega, p\rangle$	0	0	$f_1^{(2,3)} - \langle \omega_3, p\rangle$	$\dots$
	0	0	$f_0^{(2,3)}$	$\dots$

Generating functions: $\chi_0^{(1)}$, $\chi_1^{(1)}$, $\chi_0^{(2)}$, $\chi_1^{(2)}$,

Kolmogorov's normal form with knobs (for a quadratic Hamiltonian)

$$\hat{H}^{(3)}(p,q) = \langle \omega, p\rangle + \hat{f}_2^{(3,0)} + \sum_{s\ge 1} \varepsilon^s \left(\hat{f}_0^{(3,s)} + \hat{f}_1^{(3,s)} - \langle \omega_s, p\rangle + \hat{f}_2^{(3,s)} \right) \ ,$$

$\hat{f}_\ell^{(3,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K > 0$, in q.

$\hat{f}_2^{(3,0)}$	$\hat{f}_2^{(3,1)}$	$\hat{f}_2^{(3,2)}$	$\hat{f}_2^{(3,3)}$	$\ldots$
$\langle \omega, p\rangle$	0	0	$\hat{f}_1^{(3,3)} - \langle \omega_3, p\rangle$	$\ldots$
	0	0	0	$\ldots$

Generating functions: $\chi_0^{(1)}$, $\chi_1^{(1)}$, $\chi_0^{(2)}$, $\chi_1^{(2)}$, $\chi_0^{(3)}$,

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$$H^{(3)}(p,q) = \langle \omega, p\rangle + f_2^{(3,0)} + \sum_{s\geq 1} \varepsilon^s \left(f_0^{(3,s)} + f_1^{(3,s)} - \langle \omega_s, p\rangle + f_2^{(3,s)} \right) \ ,$$

$f_\ell^{(3,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K > 0$, in q.

$f_2^{(3,0)}$	$f_2^{(3,1)}$	$f_2^{(3,2)}$	$f_2^{(3,3)}$	$\ldots$
$\langle \omega, p\rangle$	0	0	0	$\ldots$
	0	0	0	$\ldots$

Generating functions: $\chi_0^{(1)}\ ,\ \chi_1^{(1)}\ ,\ \chi_0^{(2)}\ ,\ \chi_1^{(2)}\ ,\ \chi_0^{(3)}\ ,\ \chi_1^{(3)}\ ,\ \ldots$

Hamiltonian in normal form up to order $r-1$

$$H^{(r-1)} = \langle \omega, p \rangle + \sum_{s=0}^{r-1} f_2^{(r-1,s)} + \sum_{s\geq r} \varepsilon^s \left(f_0^{(r-1,s)} + f_1^{(r-1,s)} - \langle \omega_s, p \rangle + f_2^{(r-1,s)} \right)$$

$f_\ell^{(r-1,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K > 0$, in q.

Hamiltonian in normal form up to order $r-1$

$$H^{(r-1)} = \langle \omega, p \rangle + \sum_{s=0}^{r-1} f_2^{(r-1,s)} + \sum_{s \geq r} \varepsilon^s \left(f_0^{(r-1,s)} + f_1^{(r-1,s)} - \langle \omega_s, p \rangle + f_2^{(r-1,s)} \right)$$

$f_\ell^{(r-1,s)}$ is a pol. of degree ℓ in p and trig. pol. of degree sK, with $K > 0$, in q.

r-th normalization step

- Transform $H^{(r-1)}$ into $H^{(r)}$ via composition of Lie series

$$\exp\left(L_{\chi_1^{(r)}}\right) \circ \exp\left(L_{\chi_0^{(r)}}\right)$$

with $\exp\left(L_\chi\right) = \sum_{s \geq 0} \frac{1}{s!} L_\chi^s$ and $L_\chi \cdot = \{\cdot, \chi\}$.

- $\chi_0^{(r)}$, $\chi_1^{(r)}$ and ω_r kill the unwanted terms $f_0^{(r-1,r)}$ and $\hat{f}_1^{(r,r)} - \langle \omega_r, p \rangle$.

First homological equation

$$\hat{H}^{(r)} = \exp\left(L_{\chi_0^{(r)}}\right) H^{(r-1)}$$

with $\chi_0^{(r)}$ determined by solving

$$L_{\chi_0^{(r)}} \langle \omega, p \rangle + f_0^{(r-1,r)} = 0 \ .$$

First homological equation

$$\hat{H}^{(r)} = \exp\left(L_{\chi_0^{(r)}}\right) H^{(r-1)}$$

with $\chi_0^{(r)}$ determined by solving

$$L_{\chi_0^{(r)}} \langle \omega, p \rangle + f_0^{(r-1,r)} = 0 \ .$$

Solution of the homological equation

$$f_0^{(r-1,r)}(q) = \sum_{0<|k|\le rK} c_{0,k} e^{\mathbf{i}\langle k,q\rangle} \quad \Rightarrow \quad \chi_0^{(r)}(q) = \sum_{0<|k|\le rK} \frac{c_{0,k}}{\mathbf{i}\langle k,\omega\rangle} e^{\mathbf{i}\langle k,q\rangle} \ .$$

Here we introduce a *small divisor*

$$\alpha_r = \min_{0<|k|\le rK} |\langle k,\omega\rangle| \ .$$

The intermediate Hamiltonian

$$\hat{H}^{(r)}(q,p) = \langle \omega, p\rangle + \sum_{s=0}^{r-1} \hat{f}_2^{(r,s)} + \varepsilon^r \left(\hat{f}_1^{(r,r)} - \langle \omega_r, p\rangle + \hat{f}_2^{(r,r)} \right) + \sum_{s>r} \varepsilon^s \left(\hat{f}_0^{(r,s)} + \hat{f}_1^{(r,s)} - \langle \omega_s, p\rangle + \hat{f}_2^{(r,s)} \right) ,$$

$$\hat{f}_0^{(r,s)} = \begin{cases} 0 , & s \le r ; \\ f_0^{(r-1,s)} , & r < s < 2r ; \\ f_0^{(r-1,s)} + L_{\chi_0^{(r)}} \left(f_1^{(r-1,s-r)} - \langle \omega_{s-r}, p\rangle \right) + \frac{1}{2} L^2_{\chi_0^{(r)}} f_2^{(r-1,s-2r)} , & s \ge 2r . \end{cases}$$

$$\hat{f}_1^{(r,s)} = \begin{cases} 0 , & s < r ; \\ f_1^{(r-1,s)} + L_{\chi_0^{(r)}} f_2^{(r-1,s-r)} , & s \ge r . \end{cases}$$

$$\hat{f}_2^{(r,s)} = f_2^{(r-1,s)}$$

Second homological equation

$$H^{(r)} = \exp\left(L_{\chi_1^{(r)}}\right) \hat{H}^{(r)}$$

with $\chi_1^{(r)}$ determined by solving

$$L_{\chi_1^{(r)}} \langle \omega, p \rangle + \hat{f}_1^{(r,r)} - \langle \omega_r, p \rangle = 0 \ , \quad \text{with} \quad \hat{f}_1^{(r,r)} = f_1^{(r-1,r)} + L_{\chi_0^{(r)}} f_2^{(0,0)} \ .$$

Second homological equation

$$H^{(r)} = \exp\left(L_{\chi_1^{(r)}}\right) \hat{H}^{(r)}$$

with $\chi_1^{(r)}$ determined by solving

$$L_{\chi_1^{(r)}} \langle \omega, p \rangle + \hat{f}_1^{(r,r)} - \langle \omega_r, p \rangle = 0 \ , \quad \text{with} \quad \hat{f}_1^{(r,r)} = f_1^{(r-1,r)} + L_{\chi_0^{(r)}} f_2^{(0,0)} \ .$$

Solution of the homological equation

$$\hat{f}_1^{(r,r)} = \sum_{0 \le |k| \le rK} c_{0,k}(p) e^{\mathbf{i}\langle k,q \rangle} \quad \Rightarrow \quad \begin{cases} \chi_1^{(r)}(q) = \displaystyle\sum_{0 < |k| \le rK} \frac{c_{0,k}}{\mathbf{i}\langle k, \omega \rangle} e^{\mathbf{i}\langle k,q \rangle} \ , \\ \langle \omega_r, p \rangle = \overline{f_1^{(r-1,r)}} \ . \end{cases}$$

Here also we have the *small divisor*

$$\alpha_r = \min_{0 < |k| \le rK} |\langle k, \omega \rangle| \ .$$

The Hamiltonian in normal form up to order r

$$H^{(r)}(q,p) = \langle \omega, p\rangle + \sum_{s=0}^{r} f_2^{(r,s)} + \sum_{s>r} \varepsilon^s \big(f_0^{(r,s)} + f_1^{(r,s)} - \langle \omega_s, p\rangle + f_2^{(r,s)} \big) \ ,$$

$$f_0^{(r,s)} = \sum_{j=0}^{k-1} \frac{1}{j!} L^j_{\chi_1^{(r)}} \hat{f}_0^{(r,s-jr)} \ , \quad s = kr + m > r \ ; \qquad f_2^{(r,s)} = \sum_{j=0}^{k} \frac{1}{j!} L^j_{\chi_1^{(r)}} \hat{f}_2^{(r,s-jr)} \ ;$$

$$f_1^{(r,s)} = \begin{cases} {\color{red}0} \ , & {\color{red}s \le r} \ ; \\ \displaystyle\sum_{j=0}^{k-1} \frac{1}{j!} L^j_{\chi_1^{(r)}} \Big(\hat{f}_1^{(r,s-jr)} - \langle \omega_{s-jr}, p\rangle \Big) \ , & s > r \ , \ s = kr + m \ ; \\ \displaystyle\frac{k-1}{k!} L^{k-1}_{\chi_1^{(r)}} \Big(\hat{f}_1^{(r,r)} - \langle \omega_r, p\rangle \Big) & \\ \qquad + \sum_{j=0}^{k-2} \frac{1}{j!} L^j_{\chi_1^{(r)}} \Big(\hat{f}_1^{(r,s-jr)} - \langle \omega_{s-jr}, p\rangle \Big) \ , & s > r \ , \ s = kr \ . \end{cases}$$

Quantitative estimates: domains and norms

Complex domains $\mathcal{D}_{(\rho,\sigma)} = \Delta_\rho(0) \times \mathbb{T}^n_\sigma$ with

$$\Delta_\rho(0) = \left\{ z \in \mathbb{C}^n : \max_{1\le j\le n} |z_j| < \rho \right\} \quad \text{and} \quad \mathbb{T}^n_\sigma = \left\{ q \in \mathbb{C}^n : |\mathrm{Im}\, q| < \sigma \right\} .$$

For real vectors $x \in \mathbb{R}^n$ we use

$$|x| = \sum_{j=1}^{n} |x_j| .$$

For an analytic function $f(q,p)$ with $q \in \mathbb{T}^n$ we use the weighted Fourier norm

$$\|f\|_{\rho,\sigma} = \sum_{k\in\mathbb{Z}^n} |f_k|_\rho e^{|k|\sigma} , \quad |f_k|_\rho = \sup_{p\in\Delta_\rho(0)} |f_k(p)| .$$

Quantitative estimates: generating functions and Lie series

The Cauchy estimates require a restriction of domains from $\mathcal{D}_{(\rho,\sigma)}$ to $\mathcal{D}_{(1-d)(\rho,\sigma)}$.
We need an infinite sequence of restriction $d_1 < d_2 < d_3 < \ldots$ tending to $d < 1$.
We introduce the sequence

$$\delta_r = \frac{1}{2\pi^2 r^2} \ , \quad \sum_{r>0} \delta_r = \frac{1}{12} \quad \text{and we set} \quad d_0 = 1 \ , \quad d_r = \sum_{j=1}^{r} 2\delta_r \ .$$

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$$\|\chi_0^{(r)}\|_{1-d_{r-1}} \le \frac{\|f_0^{(r-1,r)}\|_{1-d_{r-1}}}{\alpha_r}\ ,\quad \|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} \le \frac{\|\hat{f}_1^{(r-1,r)}\|_{1-d_{r-1}-\delta_r}}{\alpha_r}\ .$$

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Lemma Let $d > 0$, $d' \geq 0$ and $d + d' < 1$; let χ and g be two analytic functions on $\mathcal{D}_{(1-d')(\rho,\sigma)}$. Then, for $j \geq 1$, we have

$$\frac{1}{j!} \left\| L_\chi^j g \right\|_{1-d-d'} \leq \frac{1}{e^2} \left(\frac{2e}{\rho\sigma} \right)^j \frac{1}{d^{2j}} \|\chi\|_{1-d'}^j \|g\|_{1-d'} \ .$$

Estimates for the intermediate Hamiltonian

$$G_{r,0} = \frac{2}{e\rho\sigma} \| f_0^{(r-1,r)} \|_{1-d_{r-1}}$$

$$\| \hat{f}_0^{(r,s)} \|_{1-d_{r-1}-\delta_r} \le \begin{cases} \| f_0^{(r-1,s)} \|_{1-d_{r-1}} \ , & r < s < 2r \ ; \\ \| f_0^{(r-1,s)} \|_{1-d_{r-1}} \\ \qquad + \frac{G_{r,0}}{\delta_r^2 \alpha_r} \| f_1^{(r-1,s-r)} - \langle \omega_{s-r}, p \rangle \|_{1-d_{r-1}} \\ \qquad + \left(\frac{G_{r,0}}{\delta_r^2 \alpha_r} \right)^2 \| f_2^{(r-1,s-2r)} \|_{1-d_{r-1}} \ , & s \ge 2r \ . \end{cases}$$

$$\| \hat{f}_1^{(r,s)} \|_{1-d_{r-1}-\delta_r} \le \| f_1^{(r-1,s)} \|_{1-d_{r-1}} + \frac{G_{r,0}}{\delta_r^2 \alpha_r} \| f_2^{(r-1,s-r)} \|_{1-d_{r-1}} \ , \quad s \ge r \ .$$

$$\| \hat{f}_2^{(r,s)} \|_{1-d_{r-1}-\delta_r} = \| f_2^{(r-1,s)} \|_{1-d_{r-1}}$$

Estimates for the Hamiltonian in normal form up to order r

$$G_{r,1} = \frac{2}{e\rho\sigma} \|\hat{f}_1^{(r-1,r)}\|_{1-d_{r-1}-\delta_r}$$

$$\|f_0^{(r,s)}\|_{1-d_r} \leq \sum_{j=0}^{k-1} \left(\frac{G_{r,1}}{\delta_r^2 \alpha_r}\right)^j \|\hat{f}_0^{(r,s-jr)}\|_{1-d_{r-1}-\delta_r}$$

$$\|f_1^{(r,s)}\|_{1-d_r} \leq \sum_{j=0}^{k-1} \left(\frac{G_{r,1}}{\delta_r^2 \alpha_r}\right)^j \|\hat{f}_1^{(r,s-jr)} - \langle \omega_{s-jr}, p\rangle\|_{1-d_{r-1}-\delta_r} \ ,$$

$$\|f_2^{(r,s)}\|_{1-d_r} \leq \sum_{j=0}^{k} \left(\frac{G_{r,1}}{\delta_r^2 \alpha_r}\right)^j \|\hat{f}_2^{(r,s-jr)}\|_{1-d_{r-1}-\delta_r}$$

Small divisors

All estimates exhibit a common structure: sums of powers of $G_{r,0}/(\delta_r^2\alpha_r)$ or $G_{r,1}/(\delta_r^2\alpha_r)$ multiplied by the known norm of some functions.

Thus, we are naturally led to consider the quantities

$$\beta_0 = 1\ , \quad \beta_r = \delta_r^2\alpha_r\ .$$

These are the *small divisors* that we are going to carefully analyze.

The unknown detunings...

A crucial difference with respect to the classical Kolmogorov's normal form algorithm is the presence of the corrections $\{\omega_s\}$.

The detunings $\{\omega_s\}_{s\le r}$ are determined and remain unchanged by the subsequent steps.

The detunings $\{\omega_s\}_{s>r}$ are still unknowns to be determined, but $\langle\omega_s, p\rangle$ are always paired with $f_1^{(*,s)}$ and by construction they simply cancel out the average term, thus do not play any role in the normal form estimates.

Condition $\boldsymbol{\tau}$ (non resonant condition); Giorgilli, Marmi, DCDS (2010)

$$-\sum_{r\geq 1}\frac{\ln\alpha_r}{r(r+1)}=\Gamma<\infty\ .$$

Condition $\boldsymbol{\tau}$ is weaker than the Diophantine condition and equivalent to Bruno's one, indeed $\Gamma\leq \mathrm{B}\leq 2\Gamma$.

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Small divisors and indeces

The quantities β_r are the actual **small divisors** that we must deal with.

$$\beta_{j_1}\cdots\beta_{j_q} \qquad \Longleftrightarrow \qquad j_1,\ldots,j_q\ .$$

The problem is to identify the **worst product** among them. The **key** of our argument is to focus our attention on the **indeces** rather than on the actual values of the divisors.

Selection rule

We call $I = \{j_1, \ldots, j_s\}$ a *list of indeces*.

Partial ordering: $I \triangleleft I'$ in case there is a permutation of the indeces such that the relation $j_m \le j'_m$ holds true for $m = 1, \ldots, s$.

We define the special lists with $s \ge 0$

$$I^*_s = \left(\left\lfloor \frac{s}{s} \right\rfloor, \left\lfloor \frac{s}{s-1} \right\rfloor, \ldots, \left\lfloor \frac{s}{2} \right\rfloor \right) .$$

The **allowed combinations of small divisors** are described by

$$\mathcal{J}_{r,s} = \left\{ I = \{j_1, \ldots, j_{s-1}\} \ :\ j_m \in \{0, \ldots, \min(r, \lfloor s/2 \rfloor)\} \,,\ I \triangleleft I^*_s \right\} .$$

The condition $I \triangleleft I^*_s$ is the *selection rule* $\boldsymbol{S}$.

Accumulation of small divisors

We associate to the sets of lists $\mathcal{J}_{r,s}$ the sequence of positive real numbers $T_{r,s}$ defined as

$$T_{0,s} = 1 \ , \quad T_{r,s} = \max_{I \in \mathcal{J}_{r,s}} \prod_{j \in I} \frac{1}{\beta_j} \ , \quad 0 < r \leq s \ .$$

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Selection rule for the Kolmogorov's algorithm

Function	# of indices	Selection rule	Bounded by
$f_2^{(r,s)}$	2s	$\{r\} \cup \{r\} \cup \mathcal{J}_{r,s} \cup \mathcal{J}_{r,s}$	$\frac{1}{\beta_r^2} T_{r,s}^2$
$f_1^{(r,s)}$	2s-1	$\{r\} \cup \mathcal{J}_{r,s} \cup \mathcal{J}_{r,s}$	$\frac{1}{\beta_r} T_{r,s}^2$
$f_0^{(r,s)}$	2s-2	$\mathcal{J}_{r,s} \cup \mathcal{J}_{r,s}$	$T_{r,s}^2$

Lemma

Let the sequence $\{\alpha_r\}_{r\geq 1}$ satisfy condition $\boldsymbol{\tau}$ and consider the previously defined sequence $\{\delta_r\}_{r\geq 1}$. Then, the sequence $\{T_{r,s}\}_{r\geq 0\,,\,s\geq 0}$ defined above is bounded by

$$T_{r,s} \leq \frac{1}{\alpha_s \delta_s^2} T_{r,s} \leq \left(2^{15} e^{\Gamma}\right)^s \qquad \text{for } r \geq 1\,,\ s \geq 1\ .$$

Conclusion of the proof

We have bounded geometrically the accumulation of the small divisors.

It is now just a matter of patience to perform all the needed computations:

- bound the number of summands involved in the recursive formulæ;
- bound the quantities $G_{r,0}$ and $G_{r,1}$;
- bound the norms of known functions.

All these contributions are bounded geometrically.

Hence the infinite sequence of canonical transformations produces a Hamiltonian in Kolmogorov's normal form.

Conclusions

Completely constructive approach for full dimensional invariant tori:

- Kolmogorov-like normalization algorithm;
- prescription of the *final frequencies*;
- the initial frequencies are determined *a posteriori*;
- proof based on classical expansions in powers of ε, which in this case are more convenient than the quadratic method.

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- Kolmogorov-like normalization algorithm;
- prescription of the *final frequencies*;
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The Goal

Provide a constructive proof of the existence of lower dimensional elliptic tori, improving the control on the frequencies.

Can we fix both the final *internal* and *transversal* frequencies? (stay tuned!)

Thanks for your attention