Delay perturbations of an ODE

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Overview of the talk

- We show the results in $\mathbb{R}^2$, but they can be generalized to $\mathbb{R}^n$.
- We prove the existence of limit cycles under perturbations as well as their isochrons.
- We provide *a-posteriori* theorems.
- We detail the algorithms for its implementation.
- We see the composition of functions have an important role.
- We bypass the notion of phase space in the delay context.

J. Yang and J.G. and R. de la Llave.
Parameterization method for state-dependent delay perturbation of an ordinary differential equation.
https://arxiv.org/abs/2005.06084.

J.G. and J. Yang and R. de la Llave.
Numerical computation of periodic orbits and isochrons for state-dependent delay perturbation of an ODE in the plane.
https://arxiv.org/abs/2005.06086.

Basics on DDEs

A DDEs with constant delay is a differential equation of the form

$$\dot{x}(t) = X(x(t-1)). \tag{1}$$

- To consider an IVP is not enough to give just a point.
- Several phenomena may happen e.g.:
  - Lack of smoothness, injectivity, uniqueness.
  - Termination of the solution.
  - Change of stability or boundedness.
  - Accuracy and stability properties of the underlying ODE integrator may be destroyed.
- The stability of p.o. are given by compact operators $\Rightarrow$ stable manifold is ∞-dim. and unstable manifold is finite dim.

Reviewing the parameterization method

Let us consider a smooth ODE in $\mathbb{R}^2$

$$\dot{x} = X_0(x). \tag{2}$$

Assume a periodic orbit K_0 with frequency ω_0. That is,

$$x(t) = K_0(\theta_0 + \omega_0 t) \tag{3}$$

solves (2). Explicitly,

$$\omega_0 \partial_\theta K_0(\theta) = X_0 \circ K_0(\theta), \quad \theta(t) = \theta_0 + \omega_0 t \in \mathbb{R}, \forall\, t. \tag{4}$$

Assume that K_0 is in fact an stable limit cycle

$\Downarrow$

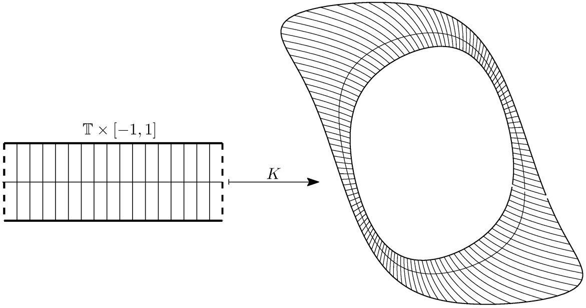
There is a neighborhood around K_0 where the orbits tend to K_0 asymptotically. $\rightsquigarrow (K, \omega_0, \lambda_0)$.

That is,

$$x(t) = K(\theta_0 + \omega_0 t, e^{\lambda_0 t} s_0), \qquad \lambda_0 < 0 \tag{5}$$

solves the ODE (2). Explicitly,

$$DK(\theta_0, s_0)\begin{pmatrix}\omega_0\\ \lambda_0 s_0\end{pmatrix} = (\omega_0\partial_\theta + \lambda_0 s_0 \partial_s)K(\theta_0, s_0) = X_0 \circ K(\theta_0, s_0),$$



Perturbation

$$\dot{x} = X_0(x) + \varepsilon P(x, \widetilde{x}, \varepsilon), \qquad 0 \leq \varepsilon \ll 1, \tag{6}$$

where

$$\widetilde{x}(t) = x(t - r(x)).$$

- (6) includes the case $\dot{x} = X(x, \varepsilon\widetilde{x})$ taking $\varepsilon P = X - X_0$.
- We assume r smooth ($\Rightarrow$ locally bounded).

We want to find (W, ω, λ) such that

$$x(t) = K \circ W(\theta + \omega t, e^{\lambda t} s) \tag{7}$$

is a solution of (6). Note that in that case

$$\widetilde{x}(t) = K \circ W(\theta + \omega t - \omega r \circ K \circ W, e^{\lambda t - \lambda r \circ K \circ W} s). \tag{8}$$

Invariance equation

Using that $DK \circ W \begin{pmatrix} \omega_0 \\ \lambda_0 W_2 \end{pmatrix} = X_0 \circ K \circ W$, then

$$
\begin{aligned}
DK \circ WDW \begin{pmatrix} \omega \\ \lambda s \end{pmatrix} &= X_0 \circ K \circ W + \varepsilon P(K \circ W, K \circ \widetilde{W}, \varepsilon) \\
&= DK \circ W \begin{pmatrix} \omega_0 \\ \lambda_0 W_2 \end{pmatrix} + \varepsilon P(K \circ W, K \circ \widetilde{W}, \varepsilon).
\end{aligned}
$$

Multiplying by $(DK \circ W)^{-1}$,

Invariance equation

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Multiplying by $(DK \circ W)^{-1}$,

$$(\omega \partial_\theta + \lambda s \partial_s) W = \begin{pmatrix} \omega_0 \\ \lambda_0 W_2 \end{pmatrix} + \varepsilon Y(K \circ W, K \circ \widetilde{W}, \varepsilon)$$

with $Y = (DK \circ W)^{-1} P$ and

$$\widetilde{W}(\theta, s) = W(\theta - \omega r \circ K \circ W, e^{-\lambda r \circ K \circ W} s). \tag{9}$$

Remarks and strategy on the invariance equation

$$(\omega\partial_\theta + \lambda s\partial_s)W = \begin{pmatrix}\omega_0 \\ \lambda_0 W_2\end{pmatrix} + \varepsilon Y(K\circ W, K\circ \widetilde{W}, \varepsilon) \qquad (10)$$

- Problems on the domain definition of W.
- The key is that Y is governed by ε.
- Y may be very complicated due to $\widetilde{W}$ and $K\circ\cdot$.
- The uniqueness is up to a shift and a scaling:

  $W(\theta, s)$ is a solution $\Rightarrow W(\theta+\sigma, \eta s)$ too.
- We fix the solution by two normalizations.

Remarks and strategy on the invariance equation

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The strategy

Plug

$$W(\theta, s) = W^0(\theta) + W^1(\theta)s + \sum_{j=2}^{N-1} W^j(\theta)s^j + W^>(\theta, s)$$

into (10) and do power matching.

0^{th} order $\rightsquigarrow (W^0, \omega)$

$$\left[\omega\partial_\theta - \begin{pmatrix} 0 & 0 \\ 0 & \lambda_0 \end{pmatrix}\right] W^0(\theta) = \begin{pmatrix} \omega_0 \\ 0 \end{pmatrix} + \varepsilon Y^0(\theta, \omega, W^0, \widetilde{W}^0, \varepsilon).$$

- Very nonlinear equation.
- Condition for ω via periodicity on W^0.

1^{st} order $\rightsquigarrow (W^1, \lambda)$

$$\left[\omega\partial_\theta + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda - \lambda_0 \end{pmatrix}\right] W^1(\theta) = \varepsilon Y^1(\theta, \lambda, W^0, W^1, \widetilde{W}^1, \varepsilon).$$

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- Eigenvalue problem.
- Condition for λ via a normalization.

0^{th} order $\rightsquigarrow (W^0, \omega)$

$$\left[\omega\partial_\theta - \begin{pmatrix} 0 & 0 \\ 0 & \lambda_0 \end{pmatrix}\right] W^0(\theta) = \begin{pmatrix} \omega_0 \\ 0 \end{pmatrix} + \varepsilon Y^0(\theta, \omega, W^0, \widetilde{W}^0, \varepsilon).$$

1^{st} order $\rightsquigarrow (W^1, \lambda)$

$$\left[\omega\partial_\theta + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda - \lambda_0 \end{pmatrix}\right] W^1(\theta) = \varepsilon Y^1(\theta, \lambda, W^0, W^1, \widetilde{W}^1, \varepsilon).$$

j^{th} order $\rightsquigarrow W^j$

$$\left[\omega\partial_\theta + \begin{pmatrix} \lambda j & 0 \\ 0 & \lambda j - \lambda_0 \end{pmatrix}\right] W^j(\theta) = \varepsilon Y^j(\theta, W^0, W^j, \widetilde{W}^j, \varepsilon) + \varepsilon R^j(\theta; \lambda).$$

Periodicity and normalization

Since W must be composed with K, then

$$W(\theta + 1, s) = W(\theta, s) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{11}$$

To fix W^1, we choose

$$\int_0^1 W_2^1(\theta)\, d\theta = 1. \tag{12}$$

Tools for the *a-posteriori* theorems

- For each order, we define spaces, norms and operators Γ.
- We see that the operators Γ map a ball in $C^{\ell+lip}$ to itself.
- We see that Γ is a C^0-contraction.
- Then by Lanford's lemma, the limit of Γ exists in $C^{\ell+lip}$.
- In the process we keep track of the conditions of ε.
- We get bounds in $C^{\ell-1}$ of the error w.r.t. the initial guess by interpolation inequality.
- We formulate the theorem in *a-posteriori* format.

⋆ The 0^{th} and 1^{st} are special.

⋆ The j^{th} is essentially induction.

⋆ The higher order requires extra work.

Numerical implementation

$$W(\theta, s) = W^0(\theta) + W^1(\theta)s + \sum_{j=2}^{N-1} W^j(\theta)s^j + W^>(\theta, s).$$

- Taylor-Fourier numerical representation.
- Cohomological equations of the form

$$\left[\omega\partial_\theta + \begin{pmatrix} \lambda s\partial_s & 0 \\ 0 & \lambda s\partial_s - \lambda_0 \end{pmatrix}\right] W = E$$

  are easy in Fourier.
- Due to the normalization we do not have small divisors.
- Solving linear system of the form $A(\theta, s)\boldsymbol{x}(\theta, s) = \boldsymbol{b}(\theta, s)$ by power matching.
- The computation $Y^j = \frac{\partial^j Y}{\partial s^j}\Big|_{s=0}$ is easy in Taylor via *automatic differentiation*.
- $K \circ W$ and $\widetilde{W}$ involve composition between Taylor-Fourier.

Composition in Taylor-Fourier

If $K(\theta, s) = \sum_{j=0}^{m-1} K^j(\theta) s^j$ and $W(\theta, s) = \sum_{j=0}^{k-1} W^j(\theta) s^j$, then

$$K \circ W(\theta, s) = \sum_{j=0}^{m-1} K^j(W_1(\theta, s))(W_2(\theta, s))^j$$

If we consider n modes of Fourier, then

- Computational complexity $\Theta(mkn^2 + mk^2 n)$ using AD.
- $\Theta(mk^4 n + mkn \log n)$ and space complexity $\Omega(kn + k^2)$ by dynamic programming.

Similarly for

$$\widetilde{W}(\theta, s) = W(\theta - \omega r \circ K \circ W, e^{-\lambda r \circ K \circ W} s)$$

and $K \circ \widetilde{W}$.

AD case

Given

$$S(\theta) = \sum_{k=0}^{n-1} a_k \cos(2\pi k\theta) + b_k \sin(2\pi k\theta).$$

and a polynomial $q(s) = q_0 + \cdots + q_l s^l$.
The composition $S \circ q$ can be computed by the recurrences $j \geq 1$,

$$s_0 = \sin 2\pi k q_0, \qquad c_0 = \cos 2\pi k q_0,$$

$$s_j = \frac{2\pi k}{j} \sum_{i=0}^{j-1} (j-i) q_{j-i} c_i, \qquad c_j = -\frac{2\pi k}{j} \sum_{i=0}^{j-1} (j-i) q_{j-i} s_i.$$

Similar recurrences for exp, log, ...

Scaling factor and well-definition of composition with K

If $W(\theta, s)$ is a solution, $W(\theta, bs)$ too with $b > 0$.

Classically we choose b in such a way all the W^j has similar size.

Since $K\colon \mathbb{T} \times [-1, 1] \to \mathbb{R}^2$, then to ensure $K \circ W$ and $K \circ \widetilde{W}$ are well-defined, we impose

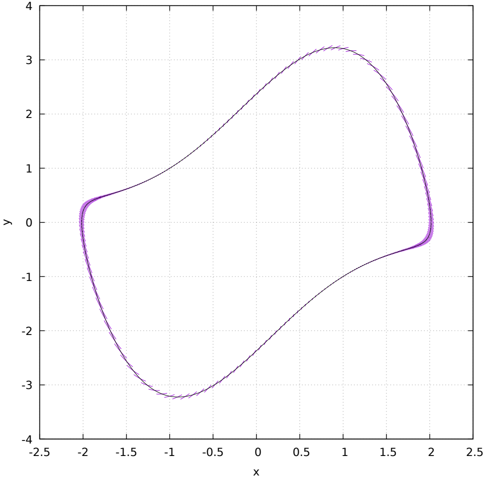
$$\|W_2(\theta, s)\| \leq 1 \qquad \text{and} \qquad \|\widetilde{W_2}(\theta, s)\| \leq 1.$$

by choosing a suitable b.

The existence of such a b is guaranteed if the 0^{th} has been computed successfully.

Example

$$\dot{x}(t) = y(t)$$
$$\dot{y}(t) = 1.5(1 - x(t)^2)y(t) - x(t) + \varepsilon x(t - r(x))$$



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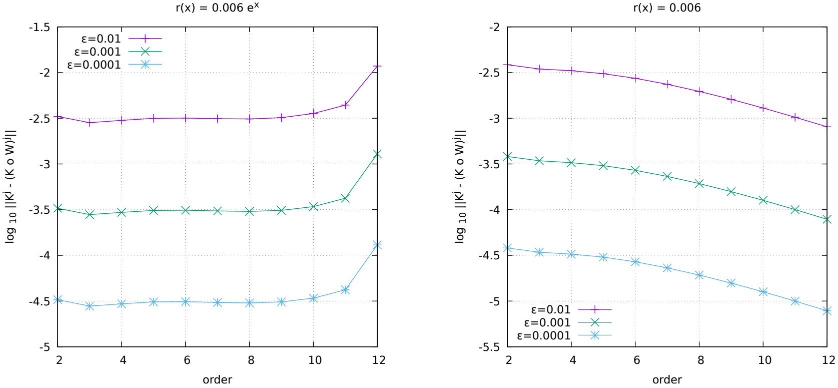

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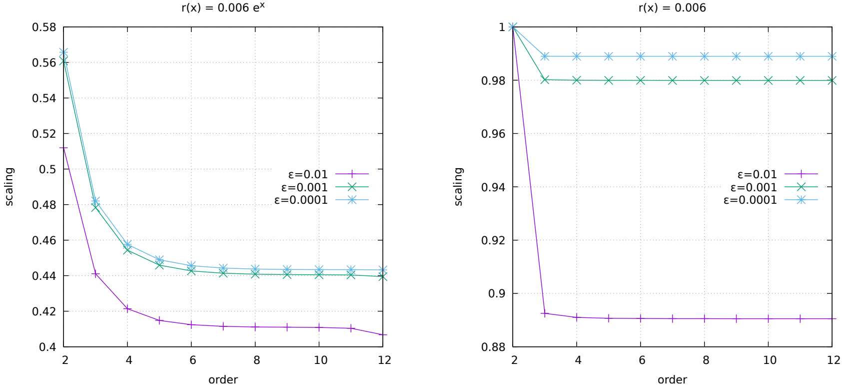

Conclusions

- ✓ It only requires mild assumptions.
- ✓ It bypasses to specify the phase space.
- ✓ It allows to study for small delays.
- ✓ It allows to perform CAPs.
- ✓ It extends the parameterization method of ODEs.
- ✓ It works for delay, advanced or even mixed differential equations.
- ✓ It can be extended for $\mathbb{R}^n$ and PDEs.
- ➔ Globalization is not immediate.
- ➔ Only finite differentiability of the solutions.

Future results

All periodic orbit in a smooth n-dim. ODE with 1 *as simple eigenvalue in its monodromy matrix persists under delay or advance perturbations.*

THANK YOU