# Librational KAM tori in the secular dynamics of Upsilon–Andromedæ planetary system

Chiara Caracciolo

In collaboration with
U. Locatelli[1], M. Sansottera[2], M. Volpi[1]
[1] University of Rome "Tor Vergata", [2] University of Milan

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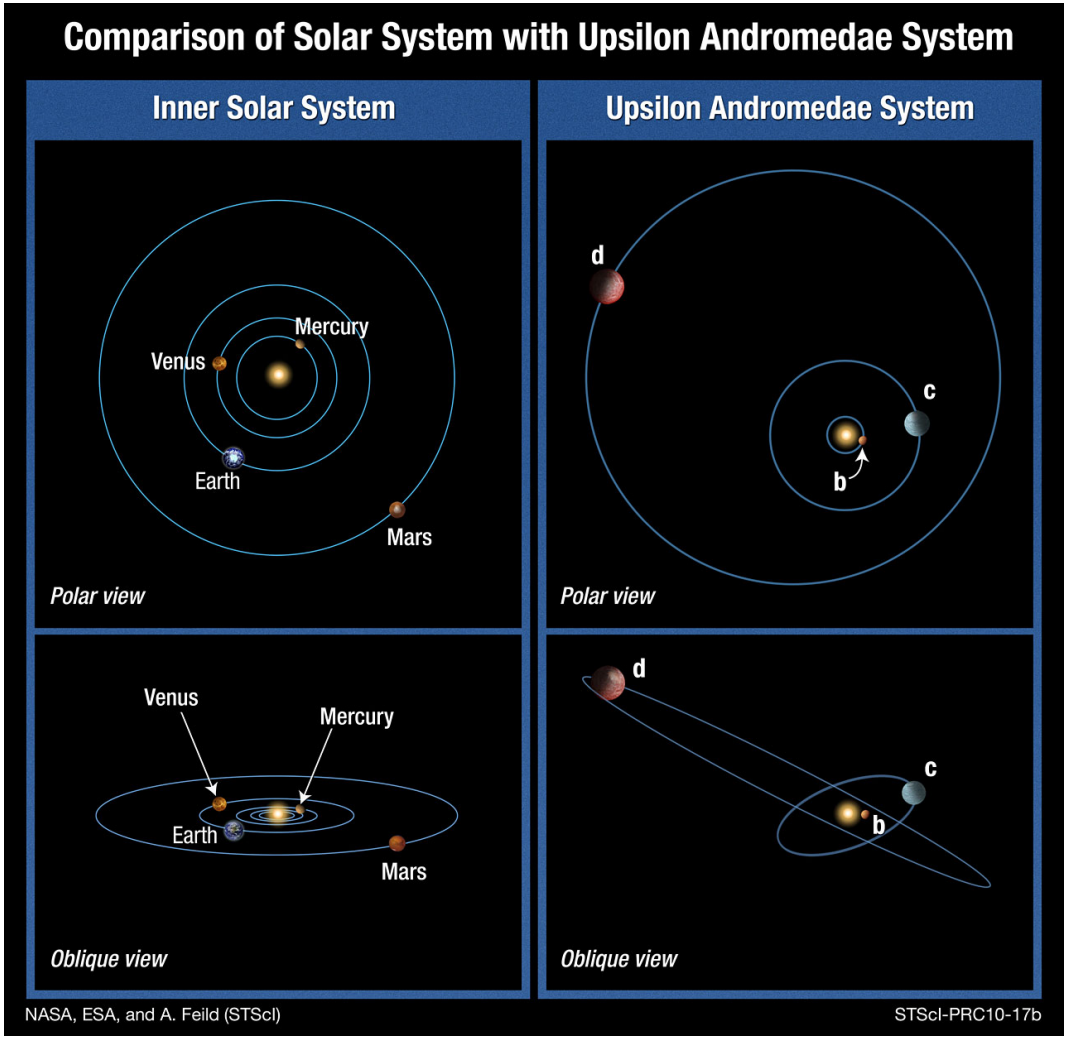
# $\upsilon$–Andromedæ planetary system

## Why this particular one?

- The two most massive Jovian planets have large masses ($m_1 \simeq 14 M_J$, $m_2 \simeq 10 M_J$) and they are orbiting relatively close to each other ($a_1 \simeq 0.8$ AU, $a_2 \simeq 2.5$ AU).
- Eccentric orbits: $e_1 \simeq 0.245$, $e_2 \simeq 0.316$.
- Large (known) mutual inclination: $i \simeq 30^\circ$.

**Is this system stable?**
**The Kepler motion is strongly perturbed!**



# Planetary three-body problem

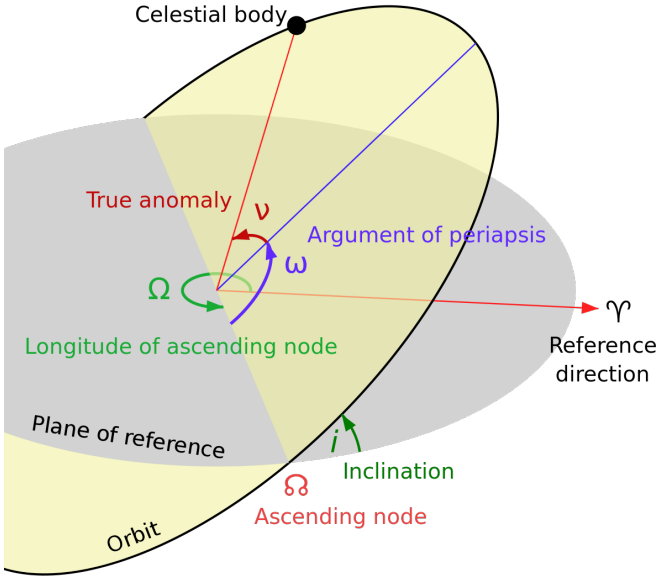
In an heliocentric reference frame, using the conservation of the total angular momentum $\boldsymbol{C} \Rightarrow 4$ degrees of freedom. Using Poincaré variables ($j = 1, 2$):

$$\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{(m_0 + m_j)a_j}, \qquad \lambda_j = M_j + \omega_j$$

$$\xi_j = \sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\cos(\omega_j)$$

$$\eta_j = -\sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\sin(\omega_j)$$

$$i_1 + i_2 = \arccos\left(\frac{C^2 - \Lambda_1^2(1 - e_1^2) - \Lambda_2^2(1 - e_2^2)}{2\Lambda_1\Lambda_2\sqrt{1 - e_1^2}\sqrt{1 - e_2^2}}\right)$$




# Hamiltonian

Expanded around fixed observed semi-axes $a_j^*$ ($\boldsymbol{L} = \boldsymbol{\Lambda} - \boldsymbol{\Lambda}^*$), it has the form

$$H(\boldsymbol{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \overbrace{K(\boldsymbol{L})}^{\text{Keplerian}} + \mu \overbrace{F(\boldsymbol{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})}^{\text{Interaction}}$$

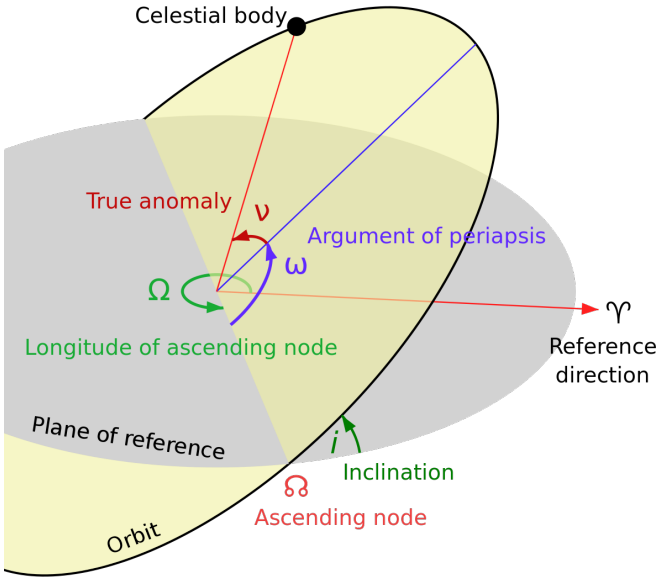
$$\Rightarrow \begin{cases} \dot{\boldsymbol{\lambda}} = \dfrac{\partial H}{\partial \boldsymbol{L}} & \text{fast variables} \\ \dot{\boldsymbol{\xi}} = \mu \dfrac{\partial F}{\partial \boldsymbol{\eta}} & \text{secular variables} \\ \dot{\boldsymbol{\eta}} = -\mu \dfrac{\partial F}{\partial \boldsymbol{\xi}} & \text{secular variables} \end{cases}$$

where $\mu = \max\{m_1/m_0, m_2/m_0\} \simeq 10^{-2}$.

# Secular model

$$H(\boldsymbol{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}) = \overbrace{\boldsymbol{n}^*\boldsymbol{L} + O(\boldsymbol{L}^2)}^{\text{Keplerian}} + \mu \overbrace{F(\boldsymbol{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta})}^{\text{Interaction}}$$

We want to study long term stability: we consider the evolution of the secular variables, averaging over the fast angles $\boldsymbol{\lambda}$ and putting $\boldsymbol{L} = 0$.




## Order one in the masses

We just cut the terms depending on the angles:

$$H_{sec}(\boldsymbol{\xi},\boldsymbol{\eta}) = \langle H(\boldsymbol{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta})|_{\boldsymbol{L}=0}\rangle_{\boldsymbol{\lambda}}$$

$\Rightarrow$ we have in the expansion terms of order $\mu$ only.

# Secular model

$$H(\boldsymbol{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}) = \overbrace{\boldsymbol{n}^*\boldsymbol{L} + O(\boldsymbol{L}^2)}^{\text{Keplerian}} + \mu \overbrace{F(\boldsymbol{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta})}^{\text{Interaction}}$$

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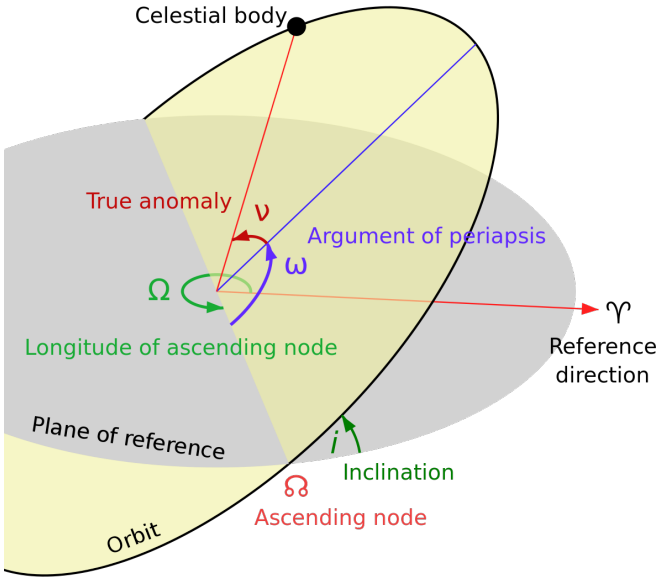



## Order two in the masses

We eliminate the dependence from the fast angles $\boldsymbol{\lambda}$ from the perturbation of order $\mu$ with a Kolmogorov's like step and take into account terms up to order $\mu^2$.
The secular Hamiltonian becomes

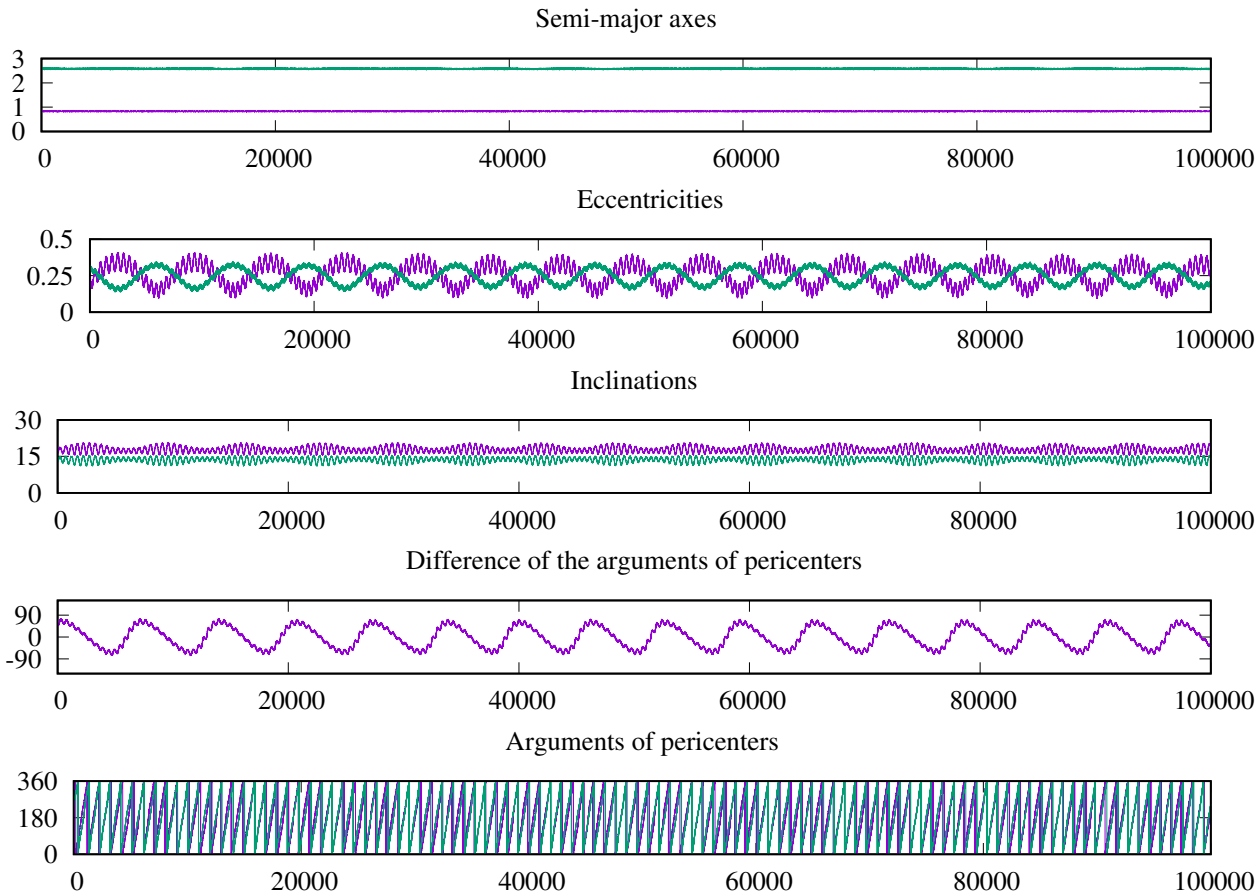
$$H_{\text{sec}}(\boldsymbol{\xi},\boldsymbol{\eta}) = \sum_{\ell=1}^{+\infty} h_{2\ell}(\boldsymbol{\xi},\boldsymbol{\eta})$$

- the expansion contains only even terms (**D'Alembert rules**);
- the origin is an elliptic equilibrium point.

# Numerical study: three-boby problem

## Apsidal locking

The difference of the arguments of the pericenters is librating around zero (with a wide angle $\simeq \pm 80°$).



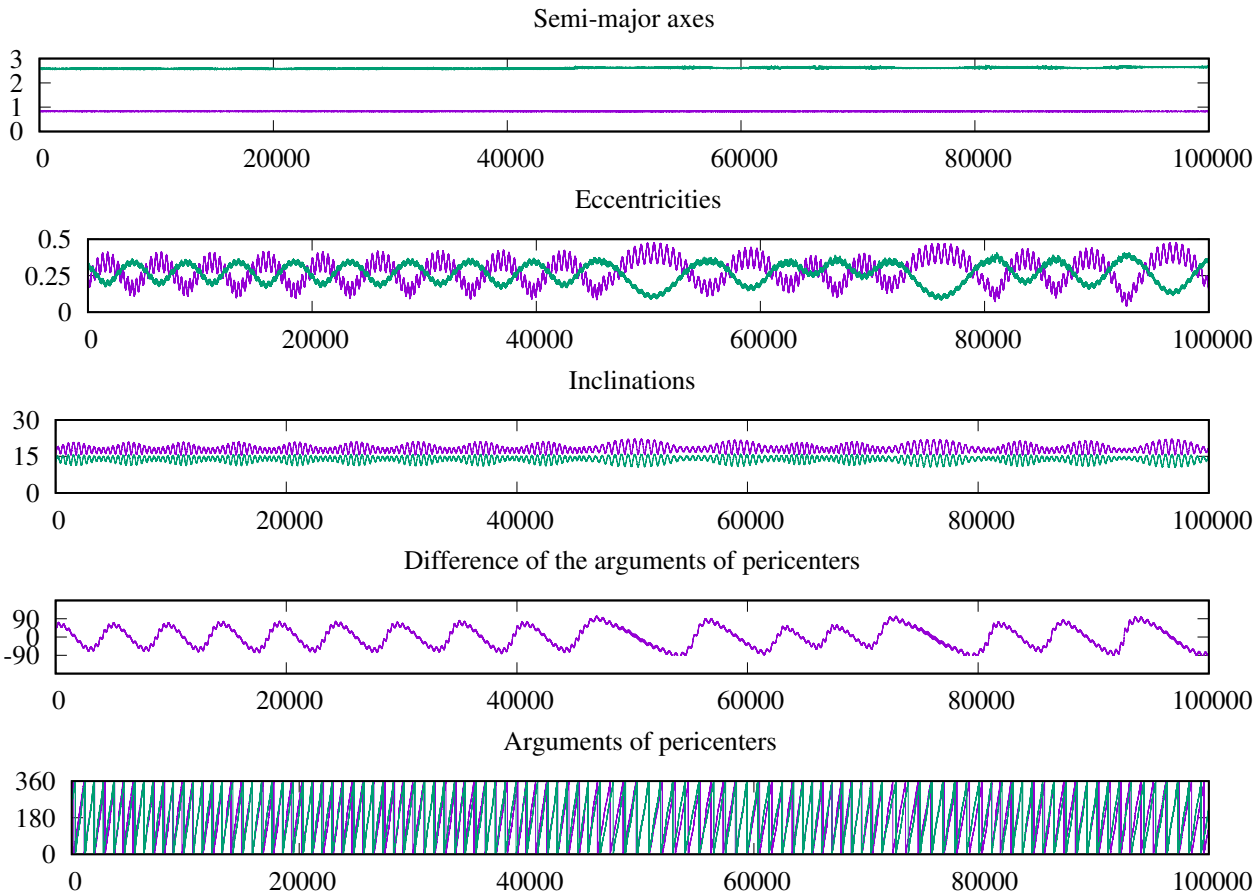
# Numerical study: three-boby problem

## Apsidal locking

The difference of the arguments of the pericenters is librating around zero (with a wide angle $\simeq \pm 80°$).

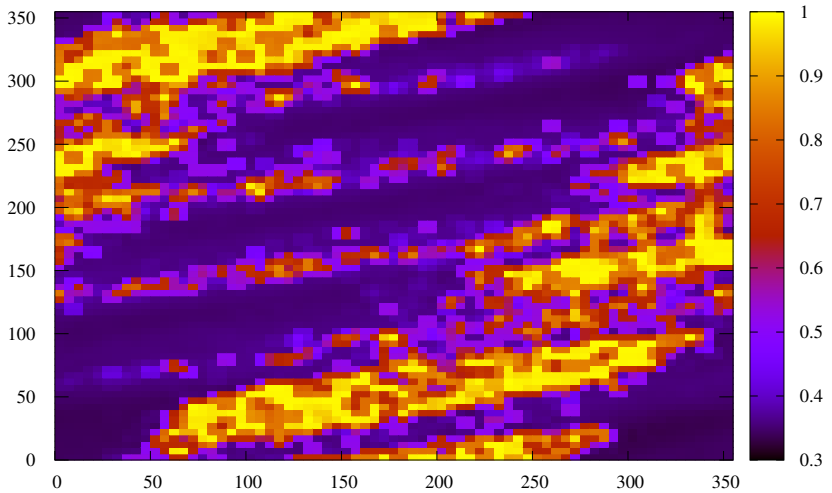
## Remark!

There are a lot of uncertainties and missing data from the observations. For some initial conditions the motion is unstable.

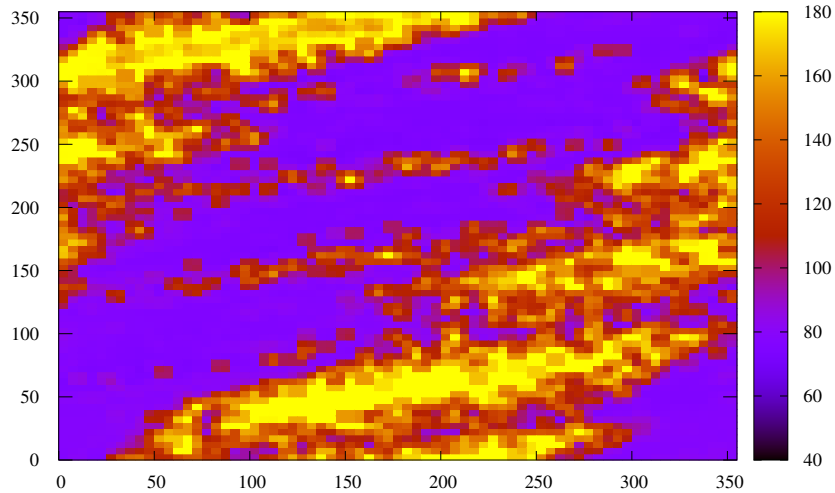


# Numerical study: search for more robust configurations

If we vary just the mean anomalies (unknown parameters) in a grid $360^\circ \times 360^\circ$ we obtain the following behavior:




Maximal eccentricity of the second planet.



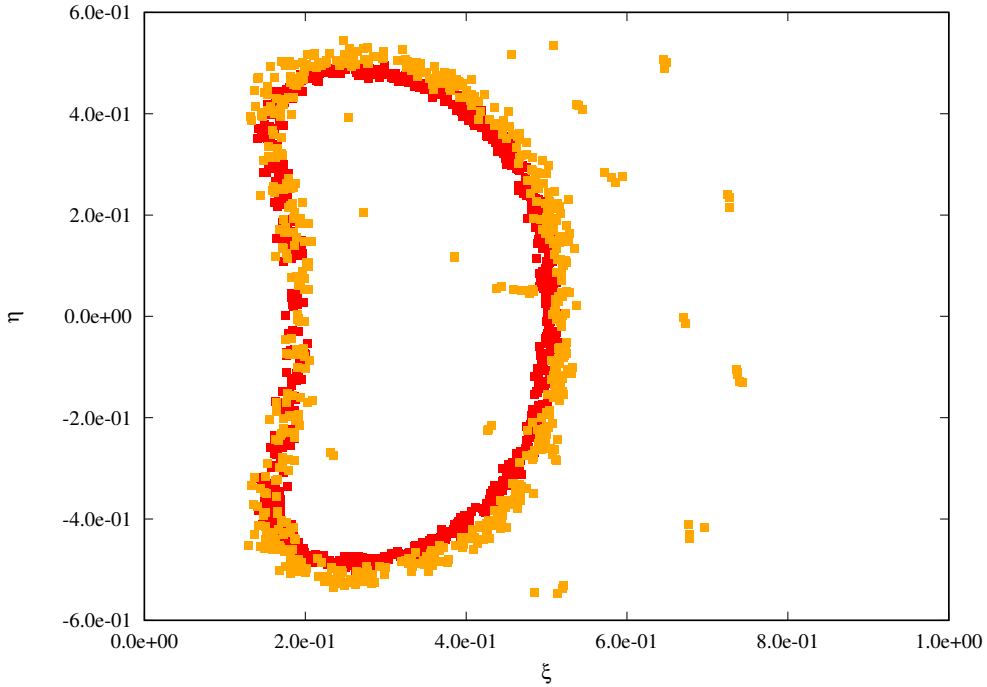

Maximal angle of libration.

# Numerical study: search for more robust configurations

$$\xi = \sqrt{2\Lambda^*}\sqrt{1-\sqrt{1-e_1^2}}\cos\left(\omega_1-\omega_2\right)$$

$$\eta = -\sqrt{2\Lambda^*}\sqrt{1-\sqrt{1-e_1^2}}\sin\left(\omega_1-\omega_2\right)$$

We expect that the configurations with a smaller libration angle and smaller variations of the eccentricities are more reliable.




## Criterion

We look for the initial conditions that minimize "area" estimated as

$$(e_{1max}-e_{1min})\frac{e_{1max}+e_{1min}}{2}2\Delta\omega_{max}.$$

being $\Delta\omega_{max}$ the maximal libration angle.

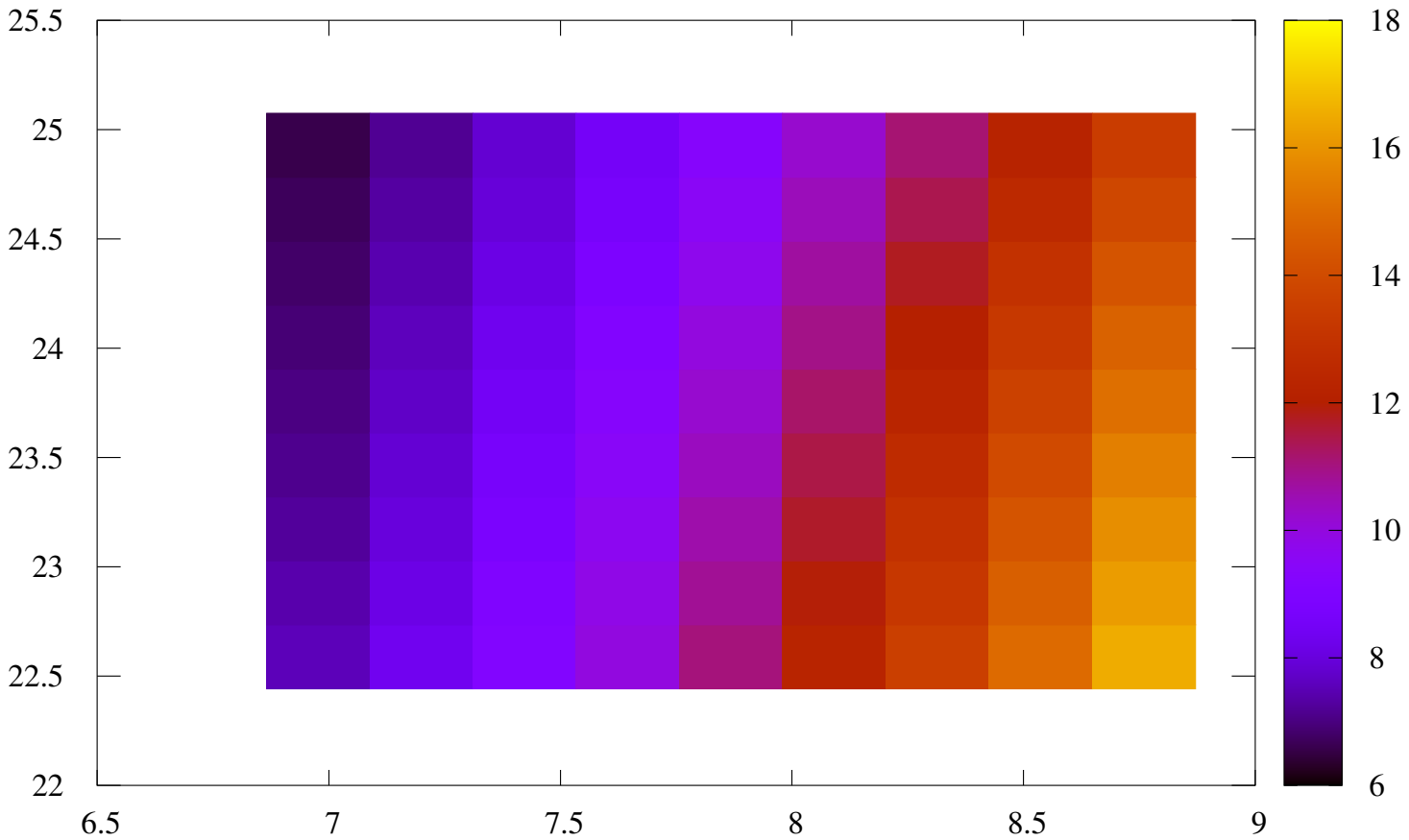
# Numerical study: search for more robust configurations

- Fixed semi-axes and eccentricities (small errors in the data):
  $a_1 = 0.829AU, \quad a_2 = 2.53AU, \quad e_1 = 0.239, \quad e_2 = 0.31.$

We vary the following initial conditions in the range of uncertainties given by the observations:

- Inclinations ($\pm 1^\circ$); as a consequence, the masses of the planets are:
  $m_1 \in (12.39, 15.98)M_J \quad m_2 \in (9.96, 11.05)M_J$
- Arguments of the pericenters and longitudes of the nodes, namely the initial relative orientation of the orbits.
  ($3 \times 3 \times 3 \times 3$ grid, with the minimal, central and maximal value of every element)
- Mean anomalies in a grid $360^\circ \times 360^\circ$.

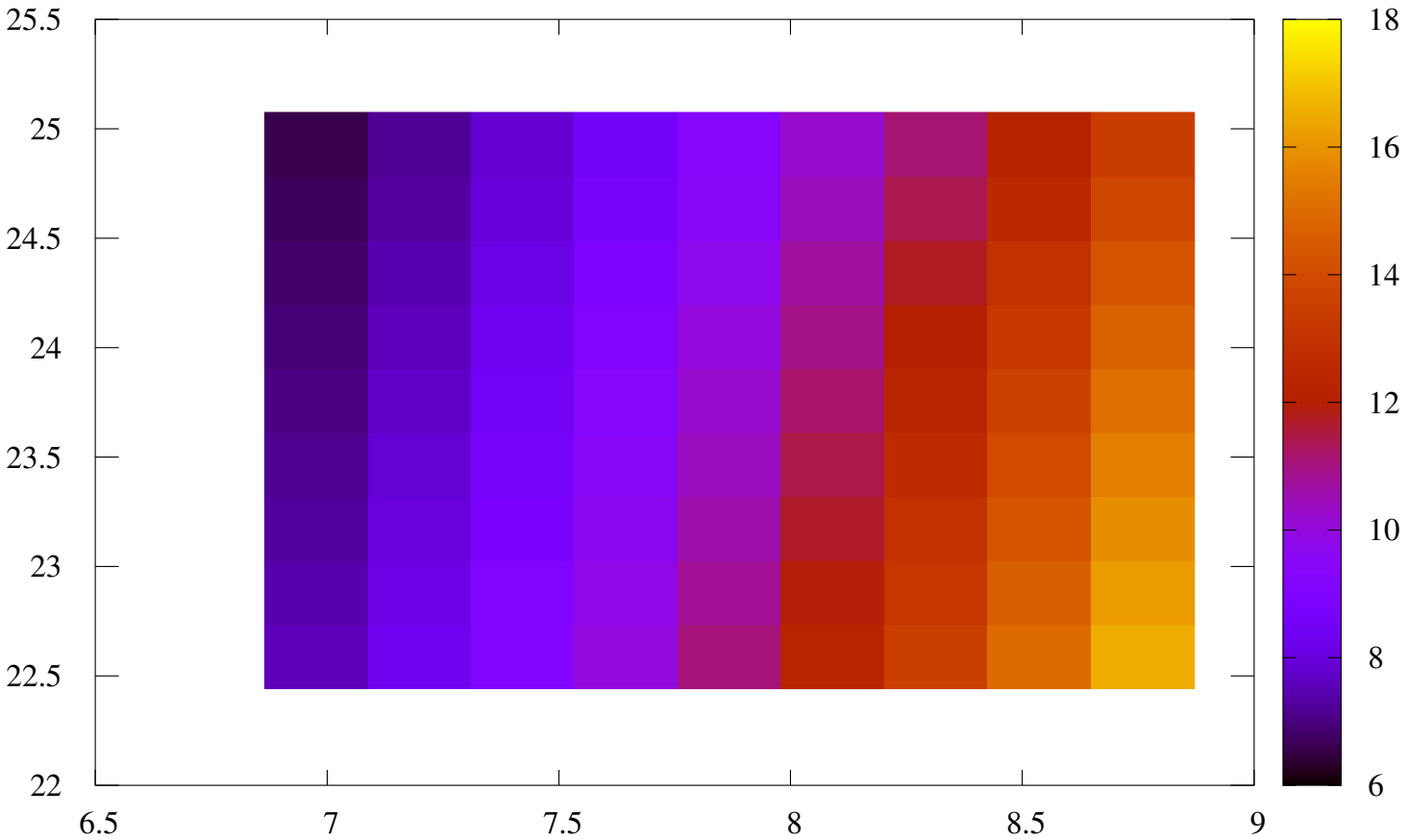
# Numerical study: search for more robust configurations




## Inclinations

It is preferable to minimize the inclination of the first planet and maximize the one of the second planet.

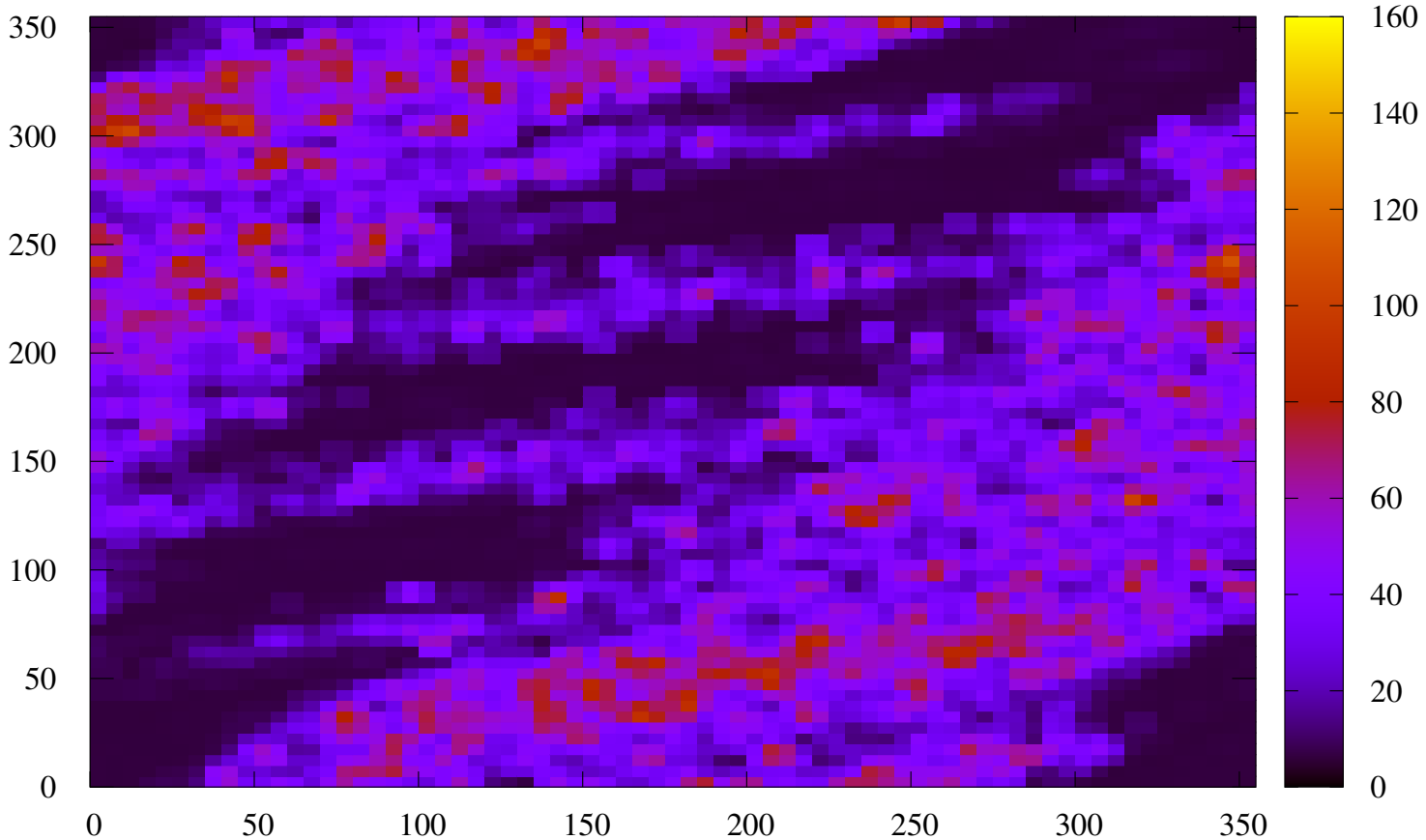
# Numerical study: search for more robust configurations




## Inclinations

This choice corresponds to a maximization of the mass of the first planet ($\simeq 16 M_J$) and a minimization of the one of the second planet ($\simeq 10 M_J$).

# Numerical study: search for more robust configurations




## Mean anomalies

We take $M_1 = 355^\circ$ and $M_2 = 335^\circ$ as initial values of the mean anomalies.

# Numerical study: search for more robust configurations

## Summary of the initial conditions

- Fixed semi-axes and eccentricities:
  $a_1 = 0.829AU, \quad a_2 = 2.53AU, \quad e_1 = 0.239, \quad e_2 = 0.31.$
- Inclinations and masses:
  $i_1 = 6.865^\circ, \quad i_2 = 25.074^\circ, \quad m_1 = 15.98 M_J \quad m_2 = 9.96 M_J.$
- Arguments of the pericenters and longitudes of the nodes:
  $\omega_1 = 245.8^\circ, \quad \omega_2 = 254.3^\circ, \quad \Omega_1 = 229.3^\circ, \quad \Omega_2 = 7.4^\circ.$
- Mean anomalies:
  $M_1 = 355^\circ, \quad M_2 = 335^\circ.$

With these initial conditions, the libration angle is $\simeq 50^\circ$ and we have decreased the excursion in eccentricity of the inner planet.

# Numerical study: secular model of order 2 in the masses

## New action-angle variables adapted to this kind of motion

If $(\boldsymbol{J}, \boldsymbol{\omega})$ is the action-angle couple related to the secular motion, being $J_j = \Lambda_j^*(1 - \sqrt{1 - e_j^2})$ and $\omega_j$ the arguments of the pericenters, we define $(\boldsymbol{I}, \boldsymbol{\varphi})$ as

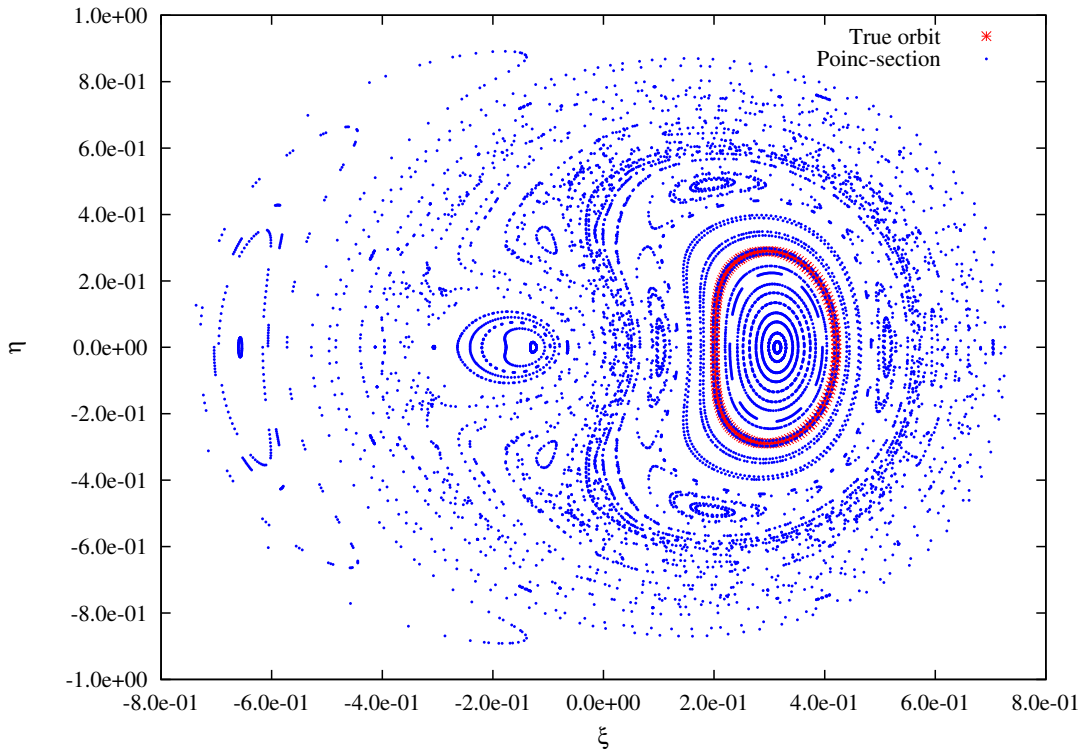
$$\begin{cases} \varphi_1 = \omega_1 - \omega_2, & \varphi_2 = \omega_2, \\ I_1 = J_1, & I_2 = J_1 + J_2. \end{cases}$$

## Poincaré section

We plot the variabiles:

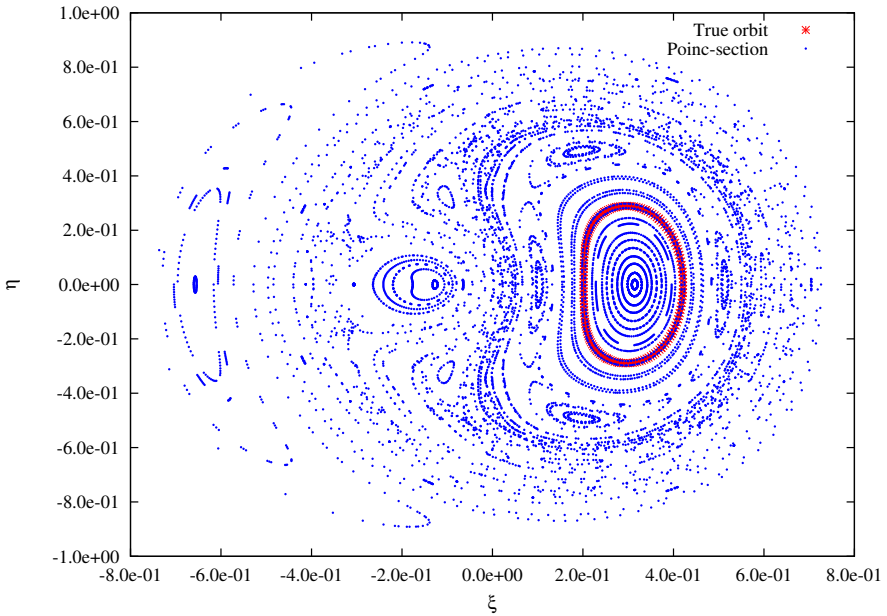
$$\begin{cases} \xi = \sqrt{2I_1}\cos\varphi_1 \\ \eta = \sqrt{2I_1}\sin\varphi_1 \end{cases}$$

every time the angle $\varphi_2 = 0$ (it is rotating).



# Our strategy

We start by constructing the lower dimensional (1−D) elliptic torus, i.e., the solution which corresponds to the configuration with the pericenters perfectly anti-aligned ($\varphi_1 \equiv 0$).



*From Arnold, Kozlov, Neishtadt, Springer ed. (2006).*

# Preliminary set up

Translations to be centered around the equilibrium point:

- $p = I_2 - I_2^*$, $q = \varphi_2$. $I_2^*$ is the value at the initial conditions.
- $\xi_1 = \xi - \xi^*$, $\eta_1 = \eta$, where $\xi^*$ is numerically determined.

# Construction of the elliptic torus

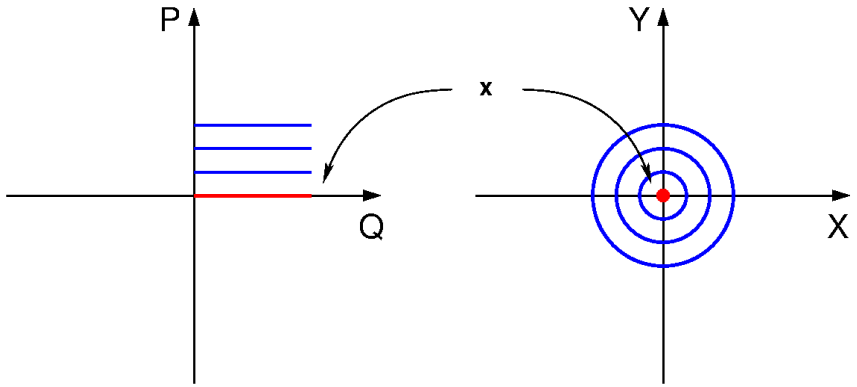
$$H(p,q,\xi_1,\eta_1) = E^{(0)} + \omega^* p + \Omega^* \frac{\xi_1^2+\eta_1^2}{2} + \varepsilon\mathcal{F}_0(q) + \varepsilon\mathcal{F}_1(q,\xi_1,\eta_1) + \\ + \varepsilon\mathcal{F}_2(p,q,\xi_1,\eta_1) + o(p + \|(\xi_1,\eta_1)\|^2).$$

## Normal form for lower dimensional elliptic tori

We look for a canonical transformation $\mathcal{C}(P,Q,X,Y) = (p,q,\xi_1,\eta_1)$ such that

$$H^{(\infty)}(P,Q,X,Y) = E^{(\infty)} + \omega^{(\infty)} P + \Omega^{(\infty)} \frac{X^2+Y^2}{2} + o(P + \|(X,Y)\|^2);$$

$\Rightarrow (0, Q_0 + \omega^{(\infty)} t, 0, 0)$ is a periodic orbit.



## **Construction of the elliptic torus: algorithm scheme,** adapting Giorgilli, Locatelli, Sansottera, CMDA(2014)

We define canonical transformations by using the formalism of the Lie series and with an iterative procedure.

$$H^{(0)}(p,q,\xi_1,\eta_1) = \overbrace{E^{(0)} + \omega^{(0)} p + \Omega^{(0)} \frac{\xi_1^2+\eta_1^2}{2}}^{\text{normal form}} + \overbrace{\varepsilon \sum_{s\geq 1} f_0^{(0,s)}(q)}^{\text{1. degree 0 in the actions}} +$$

$$+ \overbrace{\varepsilon \sum_{s\geq 1} f_1^{(0,s)}(q,\xi_1,\eta_1)}^{\text{2. degree 1/2 in the actions}} + \overbrace{\varepsilon \sum_{s\geq 1} f_2^{(0,s)}(p,q,\xi_1,\eta_1)}^{\text{3. degree 1 in the actions}} + \overbrace{\sum_{s\geq 0}\sum_{\ell>2} f_\ell^{(0,s)}(p,q,\xi_1,\eta_1)}^{\text{normal form}},$$

Every step $r$ is composed of three sub-steps,

$$\exp\left(\varepsilon^r L_{\chi_2^{(r)}}\right) \circ \exp\left(\varepsilon^r L_{\chi_1^{(r)}}\right) \circ \exp\left(\varepsilon^r L_{\chi_0^{(r)}}\right),$$

where each generating function $\chi_0^{(r)}$, $\chi_1^{(r)}$, $\chi_2^{(r)}$ aim to remove three different types of "unwanted" terms, of order $\varepsilon^r$ (up to a fixed degree in $q$).

# Construction of the elliptic torus: first step of the algorithm

## First homological equation: terms that depend only on $q$

We define the new Hamiltonian $H^{(\mathrm{I};1)} = \exp\left(L_{\chi_0^{(1)}}\right) H^{(0)}$, with $\chi_0^{(1)}(q)$ that solves:

$$\left\{\chi_0^{(1)},\ \omega^{(0)} p\right\} + f_0^{(0,1)}(q) = \langle f_0^{(0,1)}(q)\rangle_q \ .$$

$$H^{(0)}(p,q,\xi_1,\eta_1) = \sum$$

| | | | | | |
|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $f_3^{(0,0)}$ | $f_3^{(0,1)}$ | $f_3^{(0,2)}$ | $\dots$ | $f_3^{(0,s)}$ | $\dots$ |
| $\omega^{(0)} p + \Omega^{(0)} I$ | $f_2^{(0,1)}$ | $f_2^{(0,2)}$ | $\dots$ | $f_2^{(0,s)}$ | $\dots$ |
| $0$ | $f_1^{(0,1)}$ | $f_1^{(0,2)}$ | $\dots$ | $f_1^{(0,s)}$ | $\dots$ |
| $E^{(0)}$ | $f_0^{(0,1)}$ | $f_0^{(0,2)}$ | $\dots$ | $f_0^{(0,s)}$ | $\dots$ |
| $\mathcal{O}(1)$ | $\mathcal{O}(\varepsilon)$ | $\mathcal{O}(\varepsilon^2)$ | $\dots$ | $\mathcal{O}(\varepsilon^s)$ | $\dots$ |

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$$H^{(0)}(p,q,\xi_1,\eta_1) = \sum$$

| | | | | | |
|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $f_3^{(0,0)}$ | $f_3^{(0,1)}$ | $f_3^{(0,2)}$ | $\dots$ | $f_3^{(0,s)}$ | $\dots$ |
| $\omega^{(0)} p + \Omega^{(0)} I$ | $f_2^{(0,1)}$ | $f_2^{(0,2)}$ | $\dots$ | $f_2^{(0,s)}$ | $\dots$ |
| $0$ | $f_1^{(0,1)}$ | $f_1^{(0,2)}$ | $\dots$ | $f_1^{(0,s)}$ | $\dots$ |
| $E^{(0)}$ | $\langle f_0^{(0,1)}\rangle_q$ | $f_0^{(0,2)}$ | $\dots$ | $f_0^{(0,s)}$ | $\dots$ |
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$$\left\{\chi_0^{(1)},\ \omega^{(0)} p\right\} + f_0^{(0,1)}(q) = \langle f_0^{(0,1)}(q)\rangle_q \ .$$

$$H^{(\mathrm{I};1)}(p,q,\xi_1,\eta_1) = \sum$$

| | | | | | |
|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $f_3^{(\mathrm{I};1,0)}$ | $f_3^{(\mathrm{I};1,1)}$ | $f_3^{(\mathrm{I};1,2)}$ | $\dots$ | $f_3^{(\mathrm{I};1,s)}$ | $\dots$ |
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| $0$ | $f_1^{(\mathrm{I};1,1)}$ | $f_1^{(\mathrm{I};1,2)}$ | $\dots$ | $f_1^{(\mathrm{I};1,s)}$ | $\dots$ |
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# Construction of the elliptic torus: first step of the algorithm

## Second homological equation: terms linear in $(\xi_1, \eta_1)$

We define the new Hamiltonian $H^{(\mathrm{II};1)} = \exp\left(L_{\chi_1^{(1)}}\right) H^{(\mathrm{I};1)}$, with $\chi_1^{(1)}(q, \xi_1, \eta_1)$ that solves:

$$\left\{ \chi_1^{(1)} \,,\; \omega^{(0)} p + \Omega^{(0)} \frac{\xi_1^2 + \eta_1^2}{2} \right\} + f_1^{(\mathrm{I};1,1)}(q, \xi_1, \eta_1) = 0 \,.$$

$$H^{(\mathrm{I};1)}(p, q, \xi_1, \eta_1) = \sum$$

| | | | | | |
|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $f_3^{(\mathrm{I};1,0)}$ | $f_3^{(\mathrm{I};1,1)}$ | $f_3^{(\mathrm{I};1,2)}$ | $\dots$ | $f_3^{(\mathrm{I};1,s)}$ | $\dots$ |
| $\omega^{(0)} p + \Omega^{(0)} I$ | $f_2^{(\mathrm{I};1,1)}$ | $f_2^{(\mathrm{I};1,2)}$ | $\dots$ | $f_2^{(\mathrm{I};1,s)}$ | $\dots$ |
| $0$ | $f_1^{(\mathrm{I};1,1)}$ | $f_1^{(\mathrm{I};1,2)}$ | $\dots$ | $f_1^{(\mathrm{I};1,s)}$ | $\dots$ |
| $E^{(1)}$ | $0$ | $f_0^{(\mathrm{I};1,2)}$ | $\dots$ | $f_0^{(\mathrm{I};1,s)}$ | $\dots$ |
| $\mathcal{O}(1)$ | $\mathcal{O}(\varepsilon)$ | $\mathcal{O}(\varepsilon^2)$ | $\dots$ | $\mathcal{O}(\varepsilon^s)$ | $\dots$ |

# Construction of the elliptic torus: first step of the algorithm

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$$H^{(\mathrm{I};1)}(p, q, \xi_1, \eta_1) = \sum$$

| | | | | | |
|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $f_3^{(\mathrm{I};1,0)}$ | $f_3^{(\mathrm{I};1,1)}$ | $f_3^{(\mathrm{I};1,2)}$ | $\dots$ | $f_3^{(\mathrm{I};1,s)}$ | $\dots$ |
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| $0$ | $0$ | $f_1^{(\mathrm{I};1,2)}$ | $\dots$ | $f_1^{(\mathrm{I};1,s)}$ | $\dots$ |
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We define the new Hamiltonian $H^{(\mathrm{II};1)} = \exp\left(L_{\chi_1^{(1)}}\right) H^{(\mathrm{I};1)}$, with $\chi_1^{(1)}(q, \xi_1, \eta_1)$ that solves:

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$$H^{(\mathrm{II};1)}(p, q, \xi_1, \eta_1) = \sum \left| \begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \\ f_3^{(\mathrm{II};1,0)} & f_3^{(\mathrm{II};1,1)} & f_3^{(\mathrm{II};1,2)} & \dots & f_3^{(\mathrm{II};1,s)} & \dots \\ \omega^{(0)} p + \Omega^{(0)} I & f_2^{(\mathrm{II};1,1)} & f_2^{(\mathrm{II};1,2)} & \dots & f_2^{(\mathrm{II};1,s)} & \dots \\ 0 & 0 & f_1^{(\mathrm{II};1,2)} & \dots & f_1^{(\mathrm{II};1,s)} & \dots \\ E^{(1)} & 0 & f_0^{(\mathrm{II};1,2)} & \dots & f_0^{(\mathrm{II};1,s)} & \dots \\ \hline \mathcal{O}(1) & \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon^2) & \dots & \mathcal{O}(\varepsilon^s) & \dots \end{array} \right.$$

# Construction of the elliptic torus: first step of the algorithm

## Third homological equation: terms linear in the actions

We define the new Hamiltonian $H^{(1)} = \exp\left(L_{\chi_2^{(1)}}\right) H^{(\mathrm{II};1)}$, with $\chi_2^{(1)}(p, q, \xi_1, \eta_1)$ that solves:

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The non-zero average terms will be added to the normal form by defining new frequencies $\omega^{(1)}$ and $\Omega^{(1)}$ (just a correction of $\mathcal{O}(\varepsilon)$).

$$H^{(\mathrm{II};1)}(p, q, \xi_1, \eta_1) = \sum$$

| | | | | | |
|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $f_3^{(\mathrm{II};1,0)}$ | $f_3^{(\mathrm{II};1,1)}$ | $f_3^{(\mathrm{II};1,2)}$ | $\dots$ | $f_3^{(\mathrm{II};1,s)}$ | $\dots$ |
| $\omega^{(0)} p + \Omega^{(0)} I$ | $f_2^{(\mathrm{II};1,1)}$ | $f_2^{(\mathrm{II};1,2)}$ | $\dots$ | $f_2^{(\mathrm{II};1,s)}$ | $\dots$ |
| $0$ | $0$ | $f_1^{(\mathrm{II};1,2)}$ | $\dots$ | $f_1^{(\mathrm{II};1,s)}$ | $\dots$ |
| $E^{(1)}$ | $0$ | $f_0^{(\mathrm{II};1,2)}$ | $\dots$ | $f_0^{(\mathrm{II};1,s)}$ | $\dots$ |
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| | | | | | |
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| $f_3^{(\mathrm{II};1,0)}$ | $f_3^{(\mathrm{II};1,1)}$ | $f_3^{(\mathrm{II};1,2)}$ | $\dots$ | $f_3^{(\mathrm{II};1,s)}$ | $\dots$ |
| $\omega^{(0)} p + \Omega^{(0)} I$ | $\langle f_2^{(\mathrm{II};1,1)} \rangle_q$ | $f_2^{(\mathrm{II};1,2)}$ | $\dots$ | $f_2^{(\mathrm{II};1,s)}$ | $\dots$ |
| $0$ | $0$ | $f_1^{(\mathrm{II};1,2)}$ | $\dots$ | $f_1^{(\mathrm{II};1,s)}$ | $\dots$ |
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$$H^{(1)}(p, q, \xi_1, \eta_1) = \sum$$

| | | | | | |
|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $f_3^{(1,0)}$ | $f_3^{(1,1)}$ | $f_3^{(1,2)}$ | $\dots$ | $f_3^{(1,s)}$ | $\dots$ |
| $\omega^{(1)} p + \Omega^{(1)} I$ | $0$ | $f_2^{(1,2)}$ | $\dots$ | $f_2^{(1,s)}$ | $\dots$ |
| $0$ | $0$ | $f_1^{(1,2)}$ | $\dots$ | $f_1^{(1,s)}$ | $\dots$ |
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| $\mathcal{O}(1)$ | $\mathcal{O}(\varepsilon)$ | $\mathcal{O}(\varepsilon^2)$ | $\dots$ | $\mathcal{O}(\varepsilon^s)$ | $\dots$ |

# Small divisors / Convergence of the procedure

## Can we **always** solve the homological equations?

At each normalization step we introduce different divisors, containing linear combinations of the frequencies.

## Non-resonance condition

For $\gamma > 0$ and $\tau > 0$:

$$\min_{\substack{0 \leq |k| \leq 2r,\ 0 \leq |\ell| \leq 2 \\ |k|+|\ell|>0}} \left| k\omega^{(r-1)} + \ell\Omega^{(r-1)} \right| \geq \frac{\gamma}{|k|^{\tau}+1}.$$

For $\ell = 0$ this is the usual **Diophantine non-resonant inequality**, while for $|\ell| = 1, 2$ it is known as **first** and **second Melnikov condition**, respectively.

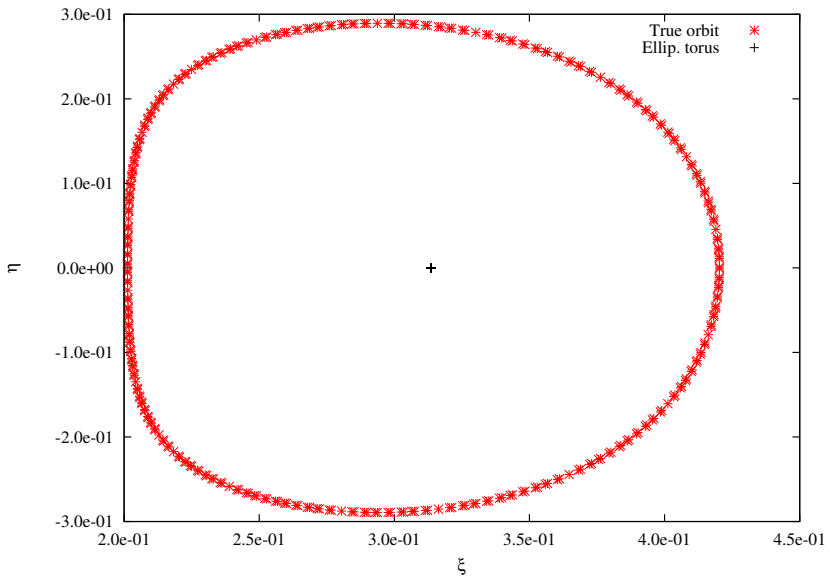
## Remarks

- The algorithm can be iterated up to infinity only if the non-resonance condition is verified at each step, but the frequencies are changing... $\Rightarrow$ measure argument.
- In practice, we explicitly perform a finite number of normalization steps, and we can control how the generating functions are decreasing (in norm).

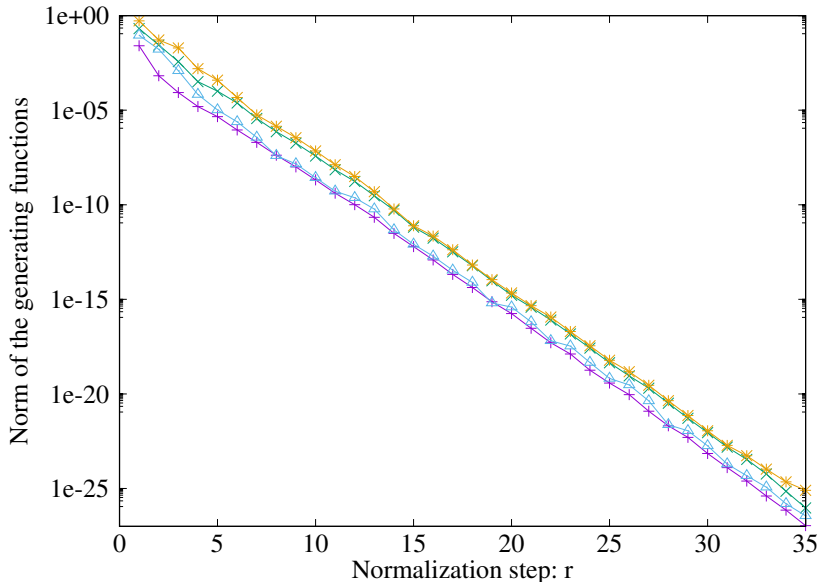
# Numerical and semi-analytical elliptic torus

## Poincaré section

of the orbit with initial conditions on the elliptic torus constructed with our algorithm.




Decrease of the norms of the generating functions.




## Towards the KAM torus

By expressing the initial conditions of the true orbit in the coordinates of the elliptic torus, we can approximately define a first guess for the translation on $I_1$, to be centered around the KAM torus we aim to construct.

# Construction of the KAM torus

## Preliminary set up

- $p_2 = I_1 - I_1^*$, $q_2$.
- We recollect the variables in $(\boldsymbol{p}, \boldsymbol{q})$.

The Hamiltonian after the translation and Taylor–Fourier series expansions:

$$H^{(0)}(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{\omega}^{(0)} \cdot \boldsymbol{p} + \varepsilon \mathcal{F}_0(\boldsymbol{q}) + \varepsilon \mathcal{F}_1(\boldsymbol{p}, \boldsymbol{q}) + \mathcal{O}(\|\boldsymbol{p}\|^2)$$

## How can we characterize the "true orbit"?

The frequencies $\boldsymbol{\omega}^*$ cannot be directly "observed", since they describe a *secular* motion
$\Rightarrow$ **frequency analysis method (Laskar (1995))** on the numerical solution.

## Kolmogorov normal form

We look for a canonical transformation $\mathcal{C}(\boldsymbol{P}, \boldsymbol{Q}) = (\boldsymbol{p}, \boldsymbol{q})$ such that the Hamiltonian will be

$$H^{(\infty)}(\boldsymbol{P}, \boldsymbol{Q}) = \boldsymbol{\omega}^* \cdot \boldsymbol{P} + \mathcal{O}(\|\boldsymbol{P}\|^2);$$

$\Rightarrow (0, \boldsymbol{Q}_0 + \boldsymbol{\omega}^* t)$ is an invariant torus and the motion is quasi-periodic on it.

# Construction of the KAM torus: algorithm

Every step is composed by two substeps, aiming at removing two different kinds of terms, up to a fixed order in $\varepsilon$:

$$H^{(0)}(\boldsymbol{p},\boldsymbol{q}) = \overbrace{E_0 + \boldsymbol{\omega}^{(0)}\cdot\boldsymbol{p}}^{\text{normal form}} + \overbrace{\varepsilon \sum_{s\geq 1} f_0^{(0,s)}(\boldsymbol{q})}^{1.} + \overbrace{\varepsilon \sum_{s\geq 1} f_1^{(0,s)}(\boldsymbol{p},\boldsymbol{q})}^{2.} + \overbrace{\sum_{s\geq 0}\sum_{\ell\geq 2} f_\ell^{(0,s)}(\boldsymbol{p},\boldsymbol{q})}^{\text{normal form}},$$

# Construction of the KAM torus: algorithm

Every step is composed by two substeps, aiming at removing two different kinds of terms, up to a fixed order in $\varepsilon$:

$$H^{(\mathrm{I};1)}(\boldsymbol{p},\boldsymbol{q}) = \overbrace{E_1 + \boldsymbol{\omega}^{(0)}\cdot\boldsymbol{p}}^{\text{normal form}} + \overbrace{\varepsilon^2 \sum_{s\geq 2} f_0^{(\mathrm{I};1,s)}(\boldsymbol{q})}^{1.} + \overbrace{\varepsilon \sum_{s\geq 1} f_1^{(\mathrm{I};1,s)}(\boldsymbol{p},\boldsymbol{q})}^{2.} + \overbrace{\sum_{s\geq 0}\sum_{\ell\geq 2} f_\ell^{(\mathrm{I};1,s)}(\boldsymbol{p},\boldsymbol{q})}^{\text{normal form}},$$

**First homological equation (1.):** terms depending just on angles $\boldsymbol{q}$.

We define the new Hamiltonian $H^{(\mathrm{I};1)} = \exp\left(\varepsilon L_{\chi_1^{(1)}}\right) H^{(0)}$, with $\chi_1^{(1)}$ that solves:

$$\left\{\chi_1^{(1)},\ \boldsymbol{\omega}^{(0)}\cdot\boldsymbol{p}\right\} + f_0^{(0,1)}(\boldsymbol{q}) = \langle f_0^{(0,1)}(\boldsymbol{q})\rangle_{\boldsymbol{q}}\ .$$

being $\chi_1^{(1)} = \chi_1^{(1)}(\boldsymbol{q})$.

# Construction of the KAM torus: algorithm

Every step is composed by two substeps, aiming at removing two different kinds of terms, up to a fixed order in $\varepsilon$:

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## Second homological equation (2.): terms linear in $\boldsymbol{p}$, depending on $\boldsymbol{q}$

We define the new Hamiltonian $H^{(1)} = \exp\left(\varepsilon L_{\chi_2^{(1)}}\right) H^{(\mathrm{I};1)}$, with $\chi_2^{(1)}$ that solves:

$$\left\{\chi_2^{(1)},\ \boldsymbol{\omega}^{(0)}\cdot\boldsymbol{p}\right\} + f_1^{(\mathrm{I};1,1)}(\boldsymbol{p},\boldsymbol{q}) = \langle f_1^{(\mathrm{I};1,1)}(\boldsymbol{p},\boldsymbol{q})\rangle_{\boldsymbol{q}}\ .$$

being $\chi_2^{(1)} = \chi_2^{(1)}(\boldsymbol{p},\boldsymbol{q})$.

# Construction of the KAM torus: algorithm

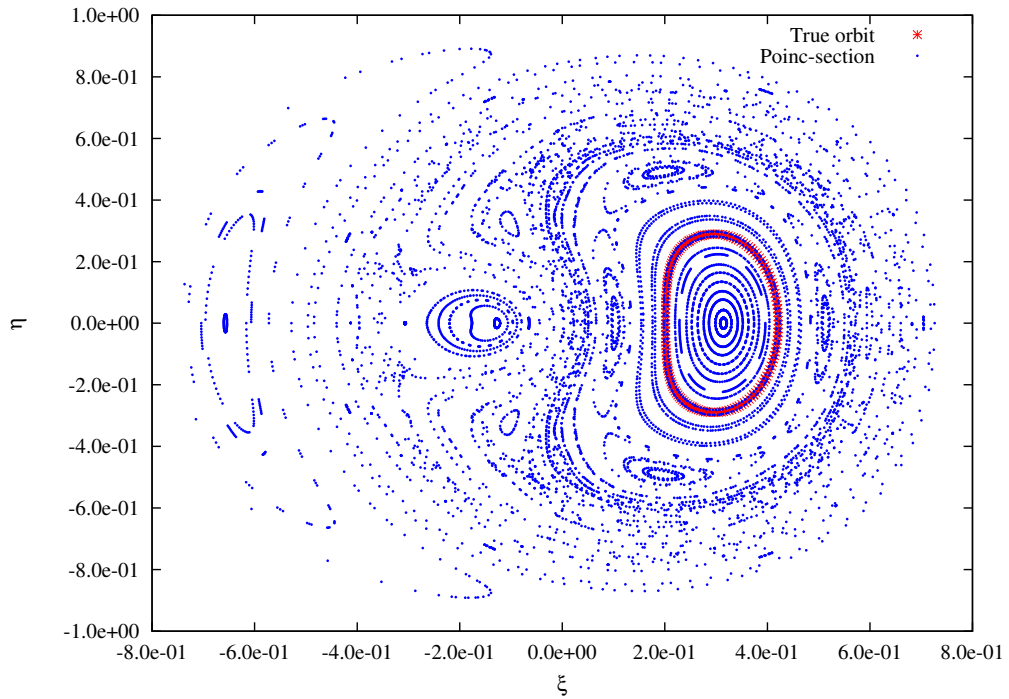
After $r$ steps, the Hamiltonian will be

$$H^{(r)}(\boldsymbol{P},\boldsymbol{Q}) = \boldsymbol{\omega}^{(r)} \cdot \boldsymbol{P} + \mathcal{O}(\|\boldsymbol{P}\|^2) + \varepsilon^{r+1}\mathcal{O}(\|\boldsymbol{P}\|);$$

- If the initial translations $I_1^*$ is accurate enough, then $\boldsymbol{\omega}^{(r)} \simeq \boldsymbol{\omega}^*$.
  $\Rightarrow$ Newton method to solve

  $$f(I_1^*, I_2^*) = \boldsymbol{\omega}^*,$$

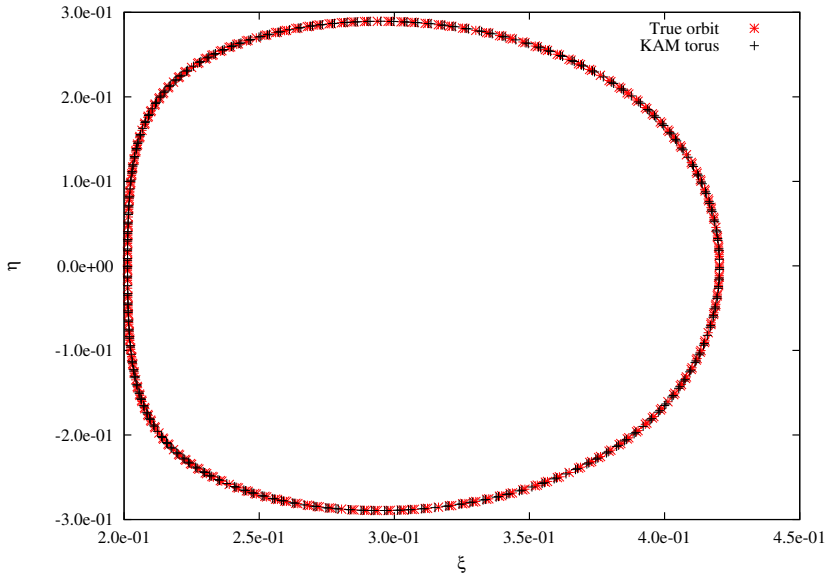
  being $f(I_1^*, I_2^*)$ the function which maps the preliminary translations to the frequencies after the normalization procedure.
- We enqueue at this Hamiltonian the same algorithm, but this time by defining a translation for fixing the frequencies.



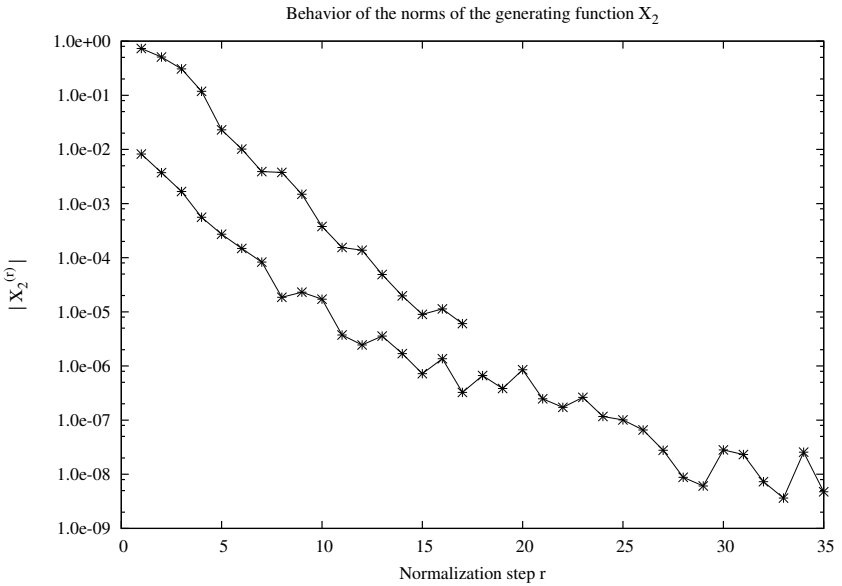
# Comparison between numerical and semi-analytic solution

## Poincaré section

Orbit with initial conditions on the torus constructed with our algorithm.



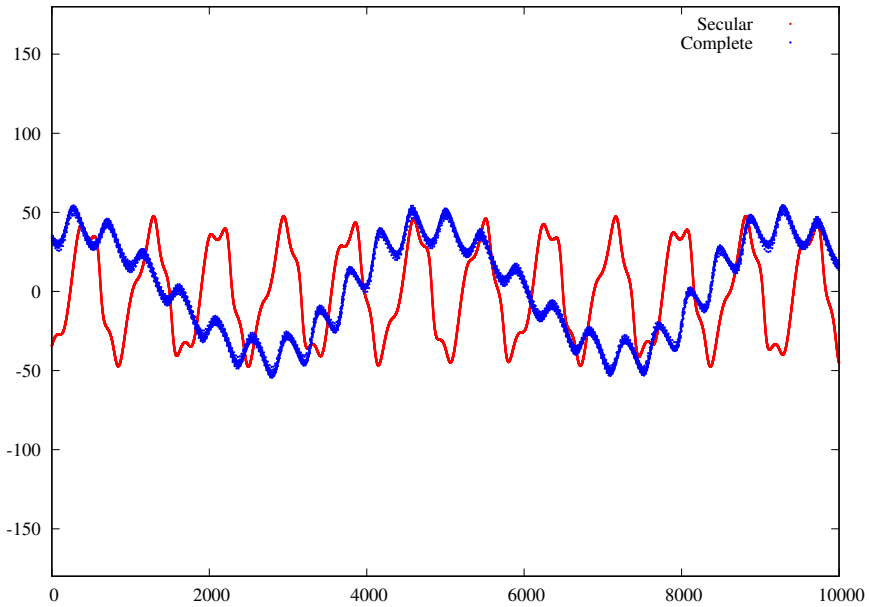

Decrease of the norms of the generating functions $\chi_2$ for the algorithm with no translations (above) and with fixed frequencies (below).



# Conclusions / Future work

- We have developed a KAM theory for the secular model that works also in exoplanetary systems with large eccentricities (improvement with respect to a previous work by **Volpi et al., CMDA (2018)**, where the limit was $e \lesssim 0.1$);
- Secular model VS three-body problem;

Difference of the arguments of the pericenters




- Check the performances for other planetary systems with large eccentricities;
- To produce a computer–assisted proof.

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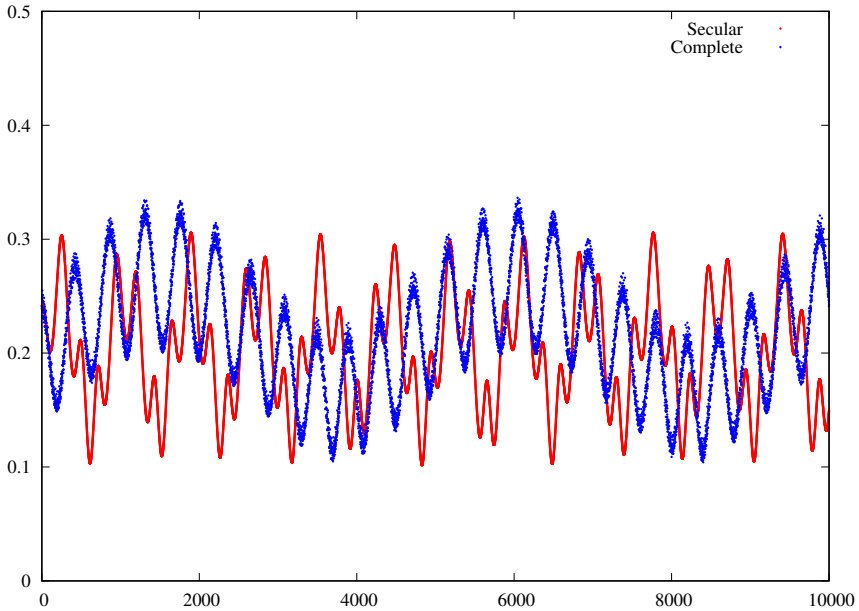
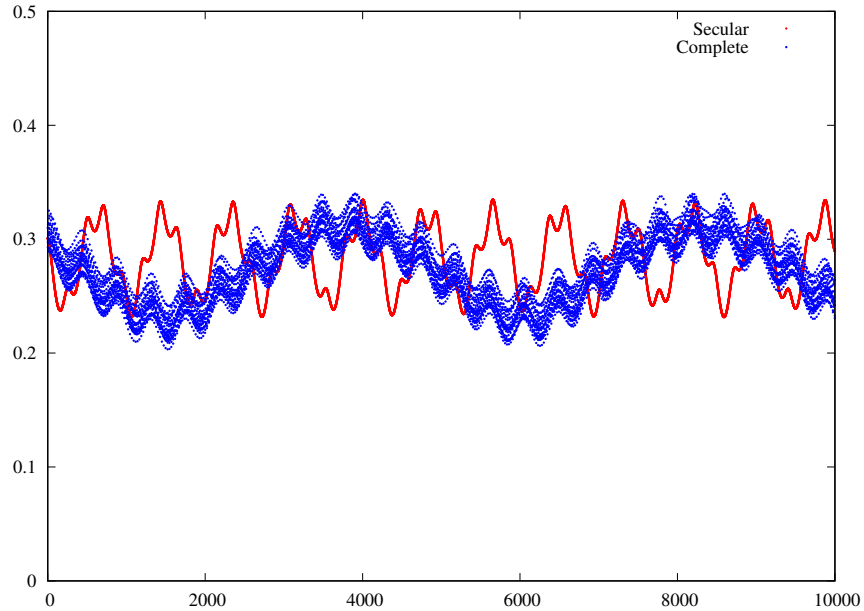
Eccentricity $e_1$




Eccentricity $e_2$

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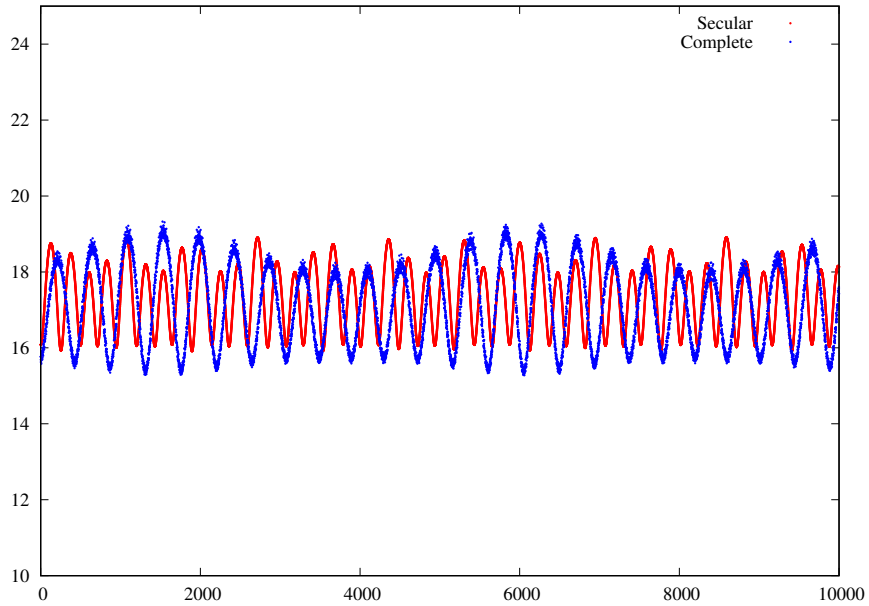
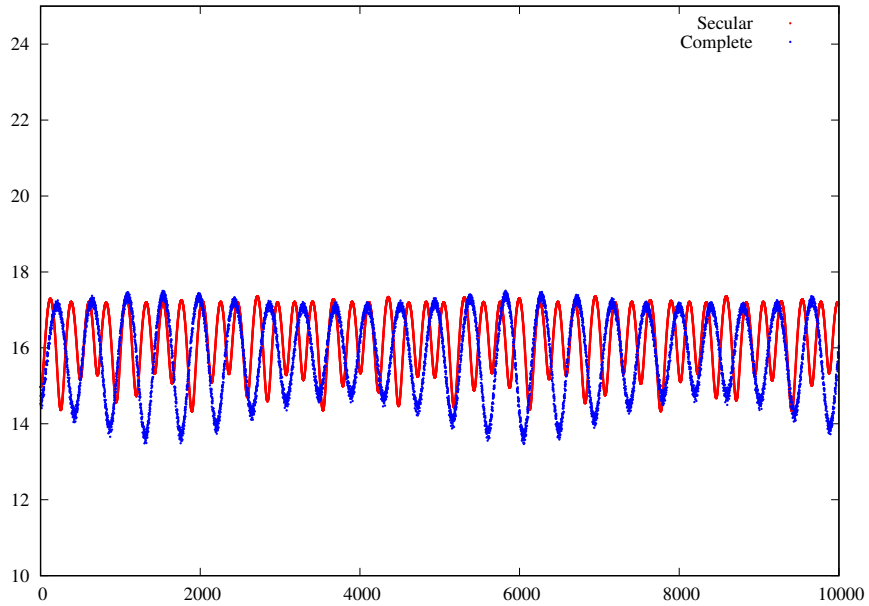



Inclination $i_2$

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Thank you for your attention!