Gevrey estimates and domains of analyticity for asymptotic expansions of tori in weakly dissipative systems

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Joint work with R. Calleja and R. de la Llave

Outline

- Background
- Numerical results
- Rigorous results
- Sketch of the proof

Program

Numerics $\rightarrow$ Conjectures/ Interesting phenomena $\rightarrow$ Rigorous results

Background

Dissipative standard map

DSM $f_{\varepsilon,\mu} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$

$$f_{\varepsilon,\mu}(\theta, y) = (\theta + \lambda y + \mu + \varepsilon V(\theta), \lambda y + \mu + \varepsilon V(\theta))$$

where $V(\theta) = \frac{1}{2\pi}\sin(2\pi\theta)$, λ acts as a dissipation ($\lambda < 1$), μ is called the *drift* parameter, and ε is the perturbation parameter.

Discrete model of **Spin Orbit problem**: Motion of an oblate satellite (as a semi-rigid body) under the gravitational influence of a planet and tidal dissipation.

$$\ddot{\theta} + \varepsilon \sin(\theta) + \lambda\dot{\theta} + \mu = 0, \quad \theta \equiv x - \frac{p}{2}t$$

Conformally symplectic systems.

Dissipative standard map satisfies

$$f_{\varepsilon,\mu}^{*}\Omega = \lambda\Omega, \tag{0.1}$$

where $\Omega = d\theta \wedge dy$ is the standard symplectic form in $\mathbb{T} \times \mathbb{R}$.

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Many problems of interest satisfy (0.1), for example,

- Hamiltonian systems with friction proportional to the velocity.
- In finance, Euler-Lagrange equations of exponentially discounted systems. $S(q) = \int_0^\infty e^{-\lambda t} L(q(t), \dot{q}(t))dt$

KAM theory for conformally symplectic

- Calleja-Celletti-de la Llave, '13. KAM theory for conformally symplectic systems, $f_\mu^*\Omega = \lambda\Omega$, *a-posteriori theorems*
- C-C-dlL, '17. Domains of analyticity and Lindstedt expansions, in ε, of KAM tori for $f_{\varepsilon,\mu_\varepsilon}^*\Omega = \lambda(\varepsilon)\Omega$, $\lambda(\varepsilon) = 1 - \varepsilon^a, a \in \mathbb{N}$, $\varepsilon \in \mathbb{C}$.

  Note this is a **singular perturbation!** Since any friction reduces drastically the set of quasi-periodic solutions of the system.

Lindstedt series

Theorem (Calleja-Celletti-de la Llave, '17)

- ω *diophantine* ($|e^{2\pi i k\cdot\omega} - 1| \geq \nu |k|^{-\tau}$) $f_{\varepsilon,\mu_\varepsilon} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$, $f^*_{\varepsilon,\mu_\varepsilon}\Omega = \lambda(\varepsilon)\Omega$, $\lambda(\varepsilon) = 1 - \varepsilon^\alpha, \alpha \geq 1$
- $\exists K_0 \in \mathcal{A}_\rho(\mathbb{T}^d, \mathbb{T}^d \times \mathbb{R}^d)$ *and* $\mu_0 \in \mathbb{C}^d$ *such that* $f_{0,\mu_0} \circ K_0(\theta) = K_0(\theta + \omega)$
- $+$ *Non-degeneracy conditions*

Then,

A) *There exist formal power series expansions* $K_\varepsilon^{[\leq N]}(\theta) = \sum_{j=0}^N K_j(\theta)\varepsilon^j$ *and* $\mu_\varepsilon^{[\leq N]} = \sum_{j=0}^N \mu_j \varepsilon^n$ *such that for any* $N \in \mathbb{N}$ *we have*

$$\|f_{\varepsilon,\mu_\varepsilon^{[\leq N]}} \circ K_\varepsilon^{[\leq N]} - K_\varepsilon^{[\leq N]} \circ T_\omega\|_{\rho_0} \leq C_N |\varepsilon|^{N+1}$$

for some $0 < \rho_0 < \rho$ *and* $C_N > 0$.

B) There exists a set $\mathcal{G} \subseteq \mathbb{C}$ and functions

$$K : \mathcal{G} \to \mathcal{A}_{\rho_0}, \qquad \mu : \mathcal{G} \to \mathbb{C}^d$$

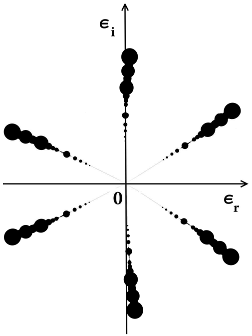
which are analytic in the interior of $\mathcal{G}$ and continuous on the boundary of $\mathcal{G}$ and such that

$$f_{\varepsilon,\mu_\varepsilon} \circ K_\varepsilon - K_\varepsilon \circ T_\omega = 0.$$

Furthermore, the solutions thus found have the formal power series provided in A) as an asymptotic expansion. That is, for any $N \in \mathbb{N}$:

$$\|K_\varepsilon^{[\leq N]} - K_\varepsilon\|_{\rho'} \leq C_N |\varepsilon|^{N+1}, \qquad \varepsilon \in \mathcal{G}$$

$$|\mu_\varepsilon^{[\leq N]} - \mu_\varepsilon| \leq C_N |\varepsilon|^{N+1} \qquad \varepsilon \in \mathcal{G}$$



Domain $\mathcal{G} \subseteq \mathbb{C}$, for $\lambda(\varepsilon) = 1 - \varepsilon^3$, which was rigorously proved to be a lower bound of the optimal domain of analyticity. It was also conjectured that this domain of analyticity is qualitatively optimal for generic families. In particular, does not contain balls with center at $\varepsilon = 0$, nor angular sectors of angle bigger than $\pi/3$.

Numerical Results

Goal: Study numerically

- Domains of analyticity of quasiperiodic orbits for DSM
- Properties of the asymptotic expansions

Quasi periodic orbits

Quasi periodic orbits for the DSM can be described by a 1-periodic function $u_\varepsilon : \mathbb{S}^1 \to \mathbb{R}$ and a constant μ_ε satisfying

$$E_{\mu_\varepsilon}[u_\varepsilon] = 0 \tag{0.2}$$

where

$$E_{\mu_\varepsilon}[u_\varepsilon(\theta)] \equiv u_\varepsilon(\theta+\omega) - (1+\lambda(\varepsilon))u_\varepsilon(\theta) + \lambda(\varepsilon)u_\varepsilon(\theta-\omega) + (1-\lambda(\varepsilon))\omega - \mu_\varepsilon + \varepsilon V'$$

Then $K_\varepsilon(\theta) = \begin{pmatrix} \theta + u_\varepsilon(\theta) \\ \omega + u_\varepsilon(\theta) - u_\varepsilon(\theta-\omega) \end{pmatrix}$ satisfies

$$f_{\varepsilon,\mu_\varepsilon} \circ K_\varepsilon(\theta) = K_\varepsilon(\theta+\omega)$$

Lindstedt expansions

Lindstedt expansions for the DSM are obtained by considering $u_\varepsilon = \sum_{k=0}^{\infty} u_k(\theta)\varepsilon^k$ and $\mu_\varepsilon = \sum_{k=0}^{\infty} \mu_k \varepsilon^k$ and solving $E_{\mu_\varepsilon}[u_\varepsilon] = 0$ at each order. At order k we have,

$$L_\omega u_k(\theta) - \mu_k = S_k(\theta), \qquad k \leq 2 \tag{0.3}$$

$$L_\omega u_k(\theta) - \mu_k + \omega = S_k(\theta), \qquad k = 3 \tag{0.4}$$

$$L_\omega u_k(\theta) - \mu_k = S_k(\theta) + u_{k-3}(\theta) - u_{k-3}(\theta - \omega), \qquad k \geq 4 \tag{0.5}$$

where $\varepsilon V' \equiv \sum_{k=0}^{\infty} S_k(\theta)\varepsilon^k$, $\lambda(\varepsilon) = 1 - \varepsilon^3$, and

$$L_\omega \varphi(\theta) \equiv \varphi(\theta + \omega) - 2\varphi(\theta) + \varphi(\theta - \omega)$$

Numerical Methods

To study the domain of convergence and its properties we use

- Padé approximants
  - Berretti, Chierchia '89
  - Falcollini, de la Llave '90
- Newton method
  - Calleja, Celletti '10

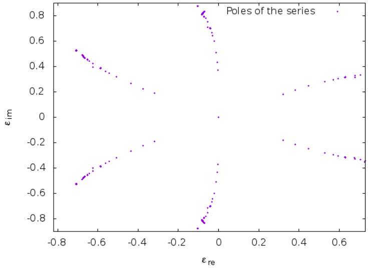
Padé approximants[1]

A Padé approximant of order [p/q] of $g_\varepsilon = \sum g_k \varepsilon^k$ is a rational function, $P(\varepsilon)/Q(\varepsilon)$ which agrees with g_ε to the highest possible order in ε, that is

$$g_\varepsilon - \frac{P(\varepsilon)}{Q(\varepsilon)} = \mathcal{O}(\varepsilon^{p+q+1})$$

where P and Q are polynomials of degree p and q respectively.
The singularities can be approximated by the roots of $Q(\varepsilon)$!!

[1] G. Baker, P.R. Graves-Morris, 1996



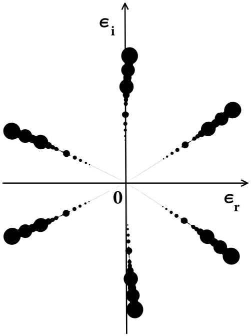


Figure: Poles in the complex plane $\varepsilon \in \mathbb{C}$. 400 digits of precision, using pari/gp.

Numerical results. B.- Calleja, '19

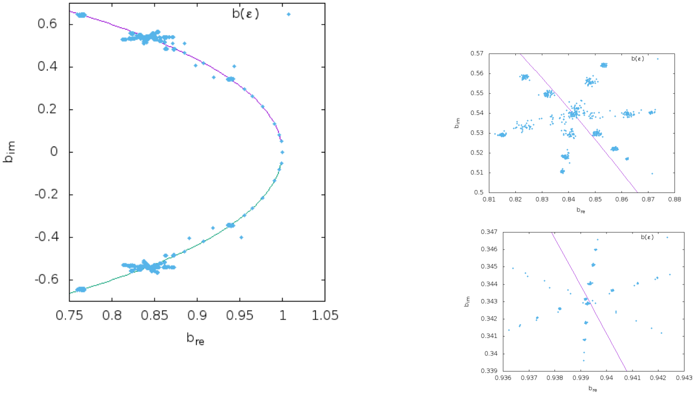


Figure: The poles compared to the unit circle. Left panel: Evaluation of the poles of the series by the function $\lambda(\varepsilon) = 1 - \varepsilon^3$. Right panels: Two zoomed in versions of the figure on the left. Denominators are zero at $\lambda(\varepsilon) = e^{2i\pi k\omega}$.

Newton method

We use Newton method, that solves $E_{\mu_\varepsilon}[u_\varepsilon] = 0$, and continuation to explore the monodromy of the solutions in the domains.

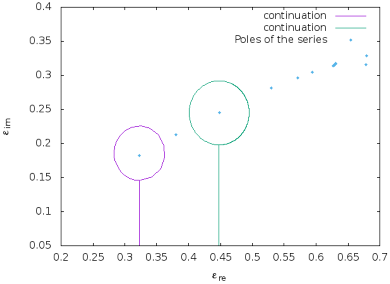
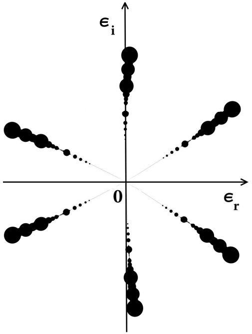


Figure: Poles of the series and two different continuations done with the Newton algorithm. The continuation is done around the pole in order to illustrate that the monodromy is trivial.

Asymptotic expansions in angular sectors

- Some class of asymptotic expansions turn out to be asymptotic to analytic functions in angular sectors

Gevrey classes!



Gevrey classes

A formal power series $g = \sum_{n \geq 0} g_n \varepsilon^n$ belongs to a Gevrey class $\sigma \geq 0$ if the coefficients satisfy

$$\|g_n\| \leq C R^n n^{\sigma n} \quad \Leftrightarrow \quad \|g_n\| \leq C R^n (n!)^{\sigma}$$

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g is σ- Borel summable !!!!

Growth of the coefficients

If $g = \sum g_k \varepsilon^k$ is σ-Gevrey, $\|g_k\| \leq C R^k k^{\sigma k}$ then

$$\frac{1}{k} \log \|g_k\| \sim \sigma \log(k) + \log(R)$$

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$$\frac{1}{k} \log \|g_k\| \sim \sigma \log(k) + \log(R)$$

$$A_\rho(k) \equiv \frac{1}{k} \log \|u_k\|_\rho$$

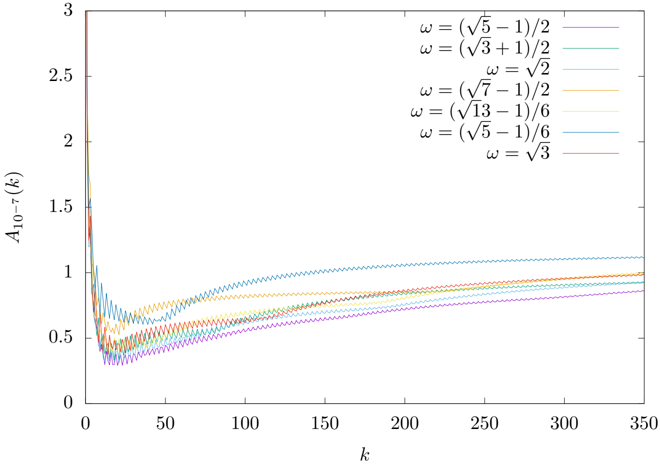
$\| \cdot \|_\rho$ analytic norm in a complex neighborhood of the torus.

Numerical fit

$A_\rho(k) = \log(R) + \sigma \log(k), \quad \omega = \frac{\sqrt{5}-1}{2}$		
	R	σ
$\rho = 0.0001$	0.575333	0.240225
$\rho = 0.00001$	0.575239	0.240243
$\rho = 0.000001$	0.575230	0.240244
$\rho = 0.0000001$	0.575229	0.240244

$$A_\rho(k) \equiv \frac{1}{k} \log \|u_k\|_\rho$$

More frequencies



$$A_\rho(k) = \frac{1}{k} \log \|u_k\|_\rho$$

Frame Title

$A_\rho(k) \approx \log(R) + \sigma\log(k), \qquad \rho = 10^{-7}$		
	R	σ
$\omega = \frac{\sqrt{5}-1}{2} = [0,1,1,1,1,1,1,...]$	0.575229	0.240244
$\omega = \frac{\sqrt{3}-1}{2} = [0,2,1,2,1,2,1,...]$	0.695887	0.225349
$\omega = \sqrt{2} = [1,2,2,2,2,2,2,...]$	0.583365	0.247799
$\omega = \sqrt{3} = [1,1,2,1,2,1,2,1,...]$	0.460186	0.307029
$\omega = \frac{\sqrt{7}-1}{2} = [0,1,4,1,1,4,1,1,...]$	1.300597	0.112924
$\omega = \frac{\sqrt{13}-1}{6} = [0,2,3,3,3,3,3,...]$	0.582937	0.258504
$\omega = \frac{\sqrt{5}-1}{6} = [0,4,1,5,1,5,1,5,...]$	1.235768	0.158503

Table: Numerical fit of a function $\log(R) + \sigma\log(k)$ to the data $A_\rho(k)$ for different values of the frequency ω and $\rho = 10^{-7}$. Computations were done using 2^{13} Fourier coefficients and 600 digits of precision. The numerical fit was made in for $100 \leq k \leq 300$.

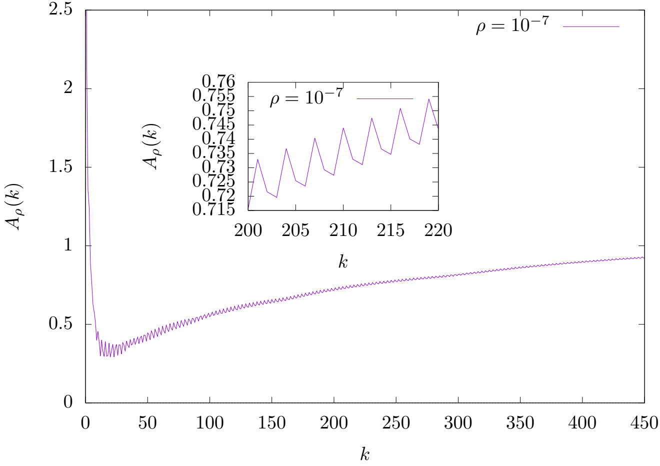
Conjecture. B. - Calleja '19

Lindstedt expansions of quasi periodic orbits for the dissipative standard map belongs to a Gevrey class.

Extended computations ('20),

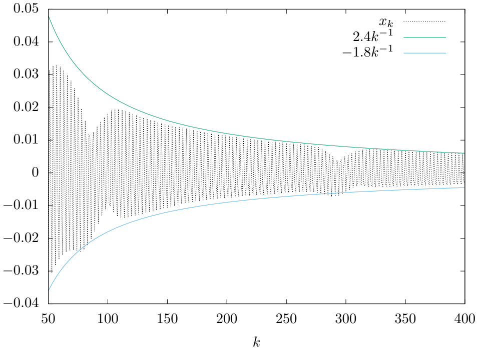
- If $\omega \in D(\nu, 1)$, the Gevrey exponent $\sigma \leq 0.307$.
- $\omega \in D(\nu, \tau)$, means $|e^{2\pi i k \cdot \omega} - 1| \geq \nu |k|^{-\tau}$,

New patterns



$$A_\rho(k) = \frac{1}{k} \log \|u_k\|_\rho$$

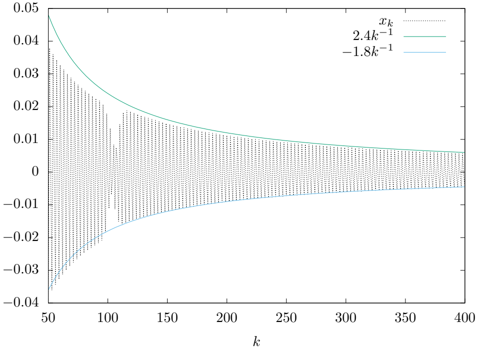
New patterns



$$\tilde{x}_k = A_\rho(k) - \frac{1}{5}\sum_{j=k-2}^{k+2} A_\rho(j), \qquad \rho = 10^{-7}$$

$$\omega = \frac{\sqrt{5}-1}{2}$$

New patterns



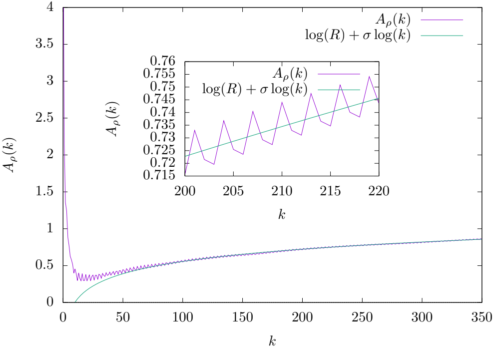
$$\tilde{x}_k = A_\rho(k) - \frac{1}{5}\sum_{j=k-2}^{k+2} A_\rho(j), \qquad \rho = 10^{-7}$$

$$\omega = \sqrt{3}$$

New Conjecture ('20)

$$A_\rho(k) \approx \log(R) + \sigma \log(k) + \frac{1}{k^\beta} f(k)$$

where f is a periodic function.



Summary of numerical results

- We verify theoretical conjectures
- Established conjectures – Gevrey character of Lindstedt series
- New interesting phenomena has been found – *Regular oscillations*
  - More conjectures

$$A_\rho(k) \approx \log(R) + \sigma \log(k) + \frac{1}{k^\beta} f(k)$$

Rigorous results

Gevrey estimates

Theorem (B. - de la Llave, 2020)

For a family of conformally symplectic maps, including

$$f_{\varepsilon,\mu_\varepsilon}(\theta,y) = (\theta + \lambda(\varepsilon)y + \mu_\varepsilon + \varepsilon V'(\theta), \lambda(\varepsilon)y + \mu_\varepsilon + \varepsilon V'(\theta))$$

with $\lambda(\varepsilon) = 1 - \varepsilon^\alpha$ *and* $V'(\theta)$ *a trigonometric polynomial. Assume that* $\omega \in D(\nu,\tau)$, *that is* $|e^{2\pi i k\cdot\omega} - 1| \geq \nu|k|^{-\tau}$, *and there* $\exists K_0 \in \mathcal{A}_\rho$ *and* $\mu_0 \in \mathbb{C}^d$ *such that* $f_{0,\mu_0} \circ K_0(\theta) = K_0(\theta+\omega)$
Then, there exist $0 < \rho_0 < \rho$ *and positive constants* L, F, N_0 *such that the Lindstedt expansions of quasiperiodic orbits* $K_\varepsilon^{[\infty]} = \sum_{n=0}^\infty K_n\varepsilon^n$ *and* $\mu_\varepsilon^{[\infty]} = \sum_{n=0}^\infty \mu_n\varepsilon^n$ *satisfy*

$$\|K_n\|_{\rho_0} \leq LF^n n^{(2\tau/\alpha)n} \quad \text{and} \quad |\mu_n| \leq LF^n n^{(2\tau/\alpha)n} \qquad \text{for any } n > N_0.$$

Numerics vs Rigourous result

For the Dissipative standard map with $\omega \in D(\nu, 1)$ and $\lambda(\varepsilon) = 1 - \varepsilon^3$

- Rigorous result:
$$\|K_n\|_{\rho_0} \leq LF^n n^{(2/3)n}$$
$$\sigma = 2/3$$
- Numerical results:
$$\sigma \leq 0.307$$

Sketch of the proof

- We developed a quasi-Newton method (with truncations) for the invariance equation $f_{\varepsilon,\mu_\varepsilon} \circ K_\varepsilon(\theta) = K_\varepsilon(\theta + \omega)$

Sketch of the proof

- We developed a quasi-Newton method (with truncations) for the invariance equation $f_{\varepsilon,\mu_\varepsilon} \circ K_\varepsilon(\theta) = K_\varepsilon(\theta+\omega)$
- Each step starts with approximations

$$K_\varepsilon^{[\leq N]}(\theta) = \sum_{n=0}^{N} K_n(\theta)\varepsilon^n \quad \mu_\varepsilon^{[\leq N]} = \sum_{n=0}^{N} \mu_n \varepsilon^n$$

satisfying

$$f_{\varepsilon,\mu_\varepsilon^{[\leq N]}} \circ K_\varepsilon^{[\leq N]}(\theta) - K_\varepsilon^{[\leq N]}(\theta+\omega) =: E_\varepsilon^N(\theta)$$

with

$$\|E_\varepsilon^N\|_\rho = \mathcal{O}(|\varepsilon|^{N+1}).$$

- Newton method takes advantage of the conformally symplectic structure (automatic reducibility).
- We truncate the error at each step, i.e., we consider

$$E_\varepsilon^{(N,2N]}(\theta) := \sum_{k=N+1}^{2N} E_k(\theta)\varepsilon^k$$

- Corrections satisfy

$$\tilde{K}_\varepsilon^{[\leq 2N]}(\theta) = K_\varepsilon^{[\leq N]}(\theta) + \sum_{n=N+1}^{2N} K_n(\theta)\varepsilon^n, \; \tilde{\mu}_\varepsilon^{[\leq 2N]} = \mu_\varepsilon^{[\leq N]} + \sum_{n=N+1}^{2N} \mu_n \varepsilon^n$$

$$\|E_\varepsilon^{2N}\|_\rho = \mathcal{O}(|\varepsilon|^{2N+1}).$$

Cohomology equations

- Two types of cohomology equations
  - Classic: $\varphi_\varepsilon(\theta) - \varphi_\varepsilon(\theta + \omega) = \eta_\varepsilon(\theta)$
  - Parametric:

$$\lambda(\varepsilon)\varphi_\varepsilon(\theta) - \varphi_\varepsilon(\theta + \omega) = \eta_\varepsilon(\theta) \tag{0.6}$$

- Small divisors depend on ε, i.e.,

$$\lambda(\varepsilon) = e^{2\pi i k \cdot \omega}$$

- If $\eta_\varepsilon(\theta)$ is trig polynomial, in θ, of degree s. Solution of (0.6) is analytic in balls centered at $\varepsilon = 0$ and radius $r = \mathcal{O}(s^{-\tau/\alpha})$

Algorithm

Algorithm

Given $K_\varepsilon^{[\le N]} : \mathbb{T}^n \to \mathcal{M}$, $\mu_\varepsilon^{[\le N]} \in \mathbb{R}^d$. *We perform the following computations:*

$$
\begin{array}{ll}
(1) & E_\varepsilon^N \leftarrow f_{\varepsilon,\mu_\varepsilon^{[\le N]}} \circ K_\varepsilon^{[\le N]} - K_\varepsilon^{[\le N]} \circ T_\omega \\
(2) & E_\varepsilon^{(N,2N]} \text{ obtained from } E_\varepsilon^N \text{ by truncation} \\
(3) & \alpha_\varepsilon \leftarrow DK_\varepsilon^{[\le N]} \\
(4) & \mathcal{N}_\varepsilon \leftarrow [\alpha_\varepsilon^\top \alpha_\varepsilon]^{-1} \\
(5) & V_\varepsilon \leftarrow J^{-1} \circ K_\varepsilon^{[\le N]} \alpha_\varepsilon \mathcal{N}_\varepsilon \\
(6) & M_\varepsilon \leftarrow [\alpha_\varepsilon | V_\varepsilon] \\
(7) & \beta_\varepsilon \leftarrow (M_\varepsilon \circ T_\omega)^{-1} \\
(8) & \tilde{E}_\varepsilon^{(N,2N]} \leftarrow \beta_\varepsilon E_\varepsilon^{(N,2N]}
\end{array}
$$

(9) $P_\varepsilon \leftarrow \alpha_\varepsilon \mathcal{N}_\varepsilon$

$\Gamma_\varepsilon \leftarrow \alpha_\varepsilon^\top J^{-1} \circ K_\varepsilon^{[\leq N]} \alpha_\varepsilon$

$S_\varepsilon \leftarrow (P_\varepsilon \circ T_\omega)^\top D f_{\mu_\varepsilon^{[\leq N]},\varepsilon} \circ K_\varepsilon^{[\leq N]} J^{-1} \circ K_\varepsilon^{[\leq N]} P_\varepsilon - \lambda(\varepsilon)(\mathcal{N}_\varepsilon \circ T_\omega)^\top$

$\tilde{A}_\varepsilon \leftarrow \beta_\varepsilon D_\mu f_{\mu_\varepsilon^{[\leq N]}} \circ K_\varepsilon^{[\leq N]}$

(10) $(B_{a,\varepsilon})^0$ solves $\lambda(\varepsilon)(B_{a,\varepsilon})^0 - (B_{a,\varepsilon})^0 \circ T_\omega = -(\tilde{E}_{\varepsilon,2}^{(N,2N]})^0$

$(B_{b,\varepsilon})^0$ solves $\lambda(\varepsilon)(B_{b,\varepsilon})^0 - (B_{b,\varepsilon})^0 \circ T_\omega = -(\tilde{A}_{\varepsilon,2})^0$

(11) Find $\overline{W_{\varepsilon,2}}, \sigma_\varepsilon$ by solving

$$\begin{pmatrix} \overline{S_\varepsilon} & \overline{S_\varepsilon (B_{b,\varepsilon})^0} + \overline{\tilde{A}_{\varepsilon,1}} \\ \varepsilon^3 \, \mathrm{Id} & \overline{\tilde{A}_{\varepsilon,2}} \end{pmatrix} \begin{pmatrix} \overline{W_{\varepsilon,2}} \\ \sigma_\varepsilon \end{pmatrix} = \begin{pmatrix} -\overline{S_\varepsilon (B_{a,\varepsilon})^0} - \overline{\tilde{E}_{\varepsilon,1}^{(N,2N]}} \\ -\overline{\tilde{E}_{\varepsilon,2}^{(N,2N]}} \end{pmatrix}$$

(12) $(W_{\varepsilon,2})^0 = (B_{a,\varepsilon})^0 + (B_{b,\varepsilon})^0 \sigma_\varepsilon$

(13) $W_{\varepsilon,2} = (W_{\varepsilon,2})^0 + \overline{W_{\varepsilon,2}} \sim \mathcal{O}(|\varepsilon|^{N+1})$

(14) $(W_{\varepsilon,1})^0$ solves $(W_{\varepsilon,1})^0 - (W_{\varepsilon,1})^0 \circ T_\omega = -(S_\varepsilon W_{\varepsilon,2})^0 - (\tilde{E}_{\varepsilon,1}^{(N,2N]})^0 -$

(15) $\overline{W_{\varepsilon,1}} = -\left(\int_{\mathbb{T}^d} \left[M_0^{-1} \alpha_\varepsilon\right]_1 d\theta\right)^{-1} \int_{\mathbb{T}^d} \left[M_0^{-1} \left(\alpha_\varepsilon (W_{\varepsilon,1})^0 + V_\varepsilon W_{\varepsilon,2}\right)\right]_1 d\theta$

(16) $W_{\varepsilon,1} = (W_{\varepsilon,1})^0 + \overline{W_{\varepsilon,1}} \sim \mathcal{O}(|\varepsilon|^{N+1})$

(17) $\Delta_\varepsilon \leftarrow M_\varepsilon W_\varepsilon$

(18) $K_\varepsilon^{[\leq 2N]} \leftarrow K_\varepsilon^{[\leq N]} + \Delta_\varepsilon^{(N,2N]}$

$$K_\varepsilon^{[\leq N]}(\theta) + \Delta_\varepsilon^{(N,2N]}(\theta)$$

- As function on θ are analytic in a complex neighborhood of the torus
- As function on ε are analytic functions in balls in the complex plane.

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At each step, h, we lose domain of analyticity in θ and ε.

$$\sup_{\varepsilon \in B_{r_{h+1}}} \|\Delta_{\varepsilon}\|_{\rho_{h+1}} \leq C_h \sup_{\varepsilon \in B_{r_h}} \|E_{\varepsilon}\|_{\rho_h}$$

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- In θ, at step h will have estimates in neighborhoods of the torus of width $\rho_{h+1} = \rho_h - \delta_h \geq \frac{\rho_0}{2}$, $\delta_h = \frac{\rho_0}{2^{h+2}}$
- In ε, at step h, will have estimates on balls with center at the origin and radius $r_h = \mathcal{O}\left((2^h N_0)^{-\tau/\alpha}\right) \to 0$

The **quasi-Newton method does not converge**, however, it allows us to construct formal power series and get estimates in balls for the coefficients, K_n and μ_n.

Gevrey estimates

At step h

$$\tilde{K}_{\varepsilon}^{[\leq 2^{h+1}N_0]}(\theta) = K_{\varepsilon}^{[\leq 2^{h}N_0]}(\theta) + \sum_{n=2^h N_0+1}^{2^{h+1}N_0} K_n(\theta)\varepsilon^n$$

we get estimates for the coefficient of order $n \in \left(2^h N_0, 2^{h+1}N_0\right] \cap \mathbb{N}$, in balls of radius $r_h = \mathcal{O}\left((2^h N_0)^{-\tau/\alpha}\right)$

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we get estimates for the coefficient of order $n \in \left(2^h N_0, 2^{h+1} N_0\right] \cap \mathbb{N}$, in balls of radius $r_h = \mathcal{O}\left((2^h N_0)^{-\tau/\alpha}\right)$ Using Cauchy estimates will have

estimates of the form

$$\|K_n\|_{\rho_0/2} = \mathcal{O}(r_h^{-n}) \sim \mathcal{O}((2^h N_0)^{(\tau/\alpha)2^{h+1}N_0}) \sim \mathcal{O}(n^{(c\tau/\alpha)n})$$

Final remarks

- Exponent, $\sigma = 2\tau/\alpha$, does not seem to be optimal. It might be improved choosing different rates r. We use $r = \mathcal{O}(s^{-\tau/\alpha})$

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- Exponent, $\sigma = 2\tau/\alpha$, does not seem to be optimal. It might be improved choosing different rates r. We use $r = \mathcal{O}(s^{-\tau/\alpha})$
- Trig. polynomial hypothesis can be relaxed?
- Method of proof might work for expansions of other type of invariant objects.
    - Functional equation $\to$ Newton method $\to$ estimates in balls.

Thank you very much!

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