Normal forms for the Laplace Resonance

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Motivation and Content

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- Alessandra Celletti
- Efsevia Karampotsiou
- Christoph Lhotka
- Mara Volpi

Scheme of the talk:

- The Laplace resonance
- The model and its normal forms
- Applications

Multi-resonant 1+3 body gravitational systems

Three-body mean motion resonances involve the orbital frequencies in the form

$$k_1\dot{\lambda}_1 + k_2\dot{\lambda}_2 + k_3\dot{\lambda}_3 \simeq 0, \quad k_1, k_2, k_3 \in \mathbb{Z},$$

where

$$\dot{\lambda}_1,\ \dot{\lambda}_2,\ \dot{\lambda}_3$$

are the *mean motions*. If there is NO integer m such that

$$k_1\dot{\lambda}_1 + (k_2 - m)\dot{\lambda}_2 \simeq 0; \quad m\lambda_2 + k_3\dot{\lambda}_3 + \simeq 0, \quad m \in \mathbb{Z}\backslash\{0\},$$

the resonance is said *pure.*

Laplace resonance

The *Laplace resonance* is given by the simplest case ($m = 1$) so that, with $k_1 = -1, k_2 = 3, k_3 = -2$, we have the 2:1 – 2:1 decomposition

$$2\dot{\lambda}_2 - \dot{\lambda}_1 \simeq 0; \quad 2\dot{\lambda}_3 - \dot{\lambda}_2 \simeq 0$$

and the Laplace relation is

$$3\dot{\lambda}_2 - 2\dot{\lambda}_3 - \dot{\lambda}_1 \simeq 0.$$

There are several other possible cases called *Laplace-like resonances*: for example, again with $m = 1$, the chain 3:2 – 2:1 and, with $m = 2$, the chain 3:2 – 3:2.

Examples

Laplace resonance (2:1 – 2:1)

- The Galilean system (Io-Europa-Ganymede)
- The GJ-876 exo-planetary system (EPS, planets c-b-e)
- The GJ-2046 EPS (HD40307, planets b-c-d)
- The Uranian system (Miranda-Ariel-Umbriel)

Laplace-like resonances

- Kepler 305 EPS (3:2 – 2:1)
- Kepler 114 EPS (3:2 – 3:2)
- YZ Cet EPS (3:2 – 3:2)

Model

Toy-model for a planar 1+3 body multi-resonant system

Hamiltonian $H = H_{Kep} + H_{Res}$ extending the works by Sinclair (1975), Henrard (1982,1984) and Malhotra (1991):

- Keplerian interaction with a 'large' central spherical m_0.
- Disturbing function (Resonant part): Coupling terms of the minor bodies ($m_k,\ k = 1, 2, 3$) truncated to 2nd order in eccentricity and averaged with respect to fast variables

$$H_{Res} = -\mathcal{R}_{avg}(L_k, P_k, \lambda_k, p_k).$$

We use modified Delaunay variables (with $m_0 \gg m_k$):

$$L_k \simeq m_k \sqrt{m_0 a_k}, \quad P_k \simeq \frac{1}{2} L_k e_k^2.$$

Keplerian part:

$$H_{Kep} = \sum_{k=1}^{3} \left[\overline{n}_k (L_k - \overline{L}_k) - \frac{3}{2} \eta_k (L_k - \overline{L}_k)^2 \right]$$

$\overline{L}_k$ are three 'nominal' values of the first actions and

$$\overline{n}_k = \sqrt{\frac{m_0}{\overline{a}_k{}^3}} = \frac{m_0^2 m_k^3}{\overline{L}_k^3}, \ (G = 1), \qquad \eta_k = \frac{\overline{n}_k}{\overline{L}_k}$$

are evaluated at nominal values. Exactly resonant mean motions such that

$$\overline{n}_1 = 2\overline{n}_2 = 4\overline{n}_3$$

would provide the set of resonant nominal actions.

Resonant part:

$$
\begin{aligned}
\mathcal{R}_{avg} \quad = \quad & \frac{m_1 m_2}{a_2} \times \{\gamma_1 \left(a_2/a_1\right) e_1 \cos(2\lambda_2 - \lambda_1 - \varpi_1) \\
& \qquad + \gamma_2 \left(a_2/a_1\right) e_2 \cos(2\lambda_2 - \lambda_1 - \varpi_2)\} + \\
& \frac{m_2 m_3}{a_3} \times \{\gamma_1 \left(a_3/a_2\right) e_2 \cos(2\lambda_3 - \lambda_2 - \varpi_2) \\
& \qquad + \gamma_2 \left(a_3/a_2\right) e_3 \cos(2\lambda_3 - \lambda_2 - \varpi_3)\} + \mathrm{O}(e_k^2) \ ,
\end{aligned}
$$

where

$$
a_k = \frac{L_k^2}{m_0 m_k^2}, \quad e_k = \sqrt{\frac{2P_k}{L_k}}, \quad \varpi_k = -p_k, \quad k = 1, 2, 3.
$$

Variables adapted to the resonance

Canonical transformation
$(L_k, P_k, \lambda_k, p_k) \longrightarrow (Q_\alpha, q_\alpha), \quad \alpha = 1, \ldots, 6$:

$$\begin{aligned}
q_1 &= 2\lambda_2 - \lambda_1 + p_1 \ , & Q_1 &= P_1 \ , \\
q_2 &= 2\lambda_2 - \lambda_1 + p_2 \ , & Q_2 &= P_2 \ , \\
q_3 &= 2\lambda_2 - \lambda_1 + p_3 \ , & Q_3 &= P_3 \ , \\
q_4 &= 3\lambda_2 - 2\lambda_3 - \lambda_1 \ , & Q_4 &= \tfrac{1}{3}(2L_1 + L_2 - L_3) \ , \\
q_5 &= \lambda_1 - \lambda_3 \ , & Q_5 &= \tfrac{1}{3}(3L_1 + L_2 + \Gamma) \ , \\
q_6 &= \lambda_3 \ , & Q_6 &= L_1 + L_2 + L_3 - \Gamma
\end{aligned}$$

$\Gamma = P_1 + P_2 + P_3$ is the *angular momentum deficit*.
Q_6 is the total angular momentum.
$q_3 = \hat{q}_3 + q_4, \ \hat{q}_3 = 2\lambda_3 - \lambda_2 + p_3$.

Keplerian part ($\Gamma = Q_1 + Q_2 + Q_3$):

$$H_{Kep} = \kappa_1 \Gamma + \kappa_4 Q_4 - \frac{3}{2} A \Gamma^2 - 3 B \Gamma Q_4 - \frac{3}{2} C Q_4^2 .$$

Resonant part:

$$\begin{aligned} H_{Res} &= -\alpha \sqrt{2Q_1} \cos q_1 - \beta_1 \sqrt{2Q_2} \cos q_2 \\ &\quad -\beta_2 \sqrt{2Q_2} \cos(q_2 - q_4) - \gamma \sqrt{2Q_3} \cos q_3 + \mathrm{O}(Q_k). \end{aligned}$$

Q_5, Q_6 are integrals of motion and provide the frequencies

$$\begin{aligned} \kappa_1 &= 6(\eta_1 + 3\eta_2) Q_5 - (\eta_1 + 6\eta_2) Q_6 \ , \\ \kappa_4 &= 3(2\eta_1 + 9\eta_2 + 2\eta_3) Q_5 - (\eta_1 + 9\eta_2 - 2\eta_3) Q_6 . \end{aligned}$$

Parameters: $A = \eta_1 + 4\eta_2$, $B = \eta_1 + 6\eta_2$, $C = \eta_1 + 9\eta_2 + 4\eta_3$, $(0 < A < B < C)$ and coupling constants:

$$\alpha, \beta_2 < 0, \quad \beta_1, \gamma > 0$$

depend on the mass ratios m_k / m_0 (to first order), the Laplace coefficients and $\overline{L}_k$.

Poincaré variables x_j, y_j ($j = 1, 2, 3$):

$$x_j = \sqrt{2Q_j}\cos q_j, \quad y_j = \sqrt{2Q_j}\sin q_j \quad \Gamma(x,y) = \frac{1}{2}\sum_{j=1}^{3}\left(x_j^2 + y_j^2\right).$$

New 'book-keeping' order of the Hamiltonian

$$H(Q_4, q_4, x_j, y_j) = H_0 + \epsilon^2 H_2 + \epsilon^4 H_4 :$$

$$\begin{aligned}
H_0 &= \kappa_4 Q_4 - \frac{3}{2} C Q_4^2, \\
H_2 &= \kappa_1 \Gamma(x,y) - 3B\Gamma(x,y)Q_4 - \alpha x_1 - \beta_1 x_2 \\
&\quad -\beta_2(x_2\cos q_4 + y_2 \sin q_4) - \gamma(x_3 \cos q_4 + y_3 \sin q_4), \\
H_4 &= -\frac{3}{2} A \Gamma(x,y)^2.
\end{aligned}$$

Equilibria at first order (de Sitter-Sinclair): $\dot{Q}_4 = 0, \dot{q}_4 = 0, \dot{x}_j = 0, \dot{y}_j = 0$ $(j = 1, 2, 3)$

$$
\begin{aligned}
Q_{4E} &= \frac{\kappa_4}{3C}, & q_{4E} = 0, \pi, \\
x_{1E} &= \frac{\alpha}{\omega}, & y_{1E} = 0, \\
x_{2E} &= \frac{\beta_1 \pm \beta_2}{\omega}, & y_{2E} = 0, \\
x_{3E} &= \pm\frac{\gamma}{\omega}, & y_{3E} = 0
\end{aligned}
$$

where

$$
\begin{aligned}
\omega &= \kappa_1 - 3BQ_{4E} = \kappa_1 - \frac{B}{C}\kappa_4 \\
&\simeq \frac{3}{C}\left[(B - C)(2\eta_2 L_2 - \eta_1 L_1) - B(2\eta_3 L_3 - \eta_2 L_2)\right].
\end{aligned}
$$

	e_1	e_2	e_3	q_1	q_2	q_3	q_4
observed	0.0042	0.0094	0.0015	0	π	rotating	π
TM1	0.0058	0.0118	0.0008	0	π	π	π
TM2	0.0042	0.0095	0.0011	0	π	π	π
AdS	0.0055	0.0113	0.0008	0	0	π	0

Table: Galilean system.

	e_1	e_2	e_3	q_1	q_2	q_3	q_4
observed	0.253	0.037	0.031	0	0	rotating?	0
TM1	0.048	0.005	0.006	π	0	0	π
NdS1	0.254	0.053	0.038	0	0	π	0

Table: GJ-876 system.

First-order toy model (TM1):

$$\dot{y}_1 = 0 \quad \longrightarrow \quad -\alpha + \omega x_1 + O(\epsilon) = 0 \longrightarrow \quad x_{1E} = \frac{\alpha}{\omega},$$
$$\dot{x}_1 = 0 \quad \longrightarrow \quad -\omega y_1 + O(\epsilon) = 0 \quad \longrightarrow \quad y_{1E} = 0.$$

TM2 (Terms of second-order in the eccentricity):

$$H_3 = \sum \alpha_{jk} x_j x_k \sim \epsilon^3, \quad j,k = 1,2,3.$$

But if we allow for 'large' eccentricities ($O(1)$), we get NEW de Sitter solutions of the cubic equation:

$$\dot{y}_1 = 0 \longrightarrow -\epsilon\alpha + \left(\hat{\omega} - k x_1^2\right) x_1 = 0, \quad k = \frac{3(AC - B^2)}{2C} > 0$$

New de Sitter (NdS) equilibria

Look for solutions of the form $x_1 = x_1^{(0)} + \epsilon x_1^{(1)}$. For $\epsilon = 0$ we have the three solutions: $x_1^{(0)} = 0, \; \pm\sqrt{\hat{\omega}/k}$ with the condition $\hat{\omega} > 0$. So we get two additional sets of solutions:

$$
\begin{aligned}
x_{1EN} &= \pm\sqrt{\frac{2C\hat{\omega}}{3(AC-B^2)}} - \frac{\alpha}{2\hat{\omega}}, \\
x_{2EN} &= \frac{\beta_1 \mp \beta_2}{\hat{\omega} + \frac{3(B^2-AC)}{2C}(x_{1EN})^2}, \\
x_{3EN} &= \frac{\mp\gamma}{\hat{\omega} + \frac{3(B^2-AC)}{2C}(x_{1EN})^2}
\end{aligned}
$$

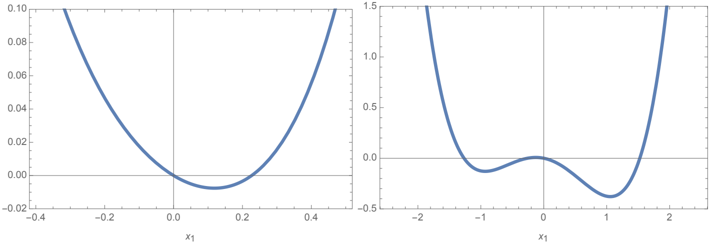


Figure: Left: de Sitter-Sinclair equilibrium. Right: New de Sitter equilibria.

Stability of the equilibria ($z = (Q_4, q_4, x_j, y_j)$):

$$\dot{\delta} z = \mathbf{J} H_{zz}\Big|_0 \delta z$$

All eigenvalues pure imaginary if $|\omega| > \omega_U(\overline{L}_k)$, Yoder and Peale (1981).

Normal form for the Laplace libration

Perform the translation $Q_4 \to \Lambda = Q_4 - Q_{4E} = Q_4 - \frac{\kappa_4}{3C}$:

$$H = (\omega - 3B\Lambda)\Gamma - \frac{3}{2}C\Lambda^2 - \frac{3}{2}A\Gamma^2 + H_{res}(Q_1, Q_2, Q_3, q_1, q_2, q_3, \lambda),$$

Assuming ω 'fast' with respect to the libration of the Laplace argument $\lambda = q_4 - q_{4E}$, we can normalise with respect to the 'isotropic oscillator'

$$H_I = \omega\Gamma = \omega(Q_1 + Q_2 + Q_3)$$

obtaining

$$K = 3B\Gamma\Lambda + \frac{3}{2}C\Lambda^2 \pm \frac{\beta_1\beta_2}{\omega}\cos\lambda.$$

Normal form for the Laplace libration

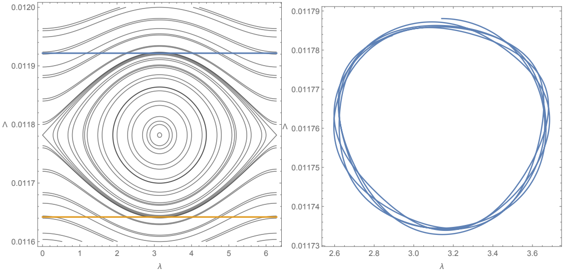


Figure: Left: Normal form around the de Sitter equilibrium for the Galilean toy model. Right: projection of numerical phase-curve.

The modified pendulum gives the libration frequency and the resonance width:

$$\Omega_L = \sqrt{\frac{3C\beta_1\beta_2}{\omega}}, \quad \Delta\Lambda = 4\sqrt{\frac{\beta_1\beta_2}{3C\omega}}.$$

Application: Oblateness of the central body

Taking into account the extra precessions of the 'peri-joves'

$$\sum_{j=1}^{3} \omega_j Q_j = \sum_{j=1}^{3} \left(\kappa_j - \frac{B}{C} \kappa_4 \right) Q_j, \quad \kappa_j = \kappa_1 + \frac{\partial H_{sec}(J_2)}{\partial Q_j},$$

due to the quadrupole of the central body, we can compute a higher-order 'detuned' normal form around de Sitter equilibria

$$K_L = \sum_{l,m} a_{lm} \Lambda^l \cos m\lambda.$$

Libration period	Toy model	Normal form	Lainey (2006)
T_1	530	406	403.5
T_2	530	479	462.7
T_3	530	490	482.2
T_L	1600	2057	2060.0

Table: The libration periods in days of the Galilean system.

Tidal coupling between central and orbiting bodies produces a slow variation of their orbital elements:

$$\frac{da_j}{dt} = \mathrm{sign}(\Omega_0 - n_j) c_j a_j, \quad \frac{de_j}{dt} = -\frac{e_j}{\tau_j},$$

where

$$c_j = \frac{9}{2} \frac{k_{20}}{Q_0} \frac{m_j}{m_0} \left(\frac{R_0}{a_j} \right)^5 n_j, \quad \tau_j = \frac{2}{21} \frac{Q_j m_j a_j^5}{k_{2j} m_0 R_j^5 n_j}.$$

Fast rotating central body (‘planet’, $\Omega_0 > n_j$): semi-major axes increase.
Slow rotating central body (‘star’, $\Omega_0 < n_j$): semi-major axes ‘usually’ decrease.

Tidal evolution in a multi-resonant system

Consider the closest orbiting body ('Io'). The evolution of the eccentricity vector can be approximated by:

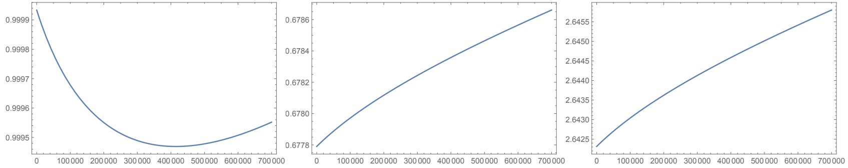
$$x_1 = \sqrt{L_1} e_1 \cos q_1 = \frac{\alpha}{\omega} + C_1 \mathrm{e}^{-t/\tau_1} \cos \omega t$$
$$y_1 = \sqrt{L_1} e_1 \sin q_1 = -\frac{\alpha}{\omega^2 \tau_1} + C_1 \mathrm{e}^{-t/\tau_1} \sin \omega t$$

Therefore, for $t \gg \tau_1 \sim 10^8\ yr$

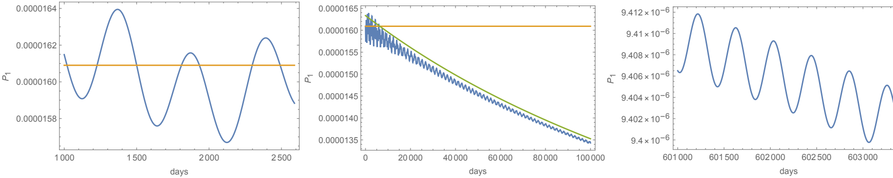
$$x_1 \to \frac{\alpha}{\omega}, \quad q_1 \simeq \frac{y_1}{x_1} \to -\frac{1}{\omega \tau_1}.$$

The small displacements from the libration centers (present also in the other angles) appear in the evolution of the semi-axes through equations of the form:

$$\frac{dL_1}{dt} = \alpha y_1 \Big|_{resonant} + \frac{dL_1}{dt}\Big|_{tidal} = -\frac{\alpha^2}{\omega^2 \tau_1} + \frac{1}{3} c_1 L_1 .$$

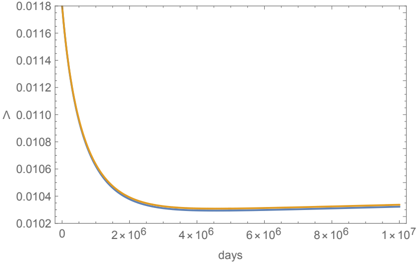


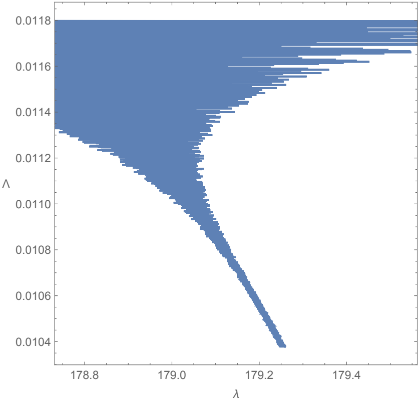
L_k of the Galilean satellites (Io, Europa, Ganymede).



$$P_1^{(FE)} = \frac{1}{2} x_{1E}^2 = \frac{\alpha^2}{2\omega^2}, \quad \omega = \omega(L_1, L_2, L_3) \quad with \quad \frac{d}{dt}|\omega| > 0.$$

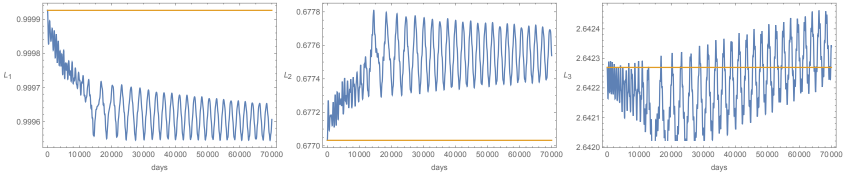
An adiabatic invariant for the tidal evolution



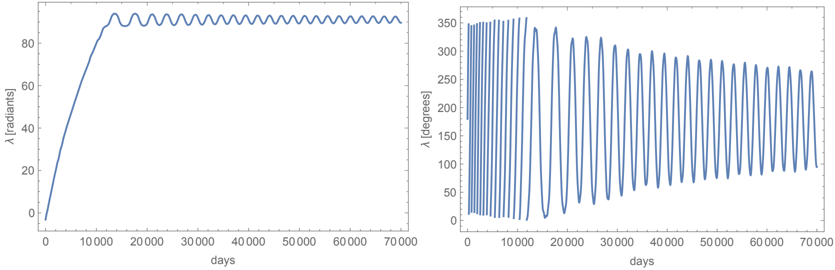


The action variable $\Lambda \simeq \frac{1}{3}(2L_1 + L_2 - L_3)$ can be used as an adiabatic invariant in the slow evolution in which the libration amplitude damps out.

Trapping in the Laplace resonance

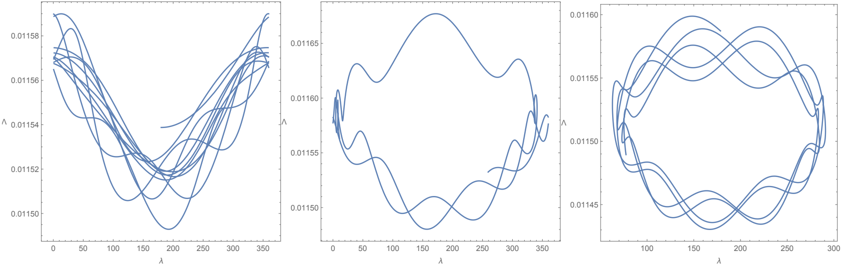


L_1, L_2, L_3 trapped in resonance



Passage of the Laplace angle λ from rotation to libration.

Trapping in the $\Lambda - \lambda$ phase plane



Following Yoder and Peale (1981) and Henrard (1983), we can compute a Probability of Capture

$$P_C = \frac{2}{1 - \frac{3\pi C L_1(0)}{84\omega_L D_1}\left(1 - 7\frac{\alpha^2 D_1}{\omega^2}\right)}, \quad D_1 = \frac{3}{7c\tau_1} = \frac{Q_0}{Q_1}\frac{k_{21}}{k_0}\left(\frac{R_1}{R_0}\right)^5 \frac{m_0^2}{m_1^2}.$$

Developments and follows up

- We are characterising the transient post-capture evolution of Io and Europa in order to better constrain the Love numbers.
- We are studying the trapping in resonance of Callisto in order to test the results by Lari, Saillenfest and Fenucci (2020) with our model.
- The evolution of exo-planetary systems in Laplace resonance can be investigated with the methods of Pichierri, Batygin and Morbidelli (2019)
- The effects of inter-satellite tides, even if very small, can be important in the resonant dynamics (B. Scoppola)

THANK YOU FOR THE ATTENTION
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