Approximate symmetries in the BepiColombo radio science experiment: a numerical approach

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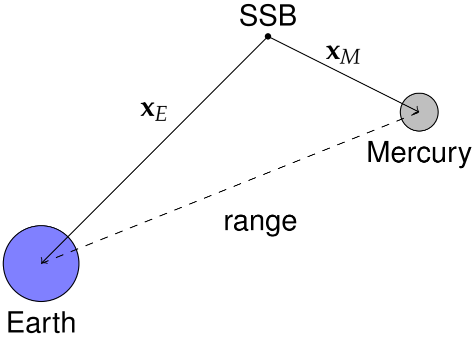
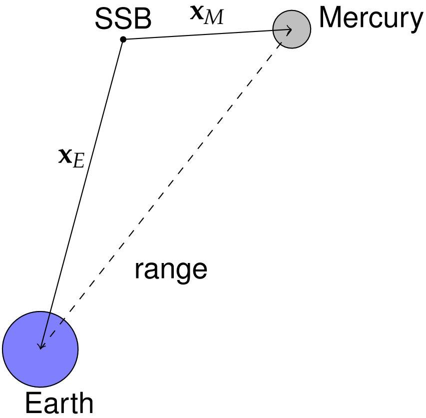
Symmetries: an example



Symmetries: an example



Orbit determination

Definition

Reconstruction of the orbit of a celestial body and determination of other parameters starting from observations of the body.

Observables $\mathbf{O} = (O_1, O_2, \ldots, O_n)$ at times $t_1, \ldots, t_n$

Computed observables $\mathbf{C}(\mathbf{u}) = (\mathscr{C}(t_1, \mathbf{u}), \mathscr{C}(t_2, \mathbf{u}), \ldots, \mathscr{C}(t_n, \mathbf{u}))$

Residuals

$$\boldsymbol{\xi}(\mathbf{u}) = \mathbf{O} - \mathbf{C}(\mathbf{u})$$

In order to determine $\mathbf{u}$, we minimize the *target function*

$$Q(\mathbf{u}) = \frac{1}{n} \boldsymbol{\xi}(\mathbf{u}) \cdot W \boldsymbol{\xi}(\mathbf{u})$$

The minimization of Q leads to the *differential corrections algorithm*:

$$C(\mathbf{u}_{k+1} - \mathbf{u}_k) = -B^T W \boldsymbol{\xi},$$

where $B = \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{u}}$ is the *design matrix* and $C = B^T W B$ is the *normal matrix* (symmetric and definite positive).

Gauss Theorem

The inverse Γ of the normal matrix C is the covariance matrix of the gaussian distribution $\mathbf{u}^*$, solution of the least squares process.

Symmetries and rank deficiencies

Definition

A group G of transformations acting on the space of parameters is called *group of exact symmetries* if it is a Lie group and

$$\boldsymbol{\xi}(g[\mathbf{u}]) = \boldsymbol{\xi}(\mathbf{u}) \quad \text{for all } g \in G.$$

Theorem

If there is a group G of exact symmetries of dimension d and the no-isotropy condition applies, then the normal matrix C has rank $N - d$.

Consequence: C is not invertible and

$$C(\mathbf{u}_{k+1} - \mathbf{u}_k) = -B^T W \boldsymbol{\xi}$$

has no unique solution.

Examples

Rotation symmetry

The $(N+1)$-body problem has an exact rank deficiency of order 3, corresponding to the group of rotations $SO(3)$.
The group leaves invariant the lagrangian and the equations of motion:

$$L = \frac{1}{2}\sum_{i=0}^{N} m_i|\dot{\mathbf{r}}_i|^2 + \sum_{0\le i<j\le N} \frac{Gm_im_j}{|r_i - r_j|},$$

in particular range and range-rate do not vary.

Scaling symmetry

If the observables are all angular observations, scaling all masses, lengths and times in such a way that the scaling factors satisfy $\lambda^3 = \mu\tau^2$, is an exact symmetry of order 1.

Curing rank deficiencies

In presence of a rank deficiency of order d:

Descoping Exactly d parameters have to be removed from the fit.

A priori information Equivalent to adding new observations, $u_i = u_i^P \pm \sigma_i$:

$$C_{\text{new}} = C + C^P, \quad Q_{\text{new}} = Q + \frac{1}{N+m}\left[(\mathbf{u} - \mathbf{u}^P)^T C^P (\mathbf{u} - \mathbf{u}^P)\right]$$

A priori constraints Same as previous method, but with a priori information involving a combination of the parameters.

Approximate rank deficiency

Definition

An approximate rank deficiency of order d occurs when $\exists K \subset \mathbb{R}^N$ of dimension d and there exists $0 < \epsilon \ll 1$ such that

$$|B\mathbf{v}| \leq \epsilon \quad \forall \in K.$$

If $|\mathbf{v}| = 1$,

$$\mathbf{v}^T C\mathbf{v} = \mathbf{v}B^T B\mathbf{v} = (B\mathbf{v})^T (B\mathbf{v}) = |B\mathbf{v}|^2 \leq \epsilon^2$$

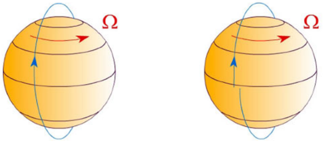
therefore C has at least d eigenvalues $\leq \epsilon^2$.

Definition

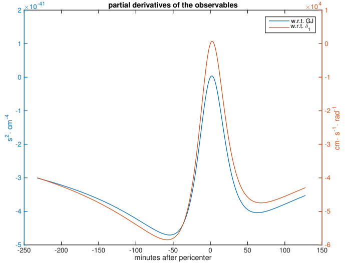
An approximate symmetry is one-parameter group G changing the residuals by $\mathcal{O}(\epsilon)$.

An example of approximate rank deficiency

An approximate rank deficiency of order 1 was found in the **Juno** radio science experiment (Serra et al., 2016), involving the pole angles of Jupiter and Jupiter's angular momentum.



The frame-dragging effect of Lense-Thirring.



Parameter	Value	Formal σ (no pole)	Formal σ (pole)
$GJ(\mathrm{cm}^5/\mathrm{s}^3)$	2.82×10^{38}	5.25×10^{36}	8.52×10^{38}

The BepiColombo mission

BepiColombo is an ESA/JAXA mission for the exploration of Mercury, launched in October 2018, scheduled to arrive in 2025.

MORE (Mercury Orbiter Radioscience Experiment) is aimed at:

- determining Mercury's gravity field,
- determining Mercury's rotational state,
- performing a relativity experiment.

The first Superior Conjunction Experiment for the determination of the Parameterized Post-Newtonian parameter γ is being performed as we speak.

The innovation of BepiColombo

Figure: Ka-band transponder onboard BepiColombo MPO spacecraft.

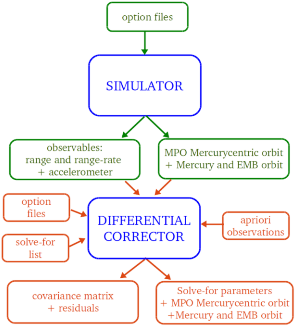
The BepiColombo spacecraft is endowed with a triple link X/X, X/Ka and Ka/Ka both for range and Doppler, ensuring:

- measurements as accurate as 15 cm at 300 s and 1.6×10^{-4} Hz at 1000 s
- the capability to cancel the plasma noise.

The role of the University of Pisa: ORBIT14

The ORBIT14 project started in 2007 for the analysis of the **BepiColombo** Radio Science experiment, soon after also for **Juno**'s.

- **11** people (4 women) have worked on the software
- **100000** lines of code have been written
- **362** updates to the software have been made
- **17** papers in journal/proceedings have been published about numerical simulations with ORBIT14



Approximate symmetries in BepiColombo: a review

- Since the first simulations in 2002, it was assumed that the BepiColombo MORE experiment suffered from an approximate **rank deficiency of order 4** (rotation+scaling).
- Initial simulations used the **descoping** technique, eliminating 4 parameters from the fit: the Earth position coordinates and one velocity component.
- In recent years it was explored the possibility of using **a priori constraints** in order to eliminate the symmetries:

$$\Delta\mathbf{x}_M \cdot R\mathbf{x}_M + \Delta\dot{\mathbf{x}}_M \cdot R\dot{\mathbf{x}}_M + \Delta\mathbf{x}_E \cdot R\mathbf{x}_E + \Delta\dot{\mathbf{x}}_E \cdot R\dot{\mathbf{x}}_E = 0,$$

  where R is a generator of the Lie Algebra of $SO(3)$.
- Observations are not angles: **no scaling symmetry?**

Results of the relativity experiment

Simulations solving for all parameters show that the accuracy on $\beta, \eta, J_{2\odot}$ does not improve much with BepiColombo:

Parameter	Reference Sim.	Current knowledge
β	3.33×10^{-5}	1.8×10^{-5}
γ	1.32×10^{-6}	2.3×10^{-5}
η	1.34×10^{-4}	4.5×10^{-4}
α_1	1.16×10^{-6}	6×10^{-6}
α_2	1.88×10^{-7}	3.5×10^{-5}
$\mu_\odot$	1.08×10^{14}	8×10^{15}
$J_{2\odot}$	3.42×10^{-9}	2.2×10^{-9}
ζ	4.03×10^{-14}	4.3×10^{-14}

It was believed that this was due to the presence of approximate rank deficiencies.

Can we see the effects of the approximate symmetries?

Formal uncertainties of the solve-for parameters in three scenarios

Parameter	Reference Sim.	Descoping	Rotation Constraints
x_M	4.78×10^2	2.12×10^0	7.89×10^0
y_M	2.86×10^2	2.95×10^1	3.76×10^1
z_M	1.67×10^3	1.27×10^3	3.48×10^0
x_{EMB}	6.25×10^1	–	8.99×10^0
y_{EMB}	1.16×10^3	–	3.39×10^1
z_{EMB}	4.46×10^3	–	7.13×10^0
$\dot{x}_M$	3.01×10^{-4}	5.58×10^{-6}	9.07×10^{-7}
$\dot{y}_M$	2.72×10^{-4}	1.02×10^{-4}	1.14×10^{-6}
$\dot{z}_M$	1.05×10^{-3}	4.67×10^{-4}	1.48×10^{-6}
$\dot{x}_{EMB}$	2.29×10^{-4}	8.93×10^{-7}	2.12×10^{-7}
$\dot{y}_{EMB}$	8.55×10^{-6}	4.16×10^{-7}	6.81×10^{-7}
$\dot{z}_{EMB}$	6.34×10^{-4}	6.28×10^{-4}	1.18×10^{-6}
β	3.33×10^{-5}	4.36×10^{-6}	3.33×10^{-5}
γ	1.32×10^{-6}	1.32×10^{-6}	1.32×10^{-6}
η	1.34×10^{-4}	1.66×10^{-5}	1.34×10^{-4}
α_1	1.16×10^{-6}	9.00×10^{-7}	1.16×10^{-6}
α_2	1.88×10^{-7}	1.87×10^{-7}	1.87×10^{-7}
$\mu_\odot$	1.08×10^{14}	8.08×10^{13}	1.08×10^{14}
$J_{2\odot}$	3.42×10^{-9}	9.35×10^{-10}	3.42×10^{-9}
ζ	4.03×10^{-14}	3.69×10^{-14}	4.02×10^{-14}

Are there approximate rank deficiencies at all?

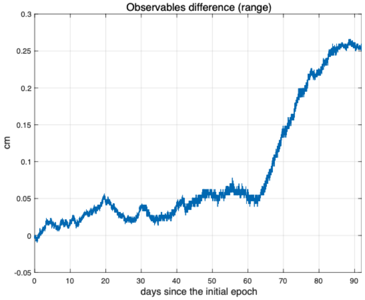
A numerical step-by-step approach

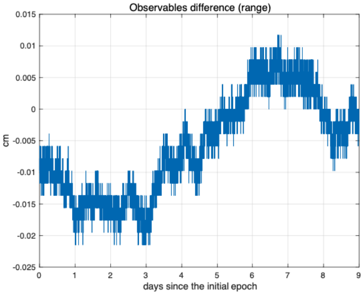
In order to understand the entity of the supposed approximate rank deficiency, we choose to analyze simple cases that are representative of the actual mission.

- We start with a basic model, where we are sure that an exact symmetry exists.
- We study the effect of single perturbations that break the symmetry in terms of singular values of the normal matrix.
- We compare the results.

Model 0

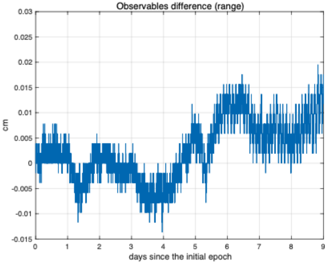
Three-body problem: Sun, Earth,Mercury.
Difference between residuals obtained rotating the initial conditions:



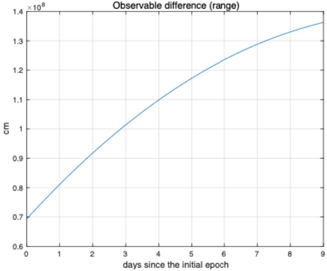


In order to be more realistic, we add the spacecraft.

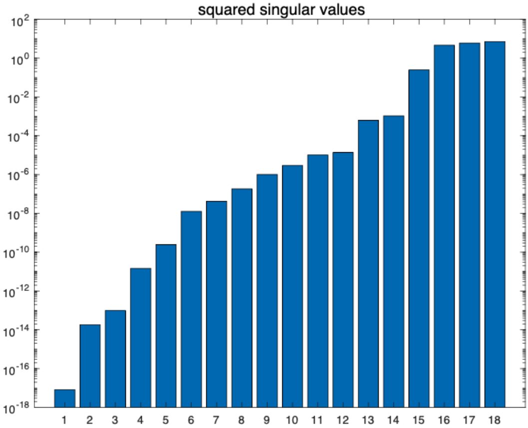
The rotation is also applied to the initial condition of the spacecraft.



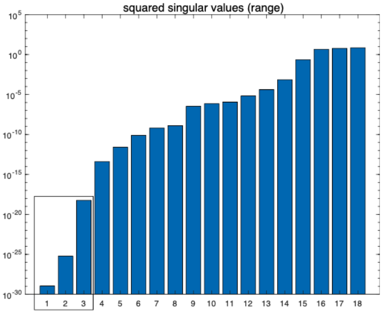
The rotation is not applied to the initial condition of the spacecraft.



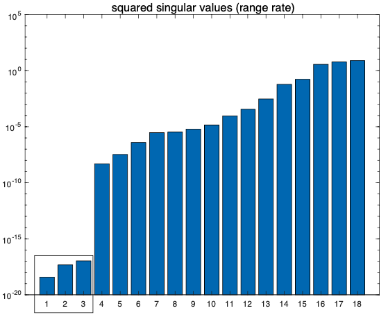
There is no sign of symmetry when we analyze the entire normal matrix:



Restriction to the range observables:

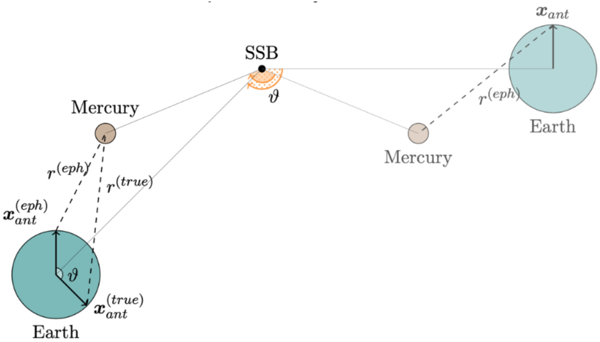


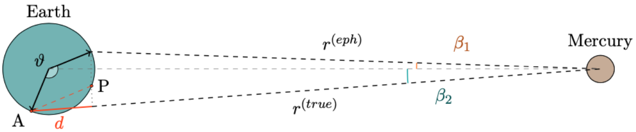
Restriction to the range rate observables:



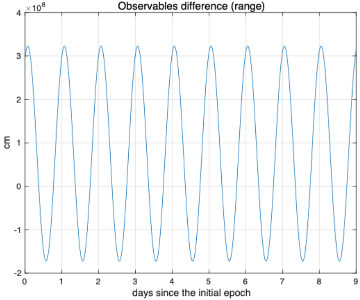
The effect of the antenna

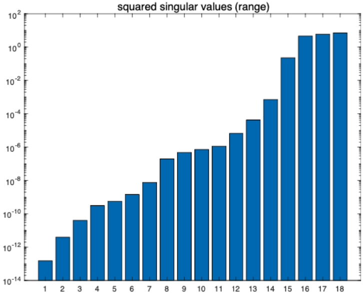
The position of the antenna with respect to the Earth barycenter is not a solve-for parameter, but it is taken by the IERS tables.





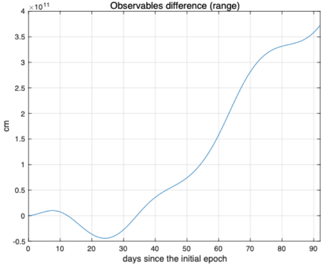
Basic model with antenna
Residuals and singular values

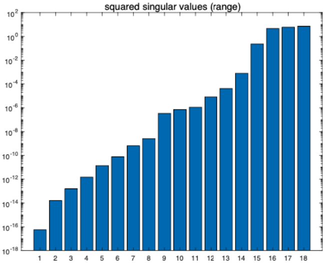




Basic model with Earth-Moon Barycenter
Residuals and singular values

We solve for the orbit of the Earth-Moon barycenter. We model it as a point mass

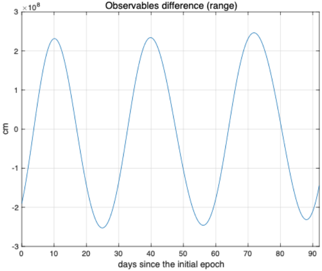


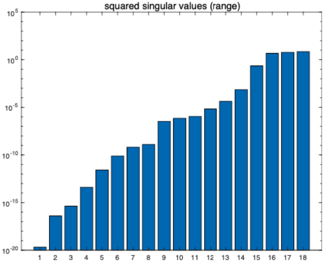


Basic model with Earth position
Residuals

In the range formulation there appears the Earth position:

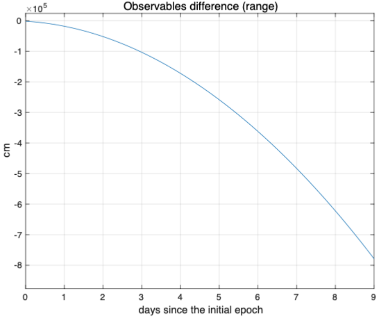
$$r = \left| \mathbf{x}_M + \mathbf{x}_{S/C} - \left(\mathbf{x}_{EMB} + \mathbf{x}_E + \mathbf{x}_{ant} \right) \right|.$$

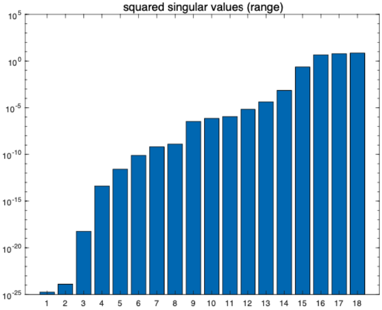




Basic model with other planets
Residuals

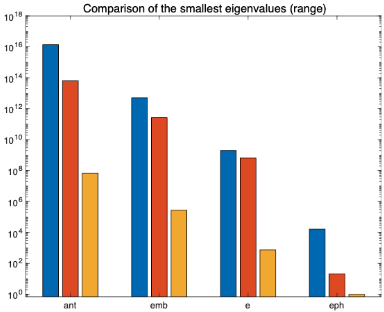
The position and velocities of the other planets of the solar system influence the motion of the EMB, Mercury and the spacecraft. Their states are taken from the JPL ephemerides.

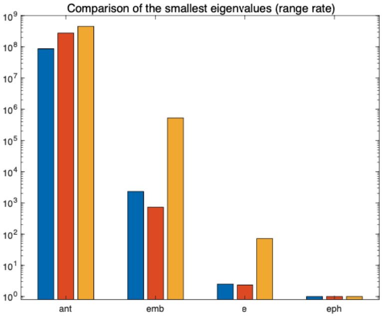




A comparison

Ratio of the three smallest eigenvalues:





Conclusions

- From the results of complete and accurate simulations BepiColombo's MORE experiment does not seem to be affected by approximate rank deficiencies.
- From the analysis of a basic toy model, it appears that putting together two different observables, each affected by the symmetry, is a source of symmetry disruption.
- Among the perturbative effects, the position of the antenna (i.e., the Earth rotation) appears to affect the symmetry, the other planets' ephemerides only in a minimum part.

Future work

- Analyze other sources of symmetry disruption: non-sphericity of the planets and the sun, relativity effects,...
- Is there a numerical explanation for the disruption of the exact symmetry in the basic model when using two kinds of observables?
- Is there a way to improve the determination of the PPN parameters?
  - De Marchi and Cascioli, 2020 propose to fit at the same time the radio science data of BepiColombo and other space missions.
  - Use independent results from other space missions as a priori information: Gaia, Lunar Laser Ranging, MESSENGER, tests on radio pulsars.