Closed-form perturbation theory with Lie Series

Christos Efthymiopoulos

Department of Mathematics Tullio Levi-Civita
University of Padua

“Closed form” perturbation theory

1. Contrary to the usual (Laplace-Lagrange) theory, closed-form theories **avoid expansions** of the disturbing function in powers of the eccentricity(ies).

2. As a result, useful for constructing secular models for highly eccentric orbits (the inversion of Kepler’s equation in powers of the eccentricity converges only up to e=0.6627…, the convergence gets quite slow for values of e beyond 0.3-0.4)

3. However: no explicit solution of the homological equation when the Kernel contains terms beyond the Keplerian ones

4. Proposed solution: **relegation**

5. Other limitation: requires multipolar (Legendre) expansion of the disturbing function in powers of the ratio of the particles’ distances

6. Alternatives to relegation: joint works with **I. Cavallari** (problem of perturbations external to the body) & **M. Rossi** (problem of perturbations internal to the body).

Some basic references:

Brouwer 1959, Kozai 1959 (several relations found in Tisserand 1889) - satellite problem (J2)

Deprit (1981) **Elimination of the parallax**

Deprit, Palacian & Deprit (2001) **Relegation**

Variants and subsequent developments: Metris & Exertier (1995), Palacian 2002, 2006, Segerman & Coffey (2000), Lara et al. (2013) (relegation revisited), Lara (and collaborators) 2014, 2018, 2019 (Earth and Lunar satellites), Ceccaroni & Biggs (2012, 2013 + Biscani 2014, relegation - motion around elongated asteroids), Sansottera & Ceccaroni (2017) (rigorous estimates on relegation) etc.

Presentation of algorithms: Jefferys 1971, Deprit (1981) & Coffey and Deprit 1982 (J2 at order 4), Metris thesis (1991) & Metris and Exertier (1995, J2+C22 + lunisolar perturbations), Healy (2000, algorithm), De Saedeleer 2006 (theory of lunar satellites)

An elementary example: the J2 satellite problem

J2 potential term

$$V_{J2} = -J_2 \frac{\mathcal{G} M_0 R^2}{2r^3} \left(1 - 3\frac{z^2}{r^2}\right)$$

transformation into elements (s=sini, c=cosi)

$$\begin{aligned} x &= r\cos(f+g)\cos h - r\sin(f+g)\cos i \sin h \\ y &= r\cos(f+g)\sin h + r\sin(f+g)\cos i \cos h \\ z &= r\sin(f+g)\sin i \end{aligned}$$

$$V_{J2} = J_2 \frac{\mathcal{G} M_0 R^2}{4r^3} \left[-2 + 3s^2 - 3s^2\cos(2f+2g)\right]$$

Series expansion (classical approach)

$$\cos f = \frac{2(1-e^2)}{e}\left(\sum_{\nu=1}^{\infty} J_\nu(\nu e)\cos(\nu M)\right) - e$$

$$\sin f = 2\sqrt{1-e^2}\sum_{\nu=1}^{\infty}\frac{1}{2}[J_{\nu-1}(\nu e) - J_{\nu+1}(\nu e)]\sin(\nu M)$$

$$r = a\left(1 + \frac{e^2}{2} - 2e\sum_{\nu=1}^{\infty}\frac{[J_{\nu-1}(\nu e) - J_{\nu+1}(\nu e)]\cos(\nu M)}{2\nu}\right)$$

Re-organized in powers of the eccentricity, these series converge up to e=0.6627…

Hamiltonian expressed in modified Delaunay variables.
Secular Hamiltonian: elimination of the fast angle (M or λ = M+g+h).

H_{sec}(g,h,L,G,H), or H_{sec}(p=-g-h,q=-h,P=L-G,Q=G-H)

The average of VJ2 with respect to the fast angle (mean anomaly) can be computed in closed form

$$< V_{J2} >= \frac{1}{2\pi}\int_0^{2\pi} J_2 \frac{\mathcal{G}M_0R^2}{4r^3}\left[-2+3s^2-3s^2\cos(2f+2g)\right] dM$$

Change of integration variable $df = \frac{a^2\eta}{r^2}dM$ where $\eta = \sqrt{1-e^2}$

After the change, there remains a factor 1/r in the integrand. However

$$\frac{1}{r} = \frac{1+e\cos f}{a\eta^2}$$

Thus, the integral becomes over purely trigonometric functions

$$< V_{J2} >= \frac{1}{2\pi}\int_0^{2\pi} J_2 \frac{\mathcal{G}M_0R^2}{4}\left(\frac{1+e\cos f}{a\eta^2}\right)\left[-2+3s^2-3s^2\cos(2f+2g)\right] \frac{df}{a^2\eta}$$

This leads to the well-known result:

$$< V_{J2} >= J_2 \frac{\mathcal{G} M_0 R^2}{2a^3(1-e^2)^{3/2}} \left(\frac{3}{2} \sin^2 i - 1 \right)$$

In this way ("scissor averaging") we can build the secular Hamiltonian of the J2 problem:

$$H_{sec,J2} = -\frac{GM_0}{2a} + J_2 \frac{\mathcal{G} M_0 R^2}{2a^3(1-e^2)^{3/2}} \left(\frac{3}{2} \sin^2 i - 1 \right)$$

However, can we compute, instead, **a canonical transformation** allowing to obtain the above Hamiltonian without scissor averaging?

Write the Hamiltonian as:

$$\mathcal{H} = -\frac{1}{2}a^2n^2 + V_{J2}$$

Is there (?) a generating function χ satisfying the homological equation:

$$\{-\frac{1}{2}a^2n^2, \chi\} + V_{J2} = < V_{J2} >$$

Answer:

$$\chi = \frac{nR^2}{8\eta^2}\Bigg(-4\phi + 6\phi\sin^2 i + 2e(-2+3\sin^2 i)\sin f$$
$$-3\sin^2 i\sin(2f+2g) - 3e\sin^2 i\sin(f+2g) - e\sin^2 i\sin(3f+2g)\Bigg)$$

How can we reach this beautiful result?

Depends on the quantity called `equation of the center' ***φ=f-M***

What happens when we consider the C22 term (kernel of the homological equation changes)

Dealing with the C22 term:

$$H = -\frac{1}{2}a^2n^2 - n_{\oplus}H + V_{J2} + V_{C22}$$

There is no problem in computing the average $<V_{C22}>$ (same as with J2)

$$V_{C22} = -C_{22}\frac{3\mathcal{G}MR^2}{r^5}\left(x^2 - y^2\right)$$

However, the homological equation will no longer be possible to solve in closed form

$$\{-\frac{1}{2}a^2n^2 - n_{\oplus}H, \chi\} + V_{J2} + V_{C22} = < V_{J2} > + < V_{C22} >$$

Possible solution: relegation

Assume the homological equation (in action-angle variables)

$$\{\omega_1 J_1 + \omega_2 J_2, \chi\} + b(J_1, J_2)\cos(k_1\phi_1 + k_2\phi_2) = 0$$

Solution

$$\chi = \frac{b(J_1, J_2)\sin(k_1\phi_1 + k_2\phi_2)}{k_1\omega_1 + k_2\omega_2}$$

Write this as

$$\chi = \frac{b(J_1, J_2)\sin(k_1\phi_1 + k_2\phi_2)}{k_1\omega_1(1 + \frac{k_2\omega_2}{k_1\omega_1})} = \frac{b(J_1, J_2)\sin(k_1\phi_1 + k_2\phi_2)}{k_1\omega_1}\left(1 - \frac{k_2\omega_2}{k_1\omega_1} + \left(\frac{k_2\omega_2}{k_1\omega_1}\right)^2 + \ldots\right)$$

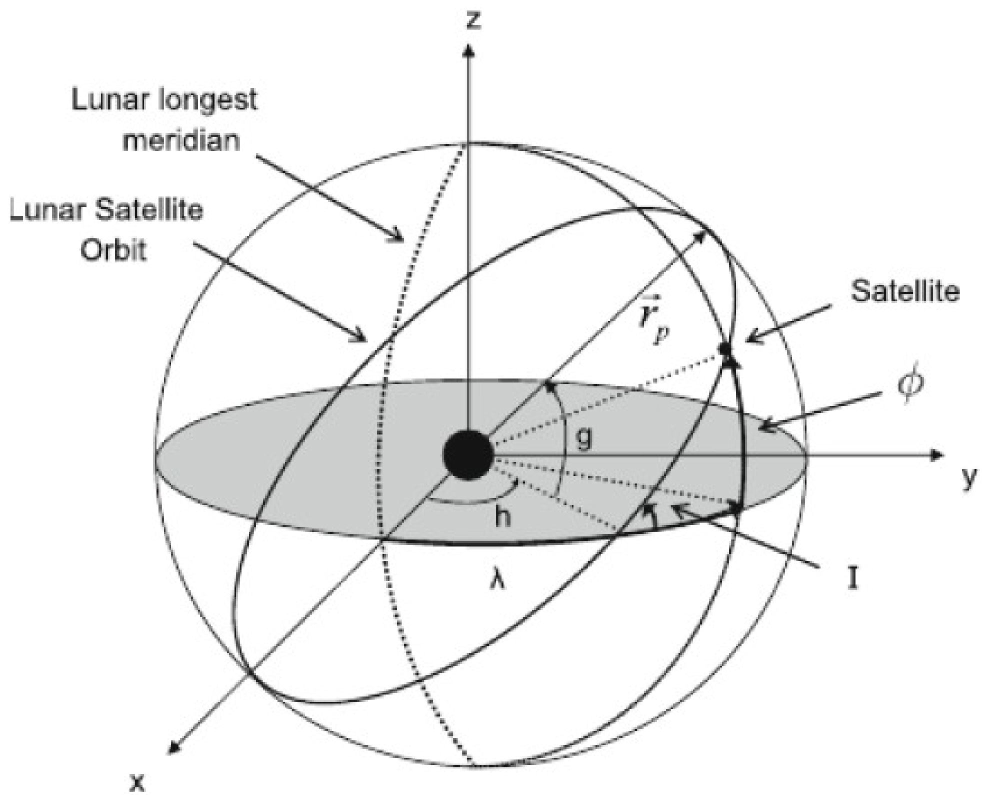
Same as solving repeatedly a homological equation with only ω1 J1 in the kernel!

$$\{\omega_1 J_1, \chi\} + b(J_1, J_2)\cos(k_1\phi_1 + k_2\phi_2) = 0$$

But why so much insistance in “closed form”?

Is it worthwhile the effort?

Secular theory for Lunar satellites (development of a semi-analytic propagator - CNES-UniPd-A



$$\mathcal{H} = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - n_{☾}(xp_y - yp_x) + V(x, y, z, t)$$

Hamiltonian

$$H = H_{Kep} - n_{☾} H + H_{☾,1} + H_{\oplus} + H_{rp}$$

$$H_{Kep} = -\frac{\mathcal{G} M_{☾}}{2a} = -\frac{\mathcal{G}^2 M_{☾}^2}{2L^2} = -\frac{1}{2} n_s^2 a^2$$

$$H_{☾,1} = -\frac{\mathcal{G} M_{☾}}{r} \sum_{n=0}^{\infty} \left(\frac{R_{☾}}{r} \right) \sum_{m=0}^{n} P_{nm}(\sin\phi)[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)]$$

$$H_{\oplus} = -\mathcal{G} M_{\oplus} \left(\frac{1}{\sqrt{r^2 + r_{\oplus}^2 - 2\mathbf{r}\cdot\mathbf{r}_{\oplus}}} - \frac{\mathbf{r}\cdot\mathbf{r}_{\oplus}}{r_{\oplus}^3} \right)$$

Expanded up to qudrupolar terms, this is

$$V_{\oplus} = \frac{\mathcal{G} M_{\oplus}}{2 r_{\oplus}^3} \left(r^2 - 3\frac{(\mathbf{r}\cdot\mathbf{r}_{\oplus})^2}{r_{\oplus}^2} + \mathcal{O}((r/r_{\oplus})^4) \right)$$

Estimate of the importance of the various terms (book-keeping scheme)

$$\frac{\Delta a(n)}{a_{Kep}} \sim \sum_{m=0}^{n} \frac{(n+1) R_{☾}^{n}}{r^{n}} \left(C_{nm}^2 + S_{nm}^2\right)^{1/2}$$

$$\frac{\Delta a_{\oplus}}{a_{Kep}} \sim \frac{M_{\oplus} r^3}{M_{☾} a_{\oplus}^3}, \qquad \frac{\Delta a_{\odot}}{a_{Kep}} \sim \frac{M_{\odot} r^3}{M_{☾} a_{\odot}^3}, \qquad \frac{\Delta a_{NI}}{a_{Kep}} \sim \frac{n_{☾}^2 r^3}{\mathcal{G} M_{☾}} = \frac{(M_{\oplus} + M_{☾}) r^3}{M_{☾} a_{\oplus}^3}$$

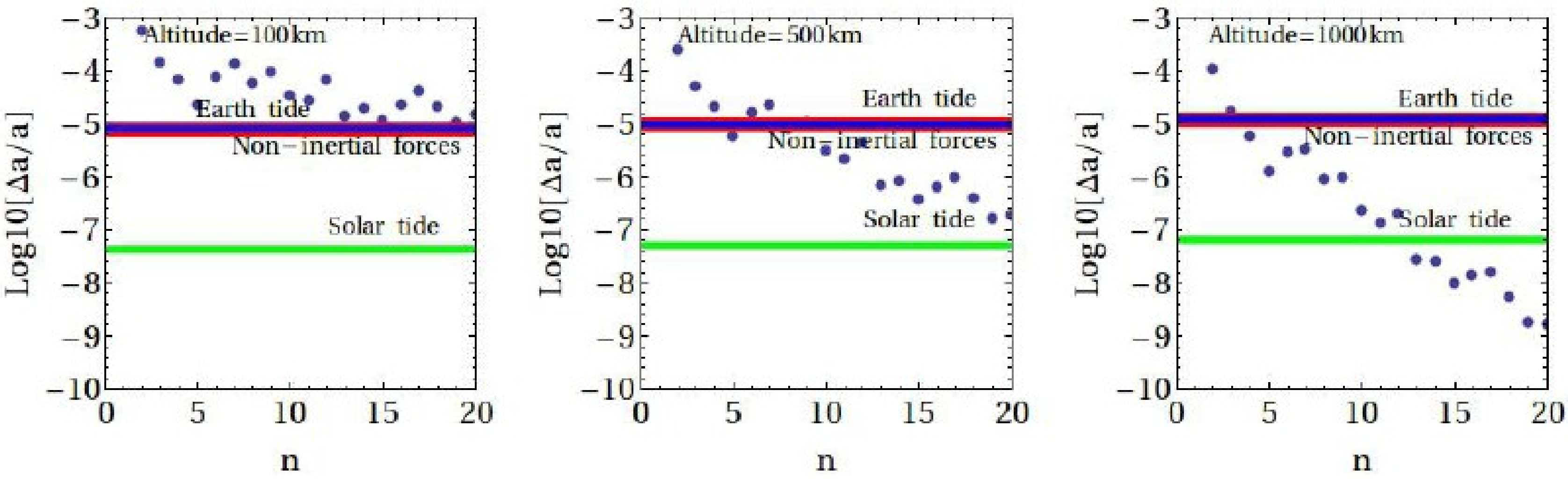
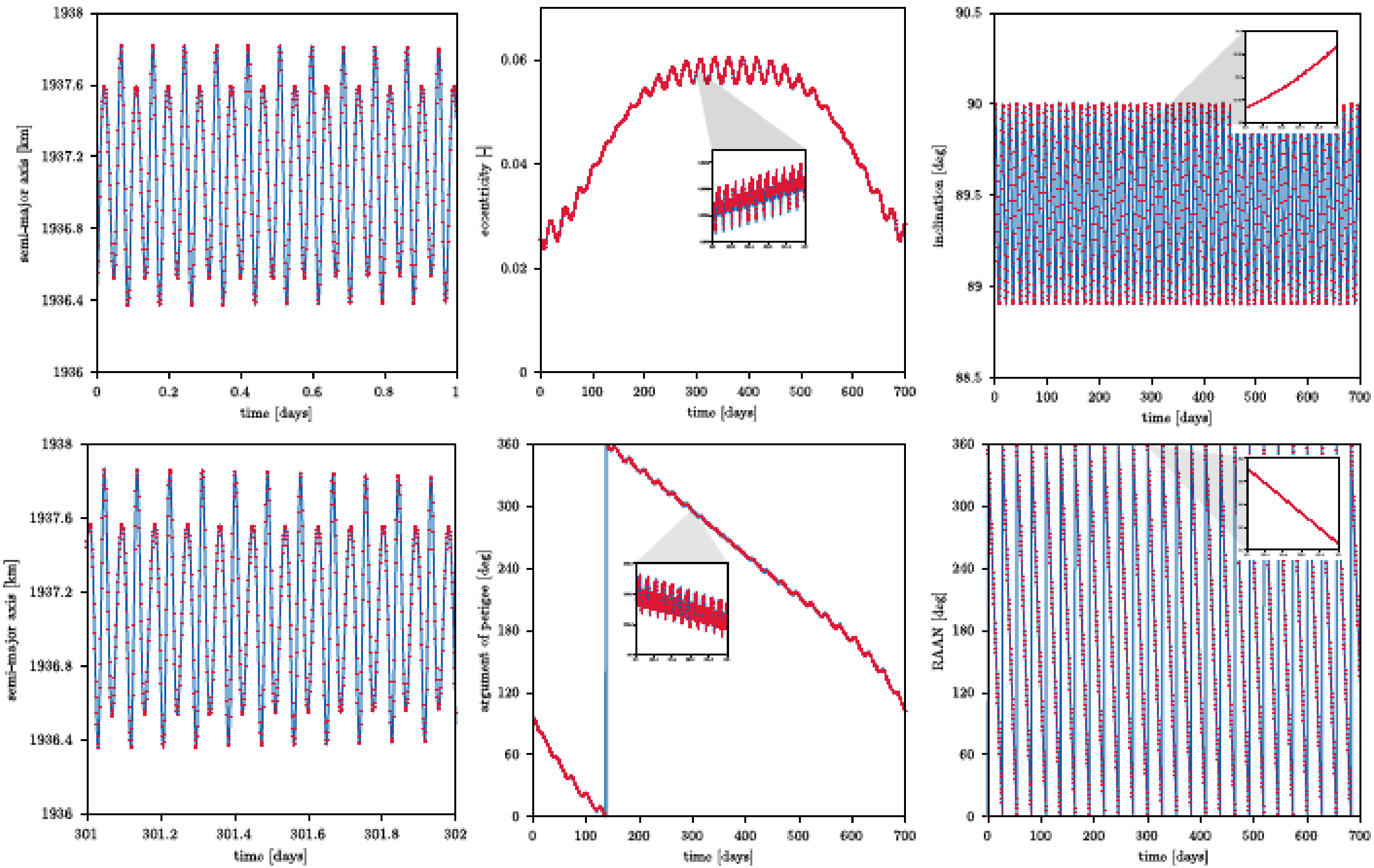


Figure 2: Blue dots show the cummulative size (zonal+tesseral) of the relative (with respect to the Keplerian term) acceleration generated by the order n terms of the lunipotential GRAIL values are used, see text). The blue line corresponds to the acceleration generated by the Earth's tide, green line the acceleration by the Solar tide. The altitude from the Lunar surface is equal to (left) 100 km, (center) 500 km, (right) 1000 km.



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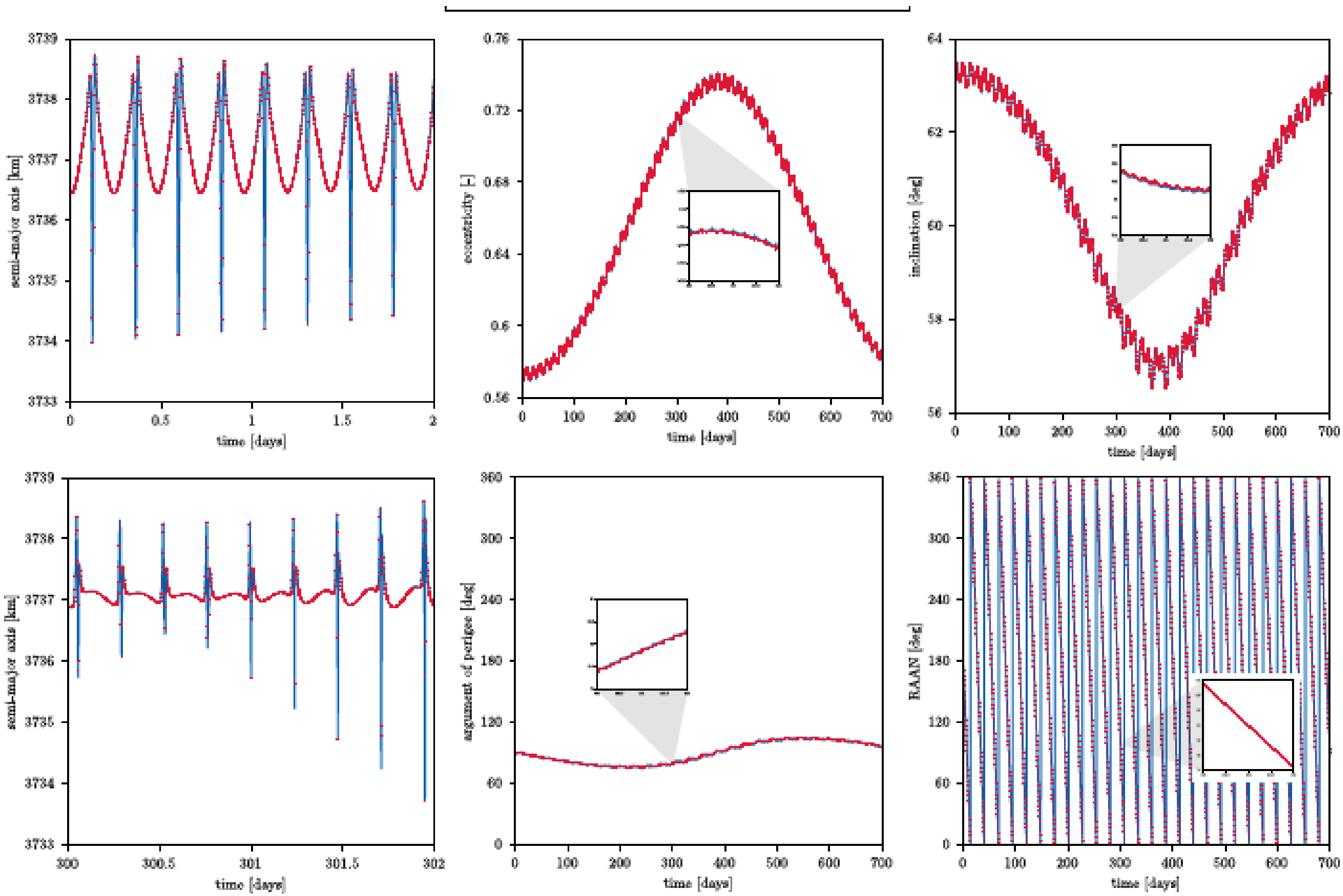

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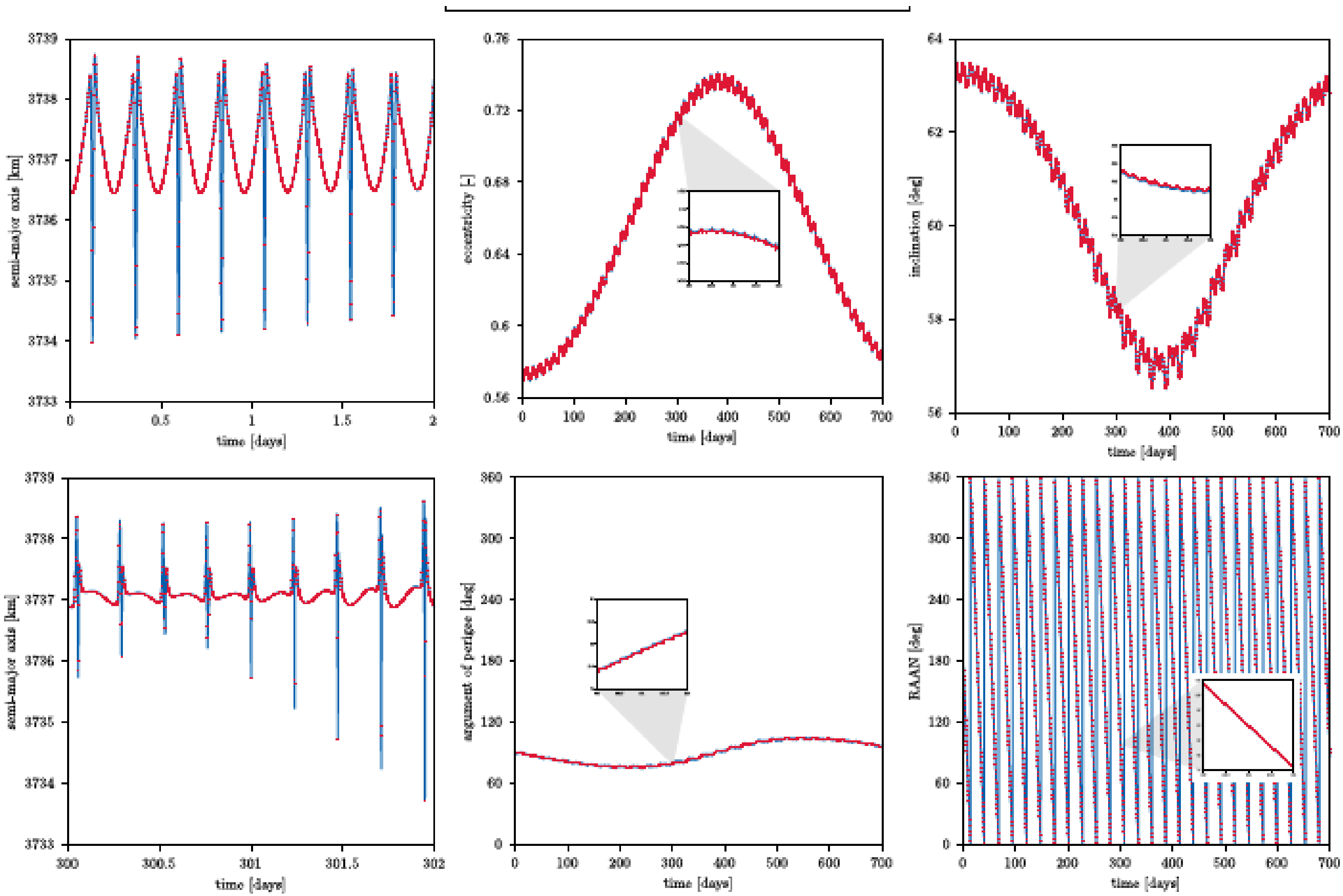

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Internal terms (caused by non-Keplerian perturbations internal to the obit)

$$V_{J2} = J_2 \frac{\mathcal{G} M_{☾} R_{☾}^2}{4 r^3} \left[-2 + 3s^2 - 3s^2 \cos(2f + 2g) \right]$$

$$\begin{aligned} V_{C22} &= C_{22} \frac{3 \mathcal{G} M_{☾} R_{☾}^2}{4 r^3} \Big[-2s^2 \cos 2h + 3(-2 + 2c + s^2) \cos(2f + 2g - 2h) \\ &\quad - 3(2 + 2c - s^2) \cos(2f + 2g + 2h) \Big] \end{aligned}$$

$$\begin{aligned} V_{C31} &= C_{31} \frac{3 \mathcal{G} M_{☾} R_{☾}^3}{16 r^4} \Big[(-4 - 11c + 5s^2 + 15c^3) \cos(f + g - h) \\ &\quad + (-4 + 11c + 5s^2 - 15c^3) \cos(f + g + h) \\ &\quad + 15(c - s^2 - c^3) \cos(3f + 3g - h) \\ &\quad - 15(c + s^2 - c^3) \cos(3f + 3g + h) \Big] \end{aligned}$$

$$\begin{aligned} V_{S31} &= S_{31} \frac{3 \mathcal{G} M_{☾} R_{☾}^3}{16 r^4} \Big[(4 + 11c - 5s^2 - 15c^3) \sin(f + g - h) \\ &\quad + (-4 + 11c + 5s^2 - 15c^3) \sin(f + g + h) \\ &\quad - 15(c - s^2 - c^3) \sin(3f + 3g - h) \\ &\quad - 15(c + s^2 - c^3) \sin(3f + 3g + h) \Big] \end{aligned}$$

External terms (caused by third-body perturbations - external to the obit)

Earth's orbit nearly circular

$$\mathbf{r}_\oplus = \mathbf{r}_E + \mathbf{d}_E = (r_E, 0, 0) + \mathbf{d}_E$$

Expand in powers of $\mathbf{d}_\mathbf{E}$

$$\begin{aligned} V_\oplus = V_{\oplus,0} + V_{\oplus,1} \quad &= \quad \frac{\mathcal{G}M_\oplus}{2r_E^3}\left(r^2 - 3\frac{(\mathbf{r}\cdot\mathbf{r}_E)^2}{r_E^2}\right) \\ &+ \quad \frac{3\mathcal{G}M_\oplus}{2r_E^5}\left(r^2(\mathbf{r}_E\cdot\mathbf{d}_E) - 2(\mathbf{r}\cdot\mathbf{d}_E)(\mathbf{r}\cdot\mathbf{r}_E) - 5\frac{(\mathbf{r}_E\cdot\mathbf{d}_E)(\mathbf{r}\cdot\mathbf{r}_E)^2}{r_E^2}\right) \end{aligned}$$

Since **r now appears in the numerator**, compute Poisson structure in terms of the eccentric rather than the true anomaly

$$\cos f = \frac{a(\cos u - e)}{r}, \quad \sin f = \frac{a\eta \sin u}{r}$$

$$V_{\oplus,0} = \frac{\mathcal{G}M_\oplus a^2}{64 r_E^3} \sum_{k,l,m} \Phi_{k,l,m}(e,i)\cos(kg + lh + mu)$$

Poisson structure to deal with all internal perturbation terms

Let $A(a,e,i,f,g,h)$, $B(a,e,i,f,g,h)$ be two such functions. We define

$$\begin{aligned}\{A,B\}_I &= \left(\frac{\partial A}{\partial M}\right)_I\left(\frac{\partial B}{\partial L}\right)_I+\left(\frac{\partial A}{\partial g}\right)_I\left(\frac{\partial B}{\partial G}\right)_I+\left(\frac{\partial A}{\partial h}\right)_I\left(\frac{\partial B}{\partial H}\right)_I\\ &- \left(\frac{\partial A}{\partial L}\right)_I\left(\frac{\partial B}{\partial M}\right)_I-\left(\frac{\partial A}{\partial G}\right)_I\left(\frac{\partial B}{\partial g}\right)_I-\left(\frac{\partial A}{\partial H}\right)_I\left(\frac{\partial B}{\partial h}\right)_I\end{aligned}$$

Compute derivarives with chain rule

$$\left(\frac{\partial F}{\partial M}\right)_I=\frac{\partial F}{\partial M}+\frac{\partial F}{\partial f}\frac{\partial f}{\partial M}+\frac{\partial F}{\partial r}\frac{\partial r}{\partial M}+\frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial M}$$

$$\left(\frac{\partial F}{\partial g}\right)_I=\frac{\partial F}{\partial g}$$

$$\left(\frac{\partial F}{\partial h}\right)_I=\frac{\partial F}{\partial h}$$

$$\left(\frac{\partial F}{\partial L}\right)_I=\frac{\partial F}{\partial L}+\frac{\partial F}{\partial a}\frac{\partial a}{\partial L}+\frac{\partial F}{\partial e}\frac{\partial e}{\partial L}+\frac{\partial F}{\partial n}\frac{\partial n}{\partial L}+\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial L}+\frac{\partial F}{\partial f}\frac{\partial f}{\partial L}+\frac{\partial F}{\partial r}\frac{\partial r}{\partial L}+\frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial L}$$

$$\left(\frac{\partial F}{\partial G}\right)_I=\frac{\partial F}{\partial G}+\frac{\partial F}{\partial e}\frac{\partial e}{\partial G}+\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial G}+\frac{\partial F}{\partial f}\frac{\partial f}{\partial G}+\frac{\partial F}{\partial r}\frac{\partial r}{\partial G}+\frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial G}+\frac{\partial F}{\partial \cos i}\frac{\partial \cos i}{\partial G}+\frac{\partial F}{\partial \sin i}\frac{\partial \sin i}{\partial G}$$

$$\left(\frac{\partial F}{\partial H}\right)_I=\frac{\partial F}{\partial \cos i}\frac{\partial \cos i}{\partial H}+\frac{\partial F}{\partial \sin i}\frac{\partial \sin i}{\partial H}$$

Intermediate formulas: required (should be written in the form prescribed below)

$$\frac{\partial a}{\partial L} = \frac{2}{an} \quad , \quad \frac{\partial n}{\partial L} = -\frac{3}{a^2}$$

$$\frac{\partial e}{\partial L} = \frac{\eta^2}{a^2 en} \quad , \quad \frac{\partial e}{\partial G} = -\frac{\eta}{a^2 en}$$

$$\frac{\partial \eta}{\partial L} = -\frac{\eta}{a^2 n} \quad , \quad \frac{\partial \eta}{\partial G} = \frac{1}{a^2 n}$$

$$\frac{\partial \cos i}{\partial G} = -\frac{\cos i}{a^2 \eta n} \quad , \quad \frac{\partial \cos i}{\partial H} = \frac{1}{a^2 \eta n}$$

$$\frac{\partial \sin i}{\partial G} = -\frac{1 - \sin^2 i}{a^2 \eta n \sin i} \quad , \quad \frac{\partial \sin i}{\partial H} = -\frac{\cos i}{a^2 \eta n \sin i}$$

$$\frac{\partial f}{\partial M} = \frac{a^2 \eta}{r^2} \quad , \quad \frac{\partial f}{\partial L} = \frac{2 \sin f}{a^2 en} + \frac{\sin 2f}{2a^2 n} \quad , \quad \frac{\partial f}{\partial G} = -\frac{2 \sin f}{a^2 e \eta n} - \frac{\sin 2f}{2a^2 \eta n}$$

$$\frac{\partial \phi}{\partial M} = \frac{a^2 \eta}{r^2} - 1 \quad , \quad \frac{\partial \phi}{\partial L} = \frac{2 \sin f}{a^2 en} + \frac{\sin 2f}{2a^2 n} \quad , \quad \frac{\partial \phi}{\partial G} = -\frac{2 \sin f}{a^2 e \eta n} - \frac{\sin 2f}{2a^2 \eta n}$$

$$\frac{\partial r}{\partial M} = \frac{ae \sin f}{\eta} \quad , \quad \frac{\partial r}{\partial L} = \frac{2r}{a^2 n} - \frac{\eta^2 \cos f}{aen} \quad , \quad \frac{\partial r}{\partial G} = \frac{\eta \cos f}{aen}$$

Poisson structure to deal with all external perturbation terms

$$\begin{aligned}\{A,B\}_E &= \left(\frac{\partial A}{\partial M}\right)_E\left(\frac{\partial B}{\partial L}\right)_E+\left(\frac{\partial A}{\partial g}\right)_E\left(\frac{\partial B}{\partial G}\right)_E+\left(\frac{\partial A}{\partial h}\right)_E\left(\frac{\partial B}{\partial H}\right)_E\\ &- \left(\frac{\partial A}{\partial L}\right)_E\left(\frac{\partial B}{\partial M}\right)_E-\left(\frac{\partial A}{\partial G}\right)_E\left(\frac{\partial B}{\partial g}\right)_E-\left(\frac{\partial A}{\partial H}\right)_E\left(\frac{\partial B}{\partial h}\right)_E\end{aligned}$$

Parametrize the radius and the `equation of center' in terms of the eccentric anomaly

$$\phi_u = u - M \qquad\qquad r_u = a(1-e\cos u)$$

Chain rule

$$\left(\frac{\partial F}{\partial M}\right)_E = \frac{\partial F}{\partial M}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial M}+\frac{\partial F}{\partial r_u}\frac{\partial r_u}{\partial M}+\frac{\partial F}{\partial \phi_u}\frac{\partial \phi_u}{\partial M} \qquad \left(\frac{\partial F}{\partial g}\right)_E = \frac{\partial F}{\partial g} \qquad \left(\frac{\partial F}{\partial h}\right)_E = \frac{\partial F}{\partial h}$$

$$\left(\frac{\partial F}{\partial L}\right)_E = \frac{\partial F}{\partial L}+\frac{\partial F}{\partial a}\frac{\partial a}{\partial L}+\frac{\partial F}{\partial e}\frac{\partial e}{\partial L}+\frac{\partial F}{\partial n}\frac{\partial n}{\partial L}+\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial L}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial L}+\frac{\partial F}{\partial r_u}\frac{\partial r_u}{\partial L}+\frac{\partial F}{\partial \phi_u}\frac{\partial \phi_u}{\partial L}$$

$$\left(\frac{\partial F}{\partial G}\right)_E = \frac{\partial F}{\partial G}+\frac{\partial F}{\partial e}\frac{\partial e}{\partial G}+\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial G}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial G}+\frac{\partial F}{\partial r_u}\frac{\partial r_u}{\partial G}+\frac{\partial F}{\partial \phi_u}\frac{\partial \phi}{\partial G}+\frac{\partial F}{\partial \cos i}\frac{\partial \cos i}{\partial G}+\frac{\partial F}{\partial \sin i}\frac{\partial \sin i}{\partial G}$$

$$\left(\frac{\partial F}{\partial H}\right)_E = \frac{\partial F}{\partial \cos i}\frac{\partial \cos i}{\partial H}+\frac{\partial F}{\partial \sin i}\frac{\partial \sin i}{\partial H}$$

The formulas for the intermediate derivatives are the same as for internal terms. For the new quantities u,ϕ_u,r_u, we have

$$\frac{\partial u}{\partial M} = \frac{a}{r_u} \;\; , \;\; \frac{\partial u}{\partial L} = \frac{\eta^2 \sin u}{aenr_u} \;\; , \;\; \frac{\partial u}{\partial G} = -\frac{\eta \sin u}{aenr_u}$$

$$\frac{\partial \phi_u}{\partial M} = \frac{a}{r_u} - 1 \;\; , \;\; \frac{\partial \phi_u}{\partial L} = \frac{\eta^2 \sin u}{aenr_u} \;\; , \;\; \frac{\partial \phi_u}{\partial G} = -\frac{\eta \sin u}{aenr_u}$$

$$\frac{\partial r_u}{\partial M} = \frac{a^2 \eta \sin u}{r_u} \;\; , \;\; \frac{\partial r_u}{\partial L} = \frac{2r}{a^2 n} - \frac{\eta^2(-e + \cos u)}{enr_u} \;\; , \;\; \frac{\partial r_u}{\partial G} = \frac{\eta(-e + \cos u)}{enr_u}$$

Important to keep all formulas expressed as above: otherwise, several simplifications do not take place

Book-keeping

$$\mathcal{H} = -\frac{1}{2}a^2 n^2 + \epsilon(V_{J2} + V_{C22}) + \epsilon^2(-n_{☾} H + V_{C31} + V_{S31} + V_{\oplus,0})$$

Normalization algorithm: a view step by step

Setting

$$Z_0 = \mathcal{H}_{kep} = -\frac{1}{2}a^2 n^2$$

the Hamiltonian can be written as

$$\mathcal{H}^{(0)} = Z_0 + \epsilon \mathcal{H}_1^{(0)} + \epsilon^2 \mathcal{H}_2^{(0)}$$

where:

$$\mathcal{H}_1^{(0)} = Z_1 + h_1^{(0)}$$

with

$$Z_1 = < V_{J2} + V_{C22} >_M$$

denoting the average of $V_{J2} + V_{C22}$ with respect to the mean anomaly, while

$$h_1^{(0)} = V_{J2} + V_{C22} - Z_1 \quad ,$$

and also

$$\mathcal{H}_2^{(0)} = -n_{☾} H + V_{C31} + V_{S31} + V_{\oplus,0} \quad .$$

Solution of the homological equation

$$\{Z_0, \chi_1\} = -h_1^{(0)} = Z_1 - \mathcal{H}_1^{(0)}$$

Since

$$\{Z_0, \chi_1\} = -\frac{\partial Z_0}{\partial L}\frac{\partial \chi_1}{\partial M} = -n\frac{\partial \chi_1}{\partial M}$$

the solution of the homological equation in closed form will be given by:

$$\chi_1 = \frac{1}{n}\int (\mathcal{H}_1^{(0)} - Z_1) dM$$

where

$$Z_1 = < \mathcal{H}_1^{(0)} >_M = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_1^{(0)} dM$$

Since $\mathcal{H}_1^{(0)} = V_{J2} + V_{C22}$, *only internal terms* are included in $\mathcal{H}_1^{(0)}$. Recalling that

$$df = \frac{a^2\eta}{r^2} dM \quad ,$$

For any function of the form

$$
\begin{aligned}
F_I &= \sum_{p\geqslant 3} \frac{1}{r^p} \sum_{k,l,m} Q_{I,k,l,m}(a,e,i)\cos(kg+lh+mf) \\
&+ \sum_{p\geqslant 3} \frac{1}{r^p} \sum_{k,l,m} K_{I,k,l,m}(a,e,i)\sin(kg+lh+mf)
\end{aligned}
$$

m≠0

introduce the parallax reduction

$$
\begin{aligned}
P[F_I] &= \frac{1}{r^2}\sum_{p\geqslant 3}\left(\frac{1+e\cos f}{a\eta^2}\right)^{p-2} \sum_{k,l,m} Q_{I,k,l,m}(a,e,i)\cos(kg+lh+mf) \\
&+ \frac{1}{r^2}\sum_{p\geqslant 3}\left(\frac{1+e\cos f}{a\eta^2}\right)^{p-2} \sum_{k,l,m} K_{I,k,l,m}(a,e,i)\sin(kg+lh+mf)
\end{aligned}
$$

Integration over M can now be given in closed form

$$
\int F_I dM = \int P[F_I] dM = \int \frac{r^2}{a^2\eta} P[F_I] df
$$

$$\begin{aligned}
\int F_I dM &= \frac{1}{a^2\eta}\int\left(\sum_{k,l,m} B_{I,k,l,m}(a,e,i)\cos(kg+lh+mf)\right)df \\
&+ \frac{1}{a^2\eta}\int\left(\sum_{k,l,m} D_{I,k,l,m}(a,e,i)\sin(kg+lh+mf)\right)df
\end{aligned}$$

Decomposition of P[F_I] in terms depending (or not) on *f*:

$$P[F_I] = \frac{1}{r^2}(F_N + F_R)$$

with

$$F_N = \sum_{k,l} B_{I,k,l,0}(a,e,i)\cos(kg+lh) + \sum_{k,l} D_{I,k,l,0}(a,e,i)\sin(kg+lh)$$

$$F_R = \sum_{k,l,m\neq 0} B_{I,k,l,m}(a,e,i)\cos(kg+lh+mf) + \sum_{k,l,m\neq 0} D_{I,k,l,m}(a,e,i)\sin(kg+lh+m.$$

we readily obtain

$$< F_I >_M = \frac{1}{2\pi}\int_0^{2\pi} F_I dM = \frac{1}{a^2\eta}F_N$$

while

$$\begin{aligned}
\int F_I dM &= \frac{1}{a^2\eta}F_N f + \frac{1}{a^2\eta}\int F_R df \\
&= \frac{1}{a^2\eta}F_N f \\
&- \sum_{k,l,m\neq 0}\left(\frac{1}{ma^2\eta}\right) B_{I,k,l,m}(a,e,i)\sin(kg+lh+mf) \\
&+ \sum_{k,l,m\neq 0}\left(\frac{1}{ma^2\eta}\right) D_{I,k,l,m}(a,e,i)\cos(kg+lh+mf)
\end{aligned}$$

Finally, we arrive at the key formula

$$\int \left(F_I - < F_I >_M \right) dM = \frac{1}{a^2\eta} F_N (f - M) + \frac{1}{a^2\eta} \int F_R df$$

that is

$$\int \left(F_I - < F_I >_M \right) dM = \frac{\phi}{a^2\eta} F_N + \frac{1}{a^2\eta} \int F_R df$$

Eventually!

$$P[\mathcal{H}_1^{(0)}] = P[V_{J2} + V_{C22}] = \frac{1}{r^2} \left(\mathcal{H}_{1,N}^{(0)} + \mathcal{H}_{1,R}^{(0)} \right)$$

$$\chi_1 = \phi \chi_{1,c} + \chi_{1,0} = \frac{\phi}{na^2\eta} \mathcal{H}_{1,N}^{(0)} + \frac{1}{na^2\eta} \mathcal{H}_{1,R}^{(0)}$$

$$\mathcal{H}^{(1)} = Z_0 + \epsilon Z_1 + \epsilon^2 \mathcal{H}_2^{(1)}$$

$$Z_0 = -\frac{1}{2} a^2 n^2, \quad Z_1 = \frac{1}{a^2\eta} \mathcal{H}_{1,N}^{(0)}$$

$$\mathcal{H}_2^{(1)} = \mathcal{H}_2^{(0)} + \{H_1^{(0)}, \chi_1\} + \frac{1}{2} \{\{Z_0, \chi_1\}, \chi_1\}$$

Second normalization step: dealing with the equation of the center

$$H_2^{(0)} = -n_{☾} H + P[V_{C31} + V_{S31}] + V_{\oplus,0}$$

$$\mathcal{H}_2^{(1)} = \mathcal{H}_2^{(0)} + \frac{1}{2}\{H_1^{(0)}, \phi\chi_{1,c} + \chi_{1,0}\} + \frac{1}{2}\{Z_1, \phi\chi_{1,c} + \chi_{1,0}\}$$

generates terms in the Hamiltonian depending on the equation of the center

Definitions

Nucleus monomial term is called a term of the form:

$$F_N = b_{k,l}(a,e,i) \begin{matrix}\cos\\ \sin\end{matrix}(kg + lh)$$

Range monomial term is called a term of the form:

$$F_R = b_{k,l,m}(a,e,i) \begin{matrix}\cos\\ \sin\end{matrix}(kg + lh + mf)$$

$\phi-$*Nucleus monomial term* is called a term of the form:

$$F_{\phi N} = \phi^s b_{k,l}(a,e,i) \begin{matrix}\cos\\ \sin\end{matrix}(kg + lh), \quad s = 1, 2, \ldots$$

Definitions (continued)

$\phi-$*Range monomial term* is called a term of the form:

$$F_{\phi R} = \phi^s b_{k,l,m}(a,e,i) \overset{\cos}{\underset{\sin}{\cdot}} (kg + lh + mf), \quad s = 1, 2, \ldots$$

$r-$*Nucleus monomial term* is called a term of the form:

$$F_{rN} = \frac{1}{r^2} b_{k,l}(a,e,i) \overset{\cos}{\underset{\sin}{\cdot}} (kg + lh)$$

$r-$*Range monomial term* is called a term of the form:

$$F_{rR} = \frac{1}{r^2} b_{k,l,m}(a,e,i) \overset{\cos}{\underset{\sin}{\cdot}} (kg + lh + mf)$$

$r\phi$-*Nucleus monomial term* is called a term of the form:

$$F_{r\phi N} = \phi^s \frac{1}{r^2} b_{k,l}(a,e,i) \overset{\cos}{\underset{\sin}{\cdot}} (kg + lh), \quad s = 1, 2, \ldots$$

$r\phi$-*Range monomial term* is called a term of the form:

$$F_{r\phi R} = \phi^s \frac{1}{r^2} b_{k,l,m}(a,e,i) \overset{\cos}{\underset{\sin}{\cdot}} (kg + lh + mf), \quad s = 1, 2, \ldots$$

Important equations

$$\begin{aligned}
\{Z_0, \phi^s \cos(kg+lh)\} &= ns\phi^{s-1}\left(1-\frac{a^2\eta}{r^2}\right)\cos(kg+lh) \\
\{Z_0, \phi^s \sin(kg+lh)\} &= ns\phi^{s-1}\left(1-\frac{a^2\eta}{r^2}\right)\sin(kg+lh)
\end{aligned}$$

$$\begin{aligned}
\{Z_0, \phi^s \cos(kg+lh+mf)\} &= ns\phi^{s-1}\left(1-\frac{a^2\eta}{r^2}\right)\cos(kg+lh+mf) \\
&+ n\phi^s\frac{a^2\eta m}{r^2}\sin(kg+lh+mf) \\
\{Z_0, \phi^s \sin(kg+lh+mf)\} &= ns\phi^{s-1}\left(1-\frac{a^2\eta}{r^2}\right)\sin(kg+lh+mf) \\
&- n\phi^s\frac{a^2\eta m}{r^2}\cos(kg+lh+mf)
\end{aligned}$$

Cancellations

$$H_1^{(0)} = H_{1,rN}^{(0)} + H_{1,rR}^{(0)}$$

$$\left\{H_{1,rN}^{(0)}, \phi\chi_{1,c}\right\} = \left\{\frac{na^2\eta}{r^2}\chi_{1,c}, \phi\chi_{1,c}\right\} = \left\{\chi_{1,c}, \phi\right\}\frac{na^2\eta}{r^2}\chi_{1,c} + \left\{\frac{na^2\eta}{r^2}, \phi\right\}\chi_{1,c}^2 + \left\{\frac{na^2\eta}{r^2}, \chi_{1,c}\right\}\chi_{1,c}\phi$$

However, again from Eq.(38) we readily find that

$$\chi_{1,c} = \frac{n}{\eta^3}\xi_{1,c}(i,h) = \frac{n}{\eta^3}\xi_{1,c}(G,H,h)$$

while $na^2\eta r^2$ is a function of (L,G,M)

$$\frac{na^2\eta}{r^2} = \gamma(L,G,M)$$

Noticing the important identity

$$\frac{\partial}{\partial L}\left(\frac{n}{\eta^3}\right) = 0 \qquad (51)$$

we thus get

$$\left\{\frac{na^2\eta}{r^2}, \chi_{1,c}\right\} = \frac{n}{\eta^3}\left\{\gamma(L,G,M), \xi_{1,c}(G,H,h)\right\} + \xi_{1,c}(G,H,h)\left\{\gamma(L,G,M), \frac{n}{\eta^3}\right\} = 0$$

implying that $\{H_{1,rN}^{(0)}, \phi\chi_{1,c}\}$ contains no terms depending on the equation of the center ϕ:

In conclusion

$$\left\{H^{(0)}_{1,rN}, \phi\chi_{1,c}\right\} \rightarrow r\text{−nucleus} + r\text{−range terms}$$

$$\left\{H^{(0)}_{1,rR}, \phi\chi_{1,c}\right\} \rightarrow r\text{−nucleus} + r\text{−range terms} + r\phi\text{-range terms}$$

$$\rightarrow r\text{−nucleus} + r\text{−range} + (r\text{-range})\phi \text{ terms}$$

$$\left\{H^{(0)}_{1,rR}, \chi_{1,0}\right\} \rightarrow r\text{−nucleus} + r\text{−range terms}$$

$$\left\{Z1, \phi\chi_{1,c}\right\} = \left\{n\chi_{1,c}, \phi\chi_{1,c}\right\} \rightarrow \text{nucleus} + r\text{−nucleus terms}$$

$$\left\{Z1, \chi_{1,0}\right\} \rightarrow \text{range} + r\text{−range terms}$$

$r\phi$-Rule

$$\text{for every term } B_{k,l,f}(a,e,i)\left(\frac{\phi}{r^2}\sin(kg+lh+mf)+\frac{1}{m}\cos(kg+lh+mf)\right) \text{ in the Hamiltonian}$$

$$\text{add the term } -B_{k,l,f}(a,e,i)\phi\cos(kg+lh+mf) \text{in the generating function}$$

and

$$\text{for every term } D_{k,l,f}(a,e,i)\left(\frac{\phi}{r^2}\cos(kg+lh+mf)-\frac{1}{m}\cos(kg+lh+mf)\right) \text{ in the Hamiltonian}$$

$$\text{add the term } -D_{k,l,f}(a,e,i)\phi\sin(kg+lh+mf) \text{in the generating function}$$

Final algorithm

i) Compute $H_2^{(1)}$ by equation (40).

ii) Split $H_2^{(1)}$ to terms of class (i)+(ii), and those of class (iii)+(iv)

$$H_2^{(1)} = H_2^{(1),i+ii} + H_2^{(1),iii+iv}$$

iii) Compute the generating function terms $\epsilon^2\xi_2$ normalizing $H_2^{(1),iii+iv}$ by the $r\phi-$Rule.

iv) Compute, up to book-keeping order 2, the transformed Hamiltonian:

$$H^{(1),i+ii} = \exp(\mathcal{L}_{\epsilon^2\xi_2})H^{(1)} = H^{(1)} + \left\{ Z_0, \epsilon^2\xi_2 \right\}$$

v) The Hamiltonian $H^{(1),i+ii}$ is of the form

$$H^{(1),i+ii} = Z_0 + \epsilon Z_1 + \epsilon^2 H_2^{(2),i+ii}$$

where the function $H_2^{(1),i+ii}$ contains no terms of class (iii) and (iv). Then, decompose $H_2^{(1),i+ii}$ in internal and external terms

$$H_2^{(1),i+ii} = \frac{1}{r^2}\left(H_{2,N}^{(1),i+ii,I} + H_{2,R}^{(1),i+ii,I}\right) + H_2^{(1),i+ii,E}$$

vi) Compute the generating function χ_2^I normalizing all internal terms in $H^{(1),i+ii}$ where to Eq.(38)):

$$\chi_2^I = \phi\chi_{2,c}^I + \chi_{2,0}^I = \frac{\phi}{na^2\eta}\mathcal{H}_{1,N}^{(1),i+ii} + \frac{1}{na^2\eta}\mathcal{H}_{1,R}^{(1),i+ii}$$

Dealing with external terms

R-reduction

$$RR[F] = \frac{a(1 - e\cos u)}{r} F(a, e, i, u, g, h)$$

Splitting into terms depending (or not) on the eccentric anomaly u:

$$RR[F] = \frac{1}{r}(F_N + F_R)$$

u

$$< RR[F] >_M = \frac{1}{2\pi}\int_0^{2\pi} \frac{a(1 - e\cos u)}{r} F(a, e, i, u, g, h)\, dM = \frac{1}{2\pi}\int_0^{2\pi} (F_N + F_R)\, du = F_N$$

Very simple solution to the homological equation!

$$\chi_{2,E} = \int \frac{1}{an} F_R\, du + \frac{1}{an} F_N \phi_u = \int \frac{1}{an} F_R\, du + \frac{1}{an} e \sin u F_N$$

Avoiding relegation (works in progress with I. Cavallari and M. Rossi)

Restricted three body problem

$$\mathscr{H} = n^*\delta L - \frac{3}{2}\frac{\delta L^2}{a^{*2}} + n_P I_P + \mathscr{R}_{[o]}$$

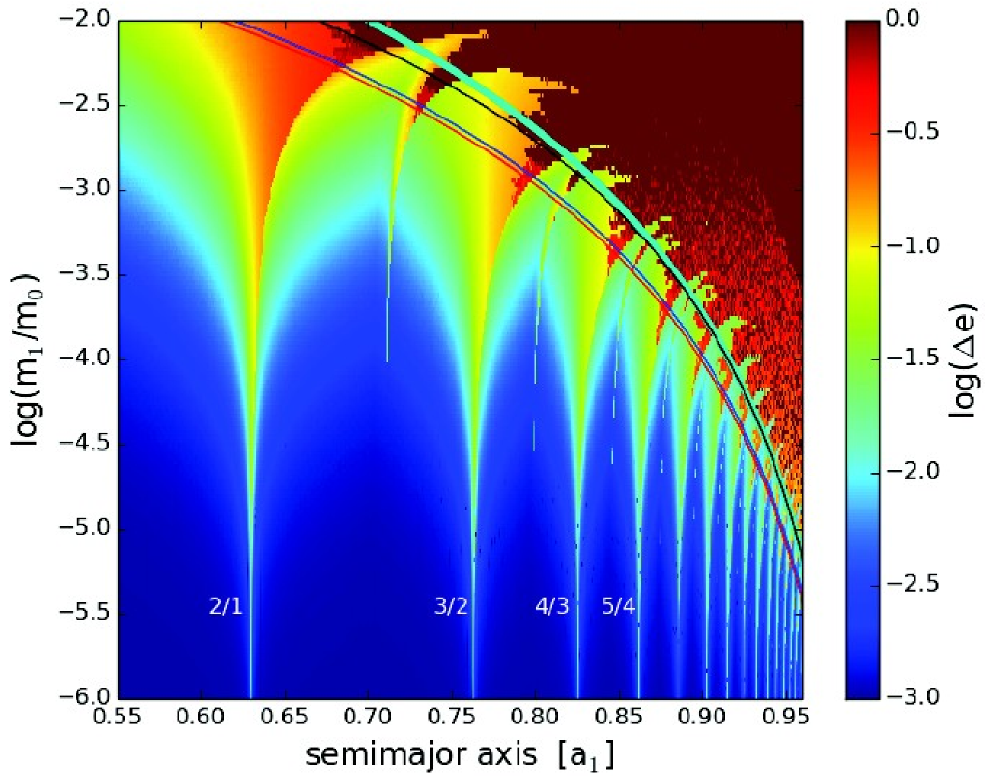
Observe that:

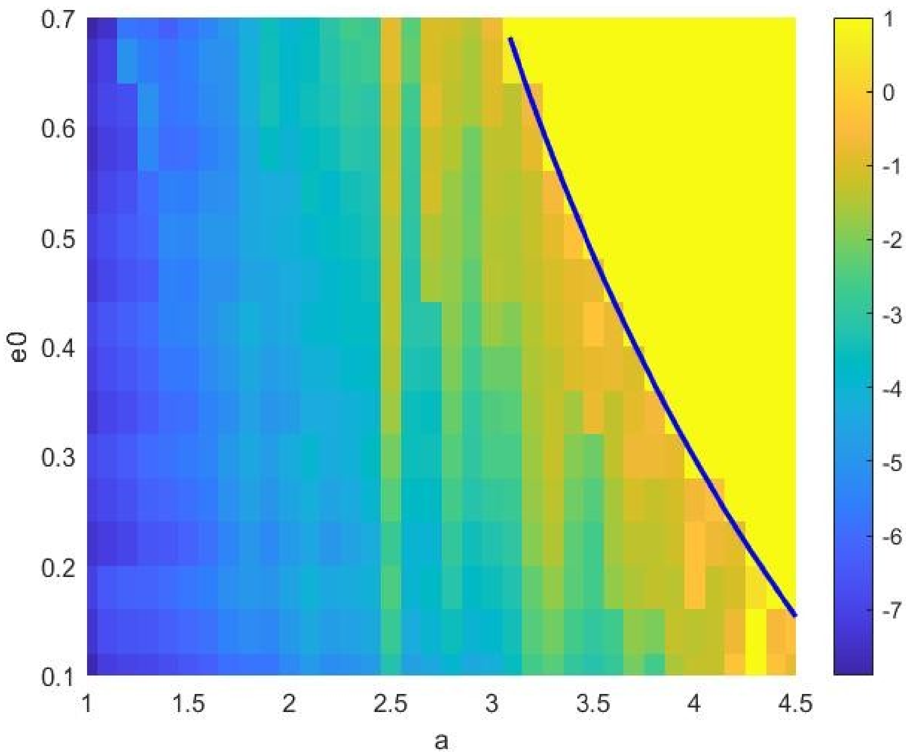
$$\{Z_0, \chi\} = n^*\left(\frac{d\chi}{dM} + \frac{a}{r}\frac{d\chi_m}{du}\right) + n_P\left(1 + \frac{2e_P\cos(f_P)}{\eta_P^3}\epsilon + \Big(\frac{1}{\eta_P^3} + \frac{e_P^2\cos^2(f_P)}{\eta_P^3} - 1\Big)\epsilon^2\right)\frac{d\chi}{df_P}$$

Solve the homological equation:

$$\{Z_0, \chi\} + \tilde{h} = \mathcal{O}(e)$$

Does it work?





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