Coleman–Oort's conjecture for degenerate irreducible curves.

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Abstract We state and prove the analogue of a conjecture of Coleman and Oort for the locus of degenerate irreducible curves.

1 Introduction.

Denote by $\mathcal{A}_{g,1}$ the moduli space of principally polarized abelian varieties of dimension g over $\mathbf{C}$. Let $\mathfrak{M}_g$ be the moduli space of smooth projective curves of genus g over $\mathbf{C}$ and denote by $\mathfrak{J}: \mathfrak{M}_g \to \mathcal{A}_{g,1}$ the Torelli map associating to a curve its Jacobian with its natural principal polarization. Define Jac_g to be its image and let $\overline{\mathrm{Jac}}_g$, the so called Torelli locus, be the schematic closure of Jac_g in $\mathcal{A}_{g,1}$. Inspired by a conjecture of Coleman's [Co, Conj. 6], Frans Oort asked in [Oo, §7] whether, for g large enough, there exists any positive dimensional locally symmetric subvariety of $\mathcal{A}_{g,1}$, which is contained in $\overline{\mathrm{Jac}}_g$ and intersects Jac_g. His expectation is that such subvarieties do not exist. Recall that a *locally symmetric variety* Y is an algebraic variety whose associated complex analytic space Y^{an} is the quotient

$$Y^{\mathrm{an}} := \Gamma \backslash G(\mathbf{R})^0/\mathbf{K},$$

where G is a semisimple algebraic group defined over $\mathbf{Q}$, $\mathbf{K} \subset G(\mathbf{R})^0$ is a maximal compact subgroup of the connected component at the identity $G(\mathbf{R})^0$ of $G(\mathbf{R})$, Γ is the intersection with $G(\mathbf{R})^0$ of an arithmetic subgroup of $G(\mathbf{Q})$ and the quotient $D := G(\mathbf{R})^0/\mathbf{K}$ is a hermitian symmetric domain of non-compact type. For example, $\mathcal{A}_{g,1}$ is the locally symmetric variety defined by $\mathrm{Sp}_{2g}(\mathbf{Z})\backslash \mathrm{Sp}_{2g}(\mathbf{R})/\mathbb{K}_g$ for a suitable maximal compact subgroup $\mathbb{K}_g \subset \mathrm{Sp}_{2g}(\mathbf{R})$. Let Y_1 and Y_2 be locally symmetric varieties with associated complex analytic spaces $Y_1^{\mathrm{an}} = \Gamma_1\backslash G_1(\mathbf{R})^0/\mathbf{K}_1$ and $Y_2^{\mathrm{an}} := \Gamma_2\backslash G_2(\mathbf{R})^0/\mathbf{K}_2$. A map $f: Y_1 \to Y_2$ of locally symmetric varieties is a morphism such that the associated map of complex analytic spaces is induced by a homomorphism of algebraic groups $\widetilde{f}: G_1 \to G_2$. A locally symmetric subvariety of Y_2 is the image of a map $Y_1 \to Y_2$ of locally symmetric varieties such that G_1 is an algebraic subgroup of G_2.

Coleman–Oort's conjecture is far from being proved. The first results are due to Johan de Jong and Rutger Noot [dJNo] who give counterexamples to the conjecture for $g = 4$ and $g = 6$ providing explicit examples of positive dimensional families of Jacobians with complex multiplication. On the positive side we refer to Richard Hain's pioneering paper [Hain] and to the works of Johan de Jong and Shou–Wu Zhang [dJZ] and of Martin Möller, Eckart Viehweg and Kang Zuo [MVZ] for partial results supporting the conjecture. These results are based on global methods. In [Hain] and [dJZ] one argues by contradiction assuming that there exists a variety Y contradicting the conjecture. Roughly speaking, the idea is to compare the fundamental group of Y^{an}, which is a

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lattice in a Lie group, and the fundamental group of $\mathfrak{M}_g^{\rm an}$ which is a mapping class group. The key ingredient is a rigidity result by Farb and Masur stating that any homomorphism from an irreducible lattice in a semisimple Lie group of real rank at least two to a mapping class group has finite image. The drawback of this approach is that the map $\mathfrak{J}$ is ramified along the hyperelliptic locus so that the comparison of the fundamental group of the complex analytic spaces Y and $\mathfrak{J}^{-1}(Y)$ is very subtle. In [MVZ] the authors prove, among other results, that for $g \geq 4$ there does not exist a curve contained in $\mathfrak{M}_g$ whose image in $\mathcal{A}_{g,1}$ is a Shimura curve. Also in this case the method is global, works for $\mathfrak{M}_g$ and not $\overline{\rm Jac}_g$, and is specific to the case of curves. It is based on a fine analysis of the discriminant loci of families of semistable curves parameterized by a base curve.

In this paper we formulate and prove an analogue of Coleman–Oort's conjecture at the boundary component of the Deligne-Mumford compactification of $\mathfrak{M}_g$ consisting of irreducible curves. In a future work we will show how this can be used to prove some cases of the original conjecture via degeneration methods. The method we propose is different from those of Hain and of Möller-Viehweg-Zuo. More precisely, denote by $\overline{\mathfrak{M}}_g$ the Deligne–Mumford compactification of $\mathfrak{M}_g$. It is the coarse moduli space of stable curves of genus g over $\mathbf{C}$. Following an idea of D. Mumford, it is proven in [Na, Cor. 18.9] that the Torelli map $\mathfrak{J}\colon \mathfrak{M}_g \to \mathcal{A}_{g,1}$ extends to a map of varieties

$$\widetilde{\mathfrak{J}}\colon \overline{\mathfrak{M}}_g \longrightarrow \widetilde{\mathcal{A}}_{g,1}^{\rm DV},$$

where $\widetilde{\mathcal{A}}_{g,1}^{\rm DV}$ is the Delaunay–Voronoi, also called second Voronoi, toroidal compactification of $\mathcal{A}_{g,1}$. Let $\overline{\mathfrak{M}}_g^{\rm irr} \subset \overline{\mathfrak{M}}_g$ be the open subscheme consisting of stable curves which are geometrically irreducible and put $\partial\overline{\mathfrak{M}}_g^{\rm irr} := \overline{\mathfrak{M}}_g^{\rm irr} \backslash \mathfrak{M}_g$. Our main result is the following:

1.1 Theorem. *Assume that $g \geq 4$. Let Y be a locally symmetric subvariety of $\mathcal{A}_{g,1}$ contained in $\overline{\rm Jac}_g$ and intersecting ${\rm Jac}_g$. Then, the closure of Y in $\widetilde{\mathcal{A}}_{g,1}^{\rm DV}$ does not intersect $\widetilde{\mathfrak{J}}\big(\partial\overline{\mathfrak{M}}_g^{\rm irr}\big)$.*

The proof of 1.1, achieved in §5.6, consists of two steps. First of all, we study the structure at the boundary of the closure of a locally symmetric subvariety of $\mathcal{A}_{g,1}$. More precisely, fix an integer $n \geq 3$. Let $\mathcal{A}_{g,n}$ be the moduli space of principally polarized abelian varieties of dimension g and full symplectic level n–structure. In fact, we will need a quotient of it in the text but let us ignore this point in the introduction for simplicity. Let Y be a locally symmetric subvariety of $\mathcal{A}_{g,n}$. Let $\widetilde{Y}$ be the closure of the image of $Y^{\rm an}$ in a (any) partial, smooth toroidal compactification $\widetilde{\mathcal{A}}_{g,n}$ of the complex analytic space $\mathcal{A}_{g,n}^{\rm an}$. Then, $\widetilde{Y}$ is "linear" at the boundary. More precisely, consider the map $q\colon \widetilde{\mathcal{A}}_{g,n} \to \mathcal{A}_{g,n}^*$ to the minimal compactification of $\mathcal{A}_{g,n}^{\rm an}$. Recall that we have a set theoretic decomposition $\mathcal{A}_{g,n}^* = \amalg_{h=0}^g \mathcal{A}_{h,n}^{\rm an}$. For every $0 \leq h \leq g-1$ the inverse image of $\mathcal{A}_{h,n}^{\rm an} \subset \mathcal{A}_{g,n}^*$ via q is the union of locally closed analytic subspaces $\{\Xi'_{h,i}\}_i$, depending on the chosen toroidal compactification, each of which has a natural structure of rigidified torus torsor over the $g-h$–th fold product A^{g-h} of the universal family

A of abelian varieties over $\mathcal{A}_{h,n}^{\mathrm{an}}$. Let $\partial\widetilde{Y}$ be the intersection of the boundary of $\widetilde{\mathcal{A}}_{g,n}$ with $\widetilde{Y}$. The "linearity" of $\widetilde{Y}$ expresses itself in two ways:

a. there is a canonical retraction of a tubular neighborhood of $\Xi'_{h,i}$ in $\widetilde{\mathcal{A}}_{g,n}$ to $\Xi'_{h,i}$. The image via this retraction of the corresponding neighborhood of $\Xi'_{h,i} \cap \widetilde{Y}$ in $\widetilde{Y}$ is $\Xi'_{h,i} \cap \widetilde{Y}$;
b. if $\partial\widetilde{Y} \cap \Xi'_{h,i}$ is non–empty, it is the complex analytic space associated to a rigidified torus subtorsor over a family of abelian subvarieties of A^{g-h} parameterized by a locally symmetric subvariety of $\mathcal{A}_{h,n}$.

This is proven in §3, especially 3.9, based on the study of (complex analytic) compactifications of locally symmetric varieties from the point of view of symmetric spaces given in [AMRT] and recalled in §2, especially 2.1.

The second step in the proof of our theorem is to show that the locus of $\widetilde{\mathcal{A}}_{g,n}$ consisting of Jacobians of irreducible curves is not linear at the boundary for $g \geq 4$. This is proved in §5, especially 5.4 and 5.5, and relies on the theory of degenerations of periods of abelian varieties given in [An] and recalled in §4.

For simplicity of exposition, at least in this introduction, we forget about the level structure n in $\mathcal{A}_{g,n}$ (needed for the existence of universal families). An indication that the Torelli locus does not behave at the boundary as a locally symmetric subvariety of $\mathcal{A}_{g,1}$ can be found in [FvdP]. The authors prove the following result. Let $\mathcal{C}_0$ be a stable curve over $\mathbf{C}$ of genus $g \geq 4$ whose $\mathrm{Pic}^0(\mathcal{C}_0/\mathbf{C})$ is not an abelian variety. Let R be the universal deformation space of $\mathcal{C}_0$ and denote by $\mathcal{C} \to S := \mathrm{Spec}(R)$ the universal stable curve over R. The Raynaud extension G of $\mathrm{Pic}^0(\mathcal{C}/S)$ is the semiabelian scheme appearing in the uniformization of $\mathrm{Pic}^0(\mathcal{C}/S)$ described in [FC]. It is an extension of an abelian scheme A over S, endowed with a natural principal polarization λ_A, and a torus over S. The fiber of (A, λ_A) over the closed point of S is the Pic^0 of the normalization of $\mathcal{C}_0$. In loc. cit., it is proven that the fiber of (A, λ_A) over the generic point of S is *not* in general isomorphic to the product of Jacobians of smooth projective curves. This would happen if the (local) retraction of $\widetilde{\mathcal{A}}_{g,1}$ to the boundary, described in (a), sent the closure of the Torelli locus in $\widetilde{\mathcal{A}}_{g,1}$ to itself.

We present other illuminating instances of the non–linearity of the Torelli locus conflicting with the second incarnation of the analytic linearity at the boundary of locally symmetric subvarieties of $\mathcal{A}_{g,1}$ (part (b) above). Let $\mathcal{C}$ be an irreducible stable curve of genus g over $\mathbf{C}$ with singular points $T_1, \ldots, T_r$. Let D be the normalization of $\mathcal{C}$ and for every i let P_i and Q_i be the inverse images of T_i in D. There is an open subspace Ξ of the fiber of a suitable (partial) toroidal compactification of $\mathcal{A}_{g,1}^{\mathrm{an}}$ over the moduli point $[\mathrm{Pic}^0(D/\mathbf{C})] \in \mathcal{A}_{g-r,1}(\mathbf{C}) \subset \mathcal{A}_{g,1}^*$ which is a torsor over a torus of dimension $r(r-1)/2$ over the abelian variety $\big(\mathrm{Pic}^0(D/\mathbf{C})\big)^r$, rigidified over the identity of $\big(\mathrm{Pic}^0(D/\mathbf{C})\big)^r$. Let $Y \subset \mathcal{A}_{g,1}$ be a locally symmetric subvariety. Let $\partial\widetilde{Y}$ be the intersection with Ξ of the closure $\widetilde{Y}$ of Y^{an}. It follows from 4.7 that the image of the degenerate periods of $\mathrm{Pic}^0(\mathcal{C}/\mathbf{C})$ in $\big(\mathrm{Pic}^0(D/\mathbf{C})\big)^r$ consist of the points $(Q_1 - P_1, \ldots, Q_r - P_r)$.

On the other hand, the image of the fiber of $\partial\widetilde{Y}$ over $[\mathrm{Pic}^0(D/\mathbf{C})]$ in $\big(\mathrm{Pic}^0(D/\mathbf{C})\big)^r$ is an abelian subvariety of $\mathrm{Pic}^0(D/\mathbf{C})^r$. The main ingredient to prove Theorem 1.1 is the fact, proven in 5.4, that for $g-r\geq 3$ there is no positive dimensional abelian subvariety contained in the surface $D-D\subset \mathrm{Pic}^0(D/\mathbf{C})$ consisting of points $P-Q$ for $P,\,Q\in D$.

Another interesting example is the case when D has genus 0 so that $g=r$. In this case the degenerate periods of $\mathrm{Pic}^0(\mathcal{C}/\mathbf{C})$ are the $g(g-1)/2$ cross ratios of the g pairs of distinct points $\big((P_1,Q_1),\ldots,(P_g,Q_g)\big)$. They define a point of $(\mathbf{C}^*)^{g(g-1)/2}$ which is the space Ξ introduced above (as an open subspace of the fiber of a suitable (partial) toroidal compactification of $\mathcal{A}_{g,1}^{\mathrm{an}}$ over the 0–dimensional cusp of $\mathcal{A}_{g,1}^*$). On the one hand Gerritzen computed in [Ge, §4&5] the equations in $(\mathbf{C}^*)^{g(g-1)/2}$ defining the closure of the locus defined by the cross ratios of g pairs of points of $\mathbf{P}^1_{\mathbf{C}}$. On the other hand, $\partial\widetilde{Y}$ is an algebraic subtorus of $\big(\mathbf{C}^*\big)^{g(g-1)/2}$. For $g=4$ Gerritzen provides us with one equation and we prove in 6.1 that there is no positive dimensional subtorus of $(\mathbf{C}^*)^6$ satisfying Gerritzen's equation and mapping non–trivially to $\mathbf{C}^*$ via each of the 6 canonical projections $(\mathbf{C}^*)^6\to\mathbf{C}^*$. In particular, no generic point of $\partial\widetilde{Y}$ can be the moduli point of a generalized Jacobian of an irreducible stable curve, of genus 4, with rational normalization. A similar analysis is done in the case when D has genus 1 and $g=4$, see 7.3, and one proves that no generic point of $\partial\widetilde{Y}$ can be the moduli point of a generalized Jacobian of an irreducible stable curve, of genus 4, with normalization of genus 1. Eventually, studying the case that D has genus 2 and $g=4$, we show in 6.3 that no generic point of $\partial\widetilde{Y}$ can be the moduli point of a generalized Jacobian of an irreducible stable curve, of genus 4, with normalization of genus 2. These three examples are the main step to prove Theorem 1.1 (for any $g\geq 4$).

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Contents.

2 Toroidal compactifications of locally symmetric varieties.

Fix a locally symmetric variety Y and write the associated complex analytic space Y^{an} as a quotient $Y^{\mathrm{an}}:=\Gamma\backslash G(\mathbf{R})^0/\mathbf{K}$. Define the *rational boundary components* of $D=$

$G(\mathbf{R})^0/\mathbf{K}$ as follows. Let T_0 be the tangent space of D at the class of $0 \in G(\mathbf{R})$. Since D has a complex structure, T_0 is a $\mathbf{C}$-vector space and we write J for multiplication by i on T_0. Decompose $T_0 \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ according to the $\pm i$–eigenspaces for J. Then, Borel and Harish–Chandra constructed holomorphic open immersions

$$D \longrightarrow \mathfrak{p}_+ \longrightarrow D^\vee,$$

where $D^\vee$ is the compact dual of D and the image of $D \to \mathfrak{p}_+$ is a bounded domain; see [AMRT, Ch. III, §2.1 Thm. 1]. One defines $\overline{D}$ to be the closure of D in $\mathfrak{p}_+$. It follows from [AMRT, Ch. III, §3.1, Thm. 1] that $\partial\overline{D} := \overline{D}\backslash D$ decomposes as a disjoint union of lower dimensional hermitian symmetric domains, called the *boundary components* of D. Fix one of them and call it F. One defines $N(F)_{\mathbf{R}}$ to be the normalizer of F in the connected component $G(\mathbf{R})^0$ of $G(\mathbf{R})$ containing the identity i. e., $N(F)_{\mathbf{R}} := \{g \in G(\mathbf{R})^0 | gF = F\}$ and we say that F is *rational* if $N(F)_{\mathbf{R}} = N(F)(\mathbf{R})$ where $N(F) \subset G$ is a subgroup defined over $\mathbf{Q}$. It is proven in [AMRT, Ch. III, §3.5] that the association $F \to N(F)$ defines a bijection between the set of rational boundary components of D and the set of *admissible* subgroups $N \subset G$, meaning proper subgroups such that, decomposing $G = G_1 \times \cdots \times G_t$ into $\mathbf{Q}$-simple factors, $N = N_1 \times \cdots \times N_t$ with $N_i = G_i$ or $N_i \subset G_i$ a maximal proper parabolic subgroup defined over $\mathbf{Q}$. For every boundary component F it is proven in [AMRT, Ch. III, §3.2, Thm. 2] and [AMRT, Ch. III, §2.2, Prop. 1 & Cor.] that there exists a unique homomorphism of algebraic groups $\varphi_F \colon \mathrm{SL}_{2,\mathbf{R}} \to G_{\mathbf{R}}$ which induces a map of symmetric spaces $f_F \colon \mathcal{H} \to D$ from the Poincaré upper half space $\mathcal{H}$ to D such that f_F sends the class of $\mathbf{1} \in \mathrm{SL}_{2,\mathbf{R}}$ to the class of the identity in $D = G(\mathbf{R})^0/\mathbf{K}$ and the extension of f_F to a map $\overline{\mathcal{H}} \to \overline{D}$ satisfies $f_F(\infty) \in F$. Let $w_F \colon \mathbf{G}_{m,\mathbf{R}} \to G_{\mathbf{R}}$ be the restriction of φ_F to the subgroup of diagonal matrices. By loc. cit. $N(F)_{\mathbf{R}} = P\big(w_F\big)(\mathbf{R})$ where $P\big(w_F\big)(\mathbf{R}) := \big\{g \in G(\mathbf{R})^0 | \lim_{t\to 0} w_{F_i}(t) g w_{F_i}(t)^{-1} \text{ exists}\big\}$. Then, F rational if and only if w_F is defined over $\mathbf{Q}$; see [AMRT, Ch. III, §3.5].

Let D^* be the union of D and its rational boundary components endowed with the Satake topology defined in [AMRT, Ch. III, §6.1]. Then, $Y^* := \Gamma\backslash D^*$ is a compact Hausdorff space, containing Y^{an} as an open dense subset, called the *Baily–Borel* or *minimal compactification* of Y^{an}; cf. [AMRT, Ch. III, §6.1, Thm. 2]. Define a *cusp* of Y^* to be an equivalence class of rational boundary components of D with respect to the action of Γ. We identify a cusp with its image in Y^*; note that two cusps are the same in Y^* if and only if they are equivalent.

We now review how to construct toroidal compactifications of Y^{an}. Let $N \subset G$ be an admissible subgroup with associated rational boundary component $F(N)$. Consider the canonical filtration $U \subset W \subset N$, over $\mathbf{Q}$, where $W \subset N$ is the unipotent radical, $U \subset W$ is the center of W and $V := W/U$ is abelian; see [AMRT, Ch. III p. 225–231]. Note that U is abelian and $U(\mathbf{R})$ (resp. $U(\mathbf{C})$) can be identified with its Lie algebra; in particular, $U(\mathbf{C}) = U(\mathbf{R}) + iU(\mathbf{R})$ so that the imaginary part of an element of $U(\mathbf{C})$ is a well defined notion. By [AMRT, Ch. III, §4.1] we get that $\mathrm{Lie}N(\mathbf{R})$ splits as $\big(\mathrm{Lie}N(\mathbf{R})\big)_0 \oplus \big(\mathrm{Lie}N(\mathbf{R})\big)_1 \oplus \big(\mathrm{Lie}N(\mathbf{R})\big)_2$ taking the eigenspaces for the adjoint action of w_F with respect to the characters $t \mapsto 1$, $t \mapsto t$ and $t \mapsto t^2$. Let $Z(w_F) \subset N$ be the centralizer of w_F. It is defined over $\mathbf{Q}$ and its Lie algebra over $\mathbf{R}$ is $\big(\mathrm{Lie}N(\mathbf{R})\big)_0$. By [AMRT, Ch. III, §3.3, Thm. 3] the connected component at the identity of its base change to $\mathbf{R}$ defines a Levi subgroup of $\big(N \otimes_{\mathbf{Q}} \mathbf{R}\big)^0$. Eventually, let $\widetilde{G}_{h,N} \subset Z(w_F)$ be the centralizer of U in $Z(w_F)$. Then,

a. by [AMRT, Ch. III, §3.5, p. 222] (see also [AMRT, Ch. III, §3.3, Thm. 3] and [AMRT, Ch. III, §4.2, Thm. 1]) the algebraic group $\widetilde{G}_{h,N}$ is semisimple and defined over $\mathbf{Q}$;
b. $\mathbf{K} \cap \widetilde{G}_{h,N}(\mathbf{R})^0$ is a maximal compact subgroup of $\widetilde{G}_{h,N}(\mathbf{R})^0$ (see [AMRT, Ch. III, §4.3, p. 233]);
c. $\widetilde{G}_{h,N}(\mathbf{R})^0/\big(\mathbf{K} \cap \widetilde{G}_{h,N}(\mathbf{R})^0\big)$ is a hermitian symmetric subdomain of $D = G(\mathbf{R})^0/\mathbf{K}$ isomorphic to $F(N)$ (see [AMRT, Ch. III, §3.1, pp. 194–196 & Thm. 1(ii)]).

As in [AMRT, Ch. III, §4.3] define the open subset $D(N) \subset D^\vee$ taking the translates of D by elements of $U(\mathbf{C})$ i. e., $D(N) := \cup_{g \in U(\mathbf{C})} gD$. Then, as explained in [AMRT, Ch. III, §4.2 & §4.3], the map $D \subset D(N)$ is an open embedding of complex analytic manifolds and one has a fibration

$$D(N) \to D(N)' \to F(N) \tag{2.1.1}$$

and a holomorphic section

$$F(N) \to D(N), \tag{2.1.2}$$

where

i) $D(N)' := D(N)/U(\mathbf{C})$ and there exists a (non–canonical) holomorphic section to the map $D(N) \to D(N)'$ so that $D(N) \cong U(\mathbf{C}) \times D(N)'$;
ii) $F(N) = D(N)'/V(\mathbf{R})$ is the rational boundary component associated to N and it is the hermitian symmetric domain defined by $\widetilde{G}_{h,N}$;
iii) the section (2.1.2) is given by $F(N) \cong \widetilde{G}_{h,N}(\mathbf{R})^0/\big(\mathbf{K} \cap \widetilde{G}_{h,N}(\mathbf{R})^0\big) \subset D \subset D(N)$ (see [AMRT, Ch. III, §3.1, pp. 194–196 & Thm. 1(ii)];
iv) $D(N)' \to F(N)$ is a complex vector bundle over $F(N)$ with zero section induced by (2.1.2), it admits a trivialization $D(N)' \cong \mathbf{C}^k \times F(N)$ as vector bundle and it is also a principal homogeneous space under $V(\mathbf{R})$ for the action

$$V(\mathbf{R}) \times \big(\mathbf{C}^k \times F(N)\big) \to \mathbf{C}^k \times F(N) \qquad \big((v,(x,a)\big) \mapsto \big(x + \lambda_v(a), a\big)$$

with $\lambda_v(a)$ holomorphic in a and linear in v;
v) there exists a homogeneous open cone $C(N) \subset U(\mathbf{R})$, self adjoint with respect to a positive definite quadratic form on $U(\mathbf{R})$, and there exists a real analytic map $h\colon \mathbf{C}^k \times \mathbf{C}^k \times F(N) \to U(\mathbf{R})$, with $h(\_,\_,z)$ real bilinear for every fixed $z \in F(N)$, such that

$$D = \big\{(x,y,z) \in D(N) \cong U(\mathbf{C}) \times \mathbf{C}^k \times F(N) | \mathrm{Im}(x) \in C(N) + h(y,y,z)\big\}.$$

Define $\Gamma_H := \Gamma \cap H(\mathbf{R})$ for $H = N$, U or V. The minimal compactification Y^* of Y^{an} has, as boundary components, the spaces $\Gamma_N \backslash F(N)$, which are the complex analytic spaces associated to locally symmetric varieties, for varying admissible subgroups N of G; see [AMRT, Ch. III, §6.1, Thm. 2]. The toroidal compactifications of Y^{an} depend on the choice of a Γ–*admissible collection of polyhedral decompositions* $\zeta = \{\zeta_N\}_N$ of the closure of the cone $C(N) \subset U(\mathbf{R})$ for all rational boundary components of D; see [AMRT, Ch. III, §5, Main Theorem I]. The quotient $T(N) := \Gamma_U \backslash U(\mathbf{C})$ is a complex torus and $U(\mathbf{R}) = \mathrm{Hom}\big(\mathbf{G}_{m,\mathbf{C}}, T(N)\big) \otimes_{\mathbf{Z}} \mathbf{R}$. Hence, ζ_N defines a torus embedding $T(N) \subset \overline{T(N)}_{\zeta_N}$. Using that $\Gamma_U \backslash D(N) \to D(N)'$ is a torsor under $T(N)$, define the relative torus embedding $\big(\Gamma_U \backslash D(N)\big)_{\zeta_N}$ over $D(N)'$ by $\big(\Gamma_U \backslash D(N)\big) \times^{T(N)} \overline{T(N)}_{\zeta_N}$. Let $\big(\Gamma_U \backslash D(N)\big)^o_{\zeta_N}$ be the interior of the closure of $\Gamma_U \backslash G(\mathbf{R})^0/\mathbf{K}$ inside $\big(\Gamma_U \backslash D(N)\big)_{\zeta_N}$. This contains the complement of the space $\Gamma_U \backslash G(\mathbf{R})^0/\mathbf{K}$ in $\big(\Gamma_U \backslash D(N)\big)_{\zeta_N}$ but its intersection with $\Gamma_U \backslash D(N)$ is $\Gamma_U \backslash D$; see [AMRT, Ch. III, pp. 157–158 & pp. 250–251]. Recall that $V(\mathbf{R})$, and hence Γ_V/Γ_U, acts faithfully on $D(N)' \cong \mathbf{C}^k \times F(N)$ by translations. Thus, $\Gamma_V \backslash D(N)' \to F(N)$ is naturally a smooth family of complex Lie groups over $F(N)$. It is compact since the rank of Γ_V/Γ_U as $\mathbf{Z}$–module is equal to the dimension of V/U. Let

$$\Gamma_N \backslash D(N) =: \Omega \subset \overline{\Omega}_{\zeta_N} := \Gamma_N \backslash \Big(\big(\Gamma_U \backslash D(N)\big)_{\zeta_N} \Big) \supset \overline{\Omega}^o_{\zeta_N} := \Gamma_N \backslash \Big(\big(\Gamma_U \backslash D(N)\big)^o_{\zeta_N} \Big).$$

For simplicity we assume that Γ is *neat* [AMRT, Ch. III, §7] so that, in particular, Γ_N, Γ_N/Γ_U and Γ_N/Γ_V act freely on the various steps of the filtration (2.1.1). This can always be achieved possibly passing to a finite index subgroup Γ' of Γ. Then, the quotient of (2.1.1) and (2.1.2) by Γ_N provides a fibration and a section

$$\overline{\Omega}^o_{\zeta_N} \subset \overline{\Omega}_{\zeta_N} \supset \Omega \longrightarrow A \longrightarrow \Gamma_N \backslash F(N), \qquad \sigma_N \colon \Gamma_N \backslash F(N) \longrightarrow \Omega \tag{2.1.3}$$

where

a) $\Gamma_N \backslash F(N)$ is the complex analytic space associated to a locally symmetric variety;
b) $A = \Gamma_N \backslash D(N)' \to \Gamma_N \backslash F(N)$ is a smooth family of compact abelian complex Lie groups of relative dimension equal to $\dim(V/U)/2$ with zero section defined by σ_N;
c) $\Omega = \Gamma_N \backslash D(N)$ is a torsor over A under the torus $T(N)$, trivialized over the zero section of A by σ_N and $\Omega \subset \overline{\Omega}_{\zeta_N}$ is a relative toroidal embedding over A;
d) $\overline{\Omega}^o_{\zeta_N}$ is an open neighborhood of $\overline{\Omega}_{\zeta_N} \backslash \Omega$ in $\overline{\Omega}_{\zeta_N}$ and $\overline{\Omega}^o_{\zeta_N} \cap \Omega = \Gamma_N \backslash D$;
e) $\overline{\Omega}^o_{\zeta_N} \to A$ is, real analytically, the product of A and a principal homogeneous space under the compact torus $\Gamma_U \backslash U(\mathbf{R})$ over the interior of the closure of $C(N)$ in $U(\mathbf{R})_{\zeta_N}$. See [AMRT, Ch. III, p. 251].

Furthermore, we have maps

$$Y^{\mathrm{an}} = \Gamma \backslash D \longleftarrow \Gamma_N \backslash D \hookrightarrow \overline{\Omega}^o_{\zeta_N} \xrightarrow{f_N} D^*/\Gamma \tag{2.1.4}$$

where $\Gamma_N \backslash D \subset \overline{\Omega}^o_{\zeta_N}$ is an open embedding with dense image, $\Gamma_N \backslash D \to Y^{\mathrm{an}}$ is a covering map and $f_N \colon \overline{\Omega}^o_{\zeta_N} \to D^*/\Gamma$ is the continuous map defined in [AMRT, Ch. III,

p. 272–274]. Let $Z \subset D^*$ be an open subset in the Satake topology containing $F(N)$ and such that $\Gamma_N Z = Z$ and $\gamma Z \cap Z = \emptyset$ if $\gamma \in \Gamma$ but $\gamma \notin \Gamma_N$. Its existence is guaranteed by the properties of the Satake topology; see [AMRT, Ch. III, §6.1, Theorem 1]. Put $Z^o := Z \cap D$. Then, $\Gamma_N \backslash Z^o \subset \Gamma_N \backslash D$ is an open subset mapping isomorphically onto its image in Y^{an} which is open. Define $Y^{\mathrm{an}}_{\zeta_N}$ to be the analytic space obtained by gluing $f_N^{-1}(Z)$ and $\overline{\Omega}^o_{\zeta_N}$ along $\Gamma_N \backslash Z^o$. This defines an analytic open neighborhood of the boundary component defined by N in the toroidal compactification $Y^{\mathrm{an}} \subset \overline{Y}_\zeta$ associated to ζ. In the non–neat case one considers a normal, finite index normal subgroup $\Gamma' \subset \Gamma$ which is neat, one constructs the partial compactification $\left(\Gamma' \backslash D\right)_{\zeta_N}$, using the group Γ', endowed with an action of Γ/Γ'. One defines $Y^{\mathrm{an}}_{\zeta_N}$ as $\left(\Gamma/\Gamma'\right) \backslash \left(\Gamma' \backslash D\right)_{\zeta_N}$.

Let Y_1 and Y_2 be locally symmetric varieties with $Y_1^{\mathrm{an}} := \Gamma_1 \backslash G_1(\mathbf{R})^0/\mathbf{K}_1$ and $Y_2^{\mathrm{an}} := \Gamma_2 \backslash G_2(\mathbf{R})^0/\mathbf{K}_2$. Let $f\colon Y_1 \to Y_2$ be a map of locally symmetric varieties; by definition it is induced by a homomorphism of algebraic groups $\widetilde{f}\colon G_1 \to G_2$. In particular, for any such $\widetilde{f}(\Gamma_1) \subset \Gamma_2$ and we let $\overline{f}\colon D_1 \to D_2$ be the induced map from $D_1 := G_1(\mathbf{R})^0/\mathbf{K}_1$ to $D_2 := G_2(\mathbf{R})^0/\mathbf{K}_2$.

2.1 Proposition. *Let $f\colon Y_1 \to Y_2$ be a map of locally symmetric varieties. Then,*

1) *$\overline{f}$ induces a natural map from the set of (rational) boundary components of D_1 to the set of (rational) boundary components of D_2;*
2) *let F_1 and F_2 be corresponding rational components of D_1 (resp. D_2). Let $N_i \subset G_i$ be the admissible subgroup corresponding to D_i for $i = 1, 2$. Then, $\widetilde{f}(N_1) \subset N_2$, $\widetilde{f}(W_1) \subset W_2$, and $\widetilde{f}(U_1) \subset U_2$;*
3) *f extends to a continuous map $\overline{f}^*\colon Y_1^* \to Y_2^*$ of the Baily–Borel compactifications;*
4) *the hypotheses are as in (2) and assume that Γ_1 and Γ_2 are neat. We have commutative diagrams*

$$\begin{array}{ccccc} \Omega_1 & \longrightarrow & A_1 & \longrightarrow & \Gamma_{N_1}\backslash F_1 =: X_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_2 & \longrightarrow & A_2 & \longrightarrow & \Gamma_{N_2}\backslash F_2 =: X_2 \end{array} \qquad \begin{array}{ccc} X_1 & \longrightarrow & \Omega_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & \Omega_2, \end{array}$$

where $X_1 \to X_2$ is the analytification of a map of locally symmetric varieties $X_1^{\mathrm{alg}} \to X_2^{\mathrm{alg}}$, the induced map $A_1 \to A_2 \times_{X_2} X_1$ is a map of Lie groups and $\Omega_1 \to \Omega_2 \times_{A_2} A_1$ is a map of torus torsors compatible with the homomorphism of tori $T(N_1) \to T(N_2)$ induced by $\widetilde{f}$.

Assume that G_1 is a subgroup of G_2 so that Y_1 defines a locally symmetric subvariety of Y_2. Fix a torus embedding $T(N_2) \subset \overline{T(N_2)}_{\zeta_{N_2}}$ and consequently a compactification $\Omega_2 \subset \overline{\Omega}_{2,\zeta_{N_2}}$. Define $\overline{T}_{1,2,\zeta_{N_2}}$ (resp. $\overline{\Omega}_{1,2,\zeta_{N_2}}$) to be the closure of the image of $T(N_1) \to T(N_2)$ in $\overline{T(N_2)}_{\zeta_{N_2}}$ (resp. of the image of $\Omega_1 \to \Omega_2$ in $\overline{\Omega}_{2,\zeta_{N_2}}$). Then,

5.a) *$X_1^{\mathrm{alg}} \to X_2^{\mathrm{alg}}$ defines a locally symmetric subvariety of X_2^{alg};*

5.b) *the image of the composite map $\overline{\Omega}_{1,2,\zeta_{N_2}} \to \overline{\Omega}_{2,\zeta_{N_2}} \to A_2$ is the image of $A_1 \to A_2$ and the latter is closed in A_2;*

5.c) $\overline{\Omega}_{1,2,\zeta_{N_2}}$ *is a fibration over the image of* $A_1 \to A_2$ *with fibers isomorphic to* $\overline{T}_{1,2,\zeta}$.

Proof: (1)–(2) Recall that $\partial\overline{D}_i$ is the disjoint union of the boundary components of D_i. Observe that $\overline{f}$ extends to holomorphic homomorphisms

$$\begin{array}{ccccc} D_1 & \longrightarrow & \mathfrak{p}_{1,+} & \longrightarrow & D_1^\vee \\ {\scriptstyle \overline{f}}\big\downarrow & & {\scriptstyle \overline{f}'}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f}''} \\ D_2 & \longrightarrow & \mathfrak{p}_{2,+} & \longrightarrow & D_2^\vee; \end{array}$$

see [AMRT, Ch. III, p. 175]. Then, we get a holomorphic map $\overline{f}'\colon\overline{D}_1 \to \overline{D}_2$. The boundary components of $\overline{D}_i$ are characterized in [AMRT, Ch. III, §3.1, Thm. 1] as the equivalence classes of points of $\overline{D}_i$ with respect to the following relation: two points x and y are equivalent if there exists a holomorphic map $\mathcal{H} \to \mathfrak{p}_{i,+}$ whose image contains x and y. In particular, $\overline{f}'$ sends equivalence classes to equivalence classes and induces a natural map from the set of boundary components of D_1 to the set of boundary components of D_2. Given two such components $F_1 \subset \overline{D}_1$ and $F_2 \subset \overline{D}_1$ such that $\overline{f}'(F_1) \subset F_2$, since $(N_i)_{\mathbf{R}} = \{g \in G_i(\mathbf{R}) | g(F_i) = F_i\}$, we have that $\widetilde{f}\big((N_1)_{\mathbf{R}}\big) \subset \big((N_2)_{\mathbf{R}}\big)$. Let $w_{F_i}\colon \mathbf{G}_{m,\mathbf{R}} \to G_{i,\mathbf{R}}$ be the homomorphism of algebraic groups associated to F_i. Note that $f_{F_2} = \overline{f} \circ f_{F_1}$ by uniqueness of f_{F_i}. Hence, $w_{F_2} = \widetilde{f} \circ w_{F_1}$. In particular, if F_1 is rational, w_{F_1}, and consequently w_{F_2}, is defined over $\mathbf{Q}$. This implies that $\big(N_2\big)_{\mathbf{R}}$ is defined over $\mathbf{Q}$ i. e., that F_2 is rational.

It follows from [AMRT, Ch. III, §4.1] that the filtration $U_i \otimes_{\mathbf{Q}} \mathbf{R} \subset W_i \otimes_{\mathbf{Q}} \mathbf{R} \subset N_i \otimes_{\mathbf{Q}} \mathbf{R}$ is characterized at the level of Lie algebras by the adjoint action of w_{F_i} i. e., $\mathrm{Lie}U_i \otimes_{\mathbf{Q}} \mathbf{R} = \big(\mathrm{Lie}N(\mathbf{R})\big)_2$ and $\mathrm{Lie}W_i \otimes_{\mathbf{Q}} \mathbf{R} = \big(\mathrm{Lie}N(\mathbf{R})\big)_1 \oplus \big(\mathrm{Lie}N(\mathbf{R})\big)_2$. This is respected by $\mathrm{Lie}\widetilde{f} \otimes_{\mathbf{Q}} \mathbf{R}$. Since U_1 and W_1 are connected and unipotent, we conclude that we have inclusions $\widetilde{f}(U_1) \subset U_2$ and $\widetilde{f}(W_1) \subset W_2$ after base change to $\mathbf{R}$ and, hence, over $\mathbf{Q}$ as well.

(3) The claim follows since $\overline{f}\colon D_1 \to D_2$ extends to a continuous map $D_1^* \to D_2^*$.

(4) Thanks to the definition of $\widetilde{G}_{h,N_i} \subset G_i$ the morphism $\widetilde{f}$ defines a morphism of algebraic groups $u\colon \widetilde{G}_{h,N_1} \to \widetilde{G}_{h,N_2}$. Thanks to (2) the map $\overline{f}''\colon D_1^\vee \to D_2^\vee$ induces a holomorphic map $\overline{f}'''\colon D(N_1) \to D(N_2)$ compatible with the actions of $U_i(\mathbf{C})$ and $V_i(\mathbf{R})$ with $i = 1$ and 2. In particular, $\overline{f}'''$ is compatible with the fibrations defined in (2.1.1). It follows from the characterization of the inclusion $F_i \subset D_i$ given in [AMRT, Ch. III, §3.1, pp. 194–196 & Thm. 1(ii)] that $\overline{f}'''$ is compatible with the sections defined in (2.1.2) and induces the holomorphic map of symmetric spaces $F_1 \to F_2$ associated to the homomorphism of algebraic groups u. In conclusion, we get commutative diagrams

$$\begin{array}{ccccccc} D_1 & \hookrightarrow & D(N_1) & \longrightarrow & D(N_1)' & \longrightarrow & F_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D_2 & \hookrightarrow & D(N_2) & \longrightarrow & D(N_2)' & \longrightarrow & F_2 \end{array}$$

and

$$\begin{array}{ccccc} F_1 & \hookrightarrow & D_1 & \hookrightarrow & D(N_1) \\ \downarrow & & \downarrow & & \downarrow \\ F_2 & \hookrightarrow & D_2 & \hookrightarrow & D(N_2), \end{array}$$

compatible with the actions of $U_i(\mathbf{C})$ and $V_i(\mathbf{R})$ for $i = 1, 2$. Taking the quotients under Γ_{1,N_1} and Γ_{2,N_2} and noting that $\widetilde{f}(\Gamma_{1,N_1}) \subset \Gamma_{2,N_2}$ we conclude that (4) holds.

(5) In this case G_1 is a closed subgroup of G_2. Since $\widetilde{f}$ is a closed immersion, using the inclusions $\widetilde{G}_{h,N_i} \subset G_i$, we get that the induced map $u\colon \widetilde{G}_{h,N_1} \to \widetilde{G}_{h,N_2}$ is a closed immersion. In particular, (5.a) holds. Furthermore, using the description of $N(F_i)(\mathbf{R})$ as $P(w_{F_i})(\mathbf{R})$, we deduce that $N_1(\mathbf{R}) = N_2(\mathbf{R}) \cap G_1(\mathbf{R})$. It follows from the characterization in the proof of (2) of the filtration $U_i \otimes_{\mathbf{Q}} \mathbf{R} \subset W_i \otimes_{\mathbf{Q}} \mathbf{R} \subset N_i \otimes_{\mathbf{Q}} \mathbf{R}$ and the fact that U_i and W_i are connected and unipotent that $W_1 = W_2 \cap G_1$ and $U_1 = U_2 \cap G_1$ after passing to $\mathbf{R}$ and, hence, we have equality as algebraic groups over $\mathbf{Q}$. Note that $\Gamma_1' := \Gamma_2 \cap G_1(\mathbb{Q}) \cap G_1(\mathbf{R})^0$ is the intersection with $G_1(\mathbf{R})^0$ of an arithmetic subgroup of $G_1(\mathbf{Q})$ so that, in particular, Γ_1 is a finite index subgroup of Γ_1'. This implies that the vertical morphisms in the displays of claim (4) of the Proposition are all finite topological coverings. To study their image we may take Γ_1' in place of Γ_1 in the definition of Y_1. Thus, we may assume that $\Gamma_1 = \Gamma_1'$ so that, in particular, $\Gamma_{N_1} = \Gamma_{N_2} \cap N_1(\mathbf{R})$ and similarly $\Gamma_{W_1} = \Gamma_{W_2} \cap W_1(\mathbf{R})$ and $\Gamma_{U_1} = \Gamma_{U_2} \cap U_1(\mathbf{R})$. This implies that the vertical morphisms in claim (4) are all injective. Since the image of $X_1 \to X_2$ is closed and the image of $A_1 \to A_2$ is closed fiberwise over X_2, claim (5.b) is clear. The map $\overline{\Omega}_{2,\zeta} \to A_2$ (resp. $\overline{Y}_{2,\zeta} \to A_2$) is locally analytically over A_2 the product of an open analytic subspace of A_2 with the torus embedding $T_2 \subset \overline{T(N_2)}_{\zeta_{N_2}}$ (resp. an open subset of $\overline{T(N_2)}_{\zeta_{N_2}}$). Claim (5.c) follows. □

3 An example: the case of $\mathcal{A}_{g,(n),\gamma}$.

Fix integers $0 < r \leq g$ and $n \geq 1$. Let $\mathcal{A}_{g,n}$ be the moduli space of principally polarized abelian varieties of dimension g and full symplectic level n–structure. The cusps of $\mathcal{A}_{g,n}$ can be described in terms of elements of $\mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$; see 3.1. Fix such a cusp E_γ, associated to some $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$, and assume it has toric rank r. We will introduce a moduli space $\mathcal{A}_{g,(n),\gamma}$ which sits in between $\mathcal{A}_{g,n}$ and $\mathcal{A}_{g,1}$ and depends on the choice of γ. We then construct a partial toroidal compactification $\widetilde{\mathcal{A}}_{g,(n),\gamma}$ of the complex analytic space $\mathcal{A}^{\mathrm{an}}_{g,(n),\gamma}$ associated to $\mathcal{A}_{g,(n),\gamma}$. It has the property that the boundary component $\partial\widetilde{\mathcal{A}}_{g,(n),\gamma}$ is the complex analytic space associated to a rigidified torus torsor over the r–th fold fibred product $A^r := A \times_{\mathcal{A}_{g-r,n}} \cdots \times_{\mathcal{A}_{g-r,n}} A$ of the universal abelian scheme A over $\mathcal{A}_{g-r,n}$. These compactifications are well behaved in order to extend the Torelli map as will be proven in §5. Incidentally, the choice of the intermediate level structure $\mathcal{A}_{g,(n),\gamma}$ is due to the fact that, for technical reasons, we need the existence of the universal family over $\mathcal{A}_{g-r,n}$.

We then consider the following problem. Let $Y \subset \mathcal{A}_{g,(n),\gamma}$ be a locally symmetric subvariety. We consider the topological closure $\widetilde{Y}_\gamma$ of Y^{an} in $\widetilde{\mathcal{A}}_{g,(n),\gamma}$. It has a natural structure of complex analytic space. Let $\partial\widetilde{Y}_\gamma$ be the intersection of $\widetilde{Y}_\gamma$ with $\partial\widetilde{\mathcal{A}}_{g,(n),\gamma}$. Assuming that $\partial\widetilde{Y}_\gamma$ is non–empty, we prove in 3.9 two things:

1) there is a canonical retraction of $\partial\widetilde{Y}_\gamma\subset\widetilde{Y}_\gamma$ in a complex analytic neighborhood of $\partial\widetilde{Y}_\gamma$;
2) $\partial\widetilde{Y}_\gamma\subset\partial\widetilde{\mathcal{A}}_{g,(n),\gamma}$ is the complex analytic space associated to a rigidified torus subtorsor over an abelian subscheme of the restriction of A^r to a locally symmetric subvariety of $\mathcal{A}_{g-r,n}$.

We start explaining how the theory, recalled in the previous section, applies to construct partial toroidal compactifications of $\mathcal{A}_{g,n}$. Recall that $\mathcal{A}_{g,n}^{\rm an}=\Gamma_n\backslash\mathrm{Sp}_{2g}(\mathbf{R})/\mathbb{K}_g$ where $\mathbb{K}_g\subset\mathrm{Sp}_{2g}(\mathbf{R})$ is a maximal compact subgroup and Γ_n is the subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$ of matrices mapping to 1 in $\mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$. Define $\mathbb{N}_{r,g}\subset\mathrm{Sp}_{2g}$ to be the admissible subgroup defined over $\mathbf{Z}$ whose R–valued points, for every ring R, are of the form

$$\begin{array}{c}g-r\\ r\\ g-r\\ r\end{array}\begin{pmatrix}A_1 & 0 & B_1 & B_{1,2}\\ A_{2,1} & A_2 & B_{2,1} & B_2\\ C_1 & 0 & D_1 & D_{1,2}\\ 0 & 0 & 0 & D_2\end{pmatrix}\in\mathrm{Sp}_{2g}(R).$$
$$\qquad g-r\quad r\quad g-r\quad r$$

One has a semisimple quotient $p_{r,g}\colon\mathbb{N}_{r,g}\to\mathrm{Sp}_{2g-2r}$ given by sending a typical element to $\begin{pmatrix}A_1 & B_1\\ C_1 & D_1\end{pmatrix}$. Set theoretically, the minimal compactification $\mathcal{A}_{g,1}^*$ of $\mathcal{A}_{g,1}^{\rm an}$ is $\cup_{0\le h\le g}\mathcal{A}_{h,1}^{\rm an}$ and the boundary component defined by $\mathbb{N}_{r,g}$ in the minimal compactification of $\mathcal{A}_{g,1}^{\rm an}$ is $\mathrm{Sp}_{2g-2r}(\mathbf{Z})\backslash\mathrm{Sp}_{2g-2r}(\mathbf{R})/\mathbb{K}_{g-r}\cong\mathcal{A}_{g-r,1}^{\rm an}$. The unipotent radical $\mathbb{W}_{r,g}\subset\mathbb{N}_{r,g}$ is defined on R–valued points by

$$\mathbb{W}_{r,g}(R):=\left\{\begin{pmatrix}1 & 0 & 0 & B_{1,2}\\ A_{2,1} & 1 & B_{1,2}^t & B\\ 0 & 0 & 1 & -A_{2,1}^t\\ 0 & 0 & 0 & 1\end{pmatrix}\Big| B=B^t+A_{2,1}B_{1,2}-B_{1,2}^tA_{2,1}^t\right\}$$

and the center $\mathbb{U}_{r,g}\subset\mathbb{W}_{r,g}$ is defined by the condition that $A_{2,1}=B_{1,2}=0$. Note that $\mathbb{V}_{r,g}=\mathbb{W}_{r,g}/\mathbb{U}_{r,g}=\left(\mathbf{G}_a^{g-r}\right)^{2r}$ and $\mathbb{U}_{r,g}$ is isomorphic to $\mathbf{G}_a^{r(r+1)/2}$. In particular, $\mathbb{U}_{r,g}(\mathbf{C})/\mathbb{U}_{r,g}(\mathbf{Z})$ is the torus $\left(\mathbf{C}^*\right)^{r(r+1)/2}$. The cone $C\left(\mathbb{U}_{r,g}\right)\subset\mathbb{U}_{r,g}(\mathbf{R})$ is defined by the positive definite matrices. Eventually, the subgroup $\widetilde{G}_{h,\mathbb{N}_{r,g}}$ is Sp_{2g-2r} via the map $\mathrm{Sp}_{2g-2r}\to\mathbb{N}_{r,g}$ given by $\begin{pmatrix}A_1 & B_1\\ C_1 & D_1\end{pmatrix}\mapsto\begin{pmatrix}A_1 & 0 & B_1 & 0\\ 0 & 1 & 0 & 0\\ C_1 & 0 & D_1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$.

Consider on $\left(\mathbf{Z}/n\mathbf{Z}\right)^{2g}$ the standard symplectic form and let $\Sigma_{g,r}\subset\left(\mathbf{Z}/n\mathbf{Z}\right)^{2g}$ be the $\mathbf{Z}$–submodule generated by the $2g-r$–basis elements $e_1,\ldots,e_g,e_{g+r+1},\ldots,e_{2g}$. Following [FC, Def. IV.6.6],

3.1 Definition. *Given an element γ of $\mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z})$ define the associated cusp of $\mathcal{A}_{g,n}$ of toric rank r, denoted E_γ, to be the transform of $\Sigma_{r,n}$ under γ.*

3.2 Lemma. *The cusps of the minimal compactification of $\mathcal{A}^*_{g,n}$ in the sense of* [AMRT] *are in one to one correspondence with the cusps defined in 3.1.*

Proof: The correspondence sends E_γ to the admissible subgroup $\mathbb{N}_{r,g,\gamma} := \tilde{\gamma}\mathbb{N}_{r,g}\tilde{\gamma}^{-1}$ where $\tilde{\gamma} \in \Gamma_1$ is a lift of $\gamma \in \mathrm{Sp}_{2g}(\mathbf{Z}/n\mathbf{Z}) \cong \Gamma_1/\Gamma_n$. $\square$

For a cusp E_γ of $\mathcal{A}^*_{g,n}$ of toric rank $r>0$, let F_γ be the rational boundary component of $\mathrm{Sp}_{2g}(\mathbf{R})/\mathbb{K}_g$ corresponding to $\mathbb{N}_{r,g,\gamma}$. Denote by $\mathbb{W}_{r,g,\gamma} \subset \mathbb{N}_{r,g,\gamma}$ its unipotent radical. Write $\Gamma_{n,\gamma}$ for $\mathbb{N}_{r,g,\gamma} \cap \Gamma_n$. Note that $\Gamma_{1,\gamma}/\Gamma_{n,\gamma} \cong \mathbb{N}_{r,g,\gamma}(\mathbf{Z}/n\mathbf{Z})$. Define

$$\mathcal{A}_{g,(n),\gamma} := \mathcal{A}_{g,n}/\mathbb{W}_{r,g,\gamma}(\mathbf{Z}/n\mathbf{Z}).$$

It is a finite cover of $\mathcal{A}_{g,1}$, endowed with a residual action of

$$\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big) \cong \mathbb{N}_{r,g,\gamma}(\mathbf{Z}/n\mathbf{Z})/\mathbb{W}_{r,g,\gamma}(\mathbf{Z}/n\mathbf{Z}).$$

Assume that $n \geq 3$ so that Γ_n is a neat subgroup. Let (A, λ_A) be the universal principally polarized abelian scheme over $\mathcal{A}_{g-r,n}$, which exists since $n \geq 3$. By abuse of notation we write A^r for the r–th fold fibred product $A \times_{\mathcal{A}_{g-r,n}} \cdots \times_{\mathcal{A}_{g-r,n}} A$. Given a scheme S over $\mathcal{A}_{g-r,n}$ and an S-valued point $s := (s_1, \ldots, s_r) \in A^r(S)$ we let $\xi_s\colon \mathbf{Z}^r \to A \times_{\mathcal{A}_{g-r,n}} S$ be the S–group scheme homomorphism sending $(n_1, \ldots, n_s) \mapsto n_1 s_1 + \cdots + n_r s_r$. In this way we may, and we will, identify A^r with the moduli space of group-scheme homomorphisms $\xi\colon \mathbf{Z}^r \to A$ over the base $\mathcal{A}_{g-r,n}$.

Let $\Theta \to A^r$ be the scheme classifying symmetric trivializations of the pull–back of the Poincaré biextension $\mathcal{P}_A$ over $A \times_{\mathcal{A}_{g-r,n}} A^\vee$ via $\xi \times \xi\colon \mathbf{Z}^r \times \mathbf{Z}^r \to A \times_{\mathcal{A}_{g-r,n}} A \cong A \times_{\mathcal{A}_{g-r,n}} A^\vee$ where the last map is the identity times λ_A. Write 1 for the section defined by the trivialization of $\mathcal{P}_A$ over $\{0\} \times A^\vee$. Then,

3.3 Proposition. *The fibration and the section σ_N in (2.1.3), associated to $F(N) = F_\gamma$ and Γ equal to the arithmetic group defining $\mathcal{A}^{\mathrm{an}}_{g,(n),\gamma}$, are the analytification of*

$$\Theta \longrightarrow A^r \longrightarrow \mathcal{A}_{g-r,n}, \qquad 1\colon \mathcal{A}_{g-r,(n)} \longrightarrow \Theta. \tag{3.3.1}$$

The morphism $\Theta \to A^r$ is a torsor under the torus U with character group $X^*(U)$ equal to the symmetric bilinear homomorphisms $\mathbf{Z}^r \times \mathbf{Z}^r \to \mathbf{Z}$. The action $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$ on $\mathcal{A}_{g-r,n}$, defined changing the level–n structure, induces an action on Θ.

Proof: This follows from the interpretation of toroidal compactifications of Shimura varieties via the theory of degenerating polarized Hodge structures. We refer to [Br, §4.2] for details. In our concrete case, this means that the analytic space Ω of (2.1.3) classifies 1–motives $\mathbb{M} := \big[u\colon \mathbf{Z}^r \to \widetilde{\mathrm{G}}\big]$ over an analytic space S where (i) $\widetilde{\mathrm{G}}$ is a family of semiabelian varieties over S, extension of a family of principally polarized abelian varieties (A, λ_A) of dimension $g-r$ and full symplectic level–n structure over S by the torus $\mathbf{G}^r_{m,S}$ and (ii) $\mathbb{M}$ is self dual. It follows from [De, Prop. 10.2.14] that u defines and is defined by a trivialization ϖ over $\mathbf{Z}^r \times \mathbf{Z}^r$ of the pull–back of the Poincaré $\mathbf{G}_m$–biextension $\mathcal{P}_A \to A \times_S A$ (identified with $A \times_S A^\vee$ via $1 \times \lambda_A$) via $\mathbf{Z}^r \times \mathbf{Z}^r \xrightarrow{u \times u} A \times_S A$. Requirement (ii) amounts to ask that ϖ is symmetric with respect to switching the two factors $\mathbf{Z}^r \times \mathbf{Z}^r$. □

3.4 *A partial toroidal compactification of* $\mathcal{A}_{g,(n),\gamma}$. Let $e_1, \ldots, e_r$ be the standard basis of $\mathbf{Z}^r$. A basis for the character group $X^*(U)$ of the torus U of 3.3 is given by the homomorphisms

$$\varphi_{i,j} \colon \mathbf{Z}^r \times \mathbf{Z}^r \to \mathbf{Z}, \qquad \varphi_{i,j}(e_s, e_t) = \begin{cases} 1 & \text{if } (s,t) = (i,j) \text{ or } (j,i), \\ 0 & \text{otherwise.} \end{cases}$$

Decompose $U = U' \times U''$ where the character group of U'' (resp. U') is spanned by the maps $\varphi_{i,j}$ such that $i = j$ (resp. $i \neq j$). Consider the smooth affine torus embedding $U'' \subset \overline{U}''$ associated to the cone in $X^*(U'') \otimes_{\mathbf{Z}} \mathbf{R}$ defined by $\varphi_{i,i} \geq 0$ for every $i = 1, \ldots, r$. Note that $U'' \subset \overline{U}''$ is $\mathbf{G}_{m,\mathbf{C}}^r \subset \mathbf{A}_{\mathbf{C}}^r$. In particular, $\overline{U}''$ is smooth. Its lowest dimensional stratum $\partial\overline{U}''$ is simply the point $(0, \ldots, 0) \in \mathbf{A}_{\mathbf{C}}^r$. Let $\overline{U} := U' \times \overline{U}''$ and $\partial\overline{U} := U' \times \partial\overline{U}''$.

Decompose $\Theta = \Theta' \times_{A^r} \Theta''$ where $\Theta' = \Theta \times^U U'$ and $\Theta'' = \Theta \times^U U''$. Note that $\Theta' \to A^r$ is a U'–torsor, rigidified over the zero section of $A^r \to \mathcal{A}_{g-r,n}$. Define

$$\overline{\Theta}'' := \Theta'' \times^{U''} \overline{U}'' \quad \text{and} \quad \overline{\Theta} := \Theta \times^U \overline{U} = \Theta' \underset{A^r}{\times} \overline{\Theta}''.$$

Since $\overline{U}$ is smooth, $\overline{\Theta}$ is a smooth scheme over A^r. Let $\partial\overline{\Theta}''$ be the lowest dimensional stratum of $\overline{\Theta}''$. It is isomorphic to A^r. Set $\partial\overline{\Theta} \cong \Theta' \times_{A^r} \partial\overline{\Theta}'' \subset \overline{\Theta}$. The action of $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$ on Θ extends to an action on $\overline{\Theta}$.

Thanks to the theory of [AMRT] recalled in the previous section, we get from $\overline{\Theta}^{\mathrm{an}}$ a smooth, partial toroidal compactification

$$\mathcal{A}_{g,(n),\gamma}^{\mathrm{an}} \subset \widetilde{\mathcal{A}}_{g,(n),\gamma}$$

of the complex analytic space $\mathcal{A}_{g,(n),\gamma}^{\mathrm{an}}$ at F_γ. More precisely, there exists an open neighborhood Z of $\partial\overline{\Theta}^{\mathrm{an}}$ in $\overline{\Theta}^{\mathrm{an}}$ such that $Z \cap \Theta^{\mathrm{an}}$ maps isomorphically to an open analytic subspace of $\mathcal{A}_{g,(n),\gamma}^{\mathrm{an}}$ and $\widetilde{\mathcal{A}}_{g,(n),\gamma}$ is obtained gluing $\mathcal{A}_{g,(n),\gamma}^{\mathrm{an}}$ and Z along $Z \backslash \partial\overline{\Theta}^{\mathrm{an}}$. The action of $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$ on $\mathcal{A}_{g,(n),\gamma}^{\mathrm{an}}$ is compatible with the action on $\overline{\Theta}^{\mathrm{an}}$. Possibly shrinking Z we may assume that Z is stable under the action of $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$. Thus, $\widetilde{\mathcal{A}}_{g,(n),\gamma}$ is endowed with a unique action of $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$ compatible with those on $\mathcal{A}_{g,(n),\gamma}$ and Z. We also have a closed analytic subspace $\partial\widetilde{\mathcal{A}}_{g,(n),\gamma} := \partial\overline{\Theta}^{\mathrm{an}}$ stable under the action of $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$. The projection $\overline{\Theta} \to \Theta'$ induces an $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$–equivariant retraction of complex analytic spaces

$$\partial\widetilde{\mathcal{A}}_{g,(n),\gamma} \hookrightarrow Z \longrightarrow \Theta'^{\mathrm{an}} = \partial\widetilde{\mathcal{A}}_{g,(n),\gamma}. \tag{3.4.1}$$

Note that $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)\backslash Z$ is an open neighborhood of a partial toroidal compactification $\widetilde{\mathcal{A}}_{g,1}$ of $\mathcal{A}_{g,1}^{\mathrm{an}}$ (see §2). Define $\partial\widetilde{\mathcal{A}}_{g,1}$ to be the image of $\partial\widetilde{\mathcal{A}}_{g,(n),\gamma}$.

3.5 Remark. It follows from [Na, Examples 2.7(ii) & 5.4(iii)] that $\widetilde{\mathcal{A}}_{g,1}$ is an open subspace of the partial toroidal compactification associated to the principal cone in $\mathcal{A}_{g,1}^{\mathrm{DV,an}}$.

3.6 *A moduli interpretation of* Θ *and* Θ'. Since $A^\vee = \mathcal{E}\mathrm{xt}(A, \mathbf{G}_m)$, the universal map $\xi: \mathbf{Z}^r \to A_{A^r}$ defines a universal semiabelian scheme $\widetilde{\mathrm{G}}$ over A^r which is an extension of $A_{A^r} := A \times_{\mathcal{A}_{g-r,n}} A^r$ by $\mathbf{G}_{m,A^r}^r$. More precisely, if $e_1, \ldots, e_r$ define the canonical basis of $\mathbf{Z}^r$, the push–forward $\widetilde{\mathrm{G}}_i$ of $\widetilde{\mathrm{G}}$ via the projection $\mathbf{G}_{m,A^r}^r \to \mathbf{G}_{m,A^r}$ onto the i–th factor is, as rigidified $\mathbf{G}_{m,A^r}$–bundle, the pull–back via of the Poincaré biextension $\mathcal{P}_A$ via $A_{A^r} \cong A_{A^r} \times \xi(e_i) \subset A_{A^r}^2 \cong A_{A^r} \times_{A^r} A_{A^r}^\vee$. As recalled in the proof of 3.3, one can view Θ as the scheme classifying self dual 1–motives $\iota: \mathbf{Z}^r \to \widetilde{\mathrm{G}}$. The correspondence between the two descriptions of Θ is that, for every $1 \leq i,\ j \leq r$, the section of $\mathcal{P}_A$ over $\big(\xi(e_j), \xi(e_i)\big)$ is the image $\alpha_{i,j}$ of $\iota(e_j)$ in $\widetilde{\mathrm{G}}_i$. The self duality is equivalent to require that $\alpha_{i,j}$ defines the same section of $\mathcal{P}_A$ as $\alpha_{j,i}$ after switching the two factors in $A_{A^r} \times_{A^r} A_{A^r}$ for every $1 \leq i < j \leq r$. This forces the composite $\mathbf{Z}^r \to \widetilde{\mathrm{G}} \to A_{A^r}$ to be ξ. In particular, we can view Θ' as the moduli space classifying for every scheme S:

1. a principally polarized abelian scheme (A, λ_A) endowed with full symplectic level–n structure over S;
2. an S–section $s = (s_1, \ldots, s_r) \in A^r(S)$. For every, i let $\widetilde{\mathrm{G}}_i \in \mathcal{E}\mathrm{xt}_S(A, \mathbf{G}_{m,S}) = A^\vee(S)$ be the extension of A by $\mathbf{G}_{m,S}$ defined by $\lambda_A(s_i) \in A^\vee(S)$ and let $\widetilde{\mathrm{G}} := \prod_{A,i=1}^r \widetilde{\mathrm{G}}_i$;
3. for every $1 \leq i < j \leq r$ an element $\alpha_{i,j} \in \widetilde{\mathrm{G}}_i(S)$ such that the projection to $A(S)$ is s_i.

For later purposes we define $\Theta'^{o} \subset \Theta'$ as the open subscheme classifying objects as above such that (1) the geometric fibers of $(A, \lambda_A) \to S$ are simple as principally polarized abelian varieties and (2) the section $\alpha_{i,j}$ does not intersect the identity section of $\widetilde{\mathrm{G}}_i(S)$ for every $1 \leq i < j \leq r$. Note that $\mathrm{Sp}_{2(g-r)}(\mathbf{Z}/n\mathbf{Z})$ acts on Θ' and preserves Θ'^{o}.

3.7 Definition. *Let $f: Y \to \mathcal{A}_{g,(n),\gamma}$ be a morphism of locally symmetric varieties whose image defines a locally symmetric subvariety of $\mathcal{A}_{g,(n),\gamma}$. Write $\widetilde{Y}_\gamma$ for the topological closure of $f(Y)^{\mathrm{an}}$ in $\widetilde{\mathcal{A}}_{g,(n),\gamma}$ and put $\partial\widetilde{Y}_\gamma := \widetilde{Y}_\gamma \cap \partial\widetilde{\mathcal{A}}_{g,(n),\gamma}$.*

Write $Y^{\mathrm{an}} := \Upsilon\backslash H(\mathbf{R})^0/\mathbf{K}$ and let $\widetilde{f}: H \hookrightarrow \mathrm{Sp}_{2g,\mathbf{Q}}$ be the closed immersion of algebraic groups associated to f. Consider the map of hermitian symmetric domains $D = H(\mathbf{R})^0/\mathbf{K} \to \mathrm{Sp}_{2g}(\mathbf{R})/\mathbb{K}_g$ associated to $\widetilde{f}$. Let F be a rational boundary component of D which maps to the rational boundary component F_γ. Take $N \subset H$ to be the admissible subgroup associated to F so that $F = F(N)$. By 2.1 we have the commutative diagrams

$$\begin{array}{ccccccccc}
Y^{\mathrm{an}} & \longleftarrow & \Upsilon\backslash D & \subset & \Omega & \longrightarrow & A' & \longrightarrow & \Upsilon_N\backslash F \\
{\scriptstyle h}\big\downarrow & & \big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{A}_{g,(n),\gamma}^{\mathrm{an}} & \longleftarrow & \Gamma_{n,\gamma}\backslash\mathrm{Sp}_{2g}(\mathbf{R})/\mathbb{K}_g & \subset & \Theta^{\mathrm{an}} & \longrightarrow & A^{r,\mathrm{an}} & \longrightarrow & \mathcal{A}_{g-r,n}^{\mathrm{an}}
\end{array}$$

and

$$\begin{array}{ccc} \Upsilon_N\backslash F & \longrightarrow & \Omega \\ \downarrow & & \downarrow \\ \mathcal{A}^{\rm an}_{g-r,n} & \stackrel{0}{\longrightarrow} & \Theta^{\rm an}. \end{array}$$

Recall that $\Upsilon_N\backslash F \to \mathcal{A}^{\rm an}_{g-r,n}$ is the analytification of a locally symmetric subvariety of $\mathcal{A}_{g-r,n}$, that $\Omega \to A'$ is a torsor under a torus $\mathbb{U}^{\rm an}$, that $\Theta^{\rm an} \to A^{r,{\rm an}}$ is a torsor under the torus $U^{\rm an}$ and that there is a morphism of algebraic tori $q\colon \mathbb{U} \to U$ such that the map $h\colon \Omega \to \Theta^{\rm an}$ is $\mathbb{U}^{\rm an}$–equivariant. The action of $\mathbb{U}^{\rm an}$ on $\Theta^{\rm an}$ is defined via q. Furthermore, 2.1 implies that the image $h(\Omega)$ of Ω in $\Theta^{\rm an}$ is a torus torsor over the image of A' in $A^{r,{\rm an}}$ under the analytification $q(\mathbb{U})^{\rm an}$ of $q(\mathbb{U})$. Let $\overline{\mathbb{U}}$ be the schematic closure of $q(\mathbb{U})$ in $\overline{U}$ and write $\partial\overline{\mathbb{U}}$ for the intersection of $\overline{\mathbb{U}}$ with $\partial\overline{U} \subset \overline{U}$. Let $\mathbb{U}'$ be the image of $\mathbb{U} \to U \to U'$. Put $\overline{\Omega} := h(\Omega) \times^{q(\mathbb{U})^{\rm an}} \overline{\mathbb{U}}^{\rm an}$ and $\partial\overline{\Omega} := \overline{\Omega} \cap \partial\overline{\Theta}^{\rm an}$.

3.8 Lemma. *(1) The subspace $\widetilde{Y}_\gamma \subset \widetilde{\mathcal{A}}_{g,(n),\gamma}$ inherits a natural structure of an irreducible and reduced, closed analytic subspace of $\widetilde{\mathcal{A}}_{g,(n),\gamma}$. Moreover, $\partial\widetilde{Y}_\gamma = \partial\overline{\Omega}$ is a closed analytic subspace of $\widetilde{Y}_\gamma$.*

(2) If $\partial\widetilde{Y}_\gamma$ is non–empty, it coincides with $h(\Omega) \times^{q(\mathbb{U})^{\rm an}} \mathbb{U}'^{\rm an}$.

Proof: (1) Recall that we constructed $\widetilde{\mathcal{A}}_{g,(n),\gamma}$ gluing $\mathcal{A}^{\rm an}_{g,n}$ and an open neighborhood Z of $\partial\overline{\Theta}^{\rm an}$ in $\overline{\Theta}^{\rm an}$ along $Z \cap \Theta^{\rm an}$; see 3.4. Then, $\widetilde{Y}_\gamma$ is obtained by gluing $f(Y)^{\rm an}$ and $Z \cap \overline{\Omega}$ along $Z \cap h(\Omega)$. In particular, $\partial\overline{\Omega} \cong \partial\widetilde{Y}_\gamma$. This proves (1).

(2) Due to (1) and the structure of $\overline{\Omega}$ as $\overline{\mathbb{U}}$-fibred space, it suffices to prove that the composite map $\partial\overline{\mathbb{U}} \subset \partial\overline{U} \cong U' \times \partial\overline{U}'' \cong U'$ defines an isomorphism $\partial\overline{\mathbb{U}} \xrightarrow{\sim} \mathbb{U}' \subset U'$. Note that ${\rm Spec}(\mathbf{C}) \cong \partial\overline{U}''$ is the 0–dimensional stratum of the affine torus embedding $U'' \subset \overline{U}''$. Then, the intersection $\partial\overline{\mathbb{U}}$ of $\overline{\mathbb{U}}$ with $\partial\overline{U} = U' \times \partial\overline{U}'' \cong U'$ is non–empty since $\partial\widetilde{Y}_\gamma$ is non–empty by assumption, it is endowed with a unique action of $\mathbb{U}$ compatible with the action of U on $\partial\overline{U}$ and it maps via the projection $\partial\overline{U} \to U'$ to the schematic closure of the image of the composite map $\mathbb{U} \to U \to U'$. Since such image is closed and coincides with $\mathbb{U}'$, the conclusion follows. □

3.9 Proposition. *Assume that $\partial\widetilde{Y}_\gamma$ is non–empty. Then,*

a) *there is an analytic open neighborhood of $\partial\overline{\Omega}$ in $\overline{\Omega}$ mapping isomorphically to an analytic open subspace of $\widetilde{Y}_\gamma$ containing $\partial\widetilde{Y}_\gamma$ so that $\partial\widetilde{Y}_\gamma \cong \partial\overline{\Omega}$;*

b) *the image of $\overline{\Omega} \to A^{r,{\rm an}}$ is the same as the image of $\partial\overline{\Omega}$. It coincides with the image of $A' \to A^{r,{\rm an}}$ where A' is a family of abelian subvarieties of the restriction of $A^{r,{\rm an}}$ to the image of $\Upsilon_N\backslash F \to \mathcal{A}^{\rm an}_{g-r,n}$;*

c) *the image of the composite map $\overline{\Omega} \subset \overline{\Theta}^{\rm an} \to \Theta'^{\rm an}$*
   - *is closed and coincides with $\partial\overline{\Omega} \subset \partial\overline{\Theta}^{\rm an} \cong \Theta'^{\rm an}$;*
   - *is a torus torsor over the image of $A' \to A^{r,{\rm an}}$ under the torus $\mathbb{U}'^{\rm an}$;*
   - *is rigidified, as a torus torsor, over the identity section of the image of $A' \to A^{r,{\rm an}}$;*

d) *the retraction $\partial\widetilde{\mathcal{A}}_{g,(n),\gamma} \to Z \to \partial\widetilde{\mathcal{A}}_{g,(n),\gamma}$ defined in (3.4.1) induces a retraction $\partial\widetilde{Y}_\gamma \longrightarrow \widetilde{Y}_\gamma \cap Z \longrightarrow \partial\widetilde{Y}_\gamma$.*

In particular, $\partial\widetilde{Y}_\gamma$ is the analytification of a closed subscheme $\partial Y_\gamma^{\tau,\mathrm{alg}}$ of Θ' which is a torus torsor under the torus $\mathbb{U}'$ over an abelian subscheme of the restriction of A^r to a locally symmetric subvariety of $\mathcal{A}_{g-r,n}$.

Proof: (a)&(d) follow from the construction. (b) follows from 2.1(4)&(5). (c) follows from 3.8. □

4 The Torelli map at the boundary.

We first review how one constructs local formal charts for toroidal compactifications of the moduli space of principally polarized abelian varieties $\mathcal{A}_{g,1}$ using Mumford–Faltings–Chai theory of uniformization of abelian varieties. We use it in 4.4 to obtain information concerning the 1–motives uniformizing degenerating Jacobians. This allows to describe the special fiber of 1–motives uniformizing Jacobians of degenerating irreducible curves. More precisely, we provide candidates for such special fibers in 4.6 and then we prove in 4.9 that indeed they come from degenerating Jacobians.

Let R be a noetherian, normal domain, complete with respect to an ideal I, with fraction field K and satisfying the condition (†) *for any étale R/I–algebra D_0 its unique lifting D as a formally étale R–algebra is normal.* Put $S=\mathrm{Spec}(R)$, $S_0:=\mathrm{Spec}(R/I)$. Following [FC] we introduce the following two categories

(ppDEG) the objects consist of pairs (G,λ) where $G\to S$ is a semiabelian scheme over S, such that $G\otimes_R K$ is an abelian scheme, and λ is a principal polarization on $G\otimes_R K$; furthermore, $G\times_S S_0$ is assumed to be an extension of an abelian scheme over S_0 by a torus over S_0;

(ppDD) the objects consist of triples $\big(\widetilde{\mathrm{G}},\lambda,\iota,\big)$ where

(1) $\widetilde{\mathrm{G}}$ is a semiabelian scheme over S extension of an abelian scheme A and a torus T. Denote by X the étale sheaf on S defined as the character group of T;

(2) λ is a principal polarization on A. Denote by $c\colon X\to A$ the homomorphism such that $\lambda\circ c$ determines the extension $\widetilde{\mathrm{G}}$ of A by T via (the negative of) push–out;

(3) $\iota\colon X\to\widetilde{\mathrm{G}}\otimes_R K$ is a homomorphism so that the composite $X\to\widetilde{\mathrm{G}}\otimes_R K\to A\otimes_R K$ is $c\otimes_R K$.

The homomorphism ι determines and is determined by a trivialization ϖ over $\big(X\times X\big)\otimes_R K$ of the pull–back of the Poincaré $\mathbf{G}_m$–biextension $\mathcal{P}_A\to A\times A^\vee$ via $X\times X\xrightarrow{c\times c}A\times A\xrightarrow{1\times\lambda}A\times A^\vee$; see the explanation in 3.6. One requires that ϖ is symmetric and is subject to a positivity condition: for every $x\in X$ the pull–back of $\mathcal{P}_A$ via $c(x)\times c(x)\colon S\to A\times A\to A\times A^\vee$ is a $\mathbf{G}_m$–bundle with a section $\varpi(x,x)$ over K; one imposes that this section extends to a section over the whole of S of the associated $\mathbf{G}_a$–bundle and that the intersection with its 0–section contains S_0. See [FC, p. 58] for details.

The objects of (ppDEG) appear naturally in the study of degenerating principally polarized abelian varieties. The objects of (ppDD) should be thought of as uniformization data of degenerating principally polarized abelian varieties. It is one of the main results of [FC], see Cor. III.7.2, that there is a functor from (ppDD) to (ppDEG), called Mumford's construction, which provides an equivalence between the two categories above. In particular, to construct formal charts of toroidal compactifications of the moduli spaces of g–dimensional principally polarized abelian varieties, possibly with level structures, one constructs universal formal moduli for objects in (ppDD); see [FC, §IV.3 & §IV.6].

4.1 Definition. *Let $\big(\widetilde{\mathrm{G}}, \lambda, \iota,\big)$ be an object of (ppDD) over S. For every character $\gamma\colon T \to \mathbf{G}_{m,S}$ consider the pushout $\widetilde{\mathrm{G}}_\gamma$ of $\widetilde{\mathrm{G}}$. Given another character $\psi\colon T \to \mathbf{G}_{m,S}$ write $T_{\gamma,\psi}$ for the $\mathbf{G}_m$–bundle over S defined as the base–change of $\widetilde{\mathrm{G}}_\gamma$ by $c\big(\psi\big)\colon S \to A$. Consider the projection $\alpha_{\gamma,\psi}$ of $\iota(\psi)$ via $\widetilde{\mathrm{G}}(K) \to \widetilde{\mathrm{G}}_\gamma(K)$. It defines a K–valued point of $T_{\gamma,\psi}$ and, hence, a Weil divisor on S, denoted $D_{\gamma,\psi}$.*

As an example, we show how locally formally around the boundary of $\widetilde{\mathcal{A}}_{g,(n),\gamma}$, defined in §3, one can construct degenerating semi–abelian schemes. Let $\mathrm{Spf}(B)$ be the formal completion of an open affine subscheme of the scheme $\overline{\Theta}$, defined in 3.4, along the ideal defining $\partial\overline{\Theta}$. Since $\overline{\Theta}$ is a smooth scheme, B is a regular integral domain. Put $K := \mathrm{Frac}(B)$. As explained in 3.6 the associated K–valued point of Θ defines on K a symmetric 1–motive $\iota\colon \mathbf{Z}^r \to \widetilde{\mathrm{G}}(K)$.

4.2 Lemma. *The 1–motive $\iota\colon \mathbf{Z}^r \to \widetilde{\mathrm{G}}(K)$ is an object of* (ppDD)*. In particular, the associated object of* (ppDEG) *provided by Mumford's construction is a degenerating, principally polarized abelian scheme $G_B \longrightarrow \mathrm{Spec}(B)$ of relative dimension g.*

Proof: Since B is the completion of a normal excellent ring, it satisfies the assumption (†). We are left to prove that the trivialization ϖ of the pull-back of the Poincaré $\mathbf{G}_m$–biextension to $\big(\mathbf{Z}^r \times \mathbf{Z}^r\big) \otimes_R K$, defined by ι, satisfies the positivity condition required for Mumford's construction. Let $e_1, \ldots, e_r$ be the canonical basis for $\mathbf{Z}^r$ and fix a and $b \in \mathbf{Z}$ and integers $1 \le i < j \le r$ such that $\varphi_{i,j} \in X^*(U'')$. Then, we have

$$\varpi\big(ae_i + be_j, ae_i + be_j\big)^2 = \left(\frac{\varpi(e_i,e_i)^2}{\varpi(e_i,e_j)}\right)^{a^2} \varpi(e_i,e_j)^{(a+b)^2} \left(\frac{\varpi(e_j,e_j)^2}{\varpi(e_i,e_j)}\right)^{b^2}$$

and each term is a regular function on $\overline{\Theta}''$ vanishing on $\partial\overline{\Theta}''$. □

4.3 *The case of Jacobians.* Let $\mathcal{C}_0$ be a stable curve of genus g over S_0, the spectrum of an algebraically closed field. Let $\Gamma := \{\mathcal{V}, \mathcal{E}\}$ be its dual graph. For every vertex v of Γ denote by C_v the corresponding irreducible component of the normalization of $\mathcal{C}_0$. Fix an orientation of Γ. For every edge e of Γ with source $s(e) = v$ (resp. target $t(e) = v$) let P_e (resp. Q_e) be the associated point of $C_v\big(S_0\big)$. Then, $\mathrm{Pic}^0(\mathcal{C}_0/S_0)$ is naturally an extension:

$$0 \longrightarrow T_0 \longrightarrow \mathrm{Pic}^0(\mathcal{C}_0/S_0) \longrightarrow A_0 \longrightarrow 0$$

of $A_0 := \prod_{v\in\mathcal{V}} \mathrm{Pic}^0(C_v/S_0)$ by the torus T_0 over S_0 with character group $\mathrm{H}_1(\Gamma, \mathbf{Z})$. We elaborate on the last point. For every $\gamma = \sum_{e\in\mathcal{E}} m_{\gamma,e} e \in \mathrm{H}_1(\Gamma, \mathbf{Z})$ write $\Gamma_\gamma = \{\mathcal{V}_\gamma, \mathcal{E}_\gamma\}$ for the subgraph of Γ whose edges are $\mathcal{E}_\gamma := \{e \in \mathcal{E} | m_{\gamma,e} \neq 0\}$ and whose vertices are their end points. Let $\mathcal{B} := \{\gamma_1, \ldots, \gamma_r\}$ be a basis of $\mathrm{H}_1(\Gamma, \mathbf{Z})$. For $1 \le i \le r$

let $\widetilde{\mathrm{G}}_{i,0}$ be the push–out of $\mathrm{Pic}^0(\mathcal{C}_0/S_0)$ via the character $\gamma_i\colon T_0 \to \mathbf{G}_{m,S_0}$ so that $\mathrm{Pic}^0(\mathcal{C}_0/S_0) = \times_{A_0,i=1}^r \widetilde{\mathrm{G}}_{i,0}$. Let v be a vertex of Γ_{γ_i}. Then, v is the target of an edge e_i of Γ_{γ_i} and it is the source of an edge f_i of Γ_{γ_i}. Denote by $C_{i,v} := C_v/\bigl(P_{f_i} \sim Q_{e_i}\bigr)$ the semistable curve obtained from C_v by gluing transversally the S_0 valued points P_{f_i} and Q_{e_i}. Then, as explained in [An, Prop. 13.16], the semiabelian scheme $\mathrm{Pic}^0\bigl(C_{i,v}/S_0\bigr)$ sits in an exact sequence

$$0 \longrightarrow \mathbf{G}_{m,S_0} \longrightarrow \mathrm{Pic}^0\bigl(C_{i,v}/S_0\bigr) \longrightarrow \mathrm{Pic}^0\bigl(C_v/S_0\bigr) \longrightarrow 0,$$

equal to the extension of $\mathrm{Pic}^0\bigl(C_v/S_0\bigr)$ by $\mathbf{G}_{m,S_0}$ defined by the point $P_{f_i} - Q_{e_i}$ of $\mathrm{Pic}^0\bigl(C_v/S_0\bigr)^\vee$ (identified with $\mathrm{Pic}^0\bigl(C_v/S_0\bigr)$ via its canonical principal polarization) and there is an exact sequence

$$0 \longrightarrow \mathbf{G}_{m,S_0} \stackrel{\Delta}{\longrightarrow} \prod_{e\in\mathcal{E}_i} \mathbf{G}_{m,S_0} \longrightarrow \prod_{v\in\mathcal{V}_i} \mathrm{Pic}^0\bigl(C_{i,v}/S_0\bigr) \longrightarrow \widetilde{\mathrm{G}}_{i,0} \longrightarrow 0.$$

Here $\mathcal{V}_i$ is the set of vertices of Γ_{γ_i} and $\mathcal{E}_i$ is the set of edges of Γ_{γ_i}, the map Δ is the diagonal embedding and the second map factors via $\prod_{e\in\mathcal{E}_i} \mathbf{G}_{m,S_0} \to \prod_{v\in\mathcal{V}_i} \mathbf{G}_{m,S_0}$ and is given at the level of cocharacter groups by the natural map $\oplus_{e\in\mathcal{E}_i} \mathbf{Z}e \to \oplus_{v\in\mathcal{V}_i} \mathbf{Z}v$ associating to $e \in \mathcal{E}$ the difference $t(e) - s(e)$. In particular, for every v the map $\mathrm{Pic}^0\bigl(C_{i,v}/S_0\bigr) \to \widetilde{\mathrm{G}}_{i,0}$ is a closed immersion.

Let $\mathrm{Spf}(B)$ be the universal deformation space of $\mathcal{C}_0$. It is a regular local ring of dimension $3g-3$ complete and separated with respect to its maximal ideal I. Denote by K its fraction field. Let $\mathcal{C} \to S := \mathrm{Spec}(B)$ be the universal stable curve. Then, $\mathcal{C}_K$ is smooth over K and $\mathcal{C}_0 \cong \mathcal{C} \times_S S_0$. For every edge $e \in \mathcal{E}$ the completion of $\mathcal{C}$ at the corresponding singular point e of $\mathcal{C}_0$ is isomorphic to the B-algebra $B[\![X,Y]\!]/(XY - q_e)$ with $q_e \in I$.

Note that $\mathrm{Pic}^0(\mathcal{C}/S)$ is a semiabelian scheme over S and its fiber over K is a principally polarized abelian variety. In particular, it admits a uniformization $\iota\colon X \to \widetilde{\mathrm{G}}(K)$ à la Mumford–Faltings–Chai. Here, $\widetilde{\mathrm{G}}$ is a semiabelian scheme over S which is an extension of a principally polarized abelian scheme (A, λ_A) over S by a torus T with character group denoted by X. Furthermore, $\widetilde{\mathrm{G}} \times_S S_0 \cong \mathrm{Pic}^0(\mathcal{C}_0/S_0)$. In particular, X is identified with the character group of $T \times_S S_0$ and, hence, with $\mathrm{H}_1(\Gamma, \mathbf{Z})$. Write $\widetilde{\mathrm{G}}_i$ for the pushout of $\widetilde{\mathrm{G}}$ via the character $\gamma_i\colon T \to \mathbf{G}_{m,S}$. Choose a full symplectic level–n structure on A_0. It extends uniquely to a full symplectic level–n structure on A. In particular, the 1–motive $\iota\colon X \to \widetilde{\mathrm{G}}(K)$ arises from a unique K–valued point $\rho \in \Theta(K)$ of the scheme Θ defined in 3.6. We have the following:

4.4 Proposition. *For every $e \in \mathcal{E}$ there exists an element $x_e \in \widetilde{\mathrm{G}}\bigl(S\bigl[q_e^{-1}\bigr]\bigr)$ such that*

1. *the image of x_e in A extends to an S–valued section $y_e\colon S \to A$;*
2. *if $e \notin \mathcal{E}_i$ the image of x_e in $\widetilde{\mathrm{G}}_i(K)$ defines an S–section of $\widetilde{\mathrm{G}}_i$;*
3. *if $e \in \mathcal{E}_i$, put $T_{i,e}$ to be the $\mathbf{G}_m$–bundle over A defined as the base–change of $\widetilde{\mathrm{G}}_i$ via y_e. The Weil divisor of S associated to the $S\bigl[q_e^{-1}\bigr]$–section of $T_{i,e}$ defined by x_e*

is exactly the ideal (q_e);

4. *for every $\psi=\sum_{e\in\mathcal{E}} n_e e\in \mathrm{H}_1(\Gamma,\mathbf{Z})$ we have $\iota(\psi)=\prod_{e\in\mathcal{E}} x_e^{n_e}\in \widetilde{\mathrm{G}}(K)$.*

Hence, ι extends to a morphism $\iota: X\to \widetilde{\mathrm{G}}\bigl(S\bigl[q_e^{-1}|e\in\mathcal{E}\bigr]\bigr)$ so that $\rho\in\Theta\bigl(S\bigl[q_e^{-1}|e\in\mathcal{E}\bigr]\bigr)$.

Proof: This is the content of [An, Prop. 18.18]. □

For every $\gamma=\sum_{i=1}^r \alpha_i\gamma_i=\sum_{e\in\mathcal{E}} m_e e$ and $\psi=\sum_{e\in\mathcal{E}} n_e e\in \mathrm{H}_1(\Gamma,\mathbf{Z})$ the divisor $D_{\gamma,\psi}$, defined in 4.1, is $\sum_{i=1}^r \alpha_i D_{\gamma_i,\psi}$. It follows from the proposition the equality of Cartier divisors

$$\mathcal{O}_S\bigl(D_{\gamma,\psi}\bigr):=\mathcal{O}_S\left(\prod_{i=1}^r\prod_{e\in\mathcal{E}_i} q_e^{\alpha_i n_e}\right)=\mathcal{O}_S\left(\prod_{e\in\mathcal{E}} q_e^{m_e n_e}\right). \tag{4.4.1}$$

This is also the content of [FC, Thm. 8.3]. We deduce:

4.5 Corollary. *Assume that $\mathcal{B}$ is a well adapted basis for Γ of type τ in the sense of 8.4. Then,*

i. *the divisor D_{γ_i,γ_i} is always non–trivial;*

ii. *if $1\le i<j\le r$, then $D_{i,j}$ is 0 if and only if $\tau(i,j)=0$.*

In particular, the $S\bigl[q_e^{-1}|e\in\mathcal{E}\bigr]$–valued point ρ of Θ extends to an S–valued point $\rho: S\to\overline{\Theta}$ if and only if $\tau(i,j)=0$ for every $1\le i<j\le r$.

Fix an integer $r\ge 1$ such that $g+2r\ge 3$. Assume that there exists a basis $\mathcal{B}:=\{\gamma_1,\ldots,\gamma_r\}$ for Γ which is well adapted of type $\tau(i,j)=0$ for every $1\le i<j\le r$ and $\tau(i,i)=1$ for every $i=1,\ldots,r$ (in the sense of 8.4). Then, with the notations and definitions of 3.6 (for the scheme Θ' and its moduli interpretation) and of 4.3 (for the definition of the semiabelian schemes $\widetilde{\mathrm{G}}_{i,0}$, the curve $C_{i,v}$ etc.):

4.6 Definition. *Define $\partial\mathfrak{J}_{\Gamma,\mathcal{B}}^{0,\tau}: S_0\longrightarrow\Theta'$ to be the S_0–valued point of Θ' such that*

(I) *the projection via $\Theta'\to\mathcal{A}_{g-r,n}$ is the S_0–valued point corresponding to A_0 (with its principal polarization and full level–n structure);*

(II) *the projection $s=(s_1,\ldots,s_r)$ to $A_0(S_0)^r$ is characterized by the property that for every $i=1,\ldots,r$ the pull–back to $A_0\cong A_0\times\{s_i\}$ of the Poincaré biextension over $A_0\times A_0\cong A_0\times_Z A_0^\vee$ is $\widetilde{\mathrm{G}}_{i,0}$, as rigidified $\mathbf{G}_{m,S_0}$–torsor over A_0;*

(III) *for $1\le i<j\le r$ the S_0–valued point $\alpha_{i,j}$ of $\widetilde{\mathrm{G}}_{i,0}$ over s_j is*
   (a) *the identity if Γ_{γ_i} and Γ_{γ_j} have no vertex in common,*
   (b) *the image of $P_{e_j}-Q_{f_j}\in C_{i,v}^{\mathrm{sm}}-C_{i,v}^{\mathrm{sm}}\subseteq \mathrm{Pic}^0\bigl(C_{i,v}/S_0\bigr)\subseteq\widetilde{\mathrm{G}}_{i,0}$ otherwise.*

4.7 Remark. In 4.6(III.b) we write $C_{i,v}^{\mathrm{sm}}-C_{i,v}^{\mathrm{sm}}\subseteq \mathrm{Pic}^0\bigl(C_{i,v}/S_0\bigr)$ for the subscheme classifying divisors on $\mathrm{Pic}^0\bigl(C_{i,v}/S_0\bigr)$ of the type $Q-T$ where Q and T are smooth sections of $C_{i,v}=C_v/\bigl(P_{f_i}\sim Q_{e_i}\bigr)$. In particular, the section $\alpha_{i,j}$ does not intersect the identity section of $\widetilde{\mathrm{G}}_{i,S_0}$.

Due to 4.6(II) the section $s_i=(t_v)_{v\in\mathcal{V}}\in A_0=\prod_v \mathrm{Pic}^0(C_v/S_0)(S_0)$ is such that (i) $t_v=0$ if v is not a vertex of Γ_{γ_i}; (ii) $t_v=P_{e_i}-Q_{f_i}\in C_v-C_v\subset\mathrm{Pic}^0(C_v/S_0)$ if v

is the source of the edge e_i of Γ_{γ_i} and the target of the edge f_i of Γ_{γ_i}. In particular, $t_v \neq 0$ in this case if C_v has genus ≥ 1.

4.8 Lemma. *The morphism $\partial\mathfrak{J}^{0,\tau}_{\Gamma,\mathcal{B}}$ factors via the open subscheme $\Theta^{'o}$ of Θ' (see 3.6) if and only if $\mathcal{C}_0$ is an irreducible curve.*

Proof: The implication $\Longleftarrow$ is clear. The difficult part is to prove the other one. It follows from 4.7 and 4.6(III) and the fact that $\mathcal{B}$ is a well adapted basis that γ_i and γ_j intersect precisely in one vertex. We then claim that all the γ_i's intersect in the same vertex which we denote by w. To prove the claim assume that γ_i intersects γ_j and γ_h in two distinct vertices v_j and v_h for some indices i, j and h. Let $\Delta \subset \Gamma$ be the subgraph with the same vertices as Γ but with two edges removed, one edge belonging to the oriented subtree of Γ_{γ_i} going from v_j to v_h and one edge belonging to the oriented subtree of Γ_{γ_i} going from v_h to v_j. Since $\mathcal{B}$ is well adapted of type τ, those edges do not belong to any other Γ_{γ_k} for $k \neq i$. In particular, Δ has cyclomatic number one less than the cyclomatic number of Γ. Since the vertices of Δ and Γ are the same, but we have removed two edges, $\mathrm{H}_0(\Delta, \mathbf{Z})$ must then have rank 2 i. e., Δ has two connected components. By construction Γ_{γ_j} and Γ_{γ_h} belong to different components of Γ and, hence, do not intersect anywhere leading to a contradiction.

Eventually, we claim that this w is the only vertex of Γ. If this is the case, $\mathcal{C}_0$ is irreducible concluding the proof of the lemma. Since $\partial\mathfrak{J}^{0,\tau}_{\Gamma,\mathcal{B}}(S_0)$ lies in $\Theta^{'o}$ the abelian part A_0 of $\mathrm{Pic}^0(\mathcal{C}_0/S_0)$ is simple as principally polarized abelian variety. Note that A_0 coincides with $\prod_{v\in\mathcal{V}_x}\mathrm{Pic}^0(C_v/S_0)$ where $\amalg_{v\in\mathcal{V}}C_v$ is the normalization of $\mathcal{C}_0$. We deduce that there exists at most one $v_0 \in \mathcal{V}$ such that C_{v_0} has positive genus. If such v_0 exists, it must be w. Otherwise, the section $s_i \in A_0 = \mathrm{Pic}^0(C_w/S_0)$ would be trivial due to 4.7 contradicting the fact that $\partial\mathfrak{J}^{0,\tau}_{\Gamma,\mathcal{B}}(S_0)$ lies in $\Theta^{'o}$. We deduce that for every $v \neq w$ the curve C_v is a rational curve.

We claim that all the vertices of Γ belong to $\cup_{i=1}^r\Gamma_{\gamma_i}$. Suppose not and let v be a vertex not belonging to any of the graphs Γ_{γ_i}. Let Γ' be the graph obtained from Γ deleting the edges belonging to one of the Γ_{γ_i}'s. It has cyclomatic number 0 so that it is the disjoint union of trees. Let T_v be the tree containing v. Since Γ is connected there is a path connecting v to w. Every edge of this path having v as end point is an edge of Γ' so that T_v does not consist only of v and, hence, T_v has at least 2 end points. Furthermore, T_v has at most one vertex belonging to $\cup_{i=1}^r\Gamma_{\gamma_i}$. If not, let v_i and v_j be distinct vertices of Γ_{γ_i} and Γ_{γ_j} lying in T_v. Let α be a subtree of T_v with end points v_i and v_j, let β be a subtree of Γ_{γ_i} with end points v_j and w and let γ be be a subtree of Γ_{γ_i} with end points w and v_i. Then, $\alpha\cup\beta\cup\gamma$ defines a cycle of Γ which is not supported on $\cup_{i=1}^r\Gamma_{\gamma_i}$. Since any cycle is a combination of the γ_i's, this is absurd. Thus, T_v has at least one endpoint u which does not belong to $\cup_{i=1}^r\Gamma_{\gamma_i}$. Thus, u is the end point of only one edge of Γ. This is impossible since C_u is a rational curve and $\mathcal{C}_0$ is a stable curve.

We claim that w is the only vertex of Γ. Note that the cycles γ_i and γ_j intersect only at w for $i \neq j$. Thus, if v is a vertex of Γ_{γ_i} different from w, then C_v is a rational curve and v is the source (resp. the target) of precisely one edge of Γ_{γ_i} and hence of Γ. The existence of such v would contradict the stability assumption of $\mathcal{C}_0$. Hence, w is the only vertex of Γ as wanted. $\square$

Since we assume that $\mathcal{B}$ is a well adapted basis for Γ of type $\tau(i,j)=0$ for every $1\leq i<j\leq r$ and $\tau(i,i)=1$ for every $i=1,\ldots,r$, the morphism $\rho: S\bigl[q_e^{-1}|e\in\mathcal{E}\bigr]\to\Theta$ introduced in 4.4 and defining the 1–motive uniformizing $\mathrm{Pic}^0(\mathcal{C}/S)$, extends to a morphism $\rho: S\to\overline{\Theta}$. Furthermore,

4.9 Proposition. *The map $\rho|_{S_0}: S_0\subset S\to\overline{\Theta}$ factors via $\partial\overline{\Theta}$ and the induced map $\partial\rho: S_0\to\partial\overline{\Theta}\cong\Theta'$ coincides with the map $\partial\widetilde{\mathfrak{J}}_{\Gamma,\mathcal{B}}: S_0\longrightarrow\Theta'$ defined in 4.6.*

Proof: First of all we reduce to the case that the geometric fibers of $\mathcal{C}\to S$ are irreducible as follows. Let $H\subset\mathcal{E}$ be a subset of edges of $\mathcal{E}$ such that the graph $\Gamma\dashrightarrow\Gamma_H$ obtained from Γ contracting the edges in H has only one vertex and the same cyclomatic number as Γ. In particular, we can number the edges of Γ_H, which are identified with $\mathcal{E}\backslash H$, as $e_1,\ldots,e_r$ so that the image of Γ_{γ_i} in Γ_H is precisely e_i. Write $I_H:=\bigl(q_{e_i}|i=1,\ldots,r\bigr)$, $\overline{S}_H:=\mathrm{Spec}(B/I_H)$ and $\overline{\mathcal{C}}_H:=\mathcal{C}\times_S\overline{S}_H$. It is a stable curve over $\overline{S}_H$ and every $e\in\mathcal{E}\backslash H$ defines an $\overline{S}_H$–valued section corresponding to a singularity. Over the generic point of $\overline{S}_H$ these are the only singularities and $\overline{\mathcal{C}}_H$ is geometrically irreducible. Define $\overline{\mathcal{C}}_{H,i}\to\overline{\mathcal{C}}_H$ to be the curve obtained normalizing all the singularities associated to $e\in\mathcal{E}\backslash H$ except e_i. Then, $\mathrm{Pic}^0\bigl(\overline{\mathcal{C}}_{H,i}/\overline{S}_H\bigr)$ is a semiabelian scheme and, as explained in 4.3, it coincides with $\widetilde{\mathrm{G}}_i\times_S S_0$ over S_0. Let P_{H,e_j} and Q_{H,e_j} be the smooth sections of $\overline{\mathcal{C}}_{H,i}$ above the singular point e_j of $\overline{\mathcal{C}}_H$. Then, $P_{H,e_j}-Q_{H,e_j}\in\mathrm{Pic}^0\bigl(\overline{\mathcal{C}}_{H,i}/\overline{S}_H\bigr)(\overline{S}_H)$ and over S_0 it coincides with the S_0–section $P_{e_j}-Q_{f_j}\in\widetilde{\mathrm{G}}_i(S_0)$.

Let $S_H:=\mathrm{Spec}(B_H)$ where B_H is the completion with respect to the ideal I_H of the localization $B\bigl[q_h^{-1}|h\in H\bigr]$. Consider the base change $\mathcal{C}_H$ of $\mathcal{C}$ to S_H. The geometric fibers of $\mathcal{C}_H\to S_H$ are geometrically irreducible and the base change of $\mathcal{C}_H$ to $\mathrm{Spec}(B_H/I_H)$ is $\overline{\mathcal{C}}_H\times_{\overline{S}_H}\overline{S}_H^o$ where $\overline{S}_H^o=\mathrm{Spec}\bigl(B/I_H[q_h^{-1}|h\in H]\bigr)$. It follows from [An, Lemma 18.16] that the image of the section $\alpha_{i,j}\in\widetilde{\mathrm{G}}_i(S)$ via $\widetilde{\mathrm{G}}_i(S)\to\widetilde{\mathrm{G}}_i(S_H)$ is obtained applying Mumford's construction to $\mathrm{Pic}^0(\mathcal{C}_H/S_H)$ using the basis of $\mathrm{H}_1(\Gamma_H,\mathbf{Z})$ given by $e_1,\ldots,e_r$. We claim that it suffices to show that $\rho|_{\overline{S}_H^o}$ factors via $\partial\overline{\Theta}$ and it is defined as in 4.6. Indeed, if this holds, since the map $B/I_H\to B_H/I_H$ is injective, the same applies to $\rho|_{\overline{S}_H}$ and hence to $\rho|_{S_0}$. It then suffices to prove the claim. Assume that the Proposition holds for S_0 the spectrum of an algebraically closed field and $\mathcal{C}_0$ irreducible. Then, it holds for every geometric point s of $\overline{S}_H^o$ and for the formal completion of S_H at s. This implies the claim.

We are reduced to prove the Proposition in the case that $\mathcal{C}_0$ is irreducible. In this case Γ_{γ_i} consist of only one edge, denoted e_i. It is proven in [An, §16.5– 16.7] that there exists a scheme $\widetilde{P}\to S$ such that $\widetilde{P}_K=\widetilde{\mathrm{G}}_K$ and an action of $\mathrm{H}_1(\Gamma,\mathbf{Z})$ on $\widetilde{P}$ extending the action by translation on $\widetilde{\mathrm{G}}_K$ via $\iota:\mathrm{H}_1(\Gamma,\mathbf{Z})\to\widetilde{\mathrm{G}}(K)$. By [An, §14.7–14.10] the scheme $\widetilde{P}\times_S S_0$ admits the following description. For every $t=1,\ldots,r$ let $\mathbb{P}_t$ be the $\mathbf{P}^1$–bundle over $A_0=A\times_S S_0$ associated to the $\mathbf{G}_m$–bundle $\widetilde{\mathrm{G}}_{i,0}=\widetilde{\mathrm{G}}_i\times_S S_0$ with 0–section 0_t and infinity section ∞_t. Let $\mathbf{P}_0$ be the compactification of $\widetilde{\mathrm{G}}\times_S S_0=\prod_{A_0,i=1}^r\widetilde{\mathrm{G}}_{i,0}$ given by the $\mathbf{P}^1 \times \cdots \times \mathbf{P}^1$–bundle $\mathbb{P}_1 \times_{A_0} \cdots \times_{A_0} \mathbb{P}_r$ over A_0. Put $\mathbf{P}_\gamma := \mathbf{P}_0$ for every $\gamma \in \mathrm{H}_1(\Gamma, \mathbf{Z})$. Then, $\widetilde{P} \times_S S_0$ is obtained from $\amalg_{\gamma \in \mathrm{H}_1(\Gamma,\mathbf{Z})} \mathbf{P}_\gamma$ gluing for every $\gamma \in \mathrm{H}_1(\Gamma, \mathbf{Z})$ and every $t = 1, \ldots, r$ the scheme $\mathbf{P}_\gamma$ along $\mathbb{P}_1 \times_{A_0} \cdots \times_{A_0} \infty_t \times_{A_0} \cdots \times_{A_0} \mathbb{P}_r$, identified with $\prod_{A_0, j \neq t} \mathbb{P}_j$, and the scheme $\mathbf{P}_{\gamma+t}$ along $\mathbb{P}_1 \times_{A_0} \cdots \times_{A_0} 0_t \times_{A_0} \cdots \times_{A_0} \mathbb{P}_r$, identified with $\prod_{A_0, j \neq t} \mathbb{P}_j$, by translation via the element $\big(P_{e_t} - Q_{e_t}\big)_{j \neq t} \in \prod_{j \neq t} \widetilde{\mathrm{G}}_j(S_0)$. The action of $\mathrm{H}_1(\Gamma, \mathbf{Z})$ is defined by the natural identification of $\mathbf{P}_\gamma$ with $\mathbf{P}_0$. In particular, the reduction of $\iota(\gamma_t) \in \prod_{j \neq t} \widetilde{\mathrm{G}}_j(S_0)$ is defined as the image of $0 \in \mathbf{P}_0$ via the action of γ_t i e., it coincides with $\prod_{j \neq t} \big(P_{e_t} - Q_{f_t}\big)$ as claimed. $\square$

5 The Torelli locus at the cusps for irreducible curves.

We work out the results of the previous section for families of irreducible curves. Fix integers $n \geq 3$, $r \geq 1$ and $g \geq 0$ such that $g + 2r \geq 3$. Let $\mathfrak{M}_{g-r,(n)}$ be the moduli space of smooth proper curves of genus g endowed with a full symplectic level–n structure on its Jacobian and a marked section (resp. 3 marked sections) if $g - r = 1$ (resp. $g - r = 0$). It is a fine moduli space, endowed with an action of $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$ given by changing the level structure. Let $D \to \mathfrak{M}_{g-r,(n)}$ be the universal curve. Write $B := \mathrm{Pic}^0\big(D/\mathfrak{M}_{g-r,(n)}\big)$ and denote by ξ its level–n structure. We consider the construction of the previous section in the special case that Γ is a graph with only one vertex and r oriented edges $e_1, \ldots, e_r$. We let γ_i be the elementary cycle defined by the edge e_i. Then, $\mathcal{B} := \{\gamma_1, \ldots, \gamma_r\}$ is a well adapted basis for Γ of type τ given by $\tau(i,i) = 1$ for every $i = 1, \ldots, r$ and $\tau(i,j) = 0$ for every $1 \leq i < j \leq r$. Consider the following situation:

(a) $S_0 := D^{2r,o} \subset D^{2r}$ is the open subscheme classifying $2r$–distinct sections on D. It is a smooth scheme endowed with an action of $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$;

(b) $D_0 := D \times_{\mathfrak{M}_{g-r,(n)}} S_0$. It is a smooth, proper curve over S_0 with geometrically irreducible fibers and $A_0 := \mathrm{Pic}^0(D_0/S_0) = \mathrm{Pic}^0\big(D/\mathfrak{M}_{g-r,(n)}\big) \times_{\mathfrak{M}_{g-r,(n)}} S_0$ is a principally polarized abelian scheme endowed with a full symplectic level–n structure;

(c) $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_r$ are the universal $2r$–sections $D(S_0)$. Note that $P_i \cap P_j = \emptyset$ for every $i \neq j$ and $P_i \cap Q_j = \emptyset$ for every i and j.

Define $\mathcal{C}_0 \to S_0$ to be the stable curve obtained from D_0 gluing P_i ($= P_{e_i}$ in the notation of the previous section) and Q_i ($= Q_{f_i}$ in the notation of the previous section) for every $i = 1, \ldots, r$. Its genus is g. Via this identification, the map $\partial\widetilde{\mathfrak{I}}_{\Gamma,\mathcal{B}}$ of 4.6 extends to a morphism

$$\mathrm{per}_{g,r} \colon D^{2r,o} \longrightarrow \Theta'.$$

More precisely,

5.1 Definition. *For every $i = 1, \ldots, r$ let $C_i \to S_0$ be the stable curve $D_0/(P_i \sim Q_i)$ obtained from D_0 identifying transversally the sections P_i and Q_i. Then, $\widetilde{\mathrm{G}}_{i,0} := \mathrm{Pic}^0\big(C_i/S_0\big)$ is the extension of A_0 by $\mathbf{G}_{m,S_0}$ defined by the section $P_i - Q_i \in A_0(S_0) \cong A_0^\vee(S_0)$ and $\mathrm{per}_{r,g}$ is the unique morphism such that*

(I) *the projection via* $\Theta' \to \mathcal{A}_{g-r,n}$ *is the* S_0*–valued point corresponding to* A_0 *(with its principal polarization and full level–*n *structure);*

(II) *the projection* $\pi_{g,r}\colon D^{2r,o} \to A^r$ *is* $s = (s_1, \ldots, s_r) = \big(P_1 - Q_1, \ldots, P_r - Q_r\big)$*;*

(III) *if* $1 \le i < j \le r$*, the* S_0*–valued point* $\alpha_{i,j}$ *of* $\widetilde{\mathrm{G}}_{i,0}$ *is* $P_j - Q_j \in C_i^{\mathrm{sm}} - C_i^{\mathrm{sm}} \subseteq \widetilde{\mathrm{G}}_{i,0}$*.*

5.2 Remark. (1) The map $\mathrm{per}_{g,r}$ is equivariant with respect to the action of the group $\mathrm{Sp}_{2(g-r)}(\mathbf{Z}/n\mathbf{Z})$.

(2) Due to (III) the map $\mathrm{per}_{g,r}$ factors via the subscheme $\Theta^{'o} \subset \Theta'$ defined in 3.6. Consider the diagram

$$\begin{array}{ccc} D^{2r,o,\mathrm{an}} & \xrightarrow{\mathrm{per}_{g,r}^{\mathrm{an}}} & \Theta^{'o,\mathrm{an}} \subset \Theta^{'\mathrm{an}} = \partial\overline{\Theta}^{\mathrm{an}} = \partial\widetilde{\mathcal{A}}_{g,(n),\gamma} \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_g^{\mathrm{an}} & \xrightarrow{\widetilde{\mathfrak{J}}^{\mathrm{an}}} & \widetilde{\mathcal{A}}_{g,1}^{\mathrm{DV},\mathrm{an}}, \end{array}$$

where the left vertical morphism is the analytification of the moduli map corresponding to the curve $\mathcal{C}_0$, the lower horizontal arrow is the extension of the Torelli map given in [Na] and the right vertical map is obtained as the composite of the quotient map $\widetilde{\mathcal{A}}_{g,(n),\gamma} \to \widetilde{\mathcal{A}}_{g,1} = \mathrm{Sp}_{2(g-r)}(\mathbf{Z}/n\mathbf{Z})\backslash\widetilde{\mathcal{A}}_{g,(n),\gamma}$ and of the open immersion $\widetilde{\mathcal{A}}_{g,1} \subseteq \widetilde{\mathcal{A}}_{g,1}^{\mathrm{DV},\mathrm{an}}$ of 3.5. Let $\partial\widetilde{\mathcal{A}}_{g,1}^{o} \subset \widetilde{\mathcal{A}}_{g,1}$ be the open analytic subspace defined by $\mathrm{Sp}_{2(g-r)}(\mathbf{Z}/n\mathbf{Z})\backslash\Theta^{'o,\mathrm{an}}$. Then,

5.3 Proposition. *(i) The diagram above is commutative.*

(ii) The space $\left(\widetilde{\mathfrak{J}}^{\mathrm{an}}\right)^{-1}\left(\partial\widetilde{\mathcal{A}}_{g,1}^{o}\right)$ *is the analytification of the image of* $D^{2r,o} \to \overline{\mathfrak{M}}_g$*.*

Let k be an algebraically closed field of characteristic 0. Let D_k be a smooth projective curve of genus $g-r$ and suppose we are given a symplectic level–n structure ξ on $A_k := \mathrm{Pic}^0(D_k/k)$ and at least 1 or 2 marked k-valued points of D_k if $g-r=1$ or $g-r=0$. Let $\mathcal{P}_{A_k}$ be the bull–back of the Poincaré biextension to $A_k^2 = A_k \times A_k$ via the principal polarization on A_k. For every $1 \le i \le j \le r$ let $p_{ij}\colon A_k^r \to A_k^2$ be the projection onto the (i,j)-factor and let $\mathcal{P}_{ij}$ be the biextension $\mathcal{P}_{ij} := p_{ij}^*\big(\mathcal{P}_{A_k}\big)$. Write $\mathcal{P}_{A_k^r} := \prod_{A_k^r}\mathcal{P}_{ij}$ where the product is taken over all $1 \le i < j \le r$. Due to 3.6 it coincides with the fiber of Θ' over the moduli point $\big[(A_k,\xi)\big] \in \mathcal{A}_{g-r,n}(k)$. It is a torus torsor under the torus U_k' over A_k^r, rigidified over the 0-section of A_k^r. Consider the map

$$\mathrm{per}_{D_k,r}\colon D_k^{2r,o} \longrightarrow \mathcal{P}_{A_k^r}$$

obtained by taking the fiber of $\mathrm{per}_{g,r}$ over the moduli points $\big[(D_k,\xi)\big] \in \mathfrak{M}_{g-r,(n)}(k)$. and $\big[(A_k,\xi)\big] \in \mathcal{A}_{g-r,n}(k)$. Due to 5.1(III) the composite

$$\pi_{D_k,r}\colon D_k^{2r,o} \longrightarrow \mathcal{P}_{A_k^r} \longrightarrow A_k^r$$

is the map $\big(P_1, Q_1, \ldots, P_r, Q_r\big) \mapsto \big(P_1 - Q_1, \ldots, P_r - Q_r\big)$. Then:

5.4 Proposition. *If* D_k *has genus* $g(D_k) \ge 3$*, there is no positive dimensional abelian subvariety of* A_k^r *contained in the image of* $\pi_{D_k,r}$*. More precisely,*

(I) *assume that D_k is not hyperelliptic. Let $G \subset D_k - D_k$ be the translate of a closed subgroup scheme of A_k by a point of A_k. Then, G is zero dimensional;*

(II) *if D_k is hyperelliptic, then $D_k - D_k$ does not contain torsion points of exact order s with $2 < s \leq g(D_k)$.*

5.5 Proposition. *If $g \geq 4$ there exists no closed subscheme Ψ of $\mathcal{P}_{A_k^r}$ such that:*

i. *Ψ is a torus torsor over an abelian subvariety B of A_k^r under a subtorus T of U_k', rigidified over the 0–section of B;*

ii. *the generic point of Ψ is contained in the image of $\mathrm{per}_{D_k,r}$.*

Before proving the propositions we show how they imply 1.1.

5.6 *Proof of Theorem 1.1.* Let Y be a locally symmetric variety such that $Y^{\mathrm{an}} = \Upsilon \backslash H(\mathbf{R})^0/\mathbf{K}$ and let $f\colon Y \to \mathcal{A}_{g,1}$ be a map of locally symmetric varieties induced by an injective homomorphism of algebraic groups $\widetilde{f}\colon H \to \mathrm{Sp}_{2g,\mathbf{Q}}$. Thus, $f(Y)$ is a locally symmetric subvariety of $\mathcal{A}_{g,1}$. Fix an integer $n \geq 3$. Replacing Υ by a finite index subgroup we may assume that f factors via a map $h\colon Y \to \mathcal{A}_{g,(n),\gamma}$, where $\mathcal{A}_{g,(n),\gamma}$ is as in 3.4. Define $\widetilde{Y}_\gamma \subset \widetilde{\mathcal{A}}_{g,(n),\gamma}$ to be the closure of $h(Y)^{\mathrm{an}}$ in $\widetilde{\mathcal{A}}_{g,(n),\gamma}$ (respectively $\partial\widetilde{Y}_\gamma := \widetilde{Y}_\gamma \cap \partial\widetilde{\mathcal{A}}_{g,(n),\gamma}$). We know from 3.9 that the closed analytic subspace $\partial\widetilde{Y}_\gamma \subset \partial\widetilde{\mathcal{A}}_{g,(n),\gamma}$ is either empty or it is the analytification of a closed integral subscheme $\partial Y_\gamma^{\mathrm{alg}}$ of the scheme Θ' introduced in 3.6. In this case, $\partial Y_\gamma^{\mathrm{alg}}$ is a torus torsor over an abelian subscheme of the restriction of A^r to a locally symmetric subvariety of $\mathcal{A}_{g-r,n}$. We argue by contradiction assuming that the image of $Y \to \mathcal{A}_{g,1}$ is contained in the closure of $\mathfrak{J}(\mathfrak{M}_g)$ and that the image of $\partial\widetilde{Y}_\gamma$ in $\widetilde{\mathcal{A}}_{g,1}^{\mathrm{DV,an}}$ has non–empty intersection with the analytic space associated to $\widetilde{\mathfrak{J}}\big(\overline{\mathfrak{M}}_g^{\mathrm{irr}}\big)$. In view of 5.5 to conclude the proof of Theorem 1.1 it suffices to show:

5.6.1 Lemma. *The generic point of $\partial Y_\gamma^{\mathrm{alg}}$ is contained in the image of* $\mathrm{per}_{g,r}$.

Proof: By assumption $\partial\widetilde{Y}_\gamma$ is non–empty. Put $\partial\widetilde{Y}_\gamma^o := \partial\widetilde{Y}_\gamma \cap \Theta'^{o,\mathrm{an}}$. Let $\overline{Y}$ be the image of $\widetilde{Y}_\gamma \to \widetilde{\mathcal{A}}_{g,1}^{\mathrm{DV,an}}$ and let $\partial\overline{Y}^o := \partial\overline{Y} \cap \partial\widetilde{\mathcal{A}}_{g,1}^o$ with the notation of 5.3. It is open in $\partial\overline{Y}$ and it coincides with the image of $\partial\widetilde{Y}_\gamma^o$. Since $\partial\overline{Y}$ intersects $\widetilde{\mathfrak{J}}\big(\overline{\mathfrak{M}}_g^{\mathrm{irr}}\big)^{\mathrm{an}}$ by assumption, then $\partial\overline{Y}^o$ is non–empty. In particular, it is dense in $\partial\overline{Y}$ since the latter is irreducible. It follows from 5.3(ii) that $\partial\overline{Y}^o$ is contained in the image via $\widetilde{\mathfrak{J}}^{\mathrm{an}}$ of the analytification of the image of $D^{2r,o} \to \overline{\mathfrak{M}}_g$. The inverse image of $\partial\overline{Y}^o$ in $\partial\widetilde{\mathcal{A}}_{g,(n),\gamma}$ is an orbit of $\partial\widetilde{Y}_\gamma^o$ under $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$. Since the map $\mathrm{per}_{g,r}$ is $\mathrm{Sp}_{2(g-r)}\big(\mathbf{Z}/n\mathbf{Z}\big)$–equivariant by construction, we deduce from 5.3(i) that $\partial\widetilde{Y}_\gamma^o$ is in the image of $\mathrm{per}_{g,r}^{\mathrm{an}}$ as wanted. □

5.7 *Proof of Proposition 5.3.* For every $\mathbf{C}$-valued point x of $\overline{\mathfrak{M}}_g$, let $\mathcal{C}_x$ be the stable curve whose moduli point is the image of x in $\overline{\mathfrak{M}}_g$. Let $\Gamma_x = \big\{\mathcal{V}_x, \mathcal{E}_x\big\}$ be the dual graph associated to $\mathcal{C}_x$. Choose a symplectic level–n structure on Pic^0 of the normalization of $\mathcal{C}_x$. Let R be the universal deformation space of $\mathcal{C}_x$. It is a regular local ring. Put $K := \mathrm{Frac}(R)$. Write $S = \mathrm{Spec}(R)$ and S_0 for its closed point. Let $\mathcal{C} \to S$ be the universal curve over it. As explained in 4.4 the 1–motive uniformizing $\mathrm{Pic}^0(\mathcal{C}/S)$ defines and is defined by a morphism $\rho\colon S\big[q_e^{-1} | e \in \mathcal{E}\big] \to \Theta$.

(i) Take x a $\mathbf{C}$–valued point of $D^{2r,o}$. Since $\mathcal{C}\times_S S_0=\mathcal{C}_x$ is irreducible, ρ extends to a map $\rho\colon S\to\overline{\Theta}$ due to 4.5. This defines a map from $\mathrm{Spf}(R)$ to the formal completion of $\widetilde{\mathcal{A}}_{g,(n),\gamma}$ at $\rho(x)$ and, hence, to the formal completion of $\widetilde{\mathcal{A}}_{g,1}^{\mathrm{DV}}$ at the image of x. Let $\alpha\colon S\to\widetilde{\mathcal{A}}_{g,1}^{\mathrm{DV}}$ be the induced map. We also have a map $\beta\colon S\to\widetilde{\mathcal{A}}_{g,1}^{\mathrm{DV}}$ induced by the composite of the moduli map $S\to\overline{\mathfrak{M}}_g$, giving $\mathcal{C}$, with $\widetilde{\mathfrak{J}}$. By construction the K–valued point of $\widetilde{\mathcal{A}}_{g,1}^{\mathrm{DV}}$ defined by α is the moduli point $\big[\mathrm{Pic}^0(\mathcal{C}_K/K)\big]\in\mathcal{A}_{g,1}(K)$ and, in particular, it coincides with the one defined by β. Hence, $\alpha=\beta$ since R is an integral domain so that $\alpha(S_0)=\beta(S_0)$. To conclude the proof it then suffices to show that $\rho|_{S_0}=\mathrm{per}_{g,r}(S_0)$. This is the content of 4.9.

(ii) Let $x\in\big(\widetilde{\mathfrak{J}}^{\mathrm{an}}\big)^{-1}\big(\partial\widetilde{\mathcal{A}}_{g,1}^{o}\big)$. Let $H\subset\mathcal{E}_x$ be a subset of edges of $\mathcal{E}_x$ and consider the map $c_{\Gamma_x,\Gamma_H}\colon\Gamma_x\dashrightarrow\Gamma_H$ defined by contracting the edges in H. Assume that Γ_H has only one vertex and the same cyclomatic number as Γ_x. Number the edges of Γ_H, which are identified with $\mathcal{E}_x\backslash H$, as $e_1,\ldots,e_r$. We simply write $e_1,\ldots,e_r$ also for the elementary cycles of Γ_H they define. Write $\gamma_i:=c_{\Gamma_x,\Gamma_H}^{-1}(e_i)$. Due to 8.6 we can choose an orientation such that $\mathcal{B}:=\{\gamma_1,\ldots,\gamma_r\}$ is a well adapted basis of Γ_x. Let $\overline{S}_H$ be the closed subscheme of S defined by $(q_e|e\in H)$. Write $S_H^o:=S\big[q_e^{-1}|e\in\mathcal{E}_x\backslash H\big]$ and $\overline{S}_H^o:=S_H^o\cap\overline{S}_H$. Since the curve $\mathcal{C}$ is irreducible over $\overline{S}_H^o$ with fibers having dual graph Γ_H, it follows from 4.4 that ρ extends to a map $S_H^o\to\overline{\Theta}$ and $\overline{S}_H^o$ maps to $\partial\overline{\Theta}$. As in the proof of (i) one concludes that $\rho|_{\overline{S}_H^o}$ is the composite of $\widetilde{\mathfrak{J}}$ with the moduli map $\overline{S}_H^o\to\overline{\mathfrak{M}}_g$. Since also $\widetilde{\mathfrak{J}}^{\mathrm{an}}(x)$ lies in the image of $\partial\overline{\Theta}^{\mathrm{an}}$ and $\overline{S}_H^o$ is open dense in $\overline{S}_H$, we conclude from 4.5 that $\mathcal{B}$ must be of type $\tau(i,j)=0$ if $i\neq j$ and 1 if $i=j$ or, equivalently, that ρ extends to a morphism $\rho\colon S\to\overline{\Theta}$. Since $\rho|_{S_0}=\mathrm{per}_{g,r}(S_0)$ by 4.9, then $\rho|_{S_0}$ factors via $\Theta^{'o}$ by (i). We then deduce from 4.8 that $\mathcal{C}_x$ is an irreducible curve. Since $\mathcal{C}_x$ has genus g and its normalization has genus $g-r$, then $x=\big[\mathcal{C}_x\big]$ lies in the image of $D^{2r,o}\to\overline{\mathfrak{M}}_g$ as claimed.

5.8 *Proof of Proposition 5.4.* We may assume that k is algebraically closed. We drop the subscript k from the notations D_k and A_k for simplicity. Put $\alpha\colon D\times D\to A$ to be the map $(P,Q)\mapsto P-Q$.

(I) We argue by contradiction assuming that G has a positive dimensional irreducible component G^0. Since $G\subset D-D$ we have $\dim G^0\leq 2$. Assume first that G^0 has dimension 2. Since $D-D$ is irreducible and closed in A, we have $G^0=D-D$. In particular, G^0 contains 0 and it is an abelian subvariety of A. Fix a point $R\in D(k)$. Since $A=\mathrm{Pic}^0(D/k)$ is the smallest subgroup scheme of A containing $\alpha\big(D\times\{R\}\big)$, we conclude that $G^0=A$ and A is of dimension 2 i. e., $g(D)=2$. This contradicts the hypothesis that $g(D)\geq 3$. We conclude that G^0 is of dimension 1 i. e., it is isomorphic to an elliptic curve. Since D is not hyperelliptic, α is injective outside the diagonal $\Delta\subset D\times D$. Since the diagonal maps to $0\in A$ via α, we have $\alpha^{-1}\big(G^0\backslash\{0\}\big)\cong G^0\backslash\{0\}$. We deduce that there exists a closed immersion $G^0\subset D\times D$. Thus, D has genus 1 contradicting the assumption that D has genus ≥ 2.

(II) Suppose that there exists $(P,Q)\in D\times D$ such that $\alpha(P,Q)\in D-D$ is a torsion point of exact order s with $1 < s \leq g(D)$. Then, $P \neq Q$ and $sP \sim sQ$. Let H be the divisor sP. We have $r := \dim_k \mathrm{H}^0(D,H) \geq 2 > s+1-g(D)$. It follows from 5.8.1 that $sP = (r-1)g_2^1 + P_1 + \cdots + P_{s-2r+2}$ with $r \geq 2$. Thus, $P_1 = \cdots = P_{s-2r+2} = P$ and $2(r-1)P = (r-1)g_2^1$. This implies that $P = \sigma(P)$ and, similarly, $Q = \sigma(Q)$ i. e., P and Q are fixed by the hyperelliptic involution. Then, $\alpha(P,Q)$ is a 2–torsion point i. e., $s = 2$.

5.8.1 Lemma. *Let D be smooth projective irreducible hyperelliptic curve over an algebraically closed field k. Let H be an effective Cartier divisor on D of degree d such that $\dim_k \mathrm{H}^0(D,H) = r > d+1-g$. Then, $d-2r+2 \geq 0$ and there exist points $P_1, \ldots, P_{d-2r+2}$ of D such that $H = (r-1)g_2^1 + P_1 + \cdots + P_{d-2r+2}$.*

Proof: Let $K := \Omega^1_{D/k}$. We deduce from Riemann–Roch that $\dim_k \mathrm{H}^0(D, K-H) = g(D) - d - 1 + r > 0$. In particular, H is a special divisor and $r-1 \leq d/2$ i.e., $d-2r+2 \geq 0$, by Clifford's theorem. For the second claim see [ACGH, Ch. I, Ex. D.9]. □

5.9 *Proof of Proposition 5.5.* We argue by contradiction assuming that such a Ψ does exist. We write h for $g(D_k) = g - r$. The projection $\pi_{D_k,r}(D_k^{2r,o})$ to the i–th factor of A_k^r does not contain the zero section of A_k and is contained in $D_k - D_k \subset A_k$. Thus, 5.5(ii) implies that the induced map $B \to A_k^r \to A_k$ is non–trivial and is contained in $D_k - D_k$. This forces the genus h of D_k to be ≤ 2 by 5.4. Note that there is a unique map $p\colon \mathcal{P}_{A_k^r} \to \mathcal{P}_{A_k^{4-h}}$ covering the projection $q\colon A_k^r \to A_k^{4-h}$ onto the first $4-h$–factors such that, if $\mathcal{P}' := \prod_{A_k^r, 1 \leq i < j \leq 4-h} \mathcal{P}_{ij}$, it produces the natural projection $\mathcal{P}_{A_k^r} \to \mathcal{P}' \cong q^*(\mathcal{P}_{A_k^{4-h}})$ of torsors over A_k^r. The image Ψ' of Ψ via the projection $\mathcal{P}_{A_k^r} \to \mathcal{P}'$ is a torus torsor over B since it can be described via a pushout of the morphism form T to the image of the composite map $T \to U_k' \to \prod_{1 \leq i < j \leq 4-h} \mathbf{G}_{m,k}$. Note that $\mathcal{P}'$ descends to A_k^{4-h} via p so that also ψ' descends to a torus torsor over the abelian subvariety $q(B) \subset A_k^{4-h}$, rigidified over the zero section of $q(B)$. A fortiori it coincides with the image $p(\Psi)$ of Ψ via p. By definition 5.1 the composite of $\mathrm{per}_{D_k,r}$ with p factors via the projection $D_k^{2r,o} \to D_k^{2(4-h),o}$ onto the first $2(4-h)$–components and the map $\mathrm{per}_{D_k,4-h}$. In particular, $p(\Psi)$ satisfies 5.5(i)&(ii) for $r = 4-h$. We may then assume that $r = 4-h$ in 5.5. We consider separately the cases $g(D_k) = 0$ and $r = 4$, $g(D_k) = 1$ and $r = 3$, $g(D_k) = 2$ and $r = 2$ in Prop. 6.1, §7.3 and §6.3 respectively.

6 The cases $(g,r) = (0,4)$ and $(2,2)$.

We first prove 5.5 in the case that $g = 0$. We drop the subscript k, for example we write D instead of D_k. Then, $D \cong \mathbf{P}^1$, $A = 0$ and Θ' is the torus $\mathbb{Q}_r := \mathbf{G}_m^{r(r-1)/2}$ over k. In the notation of 4.6 we have $\widetilde{\mathrm{G}}_{i,0} = \mathbf{G}_m$ and the point $\alpha_{i,j}$ is simply the cross ratio of the 4 points $\bigl(P_i, Q_i, P_j, Q_j\bigr)$. Hence, the map $\mathrm{per}_{D,r}$ associates to the $2r$ distinct, ordered points $(P_1, \ldots, P_r, Q_1, \ldots, Q_r)$ of $\mathbf{P}^1$ the element $\bigl(\lambda_{i,j}\bigr)_{1 \leq i < j \leq r}$ of $\mathbf{G}_m^{r(r-1)/2}$ where $\lambda_{i,j}$ is the cross ratio of $\bigl(P_i, Q_i, P_j, Q_j\bigr)$. Note that $\lambda_{i,j} \neq 1$ since the pairs of points are taken to be distinct. Gerritzen computed in [Ge, §4&5] the equations defining the schematic closure $\overline{\mathrm{Jac}}_r$ of the image of $\mathrm{per}_{D,r}$ for every r. We recall here his result for $r=4$ which is the relevant case for the proof of 5.5. Write $\mathbb{Q}_4=\prod_{1\leq i<j\leq 4}\mathbf{G}_m$, let $\gamma_{ij}\colon\mathbb{Q}_4\to\mathbf{G}_m$ be the projection onto the (i,j)–th component and let q_{ij} be the entry of the (i,j)–component of $\mathbb{Q}_4$. Then, $\overline{\mathrm{Jac}}_4$ is a divisor defined by the equation $F=\Delta H-G$ where $\Delta:=\prod_{1\leq i<j\leq 4}(q_{ij}-1)$,

$$H:=\prod_{1\leq i<j\leq 4}q_{ij}-\sum_{1\leq i\leq 4}\left(\prod_{1\leq k<\ell\leq 4,k,\ell\neq i}q_{k\ell}\right)+q_{12}q_{34}+q_{13}q_{24}+q_{14}q_{23},$$

$$\begin{aligned}G:=&q_{12}q_{34}\cdot\prod_{1\leq i<j\leq 4,(i,j)\neq(1,2),(3,4)}(q_{ij}-1)^2+q_{13}q_{24}\cdot\prod_{1\leq i<j\leq 4,(i,j)\neq(1,3),(2,4)}(q_{ij}-1)^2+\\&+q_{14}q_{23}\cdot\prod_{1\leq i<j\leq 4,(i,j)\neq(1,4),(2,3)}(q_{ij}-1)^2.\end{aligned}$$

Assume that Ψ as in Theorem 5.5 exists. It follows from 5.5(i) that it is a subtorus of $\mathbb{Q}_4$ and due to 5.5(ii) it is contained in $\overline{\mathrm{Jac}}_4$. In fact, since the generic point of Ψ is in the image of $\mathrm{per}_{C,4}$ the projection $\Psi\subset\mathbb{Q}_4\xrightarrow{\gamma_{ij}}\mathbf{G}_m$ is non–trivial for every $i\neq j$. Thus, there exists a cocharacter $\mathbf{G}_m\subset\Psi\subset\mathbb{Q}_4$ such that the composite with $\gamma_{ij}\colon\mathbb{Q}_4\to\mathbf{G}_m$ is non–trivial for every $1\leq i<j\leq 4$. To get a contradiction, and to conclude the proof of 5.5 in this case, we are reduced to show that:

6.1 Proposition. *Let T be a 1–dimensional subtorus of $\mathbb{Q}_4$ contained in $\overline{\mathrm{Jac}}_4$. Then, there exist $1\leq i<j\leq 4$ such that the projection $T\to\mathbb{Q}_4\xrightarrow{\gamma_{ij}}\mathbf{G}_m$ is trivial.*

Proof: Suppose the proposition is false. Write $T:=\mathrm{Spec}\big(\mathbf{C}[q,q^{-1}]\big)$. The map $T\to\mathbb{Q}_4\xrightarrow{\gamma_{ij}}\mathbf{G}_m=\mathrm{Spec}\big(\mathbf{C}[q_{ij},q_{ij}^{-1}]\big)$ is a homomorphism of group schemes and hence it is defined by $q_{ij}\to q^{a_{ij}}$ with a_{ij} a non–zero integer. Let W_4 be the subgroup of the group of bijections γ of the set $\{\pm e_1,\ldots,\pm e_4\}$ such that $\gamma(-e_i)=-\gamma(e_i)$ for every $i=1,\ldots,4$. If $\gamma\in W_4$, there exists a unique permutation σ of $\{1,\ldots,4\}$ such that $\gamma(e_i)=\mathrm{sgn}(\gamma(i))e_{\sigma(i)}$. We let $\gamma\in W_4$ act on $\mathbb{Q}_4$ by sending

$$q_{ij}\longmapsto\gamma(q_{ij})=q^{\mathrm{sgn}(\gamma(i))\mathrm{sgn}(\gamma(j))}q_{\sigma(i)\sigma(j)}$$

using the convention that $q_{kh}=q_{hk}$ for $k<h$. The locus $\overline{\mathrm{Jac}}_4$ is invariant under the group W_4 acting on $\mathbb{Q}_4$ by [Ge, Prop. 1.5.1]; this is explained remarking that at the level of degenerate stable curves such transformations amounts to changing the orientation of the dual graph and renumbering its edges. In particular, acting with W_4 we may assume that $a_{12}\geq|a_{ij}|$ for every $1\leq i<j\leq 4$.

By assumption we have $\Delta(q)H(q)=G(q)$. Taking q equal to a primitive a_{12}–root of unity ζ we get that $\Delta(\zeta)=0$. Hence, $G(\zeta)=0$. Since in the second and third summand defining G there is a factor $(q_{12}-1)$, which vanishes when we substitute ζ, we deduce that

$$\prod_{1\le i<j\le 4,(i,j)\ne(1,2),(3,4)} (q_{ij}-1)^2(\zeta)=0$$

i. e., there exist $1 \le i < j \le 4$ such that $(i,j) \neq (1,2)$, $(3,4)$ and a_{12} divides a_{ij}. Hence, $a_{12} = |a_{ij}|$ and $i = 1$ or 2. If $i = 2$, we act with the element of W_4 defined by $e_1 \mapsto e_2$ and $e_t \mapsto e_t$ if $t > 2$. Thus, we may assume that $i = 1$. If $j = 4$, we use the element of W_4 defined by $e_3 \mapsto e_4$ and $e_t \mapsto e_t$ if $t < 3$ to get that $j = 3$. If $a_{13} < 0$, we act with the element of W_4 defined by $e_3 \mapsto -e_3$ and $e_t \mapsto e_t$ if $t \neq 3$ so that $a_{13} > 0$. Then, $a_{12} = a_{13} =: a$.

Assume that at most one of the integers a_{ij}'s is negative. Let

$$Q := \prod_{ij} q^{a_{ij}} \prod_{ij,a_{ij}>0} q^{a_{ij}};$$

it is the term, up to sign, of highest degree in $\Delta \cdot \prod_{ij} q^{a_{ij}}$. Its degree, denoted by t, is 2 times the sum of all positive a_{ij}'s plus, if it exists, the only negative term. For $i = 1,\dots,4$ let

$$Q_i := \prod_{1\le k<\ell\le 4,k,\ell\ne i} q^{a_{k\ell}} \prod_{ij,a_{ij}>0} q^{a_{ij}}$$

and put

$$t_i := \sum_{1\le k<\ell\le 4,k,\ell\ne i} a_{k\ell} + \sum_{ij,a_{ij}>0} a_{ij}$$

to be the degree of Q_i. Since $t - t_1 = a_{12} + a_{13} + a_{14} > 0$, $t - t_2 = a_{13} + a_{14} + a_{34} > 0$ and $t - t_3 = a_{12} + a_{14} + a_{24} > 0$, the terms of highest degree in ΔH are Q and Q_4 (up to sign). Note that $t - t_4 = a_{14} + a_{24} + a_{34}$.

Let $q_{\alpha\beta}q_{\gamma\delta}\prod_{(i,j)\ne(\alpha,\beta),(\gamma,\delta)}(q_{ij}-1)^2$, with the set $\{\alpha,\beta\}$ disjoint from the set $\{\gamma,\delta\}$, be one of the three factors composing G. The degree $t(\alpha\beta,\gamma\delta)$ of its highest term $q_{\alpha\beta,\gamma\delta}$ in q appearing in it is

$$(1) \qquad t(\alpha\beta,\gamma\delta) = a_{\alpha\beta} + a_{\gamma\delta} + 2\sum_{(i,j)\ne(\alpha,\beta),(\gamma,\delta)} a_{ij}$$

if $a_{ij} > 0$ for all $(i,j) \neq (\alpha,\beta),(\gamma,\delta)$ and it is

$$(2) \qquad t(\alpha\beta,\gamma\delta) = a_{\alpha\beta} + a_{\gamma\delta} + 2\sum_{(i,j)\ne(\alpha,\beta),(\gamma,\delta),(i',j')} a_{ij}$$

if there exists $(i',j') \neq (\alpha,\beta)$, (γ,δ) such that $a_{i'j'} < 0$. In case (1) the degree t of Q is $a_{\alpha\beta}+a_{\gamma\delta}+2\sum_{(i,j)\ne(\alpha,\beta),(\gamma,\delta)} a_{ij}$ plus those factors among $\{a_{\alpha\beta}, a_{\gamma\delta}\}$ which are positive. Since there exists at least one such factor, t is bigger than $t(\alpha\beta,\gamma\delta)$. In case (2) the degree of Q is

$$t = 2a_{\alpha\beta} + 2a_{\gamma\delta} + a_{i'j'} + 2\sum_{(i,j)\ne(\alpha,\beta),(\gamma,\delta),(i',j')} a_{ij}.$$

Note that $a_{\alpha\beta} + a_{\gamma\delta} + a_{i'j'} > 0$ if $a_{\alpha\beta} = a$ or $a_{\gamma\delta} = a$. Hence, $t > t(12, 34)$ and $t > t(13, 24)$ in any case.

If t or t_4 or $t(14, 23)$ is strictly higher than the other two degrees, the corresponding term in $\Delta(q)H(q) - G(q)$ is the only term of highest degree and this is impossible since $\Delta(q)H(q) - G(q) = 0$. If $t = t(14, 23)$, then $t \geq t_4$ and $-Q$ and $-q_{14,23}$ appear as the factors of highest degree in $\Delta(q)H(q) - G(q)$. We get again a contradiction.

We are left with the following two possibilities. The first possibility is that $t_4 = t(14, 23) > t$. In this case the terms Q_4 and $-q_{14,23}$ are the terms of highest degree in $\Delta(q)H(q) - G(q)$ and cancel out. This happens only if either a_{24} or a_{34} are negative (otherwise $t > t(14, 23)$). Acting with the element of W_4 defined by $e_2 \mapsto e_3$ and $e_t \mapsto e_t$ if $t \neq 2, 3$ we may assume that $a_{34} < 0$. Since there is only one negative term, a_{14}, a_{23} and a_{24} are positive. The terms of highest degree in

$$q_{14,23} - q^{a_{14}} q^{a_{23}} \cdot \prod_{1 \leq i < j \leq 4, (i,j) \neq (1,4),(2,3)} (q^{a_{ij}} - 1)^2$$

are the sum of the terms $2q^{a_{14}+a_{23}} \prod_{(i,j) \neq (1,4),(2,3),(3,4),(k,\ell)} q^{a_{i,j}}$, taken over all $(k, \ell) \neq (1, 4)$, $(2, 3)$ such that $a_{k\ell}$ is maximal. This contribution can never cancel with $-Q$. This excludes this case.

We are left with the second possibility that $t = t_4 > t(14, 23)$. Then, $-Q+Q_4 = 0$ and $t-t_4 = a_{14}+a_{24}+a_{34} = 0$. In particular, since only one of the a_{ij}'s is negative, $a_{23} > 0$. In this case either $a_{14} > 0$ and then $t(14, 23) - t_3 > -a_{14} - a_{24} + a_{13} = a_{34} + a_{13} \geq 0$ or $a_{14} < 0$ and then $t(14, 23) - t_3 = a_{13} + a_{34} > 0$. Acting with the element υ of W_4 defined by $e_2 \mapsto e_3$ and $e_t \mapsto e_t$ if $t \neq 2, 3$, the roles of t_2 and t_3 are interchanged and one gets that $t(14, 23) > t_2$. If $a_{24} < 0$, possibly acting with υ, we may assume that $a_{24} > 0$. Then, $t(13, 24) - t_1 > a_{12} - a_{24} - a_{34} = a_{12} + a_{14} \geq 0$. In any case, the term of highest degree in $G(q)$ is higher than the term of highest degree in $\Delta(q)H(q)$. This gives a contradiction. *We conclude that at least two a_{ij}'s are negative.*

Assume that $a_{23} > 0$. Then, the integers a_{i4} for $i = 1, 2, 3$ can not be all positive. Thus, possibly acting with $\gamma \in W_4$ defined by $e_i \mapsto e_i$ if $1 \leq i \leq 3$ and $e_4 \mapsto -e_4$, we may assume that at least two of the integers a_{i4} for $i = 1, 2, 3$ are positive. We have seen that this is not possible. Hence, $a_{23} < 0$. Possibly acting with γ we may assume that $a_{14} > 0$. If a_{24} and a_{34} are both negative, then $q^{2a_{23}} q^{2a_{24}} q^{2a_{34}}$ appears as a term of $\Delta(q)H(q)$ and its degree is strictly less than the degree of any other term of $\Delta(q)H(q)$ or of $G(q)$. This is impossible. Hence, either a_{24} or a_{34} is positive. Let $\delta \in W_4$ be defined by $e_i \mapsto e_i$ if $i = 1$ or 4 and by $e_2 \mapsto e_3$. Possibly acting with δ we may assume that $a_{34} > 0$ and $a_{24} < 0$. We summarize

$$a_{12} = a_{13} = a > 0, \quad a_{14} > 0, \quad a_{23} < 0, \quad a_{24} < 0, \quad a_{34} > 0.$$

Acting with $\rho \in W_4$ defined by $e_1 \mapsto -e_1$ and $e_i \mapsto e_i$ if $i \neq 1$ and replacing q with q^{-1} we get that a_{34} is the only negative factor. This is not allowed as we have seen before. The proposition follows. □

6.3 *The case:* $g = 2$ *and* $r = 2$. We prove 5.5 in the case that $g = 2$ and $r = 2$. We drop the subscript k. Assume that there exists $\Psi \subset \mathcal{P}_{A^2}$ satisfying 5.5(i)&(ii). Note that $\mathcal{P}_{A^2}$ is a $\mathbf{G}_m$–bundle over A^2. The map

$$D \underset{k}{\times} D \longrightarrow A = \mathrm{Pic}^0(D/k), \qquad (P, Q) \mapsto P - Q,$$

is quasi–finite since D has genus 2. We deduce that $\pi_{D,2}: D^{4,o} \to A^2$ is quasi–finite. This and 5.5(ii) imply that the projection $\Psi \to A^2$ is quasi–finite. The inverse image of the identity section of A^2 is the torus T. We conclude that $T = 0$ and, hence, the projection $\Psi \to B$ is an isomorphism. The map $\Psi \to \mathcal{P}_{A^2}$ amounts in this case to a map $B \to \mathcal{P}_{A^2}$ such that the projection to A^2 is an injective homomorphism of abelian varieties. Since the Poincaré $\mathbf{G}_m$–bundle is non–trivial, $\mathcal{P}_{A^2}$ is non–trivial either and we deduce that $\dim B \leq 3$.

Let σ be the hyperelliptic involution on D. Given distinct points P and Q of D over k, the curve $D/(P \sim Q)$ is hyperelliptic if and only if $Q = \sigma(P)$. Thus, the closure of the locus of points $0 \neq P - Q \in D - D \subset A$ such that the curve $D/(P \sim Q)$ is hyperelliptic is the image of $\zeta: D \to A$ sending $P \mapsto P - \sigma(P)$. Such map is injective outside the fixed points of σ and maps those fixed points to 0. In particular, the image of ζ is not an abelian subvariety of A. Let

$$\pi_i: B \longrightarrow A$$

be the composite of the map $B \to A^2$ with the i–th projection to A. It is non constant, since its image contains non–zero points due to 5.5(ii) and 4.7. It is not contained in the image of ζ because the latter is irreducible, 1–dimensional and it is not an abelian subvariety. Possibly interchanging the indices 1 and 2 we may assume that the image of π_2 has dimension less or equal to the dimension of the image of π_1. Let k' be the residue field of the generic point of the image of π_2 and let $\iota_2: \mathrm{Spec}(k') \to A$ be the induced map. Denote by $D' := D \times_k k'$ and let B' be the fiber product of B and ι_2 via π_2. Let $\big(P_2, Q_2\big)$ be two points of D' such that $P_2 - Q_2$ is defined by ι_2. We have just seen that $P_2 - Q_2$ is not in the image of ζ. Hence, the map

$$(D' \backslash \{P_2, Q_2\}) \times (D' \backslash \{P_2, Q_2\}) \longrightarrow (D' \backslash \{P_2, Q_2\}) - (D' \backslash \{P_2, Q_2\})$$

is injective outside the diagonal since $C := D'/\big(P_2 \sim Q_2\big)$ is not hyperelliptic. Thus, the inclusion $B \subset \mathcal{P}_{A^2}$ induces a closed immersion from B' to the pull–back $\mathrm{Pic}^0(C/k')$ of $\mathcal{P}_{A^2}$ to $\mathrm{Id} \times \iota_2: A \times_k k' \to A^2$. By assumption it intersects $C^{\mathrm{sm}} - C^{\mathrm{sm}} \subset \mathrm{Pic}^0(C/k')$ and, hence, the inclusion $B' \subset \mathrm{Pic}^0(C/k')$ lifts to a rational map $\alpha: B' \dashrightarrow D' \times D'$ from which we get a rational map

$$\beta: B \dashrightarrow D \times D.$$

The composite of β with $D \times D \to A$, given by $(P, Q) \mapsto P - Q$, is π_1 which is non–constant. In particular, β is non–constant. Since D has genus 2, then B can not be an elliptic curve either i. e., $\dim B \geq 2$.

Note that B' is a torsor under $\operatorname{Ker}(\pi_2)$ which is mapped injectively to A via π_1. Hence, α is injective. Note that $\operatorname{Ker}(\pi_2)$ is not 1–dimensional, otherwise $\operatorname{Ker}(\pi_2)^0$ would be an elliptic curve which is impossible since α is injective and since D' has genus 2. The first possibility is that $\operatorname{Ker}(\pi_2)$ has dimension 2. This and the injectivity of α imply that α, and hence β, is dominant. The second possibility is that $\operatorname{Ker}(\pi_2)$ has dimension 0. Then, π_2 is dominant since $\dim B \geq 2$ and π_1 is dominant as well since the dimension of its image is greater or equal to the dimension of the image of π_2 by assumption. Also in this case β is dominant. Since $D \times D$ is proper and B is smooth, β can be extended in codimension 1 and, being dominant, induces an injective map at the level of global differential forms. This is impossible since $\dim B$ is ≤ 3 and D has genus 2.

7 The case $g = 1$ and $r = 3$.

We prove 5.5 in the case that $g = 1$ and $r = 3$. We write D for D_k. Then, D is a smooth proper curve of genus 1 and $A := \operatorname{Pic}^0(D/k)$ is an elliptic curve. The latter is a subgroup scheme of the group of automorphisms of D: for every $P \in D(k)$ and $Q \in A(k)$ we let $P + Q$ be the unique k-valued point of D such that Q is the degree 0, invertible sheaf associated to the Weil divisor $(P + Q) - (P)$. Note that $\operatorname{per}_{D,r}$ is invariant with respect to the action of the automorphism group of D. Hence, $\operatorname{per}_{D,r}\big(P_1 + \alpha, \ldots, P_6 + \alpha\big) = \operatorname{per}_{D,r}\big(P_1, \ldots, P_{2r}\big)$ for every point α of A and every point $(P_1, \ldots, P_6)$ of $D^{6,o}$.

Fix a point $t := (t_1, t_2, t_3) \in A^3$. The fiber $\mathcal{P}_{A^3}|_t$ of $\mathcal{P}_{A^3}$ over t is a torus torsor under $\mathbf{G}_m^3$ (note that by construction basis elements for the character group of the latter are given, up to sign). We start with the following:

7.1 *A description of* $\operatorname{per}_{D,3}|_t$. Choose a point $\big((P_1, Q_1), (P_2, Q_2), (P_3, Q_3)\big) \in \pi_{D,3}^{-1}(t)$ mapping to Ψ. Such a choice defines an isomorphism of $\pi_{D,3}^{-1}(t)$ with the open subscheme

$$\begin{aligned} A^{3,o} := \big\{\big(\alpha_1, \alpha_2, \alpha_3\big) \in A^3 | \alpha_j + Q_j \neq \alpha_i + P_i \neq \alpha_j + P_j \text{ and} \\ \alpha_j + Q_j \neq \alpha_i + Q_i \neq \alpha_j + P_j \,\forall\, 1 \leq i < j \leq 3\big\} = \\ = \big\{\big(\alpha_1, \alpha_2, \alpha_3\big) \in A^3 | \alpha_i - \alpha_j \neq P_j - P_i, \neq Q_j - P_i, \neq P_j - Q_i, \neq Q_j - Q_i\big\}. \end{aligned}$$

The rigidification $\operatorname{per}_{D,3}\big((P_1, Q_1), (P_2, Q_2), (P_3, Q_3)\big)$ defines an isomorphism of $\mathcal{P}_{A^3}|_t$ with the trivial torsor $\mathbf{G}_m^3$ and of $\Psi|_t$ with a subtorus of $\mathbf{G}_m^3$. Write

$$\lambda := \prod_{(i,j)=(1,2),(2,3),(1,3)} \lambda_{i,j} \colon A^{3,o} \longrightarrow \mathbf{G}_m^3, \qquad \lambda_{i,j} \colon A^{3,o} \longrightarrow \mathbf{G}_m$$

for $\operatorname{per}_{D,3}|_t$, respectively the composite of $\operatorname{per}_{D,3}|_t$ and the projection $\mathcal{P}_{A^3}|_t \cong \mathbf{G}_m^3 \to \mathbf{G}_m$ on the (i,j)–factor for $1 \leq i < j \leq 3$. By construction $\lambda_{i,j}$ factors via the projection $p_{i,j} \colon A^3 \to A^2$ onto the i–th and j–th components. Since $\lambda_{i,j}$ is invariant with respect to the diagonal action of A on A^2, the map $\lambda_{i,j}$ factors via the difference map $d \colon A^2 \to A$ sending $(P, Q) \mapsto P - Q$.

For fixed (P_i, Q_i) let σ_{P_i,Q_i} be the hyperelliptic involution on $C_i := D/(P_i \sim Q_i)$: on the normalization D of C_i it is given by $P \mapsto (P_i - P) + Q_i$. The points (B, H)

in D^2 whose image $\mathcal{O}_{C_i}\big(B-H\big)$ in $\mathrm{Pic}^0(C_i/t)$ is $\mathcal{O}_{C_i}\big(B-H\big)$ are the points (B,H) and $\big(\sigma_{P_i,Q_i}(H),\sigma_{P_i,Q_i}(B)\big)$. Note that $\sigma_{P_i+\alpha_i,Q_i+\alpha_i}(Q_j+\alpha_j)=P_i-Q_j+Q_i-\alpha_j+2\alpha_i=P_j+\alpha'_j$ with $\alpha'_j=P_i+Q_i-P_j-Q_j-\alpha_j+2\alpha_i$ and similarly $\sigma_{P_i+\alpha_i,Q_i+\alpha_i}(P_j+\alpha_j)=Q_j+\alpha'_j$. Since the projection of $\mathrm{per}_{D,3}$ onto the component (i,j) of $\mathcal{P}_{A^3}$ is invariant with respect to σ_{P_i,Q_i}, we conclude that, via the identifications above, $\lambda_{i,j}$ is invariant with respect to the involution $A^2\ni(\alpha_i,\alpha_j)\mapsto\big(\alpha_i,P_i+Q_i-P_j-Q_j+2\alpha_i-\alpha_j\big)$. We deduce that $\lambda_{i,j}$ is the restriction to the open subscheme $A^{3,o}$ of A^3 of the composite

$$\lambda_{i,j}|_t\colon A^3\xrightarrow{p_{i,j}}A^2\xrightarrow{d}A\longrightarrow A/\langle\widetilde{\sigma}_{i,j}\rangle\xrightarrow{\sim}\mathbf{P}^1,$$

where $\widetilde{\sigma}_{i,j}\colon A\xrightarrow{\sim}A$ is the involution $\alpha\mapsto P_j+Q_j-P_i-Q_i-\alpha$. Here, $\mathbf{P}^1$ is the standard compactification of $\mathbf{G}_m$ and the isomorphism $A/\langle\widetilde{\sigma}_{i,j}\rangle\cong\mathbf{P}^1$ is such that the composite

$$\tau_{i,j}\colon A\longrightarrow A/\langle\widetilde{\sigma}_{i,j}\rangle\cong\mathbf{P}^1$$

sends $0\mapsto1$ and $\tau_{i,j}^{-1}(0)$, (resp. $\tau_{i,j}^{-1}(\infty\}$) is $\{P_j-P_i,\widetilde{\sigma}_{i,j}(P_j-P_i)=Q_j-Q_i\}$ (resp. $\{P_j-Q_i,\widetilde{\sigma}_{i,j}(P_j-Q_i)=Q_j-P_i\}$) or $\{P_j-Q_i,Q_j-P_i\}$ (resp. $\{P_j-P_i,Q_j-Q_i\}$). These conditions determine the isomorphism $A/\langle\widetilde{\sigma}_{i,j}\rangle\cong\mathbf{P}^1$ up to the automorphism $(x,y)\mapsto(y,x)$ of $\mathbf{P}^1$. This ambiguity comes from the ambiguity in the choice of sign for a basis of the character group of $\mathbf{G}_m$. We denote by $\{s_{i,j}^{(1)},\ldots,s_{i,j}^{(4)}\}$ to be the 4 ramification points of $\widetilde{\sigma}_{i,j}$ (considered in A or by abusing notation in $\mathbf{P}^1$). We conclude that the map λ is the restriction to $A^{3,o}$ of the composite of $A^3\to A^2$, given by $(P,Q,R)\mapsto(P-Q,Q-R)$, with the finite map

$$\tau\colon A^2\longrightarrow\big(\mathbf{P}^1\big)^3,\qquad(\alpha,\beta)\mapsto\big(\tau_{1,2}(\alpha),\tau_{1,3}(\alpha+\beta),\tau_{2,3}(\beta)\big).$$

Let K_A be the image of τ. Note that $\widetilde{\sigma}_{1,3}(\alpha+\beta)=\widetilde{\sigma}_{1,2}(\alpha)+\widetilde{\sigma}_{2,3}(\beta)$. Thus, two points (α,β) and $(\alpha',\beta')\in A^2$ have the same image via τ if and only if (1) $\alpha'=\alpha$ or $\alpha'=\widetilde{\sigma}_{1,2}(\alpha)$ and (2) $\beta'=\beta$ or $\beta'=\widetilde{\sigma}_{2,3}(\beta)$ and (3) $\alpha'+\beta'=\alpha+\beta$ or $\alpha'+\beta'=\widetilde{\sigma}_{1,3}(\alpha+\beta)=\widetilde{\sigma}_{1,2}(\alpha)+\widetilde{\sigma}_{2,3}(\beta)$. This implies that either $(\alpha',\beta')=(\alpha,\beta)$ or $(\alpha',\beta')=\big(\widetilde{\sigma}_{1,2}(\alpha),\widetilde{\sigma}_{2,3}(\beta)\big)$. We conclude that the map $\tau\colon A^2\to K_A$ identifies K_A with the quotient of A^2 by the involution $\widetilde{\sigma}_{1,2}\times\widetilde{\sigma}_{2,3}$. In particular, K_A is isomorphic to the Kummer variety of A^2. Such map is étale of degree 2 outside the 16 points fixed by $\widetilde{\sigma}_{1,2}\times\widetilde{\sigma}_{2,3}$ which map to the 16 singular points of K_A.

7.2 Proposition. *Assume that t_i is not a torsion point for any i. Let $Z\subset\mathcal{P}_{A^3}|_t$ be torus torsor under a subtorus T of $\mathbf{G}_m^3$ contained in the closure of the image of $\mathrm{per}_{D,3}|_t\colon\pi_{D,3}^{-1}(t)\to\mathcal{P}_{A^3}|_t$. Then, Z is of dimension 0.*

Proof: We argue by contradiction. We may assume that Z is 1–dimensional. Let $T\cong\mathbf{G}_m\subset\mathbf{G}_m^3$ be given by the cocharacter $\big(a_{1,2},a_{1,3},a_{2,3}\big)$. Then, $a_{i,j}\neq0$ for some (i,j).

Given a point $\big((P_1,Q_1),(P_2,Q_2),(P_3,Q_3)\big)\in\mathrm{per}_{D,3}^{-1}(Z)$ and using 7.1, the image of $\mathrm{per}_{D,3}|_t$ is an open subscheme of K_A and the torsor Z is identified with $\mathbf{G}_m\subset\mathbf{G}_m^3$. Extend it to a closed immersion $\iota\colon\mathbf{P}^1\subset K_A\subset\big(\mathbf{P}^1\big)^3$. Let $C\subset A^2$ be the normalization of the inverse image via τ of $\iota\colon \mathbf{P}^1 \subset K_A$. Then, the natural map $C \to \mathbf{P}^1$ is $2:1$ and, since there is no non–constant map $\mathbf{P}^1 \to A^2$, the curve C is irreducible.

Step I: we have $a_{i,j} \neq 0$ for every $1 \leq i < j \leq 3$. Note that the image of $\mathrm{per}_{C,3}|_t$ is equivariant with respect to the action of the group S_3 acting on the $\mathbf{G}_m^3$–torsor $\mathcal{P}_{A^3}|_t$ by permuting the components $\mathbf{G}_m^3$. Thus, if $a_{i,j} = 0$ for some (i,j) we may assume, after permuting the indices, that $(i,j) = (1,2)$.

Since $(1,1,1) \in \iota(\mathbf{P}^1)$ and $\tau_{1,2}(0) = 1$, we deduce that C maps to $\{0\} \times A$ and, hence, that $C = \{0\} \times A$. The map $\tau\colon A \to \big(\mathbf{P}^1\big)^3$ is $2:1$ outside the 16 points fixed by $\widetilde{\sigma}_{1,2} \times \widetilde{\sigma}_{2,3}$. The induced map $C \to \mathbf{P}^1$ is also $2:1$ and, hence, is ramified at 4 points by the Riemann–Hurwitz formula. We conclude that $0 \in A$ is a fixed point of $\widetilde{\sigma}_{1,2}$. This is equivalent to the equality $P_i + Q_i = P_j + Q_j$. In particular, for every α in an open dense subset of A, the image of $\big((P_1', Q_1'), (P_2', Q_2'), (P_3', Q_3')\big) := \big((P_1, Q_1), (P_2 + \alpha, Q_2 + \alpha), (P_3 + \alpha, Q_3 + \alpha)\big)$ via $\mathrm{per}_{C,3}$ is in Z. Then, $P_1 + Q_1 \neq P_2 + \alpha + Q_2 + \alpha$ for α not a 2–torsion point. Choosing the rigidification of $\mathrm{per}_{C,3}^{-1}(Z)$ defined by $\big((P_1', Q_1'), (P_2', Q_2'), (P_3', Q_3')\big)$ we deduce that $0 \in A$ can not be a fixed point for the associated $\widetilde{\sigma}_{1,2}$ so that $a_{1,2} \neq 0$. This leads to a contradiction.

Step II: the integers $a_{1,2}$, $a_{2,3}$ and $a_{1,3}$ are pairwise coprime.

Take two distinct pairs of integers (ℓ, h) and (ℓ', h') in $\{(1,2),(1,3),(2,3)\}$. Consider the composite $A^2 \to K_A \subset \big(\mathbf{P}^1\big)^3 \to \mathbf{P}^1 \times \mathbf{P}^1$ where the last map is the projection onto the components (ℓ, h) and (ℓ', h'). Consider $\iota\colon \mathbf{P}^1 \subset K_A \to \mathbf{P}^1 \times \mathbf{P}^1$. The restriction to $\mathbf{G}_m \to \mathbf{P}^1 \times \mathbf{P}^1$ factors via $\mathbf{G}_m \times \mathbf{G}_m$ and is given by the cocharacter $\big(a_{\ell,h}, a_{\ell',h'}\big)$. Hence, the image of $\iota\colon \mathbf{P}^1 \subset K_A$ in $\mathbf{P}^1 \times \mathbf{P}^1$ is $\mathbf{P}^1$. Let $\rho\colon \mathbf{P}^1 \to \mathbf{P}^1$ be the induced map. We have the following diagram:

$$\begin{array}{ccccccc} A^2 & \longrightarrow & K_A & \hookrightarrow & \big(\mathbf{P}^1\big)^3 & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^1 \\ & & & & \uparrow & & \uparrow \\ & & & & \mathbf{P}^1 & \xrightarrow{\rho} & \mathbf{P}^1. \end{array}$$

The induced map $A^2 \to \mathbf{P}^1 \times \mathbf{P}^1$ identifies $\mathbf{P}^1 \times \mathbf{P}^1$ with the quotient of A^2 by the group $\langle \widetilde{\sigma}_{h,\ell}, \widetilde{\sigma}_{h',\ell'} \rangle \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Then, the map $K_A \to \mathbf{P}^1 \times \mathbf{P}^1$ is étale, $2:1$ outside the 8 lines $\{s_{h,\ell}^{(i)}\} \times \mathbf{P}^1$ and $\mathbf{P}^1 \times \{s_{h',\ell'}^{(i)}\}$ for $i = 1, \ldots, 4$ and it is $1:1$ over these lines. Note that $\mathbf{P}^1 \subset K_A$ is not contained in the union of these lines since otherwise $a_{h,\ell}$ or $a_{h',\ell'}$ would be 0 contradicting Step I. Thus, the map ρ is at most of degree 2. Note that $\mathbf{P}^1 \subset \mathbf{P}^1 \times \mathbf{P}^1$ intersects all the 8 ramification lines above because $a_{ij} \neq 0$ for every (i,j). On the other hand, ρ is the compactification of a cocharacter $\mathbf{G}_m \to \mathbf{G}_m^2$, so that it is ramified at most in 2 points i.e., 0 and ∞ leading to a contradiction. We deduce that ρ can not be of degree 2. Thus, ρ is an isomorphism i. e., $a_{h,\ell}$ and $a_{h',\ell'}$ are coprime as claimed.

Step III: $|a_{i,j}| = 1$ for every $1 \leq i < j \leq 3$. Consider the following diagram with commutative squares:

$$\begin{array}{ccccccc}
C & \longrightarrow & A^2 & & \xrightarrow{\;p_{i,j}\;} & & A \\
{\scriptstyle 2:1}\big\downarrow & & {\scriptstyle \tau}\big\downarrow & & & & \big\downarrow{\scriptstyle \tau_{i,j}} \\
\mathbf{P}^1 & \xrightarrow{\;\iota\;} & K_A & \hookrightarrow & \left(\mathbf{P}^1\right)^3 & \xrightarrow{\pi_{i,j}} & \mathbf{P}^1,
\end{array}$$

where $p_{1,2}$ (resp. $p_{2,3}$) is the projection onto the first (resp. second) component, $p_{1,3}$ is the sum and $\pi_{i,j}$ is the projection on the (i,j)–component. The induced map $\mathbf{P}^1 \to \mathbf{P}^1$ obtained from the bottom row extends the map $\mathbf{G}_m \to \mathbf{G}_m$ given by $x \mapsto x^{a_{i,j}}$. In particular, the induced map

$$r_{i,j}: C \longrightarrow A$$

is Galois with group $\mathbf{Z}/a_{i,j}\mathbf{Z}$, of degree $|a_{i,j}|$ and unramified outside $\tau_{i,j}^{-1}(0)$ and $\tau_{i,j}^{-1}(\infty)$. Note that 0 or ∞ can not be both ramification points of $\tau_{i,j}$; otherwise $2(P_i - Q_i) = 0$ which contradicts our assumption in 7.2 that $t_i = P_i - Q_i$ is not a torsion point. Possibly composing $\pi_{i,j}$ with the automorphism $(x, y) \mapsto (y, x)$ of $\mathbf{P}^1$ we may assume that 0 is not a ramification point of $\tau_{i,j}$. Applying the Riemann–Hurwitz formula to $C \to A$ and since A has genus 1, we get that the genus g_C of C satisfies $2g_C = 2 + \sum_{x \in \tau_{i,j}^{-1}(0)} \sharp r_{i,j}^{-1}(x)(e_x - 1) + \sum_{x \in \tau_{i,j}^{-1}(\infty)} \sharp r_{i,j}^{-1}(x)(e_x - 1)$ where e_x is the ramification index at x. But $\sharp r_{i,j}^{-1}(x) e_x = |a_{i,j}|$. Hence, $\sharp r_{i,j}^{-1}(x)(e_x - 1) = |a_{i,j}| - \sharp r_{i,j}^{-1}(x)$. Since $\tau_{i,j}^{-1}(0)$ consist of two points which are totally ramified w.r.t. $r_{i,j}$, we have $\sum_{x \in \tau_{i,j}^{-1}(0)} \sharp r_{i,j}^{-1}(x)(e_x - 1) = 2|a_{i,j}| - 2$. We conclude that $g_C = |a_{i,j}| + \frac{1}{2} \sum_{x \in \tau_{i,j}^{-1}(\infty)} \big(|a_{i,j}| - \sharp r_{i,j}^{-1}(x)\big)$. Hence,

1) $g_C := 2|a_{i,j}| - 1$ if also ∞ is not a ramification point of $\tau_{i,j}$;
2) $g_C := 3/2|a_{i,j}| - 1/2$ if ∞ is a ramification point of $\tau_{i,j}$ and of $\tau: C \to \mathbf{P}^1$;
3) $g_C := 3/2|a_{i,j}| - 1$ if ∞ is a ramification point of $\tau_{i,j}$ but not of $\tau: C \to \mathbf{P}^1$.

Note that there do not exist positive integers α and β for which $\frac{1}{2}(3\alpha - 1)$ and $\frac{1}{2}(3\beta - 2)$ are both positive integers and they are equal. We conclude that at least two of the integers $|a_{1,2}|$, $|a_{1,3}|$, $|a_{2,3}|$ coincide. Due to Step II they must then be equal to 1 so that $g_C = 1$. We conclude that also the third integer is 1 i.e., $|a_{1,2}| = |a_{1,3}| = |a_{2,3}| = 1$ as claimed.

Step IV: conclusion of the argument. It follows from Step III that $r_{i,j}$ is an isomorphism for every $1 \leq i < j \leq 3$. By construction $r_{1,3} = r_{1,2} + r_{2,3}$. Furthermore, if we write $\gamma := r_{2,3} \circ r_{1,2}^{-1}$, we have that

i. $\gamma: A \to A$ is an automorphism of elliptic curves, since $\gamma(0) = 0$, sending the sets $\{P_2 - P_1, Q_2 - Q_1\}$, $\{P_2 - Q_1, Q_2 - P_1\}$ respectively to the sets $\{P_3 - P_2, Q_3 - Q_2\}$, $\{P_3 - Q_2, Q_3 - P_2\}$ (or viceversa);
ii. $1 + \gamma = r_{1,3} r_{1,2}^{-1}: A \to A$ is an isomorphism, since $0 \mapsto 0$, sending the sets $\{P_2 - P_1, Q_2 - Q_1\}$, $\{P_2 - Q_1, Q_2 - P_1\}$ to the sets $\{P_3 - P_1, Q_3 - Q_1\}$, $\{P_3 - Q_1, Q_3 - P_1\}$ (or viceversa).

Note that End$A$ is either $\mathbf{Z}$ or an order in a quadratic imaginary field L. Thus, γ and $1 + \gamma$ define units in $\mathbf{Z}$ or in the ring of integers $\mathcal{O}_L$ of L. The first case can not hold. In the second case, it follows from Dirichlet's units theorem that such group consists only of the roots of unity lying in L. Since L is quadratic, such roots of unity are, up to sign, at most the primitive roots of unity of order 1, 2, 3 and 6. For example, as remarked by the referee, L could be $L = \mathbf{Q}(\zeta_3)$ with $\gamma = \zeta_3$ a primitive root of unity and then $1 + \gamma = \zeta_6$ is a 6-th root of unity. To rule out this possibility we have to use the finer information on γ and $1 + \gamma$ given in (i) and (ii).

We obtain from (i) that $\gamma(P_2 - Q_2) = \gamma(P_2 - Q_1) - \gamma(Q_2 - Q_1) = \pm(P_3 - Q_3)$ or $\pm(P_2 - Q_2)$. Note that $(\gamma+1)(P_2 - Q_2) \neq 0$ since $P_2 \neq Q_2$. Similarly, $(\gamma-1)(P_2 - Q_2) \neq 0$ otherwise either $\gamma - 1$ is an isogeny and then $P_2 - Q_2 = t_2$ is a torsion point or $\gamma = \mathrm{Id}$ and $1 + \gamma$ is multiplication by 2 which is not an isomorphism. The conclusion is that $\gamma(P_2 - Q_2) = \pm(P_3 - Q_3)$. This and (i) forces the following combinations for $\big(\gamma(P_2 - Q_1), \gamma(Q_2 - Q_1), \gamma(P_2 - P_1), \gamma(Q_2 - P_1)\big)$

$$
\begin{array}{ll}
\big(P_3 - Q_2, Q_3 - Q_2, P_3 - P_2, Q_3 - P_2\big) & (1) \\
\big(Q_3 - P_2, P_3 - P_2, Q_3 - Q_2, P_3 - Q_2\big) & (2) \\
\big(P_3 - P_2, Q_3 - P_2, P_3 - Q_2, Q_3 - Q_2\big) & (3) \\
\big(Q_3 - Q_2, P_3 - Q_2, Q_3 - P_2, P_3 - P_2\big) & (4).
\end{array}
$$

In particular, $\gamma(P_1 - Q_1) = \gamma(P_2 - Q_1) - \gamma(P_2 - P_1) = \pm(P_2 - Q_2)$ in each case. Analogously, using (ii) one proves that $(1 + \gamma)(P_1 - Q_1) = (1 + \gamma)(P_2 - Q_1) - (1 + \gamma)(P_2 - P_1) = \pm(P_3 - Q_3)$ and one deduces that the only possible combinations for $\big((1 + \gamma)(P_2 - Q_1), (1 + \gamma)(P_2 - P_1), (1 + \gamma)(Q_2 - Q_1), (1 + \gamma)(Q_2 - P_1)\big)$ are

$$
\begin{array}{ll}
\big(P_3 - Q_1, Q_3 - Q_1, P_3 - P_1, Q_3 - P_1\big) & (1') \\
\big(Q_3 - P_1, P_3 - P_1, Q_3 - Q_1, P_3 - Q_1\big) & (2') \\
\big(P_3 - P_1, Q_3 - P_1, P_3 - Q_1, Q_3 - Q_1\big) & (3') \\
\big(Q_3 - Q_1, P_3 - Q_1, Q_3 - P_1, P_3 - P_1\big) & (4').
\end{array}
$$

We compute the combinations of $\big((1 + \gamma)(P_2 - Q_1), (1 + \gamma)(P_2 - P_1), (1 + \gamma)(Q_2 - Q_1), (1 + \gamma)(Q_2 - P_1)\big)$ using the first table

$$
\begin{array}{ll}
\big(P_2 - Q_1 + P_3 - Q_2, P_3 - P_1, Q_3 - Q_1, Q_2 - Q_1 + Q_3 - P_2\big) & (1'') \\
\big(Q_3 - Q_1, P_2 - P_1 + Q_3 - Q_2, Q_2 - Q_1 + P_3 - P_2, P_3 - Q_1\big) & (2'') \\
\big(P_3 - Q_1, P_2 - P_1 + P_3 - Q_2, Q_2 - Q_1 + Q_3 - P_2, Q_3 - Q_1\big) & (3'') \\
\big(P_2 - Q_1 + Q_3 - Q_2, Q_3 - P_1, P_3 - Q_1, Q_2 - Q_1 + P_3 - P_2\big) & (4'').
\end{array}
$$

Note that $\pm(P_3 - Q_3) \neq P_1 - Q_1$ since $(1 + \gamma)(P_1 - Q_1) = \pm(P_3 - Q_3)$ so that $(1 + \gamma)(P_1 - Q_1) = \pm(P_1 - Q_1)$ which leads to a contradiction as above. Similarly, $P_1 - Q_1 \neq \pm(P_2 - Q_2)$, otherwise $\gamma(P_1 - Q_1) = \pm(P_1 - Q_1)$ leading to a contradiction, and $P_3 - Q_3 \neq \pm(P_2 - Q_2)$ otherwise $\pm\gamma(P_1 - Q_1) = P_3 - Q_3$ and $(1 + \gamma)(P_1 - Q_1) = \pm\gamma(P_1 - Q_1)$ which is impossible. Since $P_i \neq Q_i$ for every i we conclude that

case 1”): looking at the second column it can agree only with case (2’). Comparing the last column we then have $Q_2 - Q_1 + Q_3 - P_2 = P_3 - Q_1$ i. e., $P_3 - Q_3 = Q_2 - P_2$. This can not hold.

case 2”): looking at the last column it can agree only with case (2’). Comparing the first columns, one deduces that this case can not hold.

case 3”): comparing the last columns it can agree only with case (3’). Comparing the first columns, one rules out this case.

case 4”): looking at the third column it can agree only with case (3’). Comparing the fourth columns, one concludes that this case can not hold either.

Thus, none of these cases can hold getting to a contradiction. The lemma follows. □

7.3 *End of proof of Proposition 5.5 if $g = 1$ and $r = 3$.* Assume that there exists $\Psi \subset \mathcal{P}_{A^3}$ satisfying 5.5(i)&(ii). It is a torus torsor under a torus T over an abelian subvariety $B \subset A^3$. Since its generic point is contained in the image of $\mathrm{per}_{D,3}$, then Ψ intersects non–trivially $\mathrm{per}_{D,3}\big(D^{6,o}\big)$. In particular, B has non–empty intersection with the image of $\pi_{D,3}$. Due to 4.7 the projection of the image of $\pi_{D,3}\big(D^{6,o}\big)$ onto each of the three factors of A^3 does not contain the 0–section of A^3. Hence, for every $i = 1,\ldots,3$, the composite map $\varphi_i\colon B \to A^3 \to A$, where the latter is the projection onto the i–th factor, is non–trivial. In particular, B must be a positive dimensional abelian variety and φ_i is surjective. Furthermore, thanks to 7.2, we have $T = 0$. We conclude that the projection $\Psi \to B$ is an isomorphism. In particular, the pull–back of $\mathcal{P}_{A^3}$ to B is trivial.

Let $\mathcal{L}$ be the invertible sheaf $\mathcal{O}_A(0)$ where $0 \in A$ is the origin. Put $\Lambda(\mathcal{L}) := m^*(\mathcal{L}) \otimes p_1^*(\mathcal{L})^{-1} \otimes p_2^*(\mathcal{L})^{-1}$ where $m\colon A^2 \to A$ is the multiplication. It is the pull–back of the Poincaré biextension over $A \times_k A^\vee$ to A^2 via the map $\mathrm{Id} \times \lambda_A$ where $\lambda_A\colon A \to A^\vee$ is the canonical principal polarization. In particular, $\mathcal{P}_{A^3}$ is the rigidified $\mathbf{G}_m^3$–torsor over A^3 given by the fibred product over A^3 of the pull–back of $\Lambda(\mathcal{L})$ via the projections $\pi_{ij}\colon A^3 \to A^2$ onto the (i,j)–th factors for $1 \leq i < j \leq 3$. Let B_{ij} be the quotient of B by the connected component of the identity of the kernel of $B \to A^3 \xrightarrow{\pi_{ij}} A^2$. Then, the composite $B_{ij} \to A^2$ is finite and the pull–back of $\Lambda(\mathcal{L})$ to B_{ij} is trivial. This implies that $\dim B_{ij} < 2$. Otherwise, since the map $\mathrm{Pic}\big(A^2/k\big) \to \mathrm{Pic}\big(B_{ij}/k\big)$ has a right quasi–inverse defined by the norm map and, hence, has finite kernel, we get that $\Lambda(\mathcal{L}^n)$ is trivial on A^2 for some $n > 0$ which is absurd. On the other hand $\dim B_{ij}$ can not be 0 since the image of B_{ij} via the first projection to A is the image of φ_i which is surjective. Hence, $\dim B_{ij} = 1$. The only possibility is that B is an elliptic curve and φ_i is an isogeny.

Denote by ψ_i the isogeny $\varphi_i \circ \varphi_1^\vee\colon A \to A$. The triviality of the pull–back of $\mathcal{P}_{A^3}$ to B implies the triviality of the pull–back of $\mathcal{P}_{A^3}$ via $(\psi_1,\psi_2,\psi_3)\colon A \to A^3$. This is equivalent to the triviality of the pull–back of $\Lambda(\mathcal{L})$ via $\psi_i \times \psi_j$ for every $1 \leq i < j \leq 3$ i. e., $\psi_i^*(\mathcal{L}) \otimes \psi_j^*(\mathcal{L}) \cong \big(\psi_i + \psi_j\big)^*(\mathcal{L})$. In particular, $\deg\psi_i + \deg\psi_j = \deg(\psi_i + \psi_j)$. Since $\deg[n] = n^2$ for every $n \in \mathbf{Z}$, we deduce that A is a CM elliptic curve. Let $\mathrm{End}(A) \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}\big(\sqrt{-\Delta}\big)$ with Δ square–free positive integer. If $\psi := a + b\sqrt{-\Delta}$ with a, $b \in \mathbf{Q}$, one has $\deg\psi = a^2 + \Delta b^2$. Let $\psi_i := a_i + b_i\sqrt{-\Delta}$ for $i = 1,\ldots,3$. Then, $\deg\psi_i + \deg\psi_j = \deg(\psi_i + \psi_j)$ is equivalent to require that $a_i a_j + \Delta b_i b_j = 0$. Note that ψ_1 is multiplication by $\deg\varphi_1$ and, hence, $b_1 = 0$ and $a_1 = \deg\varphi_1 \neq 0$. We then get that $a_j = 0$ for $j \neq 1$ and, hence, $b_2 b_3 = 0$. Hence, for $j = 2$ or 3 we have $a_j = 0$ and $b_j = 0$ i. e., $\psi_j = 0$ which is absurd.

8 Appendix: some graph theory.

In this section we briefly review some results concerning connected graphs. We refer to [OS, §4] for proofs, references and more details. The key example to keep in mind is the dual graph associated to a stable curve.

A finite graph Γ is a pair $\{\mathcal{V}, \mathcal{E}\}$ of finite sets and a map $\tau: \mathcal{E} \to \mathcal{V}^{(2)}$ from $\mathcal{E}$ to the set of unordered pairs of not necessarily distinct elements of $\mathcal{V}$. The set $\mathcal{E}$ is the set of edges, the set $\mathcal{V}$ is the set of vertices and τ is the map associating to an edge two vertices, called its end points. For example, for every $n \in \mathbf{N}$ the standard graph I_n with n vertices is given by the vertices $\mathcal{V}_n := \{1, \ldots, n\}$ and by the edges $\mathcal{E}_n := \{[1,2], \ldots, [n-1, n]\}$ where the end points of $[i, i+1]$ are $\{i, i+1\}$. A map of graphs from the graph $\Gamma = \{\mathcal{V}, \mathcal{E}\}$ to the graph $\Gamma' = \{\mathcal{V}', \mathcal{E}'\}$ is a pair of maps $\alpha: \mathcal{V} \to \mathcal{V}'$ and $\beta: \mathcal{E} \to \mathcal{E}'$ such that for every $e \in \mathcal{E}$ the images of the end points of e via α coincide with the end points of $\alpha(e)$. A *path* is a map of graphs $\delta: I_n \to \Gamma$, for some $n \in \mathbf{N}$. We say that a finite graph Γ is *connected* if, given two vertices, there is path connecting them i. e., if, given two vertices v and v', there is a path $\delta: I_n \to \Gamma$ such that the vertex 1 maps to v and the vertex n is mapped to v'. A *subgraph* $\Gamma' = \{\mathcal{V}', \mathcal{E}'\}$ of a graph $\Gamma = \{\mathcal{V}, \mathcal{E}\}$ is given by a subset $\mathcal{E}' \subset \mathcal{E}$ and a subset $\mathcal{V}' \subset \mathcal{V}$ such that the end points of $e \in \mathcal{E}'$ in the graph Γ' are the end points of $e \in \mathcal{E}$ in the graph Γ. A *spanning subgraph* Γ' of a graph Γ is a subgraph such that the set of vertices of Γ' coincide with the set of vertices of Γ.

An *orientation* on Γ is a pair of maps (s, t), where $s: \mathcal{E} \to \mathcal{V}$ is the source and $t: \mathcal{E} \to \mathcal{V}$ is the target, such that for every $e \in \mathcal{E}$ we have $\tau(e) = \{s(e), t(e)\}$. Fix a finite graph Γ with an orientation (s, t). Let $\mathrm{C}_0(\Gamma, \mathbf{Z}) := \oplus_{v \in \mathcal{V}} \mathbf{Z} v$ and $\mathrm{C}_1(\Gamma, \mathbf{Z}) := \oplus_{e \in \mathcal{E}} \mathbf{Z} e$. Let $\partial: \mathrm{C}_1(\Gamma, \mathbf{Z}) \to \mathrm{C}_0(\Gamma, \mathbf{Z})$ be the $\mathbf{Z}$–linear map defined by $\partial(e) = t(e) - s(e)$. Let $\mathrm{H}_1(\Gamma, \mathbf{Z}) := \mathrm{Ker}(\partial)$ and $\mathrm{H}_0(\Gamma, \mathbf{Z}) := \mathrm{Coker}(\partial)$. One defines the *cyclomatic number* $h(\Gamma)$ of Γ to be the rank of $\mathrm{H}_1(\Gamma, \mathbf{Z})$ as $\mathbf{Z}$–module. Note that $\mathrm{H}_1(\Gamma, \mathbf{Z})$ and $\mathrm{H}_0(\Gamma, \mathbf{Z})$ do not depend on the choice of an orientation. A *tree* is a connected graph with cyclomatic number 0. Given a connected graph Γ we define a *spanning tree* to be a spanning subgraph which is a tree.

Consider the $\mathbf{Z}$–bilinear, symmetric, positive–definite pairing $\mathrm{C}_1(\Gamma, \mathbf{Z}) \times \mathrm{C}_1(\Gamma, \mathbf{Z}) \to \mathbf{Z}$ given by $\langle e, f \rangle = 0$ if $e \neq f$ and $\langle e, f \rangle = 1$ if $e = f$. One defines by restriction the $\mathbf{Z}$–bilinear, symmetric, positive definite pairing $\langle\, ,\, \rangle: \mathrm{H}_1(\Gamma, \mathbf{Z}) \times \mathrm{H}_1(\Gamma, \mathbf{Z}) \to \mathbf{Z}$. An element $\gamma \in \mathrm{H}_1(\Gamma, \mathbf{Z})$ is called a *cycle* if, for every $e \in \mathcal{E}$, we have $\langle \gamma, e \rangle = 0$ or ± 1. Denote by $\Gamma_\gamma = \{\mathcal{V}_\gamma, \mathcal{E}_\gamma\}$ the subgraph of Γ whose edges are $\mathcal{E}_\gamma := \{e \in \mathcal{E} | \langle \gamma, e \rangle \neq 0\}$ and whose vertices are their end points. A cycle γ is called an *elementary cycle* if Γ_γ has cyclomatic number 1. We say that a subgraph of Γ is a cycle or an elementary cycle if it coincides with Γ_γ for $\gamma \in \mathrm{H}_1(\Gamma, \mathbf{Z})$ a cycle (resp. an elementary cycle).

8.1 Definition. *Given a type* $\tau: \{(i, j) | 1 \leq i < j \leq r\} \to \{0, 1\}$, *we say that* Γ *is of type* τ *if there exist an orientation of* Γ *and a* $\mathbf{Z}$*–basis* $\gamma_1, \ldots, \gamma_r$ *of* $\mathrm{H}_1(\Gamma, \mathbf{Z})$ *such that*

a. $\langle\gamma_i, e\rangle = 0$ *or* 1 *for every* $e \in \mathcal{E}$*;*

b. Γ_{γ_i} *and* Γ_{γ_j} *have an edge in common if* $\tau(i,j) = 1$ *and have at most one vertex in common if* $\tau(i,j) = 0$.

We prove that a connected graph has always a type (not necessarily unique).

8.2 Proposition. *Let* Γ *be a connected graph.*

I. *A spanning tree* $\{\mathcal{V}, \mathcal{E}'\}$ *for* Γ *exists.*

II. *Let* $\{\mathcal{V}, \mathcal{E}'\}$ *be a spanning tree for* Γ*;*

II.a for every $e \in \mathcal{E}\backslash\mathcal{E}'$ *the spanning graph* $\{\mathcal{V}, \mathcal{E}' \cup \{e\}\}$ *has cyclomatic number* 1. *Let* γ_e *be the unique cycle in this subgraph satisfying* $\langle e, \gamma_e\rangle = 1$*;*

II.b the set $\{\gamma_e | e \in \mathcal{E}\backslash\mathcal{E}'\}$ *is a* $\mathbf{Z}$*–basis for* $\mathrm{H}_1(\Gamma, \mathbf{Z})$.

Proof: See [OS, §4]. □

Let $\{\mathcal{V}, \mathcal{E}'\}$ be a spanning tree for a graph $\Gamma = \{\mathcal{V}, \mathcal{E}\}$. For every $e \in \mathcal{E}\backslash\mathcal{E}'$ denote by Γ_e the graph $\{\mathcal{V}, \mathcal{E}' \cup \{e\}\}$.

8.3 Corollary. *Let* $\{\mathcal{V}, \mathcal{E}'\}$ *be a spanning tree for a graph* $\Gamma = \{\mathcal{V}, \mathcal{E}\}$. *Let* a *and* b *be distinct elements of* $\mathcal{E}\backslash\mathcal{E}'$. *Then,* Γ_{γ_a} *and* Γ_{γ_b} *are connected, elementary cycles and* $\Gamma_{\gamma_a} \cap \Gamma_{\gamma_b}$ *is either empty or it is a tree.*

Proof: If Γ' and Γ'' are subgraphs of Γ one has an exact sequence

$$0 \longrightarrow \mathrm{H}_1\big(\Gamma' \cap \Gamma'', \mathbf{Z}\big) \longrightarrow \mathrm{H}_1\big(\Gamma', \mathbf{Z}\big) \oplus \mathrm{H}_1\big(\Gamma'', \mathbf{Z}\big) \longrightarrow \mathrm{H}_1\big(\Gamma' \cup \Gamma'', \mathbf{Z}\big)$$
$$\longrightarrow \mathrm{H}_0\big(\Gamma' \cap \Gamma'', \mathbf{Z}\big) \longrightarrow \mathrm{H}_0\big(\Gamma', \mathbf{Z}\big) \oplus \mathrm{H}_0\big(\Gamma'', \mathbf{Z}\big) \longrightarrow \mathrm{H}_0\big(\Gamma' \cup \Gamma'', \mathbf{Z}\big) \longrightarrow 0. \quad (8.3.1)$$

Consider Γ_{γ_a}. Its connected components $\Gamma_{\gamma_a,1}, \ldots, \Gamma_{\gamma_a,n}$ define cycles of Γ and, hence, have cyclomatic number ≥ 1. Using (8.3.1) we deduce that $\mathrm{H}_1(\Gamma_{\gamma_a}, \mathbf{Z})$ has rank $\geq n$. Since $\Gamma_{\gamma_a} \subset \Gamma_a$ and the latter has cyclomatic number 1, we have $n = 1$ and Γ_{γ_a} is connected and it is an elementary cycle.

Consider the subgraphs Γ_a and Γ_b of Γ. Their intersection is by assumption a spanning tree and, hence, has trivial H_1 and H_0 of rank 1 and $\Gamma_a \cup \Gamma_b$ is connected. Furthermore, Γ_a and Γ_b are connected, and therefore have $\mathrm{H}_0 \cong \mathbf{Z}$, and have cyclomatic number 1 by assumption. By (8.3.1) we get that the map $\mathrm{H}_1\big(\Gamma_a, \mathbf{Z}\big) \oplus \mathrm{H}_1\big(\Gamma_b, \mathbf{Z}\big) \to \mathrm{H}_1\big(\Gamma_a \cup \Gamma_b, \mathbf{Z}\big)$ is an isomorphism. Consider the subgraphs $\Gamma_{\gamma_a} \subset \Gamma_a$ and $\Gamma_{\gamma_b} \subset \Gamma_b$. Since $\Gamma_{\gamma_a} \cup \Gamma_{\gamma_b} \subset \Gamma_a \cup \Gamma_b$, we get the following commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_1\big(\Gamma_{\gamma_a}, \mathbf{Z}\big) \oplus \mathrm{H}_1\big(\Gamma_{\gamma_b}, \mathbf{Z}\big) & \xrightarrow{\sim} & \mathrm{H}_1\big(\Gamma_a, \mathbf{Z}\big) \oplus \mathrm{H}_1\big(\Gamma_b, \mathbf{Z}\big) \\
\downarrow & & \downarrow \\
\mathrm{H}_1\big(\Gamma_{\gamma_a} \cup \Gamma_{\gamma_b}, \mathbf{Z}\big) & \hookrightarrow & \mathrm{H}_1\big(\Gamma_a \cup \Gamma_b, \mathbf{Z}\big).
\end{array}$$

We have seen that the right vertical map is an isomorphism. We conclude from (8.3.1) that the map $\mathrm{H}_1\big(\Gamma_{\gamma_a}, \mathbf{Z}\big) \oplus \mathrm{H}_1\big(\Gamma_{\gamma_b}, \mathbf{Z}\big) \to \mathrm{H}_1\big(\Gamma_{\gamma_a} \cup \Gamma_{\gamma_b}, \mathbf{Z}\big)$ is an isomorphism. We deduce that $\mathrm{H}_1\big(\Gamma_{\gamma_a} \cap \Gamma_{\gamma_b}, \mathbf{Z}\big) = 0$ and that $\mathrm{H}_0\big(\Gamma_{\gamma_a} \cap \Gamma_{\gamma_b}, \mathbf{Z}\big)$ coincides with the kernel of $\mathrm{H}_0(\Gamma_{\gamma_a}, \mathbf{Z}) \oplus \mathrm{H}_0(\Gamma_{\gamma_b}, \mathbf{Z}) \longrightarrow \mathrm{H}_0(\Gamma_{\gamma_b} \cup \Gamma_{\gamma_a}, \mathbf{Z})$. In particular, $\mathrm{H}_0(\Gamma_{\gamma_a} \cap \Gamma_{\gamma_b}, \mathbf{Z})$ has rank ≤ 1. The conclusion follows. □

8.4 Corollary. *Let $\Gamma = \{\mathcal{V}, \mathcal{E}\}$ be a connected graph with cyclomatic number r. Then, there exists an orientation of Γ and a $\mathbf{Z}$–basis $\{\gamma_1, \ldots, \gamma_r\}$ of $\mathrm{H}_1(\Gamma, \mathbf{Z})$ such that:*

1. *$\langle \gamma_i, e \rangle$ is 0 or 1 for every $i = 1, \ldots, r$ and every $e \in \mathcal{E}$;*
2. *for $1 \leq i < j \leq r$ the intersection $\Gamma_{\gamma_i} \cap \Gamma_{\gamma_j}$ is a tree;*
3. *γ_i is an elementary cycle for every $i = 1, \ldots, r$.*

Proof: Left to the reader. □

A $\mathbf{Z}$–basis of $\mathrm{H}_1(\Gamma, \mathbf{Z})$, such that there is an orientation of Γ for which (1)–(3) of 8.4 hold, is called a *well adapted basis*. Given any such we define the *associated type* to be the map $\tau\colon \{(i,j) | 1 \leq i < j \leq r\} \to \{0, 1\}$ given by

$$\tau(i,j) = \begin{cases} 1 & \text{if } \Gamma_{\gamma_i} \text{ and } \Gamma_{\gamma_j} \text{ have at least one edge in common;} \\ 0 & \text{if } \Gamma_{\gamma_i} \cap \Gamma_{\gamma_j} \text{ is either empty or consists of one vertex.} \end{cases}$$

For later reference we briefly recall the process of contracting graphs.

8.5 Definition. *Let Δ and Γ be connected graphs. A contraction $\Delta \dashrightarrow \Gamma$ is defined by a triple $\{c_{\mathcal{V}}, \mathcal{E}', c_{\mathcal{E}'}\}$ where $c_{\mathcal{V}}$ is a surjective map from the set of vertices of Δ to the set of vertices of Γ, $\mathcal{E}'$ is a subset of edges of Δ and $c_{\mathcal{E}'}$ is a bijective map from $\mathcal{E}'$ to the set of edges of Γ such that*

(1) *for every $e \in \mathcal{E}'$ the images via $c_{\mathcal{V}}$ of the end points of e are sent to the end points of $c_{\mathcal{E}'}(e)$;*

(2) *if e is an edge of Δ not belonging to $\mathcal{E}'$ the images via $c_{\mathcal{V}}$ of its end points coincide.*

We say that Γ is obtained from Δ contracting the edges which are not in $\mathcal{E}'$. We say that Δ is a dilatation of Γ.

Note that identifying the edges of Γ with the edges of Δ via the map $c_{\mathcal{E}'}$, an orientation on Δ induces an orientation on Γ. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{C}_1(\Delta, \mathbf{Z}) & \xrightarrow{\partial} & \mathrm{C}_0(\Delta, \mathbf{Z}) \\ \downarrow c_{\mathcal{E}'} & & \downarrow c_{\mathcal{V}} \\ \mathrm{C}_1(\Gamma, \mathbf{Z}) & \xrightarrow{\partial} & \mathrm{C}_0(\Gamma, \mathbf{Z}), \end{array}$$

where $c_{\mathcal{V}}$ sends a vertex v of Δ to $c_{\mathcal{V}}(v)$, while $t_{\mathcal{E}'}$ sends an edge e to $t_{\mathcal{E}'}(e)$ if $e \in \mathcal{E}'$ and to 0 otherwise. One then gets an induced map $c_{\Delta,\Gamma}\colon \mathrm{H}_1(\Delta, \mathbf{Z}) \longrightarrow \mathrm{H}_1(\Gamma, \mathbf{Z})$.

8.6 Lemma. *The map $c_{\Delta,\Gamma}$ is surjective. Assume that it is an isomorphism. Then,*

a. *if γ is an elementary cycle of Γ, also $c_{\Delta,\Gamma}^{-1}(\gamma)$ is an elementary cycle of Δ;*

b. *let $\mathcal{C} := \{\delta_1, \ldots, \delta_r\}$ be a well adapted basis of Δ. Then, $\{c_{\Delta,\Gamma}(\delta_1), \ldots, c_{\Delta,\Gamma}(\delta_r)\}$ is a well adapted basis of Γ for the orientation induced from Δ if and only if the image of Δ_{δ_i} in Γ is not contained in the image of Δ_{δ_j} in Γ for every $i \neq j$;*

c. *let* $\mathcal{B} := \{\gamma_1, \dots, \gamma_r\}$ *be a well adapted basis of* Γ. *There is an orientation of* Γ *and elements* $\mathrm{sgn}_1, \dots, \mathrm{sgn}_r \in \{\pm 1\}$ *such that* $\mathcal{B}' := \{\mathrm{sgn}_1\gamma_1, \dots, \mathrm{sgn}_r\gamma_r\}$ *is a well adapted basis of* Γ *and there is an orientation of* Δ, *compatible with the one on* Γ *via* $c_{\mathcal{E}'}$, *such that the inverse image* $\mathcal{C} := \{c_{\Delta,\Gamma}^{-1}(\mathrm{sgn}_1\gamma_1), \dots, c_{\Delta,\Gamma}^{-1}(\mathrm{sgn}_r\gamma_r)\}$ *is a well adapted basis of* Δ.

Furthermore, if Γ *is of type* τ_Γ *relatively to the basis* $\mathcal{B}$, *then the type* τ_Δ *of* Δ *relatively to* $\mathcal{C}$ *satisfies* $\tau_\Delta(i,j) \geq \tau_\Gamma(i,j)$ *for every* $1 \leq i < j \leq r$. *If we further assume that* $\tau_\Delta = \tau_\Gamma$, *then one can take* $\mathcal{B} = \mathcal{B}'$.

Proof: Consider $\mathcal{E}' = \mathcal{E}'_0 \subset \mathcal{E}'_1 \subset \cdots \subset \mathcal{E}'_n = \mathcal{E}_\Delta$ where $\mathcal{E}_\Delta$ is the set of edges of Δ and $\mathcal{E}'_{q+1} \backslash \mathcal{E}'_q$ has cardinality 1. Let Δ_q be the graph obtained from Δ contracting the edges not in $\mathcal{E}'_q$.

If $q = 0$, the statements are obvious. If not, arguing by induction on q, we may assume that Γ is obtained from Δ contracting exactly one edge e. In particular, $\mathrm{Ker}(c_{\mathcal{E}'}) = \mathbf{Z}e$ and $\mathrm{Ker}(c_{\mathcal{V}}) = \mathbf{Z}\big(t(e) - s(e)\big)$. Note that $\mathrm{Ker}(c_{\mathcal{V}}) = 0$ if and only if $t(e) = s(e)$ if and only if $\partial e = 0$. Since $c_{\mathcal{E}'}$ is surjective by construction, one shows by diagram chasing that both in the case that $\partial e = 0$ and in the case that $\partial e \neq 0$, the map $c_{\Delta,\Gamma}$ is surjective. Such map is an isomorphism if and only if $\partial e \neq 0$. Assume that this is the case. Let v be the vertex of Γ image of the vertices of e.

(a) Let $\gamma = \sum e'$ be an elementary cycle of Γ. Let $\gamma' \in C_1(\Delta, \mathbf{Z})$ be the element $\sum c_{\mathcal{E}'}^{-1}(e')$. If $\partial\gamma' = 0$, then γ' is an elementary cycle and it coincides with $c_{\Delta,\Gamma}^{-1}(\gamma)$. If $\partial\gamma' \neq 0$, then v is a vertex of Γ_γ. Let a be an edge of which v is the source. Since Γ_γ has cyclomatic number 1 and is a cycle, $\Gamma_\gamma \backslash \{a\}$ is connected and has cyclomatic number 0 i. e., it is a tree. Hence, there is a unique edge b of which v is the target. Then, $\partial\gamma'$ is the difference between the target of $c_{\mathcal{E}'}^{-1}(b)$ and the source of $c_{\mathcal{E}'}^{-1}(a)$. Since $\partial\gamma'$ is non–trivial, $\partial\gamma'$ is then equal to $\pm\partial e$ (and not a multiple of it) so that there is a unique orientation of e such that $\partial(\gamma' + e) = 0$ and then $c_{\Delta,\Gamma}^{-1}(\gamma) = \gamma' + e$. This proves that $c_{\Delta,\Gamma}^{-1}$ sends an elementary cycle to an elementary cycle.

(b) Certainly $c_{\Delta,\Gamma}$ sends cycles to cycles considering the induced orientation on Γ. Thus, 8.4(1) always holds. Furthermore, $\Gamma_{c_{\Delta,\Gamma}(\delta_i)} \cap \Gamma_{c_{\Delta,\Gamma}(\delta_j)}$ is the image of $\Delta_{\delta_i} \cap \Delta_{\delta_j}$ in Γ.

$\Longrightarrow$ This implication follows remarking that if $\{c_{\Delta,\Gamma}(\delta_1), \dots, c_{\Delta,\Gamma}(\delta_r)\}$ is a well adapted basis of Γ then $\Gamma_{c_{\Delta,\Gamma}(\delta_i)} \cap \Gamma_{c_{\Delta,\Gamma}(\delta_j)}$ must be a tree.

$\Longleftarrow$ First of all we prove that 8.4(3) holds for every cycle $c_{\Delta,\Gamma}(\delta_i)$. We argue by contradiction assuming that there exists i such that $c_{\Delta,\Gamma}(\delta_i)$ is not an elementary cycle i. e., it has cyclomatic number ≥ 2. This happens if and only if Δ_{δ_i} contains the vertices of the edge e but not e itself. Let Δ' be the graph having the same vertices as Δ and $\mathcal{E}_\Delta \backslash \{e\}$ as edges. It is a connected graph since the vertices of e are connected by Δ_{δ_i}. Hence, $\mathrm{H}_1(\Delta', \mathbf{Z})/\mathrm{H}_1(\Delta, \mathbf{Z}) \cong \mathbf{Z}e$. In particular, there exists $j \neq i$ such that Δ_{δ_j} contains e. Since $\Delta_{i,j} := \Delta_{\delta_j} \cap \Delta_{\delta_i}$ is a tree and contains the vertices of e, we conclude that $\Delta_{\delta_j} = \Delta_{i,j} \cup \{e\}$. Then, the image of Δ_{δ_j} in Γ is contained in the image of Δ_{δ_i} contradicting the assumptions.

Eventually, we show that 8.4(2) holds. Assume that it is not the case and let $i \neq j$ be such that $W := \Gamma_{c_{\Delta,\Gamma}(\delta_i)} \cap \Gamma_{c_{\Delta,\Gamma}(\delta_j)}$ is not a tree. Since $\Delta_{\delta_i} \cap \Delta_{\delta_j}$ is a tree, the only possibility is that W does not contain e but it contains the vertices of e. Since $c_{\Delta,\Gamma}(\delta_i)$ and $c_{\Delta,\Gamma}(\delta_j)$ have cyclomatic number 1 by the previous argument, e must be an edge both of Δ_{δ_i} and of Δ_{δ_j}. This is absurd.

(c) Assume that the two cycles γ_i and γ_j have in common an edge. Since the elementary cycles $c_{\Delta,\Gamma}^{-1}(\gamma_i)$ and $c_{\Delta,\Gamma}^{-1}(\gamma_j)$ have in common a tree by (a), the orientations on the tree determined by the two cycles coincide. This implies the last assertion of the lemma.

Possibly renumbering $\{\gamma_1, \ldots, \gamma_n\}$ we may find a partition $I_1 \amalg \cdots \amalg I_m$ of $\{1, \ldots, n\}$ such that (1) for every $1 \leq a < b \leq m$ and every $s \in I_a$ and $t \in I_b$, the cycles γ_s and γ_t have no edge in common and (2) for every $1 \leq a \leq m$ and every s, $t \in I_a$ there exists $h_1, \ldots, h_z \in I_a$ such that $\gamma_s = \gamma_{h_1}$, $\gamma_t = \gamma_{h_z}$ and γ_{h_i} and $\gamma_{h_{i+1}}$ have an edge in common for every $i = 1, \ldots, z-1$. The observation above implies that for every h the cycles $\left\{c_{\Delta,\Gamma}^{-1}(\gamma_i) | i \in I_h\right\}$ have compatible orientations. The oriented subgraphs of Δ defined by the support of the cycles $\left\{c_{\Delta,\Gamma}^{-1}(\gamma_i) | i \in I_h\right\}$ and $\left\{c_{\Delta,\Gamma}^{-1}(\gamma_j) | j \in I_{h+1}\right\}$ have in common at most the edge e. The induced orientations on e could be different. If this holds, $\left\{c_{\Delta,\Gamma}^{-1}(\gamma_i) | i \in I_h\right\}$ and $\left\{-c_{\Delta,\Gamma}^{-1}(\gamma_j) | j \in I_{h+1}\right\}$ have in common e with the same orientation. Proceeding by induction on h, we get the choice of signs in (b) and the orientation on the subgraph $\Delta' \subset \Delta$ defined by the support of $c_{\Delta,\Gamma}^{-1}(\gamma_1) \cup \cdots \cup c_{\Delta,\Gamma}^{-1}(\gamma_n)$ (and on the support $\Gamma' \subset \Gamma$ of $\gamma_1 \cup \ldots \cup \gamma_n$). The complements of Δ' in Δ and Γ' in Γ are disjoint union of trees on which compatible orientations can be chosen arbitrarily.$\square$

8.7 Remark. Let R be a complete, local domain with fraction field K and residue field k. Let $C \to \mathrm{Spec}(R)$ be a stable curve. Then, the dual graph of C_K is obtained from the dual graph of C_k by contracting the edges corresponding to the singularities of C_k that are smoothened in C_K.

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