A.A. 2012/2013 - Exercise sheet
Commutative Algebra

1. Let $K \subset L$ be a finite field extension of degree n. Prove that it is Galois if and only if $L \otimes_K L \cong L^n$, as L-algebras, considering on $L \otimes_K L$ the left structure of L-algebra $L \to L \otimes_K L$, $a \mapsto a \otimes 1$, and considering on the n-th fold product L^n the structure of L-algebra $L \to L^n$, $a \mapsto (a, \dots, a)$.

2. Let k be a field. Consider the ideal $I = (Z^3, XZ, XY)$ in the ring $k[X, Y, Z]$.

(i) Find the minimal prime ideals, the associated primes and the primary decomposition I.

(ii) Let $R := k[X, Y, Z]/I$. Establish whether R is an integral domain, whether it is reduced, the dimension of R. Is the local ring at the maximal ideal (X, Y, Z) regular?

(iii) Describe the set of zero divisors in R.

3. (i) Let $f\colon A \to B$ be an *injective* ring homomorphism sending $1 \mapsto 1$ such that B is integral over A. Show that the induced map on spectra $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective by proving that the fiber over any point of $\mathrm{Spec}(A)$ is non-empty.

(ii) Find a counterexample to the statement in (i) if one drops the condition that f is injective.

4. Let k be an algebraically closed field of characteristic $\neq 2$ and let $f(X) \in k[X]$ be a monic polynomial of degree n. Let $R = k[X, Y]/\big(Y^2 - f(X)\big)$. Give necessary and sufficient conditions to guarantee that

(i) R is an integral domain;

(ii) R is a normal integral domain;

(iii) if R is an integral domain but not normal, compute its normalization.

5. Let k be a field and let $I = (f_1, \subset, f_s) \subset k[X_1, \dots, X_n]$ be an ideal contained in the maximal ideal $(X_1, \dots, X_n)$. Let M be the matrix $s \times n$ with coefficients in k defined by $M := \left(\frac{\partial f_i}{\partial X_j}(0)\right)_{i=1,\dots,s, j=1,\dots,n}$. Let R be the localization of $k[X_1, \dots, X_n]/I$ at $(X_1, \dots, X_n)$. Let m be the dimension of R.

Show that R is a regular local ring if and only if M has a maximal minor of order $n-m$, i.e., a submatrix $(n-m) \times (n-m)$ of determinant $\neq 0$ and every other submatrix $(n-m+1) \times (n-m+1)$ has determinant 0.

6. Let k be a field and let A be a k-algebra. Let $S \subset A$ be a multiplicative set. Show that there is a natural isomorphism of $S^{-1}A$-modules $S^{-1}\Omega^1_{A/k} \cong \Omega^1_{S^{-1}A/k}$.

7. Let R be a ring whose underlying additive group is a finitely generated abelian group. Shoe that if $P \subset R$ is a maximal ideal, then R/P is a finite field. Show that if $P \subset R$ is a prima ideal which is not maximal, then P is a minimal prime ideal.

8. Let k be a field. A *group functor* is a covariant functor from the category of k-algebras to the category of groups. A *k-Hopf algebra* is a k-algebra representing a group functor.

(I) Prove that to give a k-Hopf algebra it is equivalent to give k-algebra homomorphisms

i) $c\colon A \to A \otimes_k A$ (comultiplication);

ii) $\iota\colon A \to A$ (coinverse);

iii) $\varepsilon\colon A \to k$ (counit);

such that

a) $(\mathrm{Id} \otimes \varepsilon) \circ c = \mathrm{Id} = (\varepsilon \otimes \mathrm{Id}) \circ c$ as maps $A \to A$;

b) $(c \otimes \mathrm{Id}) \circ c = (\mathrm{Id} \otimes c) \circ c$ as maps $A \to A \otimes_k A \otimes_k A$;

c) $m \circ (\mathrm{Id} \times \iota) \circ c = \varepsilon \circ \alpha$ and $m \circ (\iota \times \mathrm{Id}) \circ c = \varepsilon \circ \alpha$ as maps $A \to A$. Here $m\colon A \otimes_k A \to A$ is the multiplication map and $\alpha\colon k \to A$ is the structural morphism.

(II) Show that the group functors $\mathbf{G}_a$ and $\mathbf{G}_m$ associating to a k-algebra R the additive group R, respectively the group of invertible elements R^* in R, are representable. Describe the corresponding Hopf algebras and the structural maps c, ι and ε in the two cases.

9. Provide an example of a field extension $K \subset L$ and of a K-algebra R which is lcoal and regular and such that $R \otimes_K L$ is local but not regular.

10. Let A be a noetherian ring and let I be an ideal contained in the Jacobson radical of A. Set S to be the multiplicative set $S = 1 + I$ and write $\widehat{A}$ for the I-adic completion of A. Prove that there is a natural injective map $S^{-1}A \to \widehat{A}$ of A-algebras.

11. Let $\mathbf{Z}_7$ be the completion of $\mathbf{Z}$ with respect to the ideal generated by the prime 7. Show that 2 is a square in $\mathbf{Z}_7$.