Universal extension crystals of 1–motives and applications

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Abstract

We use the crystalline nature of the universal extension of a 1-motive $\mathbb{M}$ to define a canonical Gauss-Manin connection on the de Rham realization of $\mathbb{M}$. As an application we provide a construction of the so-called *Manin map* from a motivic point of view.

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1. Introduction

One of the main goals of this paper is to understand, and generalize, the construction of Manin's map (*cf.* [18] [12, §4]) from a motivic point of view. This generalization permits first of all the definition of such map in the context of 1–motives. Secondly, it provides realizations of the motivic Manin map in the de Rham context, in the context of variations of MHS and in the crystalline context.

Fix a smooth, connected affine curve S over a field k of characteristic 0 and let $\mathbb{A}$ be an abelian scheme over S. Manin's map (*cf.* [12, §4] is a homomorphism

$$\mathcal{M}_{\mathbb{A},\mathrm{Man}} \colon \mathbb{A}(S) \longrightarrow \mathrm{Ext}^1_S\left(\left(\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S), \nabla_{\mathbb{A}}\right), \left(\mathcal{O}_S, d\right)\right), \tag{1}$$

where Ext^1_S is the group of isomorphism classes of $\mathcal{O}_S$-modules endowed with an integrable connection, extension of $\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S)$ with its Gauss-Manin connection $\nabla_{\mathbb{A}}$ by $\mathcal{O}_S$ with the usual derivation d. Such map, more precisely the study of its kernel, plays a crucial role in Manin's proof of the geometric Mordell conjecture.

The motivic Manin map. Following [14, 10.1.10], a smooth 1–motive $\mathbb{M} = [\mathbb{X} \stackrel{\mathrm{u}}{\to} \mathbb{G}]$ over a scheme S is the datum of (a) an S-group scheme $\mathbb{X}$ defined, locally for the étale topology, by a finite and free $\mathbf{Z}$-module, (b) an S-group scheme $\mathbb{G}$, extension over S of an abelian scheme $\mathbb{A}$ by a torus $\mathbb{T}$; (c) a homomorphism of S-group schemes $\mathrm{u} \colon \mathbb{X} \to \mathbb{G}$. In this paper we will call a smooth 1–motive, simply a 1–motive. A morphism of 1–motives is a morphism as complexes. A universal extension of $\mathbb{M}$ is an extension $\mathbb{E}(\mathbb{M}) = [\mathbb{X} \stackrel{\mathrm{v}}{\to} \mathbb{E}(\mathbb{M})_{\mathbb{G}}]$ of $\mathbb{M}$ by a vector group $\mathbb{W}(\mathbb{M})$ (i.e., an extension $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$ of $\mathbb{G}$ by $\mathbb{W}(\mathbb{M})$ over S together with a homomorphism v lifting u) such that the homomorphism of push-out

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathbb{W}(\mathbb{M}), \mathbb{W}) \longrightarrow \mathrm{Ext}(\mathbb{M}, \mathbb{W})$$

is an isomorphism for all vector groups $\mathbb{W}$ over S (cf. [14, 10.1.7]). (For the existence of universal extensions see also [1], [19], [5]). Note that $\mathbb{E}(\mathbb{M})$ is endowed with a weight filtration where $W_0\big(\mathbb{E}(\mathbb{M})\big) = \mathbb{E}(\mathbb{M})$,

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$W_{-1}\big(\mathbb{E}(\mathbb{M})\big) = \mathbb{E}(\mathbb{G})$ and $W_{-2}\big(\mathbb{E}(\mathbb{M})\big) = \mathbb{E}(\mathbb{T}) = \mathbb{T}$ and the graded pieces are given by $\mathrm{Gr}_0\big(\mathbb{E}(\mathbb{M})\big) = \mathbb{E}\left([\mathbb{X} \to 0]\right)$, $\mathrm{Gr}_{-1}\big(\mathbb{E}(\mathbb{M})\big) = \mathbb{E}(\mathbb{A})$ and $\mathrm{Gr}_{-2}\big(\mathbb{E}(\mathbb{M})\big) = \mathbb{T}$.

Consider the locally free $\mathcal{O}_S$-module $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}) := \mathbf{Lie}\,\mathbb{E}(\mathbb{M})_{\mathbb{G}}$, called the *de Rham realization* of $\mathbb{M}$. It is endowed with the *weight filtration*:

$$W_0\big(\mathbf{T}_{\mathrm{dR}}(\mathbb{M})\big) = \mathbf{T}_{\mathrm{dR}}(\mathbb{M}), \quad W_{-1}\big(\mathbf{T}_{\mathrm{dR}}(\mathbb{M})\big) = \mathbf{T}_{\mathrm{dR}}(\mathbb{E}(\mathbb{G})), \quad W_{-2}\big(\mathbf{T}_{\mathrm{dR}}(\mathbb{M})\big) = \mathbf{Lie}\,\mathbb{T}. \tag{2}$$

Its graded pieces are given by

$$\mathrm{Gr}_0\big(\mathbf{T}_{\mathrm{dR}}(\mathbb{M})\big) = \mathbf{T}_{\mathrm{dR}}\left(\mathbb{E}\left([\mathbb{X} \to 0]\right)\right) = \mathbb{X} \otimes \mathbf{G}_a, \quad \mathrm{Gr}_{-1}\big(\mathbf{T}_{\mathrm{dR}}(\mathbb{M})\big) = \mathbf{Lie}\,\mathbb{E}(\mathbb{A}), \qquad \mathrm{Gr}_{-2}\big(\mathbb{E}(\mathbb{M})\big) = \mathbf{Lie}\,\mathbb{T}.$$

Furthermore, $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ is endowed with the *Hodge filtration* putting

$$\mathrm{F}^{-1}\mathbf{T}_{\mathrm{dR}}(\mathbb{M}) = \mathbf{T}_{\mathrm{dR}}(\mathbb{M}), \quad \mathrm{F}^0\mathbf{T}_{\mathrm{dR}}(\mathbb{M}) = \mathbf{Lie}\,\mathbb{W}(\mathbb{M}), \quad \mathrm{F}^1\mathbf{T}_{\mathrm{dR}}(\mathbb{M}) = 0.$$

Let $\mathbb{M} := [\mathbb{X} \to \mathbb{G}]$ be a 1–motive over a scheme S and write $\mathbb{M}^\vee$ for its Cartier dual (cf. §2.4). We define the S-valued points of $\mathbb{M}$, denoted by $\mathbb{M}(S)$, to be the elements $x \in \mathrm{Hom}_{D^b(fppf)}(\mathbf{Z}, \mathbb{M})$ with $D^b(fppf)$ the bounded derived category of fppf sheaves on S. Denote by $\mathrm{Ext}^1_{1-\mathrm{Mot}}\big(\mathbb{M}^\vee, \mathbf{G}_m\big)$ the isomorphism classes of 1–motives over S which are extensions of $\mathbb{M}^\vee$ by $\mathbf{G}_m$. By Cartier duality we have an isomorphism of abelian groups

$$\mathcal{M}_{\mathbb{M}} \colon \mathbb{M}(S) \xrightarrow{\sim} \mathrm{Ext}^1_{1-\mathrm{Mot}}\big(\mathbb{M}^\vee, \mathbf{G}_m\big)$$

functorial in $\mathbb{M}$ and S; see §7.1 for details.

De Rham realization. Let $q \colon S \to T$ be a smooth morphism and consider a 1–motive $\mathbb{M}$ over S. One of the main results of this paper is the construction in §4.2 of a connection $\nabla_{\mathbb{M}}$, called the *Gauss-Manin connection*, on $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ which is functorial in $\mathbb{M}$ and preserves the weight filtration on $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$. Composing $\mathcal{M}_{\mathbb{M}}$ with the "de Rham realization" of a 1–motive, endowed with its Gauss-Manin connection, we get a homomorphism

$$\mathcal{M}_{\mathbb{M},\mathrm{dR}} \colon \mathbb{M}(S) \longrightarrow \mathrm{Ext}^1_S\left(\big(\mathbf{T}_{\mathrm{dR}}(\mathbb{M}^\vee), \nabla_{\mathbb{M}^\vee}\big), \big(\mathcal{O}_S, d\big)\right).$$

The key issue for the construction of a Gauss-Manin connection $\nabla_{\mathbb{M}}$ of $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ is the proof of the crystalline nature of the universal extension of $\mathbb{M}$; see Theorem 2.1 and Corollary 2.3. This was conjectured in some cases by Buium (see Proposition 4.7), and it was assumed without proof in [10, (2.2.2.1)]: "*La construction de* $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ *se généralise à un* 1*-motif lisse* $\mathbb{M}$ *au-dessus d'un schéma* S *; elle fournit un faisceau localement libre sur* S*, muni d'une connexion, la connexion de Gauss-Manin (ce dernier fait résulte de Mazur et Messing [M-M], qui doit permettre de définir* $M^\natural$*, comme faisceau en groupes sur le site cristallin)*".

Hodge realization. Let $q \colon S \to T = \mathrm{Spec}\,(\mathbf{C})$ be a smooth morphism. In Proposition 2.4 we construct a Hodge realization $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})$ which is a variation of MHS on S^{an} and is horizontal for $\nabla_{\mathbb{M}}$; see Proposition 2.4. This proves [10, (2.2.2.2)]:"*Si* $\mathbb{M}$ *est un* 1*-motif sur* $\mathbf{C}$*, il existe un isomorphisme canonique* $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}) \cong \mathbf{T}_{\mathbf{C}}(\mathbb{M})$*, compatible aux filtrations* F^0 *et* W*. La connexion de Gauss Manin est telle que dans une famille de* 1*-motifs lisse au-dessus d'un schéma* $S/\mathbf{C}$*, les sections continues de* $\mathbf{T}_{\mathbf{Q}}(\mathbb{M}_S)$ *sont horizontales pour cette connexion.*" Composing $\mathcal{M}_{\mathbb{M}}$ with $\mathbf{T}_{\mathbf{Z}}$ instead of $\mathbf{T}_{\mathrm{dR}}$ immediately gives the definition of a Hodge-Manin map

$$\mathcal{M}_{\mathbb{M},\mathbf{Z}} \colon \mathbb{M}(S) \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHS}/S^{\mathrm{an}}}\left(\mathbf{T}_{\mathbf{Z}}(\mathbb{M}^\vee), \mathbf{Z}\right)$$

compatible with $\mathcal{M}_{\mathbb{M},\mathrm{dR}}$ and functorial in $\mathbb{M}$ and S. Here, the target is the category of extensions of variations of MHS of $\mathbf{T}_{\mathbf{Z}}(\mathbb{M}^\vee)$ by $\mathbf{Z}$ on S^{an}.

Crystalline realizations. The motivic construction of Manin's map allows one to generalize it to the crystalline context. Let S be a scheme. We show in Corollary 2.3 that the crystalline nature of the universal extension of $\mathbb{M}$ provides a functor $\mathbf{T}_{\mathrm{crys}}$ from the category of 1–motives over S to the category of filtered crystals on the locally nilpotent crystalline site S^{crys}. Composing $\mathcal{M}_{\mathbb{M}}$ with $\mathbf{T}_{\mathrm{crys}}$ gives a homomorphism, the crystalline Manin map,

$$\mathcal{M}_{\mathbb{M},\mathrm{crys}}\colon \mathbb{M}(S) \longrightarrow \mathrm{Ext}^1_{S^{\mathrm{crys}}}\left(\mathbf{T}_{\mathrm{crys}}(\mathbb{M}^\vee), \mathcal{O}_S^{\mathrm{crys}}\right)$$

functorial in $\mathbb{M}$ and S, where the target category is the category of extensions of filtered crystals on S^{crys}.

One has variants of this definition. For example, one can consider the case in which $q\colon S \to \mathrm{Spec}\,(k)$ is a smooth scheme over a perfect field k of positive characteristic $p > 2$. Here one replaces S^{crys} with the site $\big(S/\mathbf{W}(k)\big)^{\mathrm{crys}}$ of locally nilpotent PD thickenings of S relative to $\mathbf{W}(k)$ with its natural divided power structure on the ideal $p\mathbf{W}(k)$. (The hypothesis on p assures that the natural divided power structure on $p\mathbf{W}(k)/p^n\mathbf{W}(k)$ is nilpotent for every n. This is needed to apply our results). Another option is to assume that S is a smooth (formal) scheme over $\mathbf{W}(k)$ and that $\mathbb{M}$ is a formal 1-motive over S. More precisely, assume that S is the limit of smooth schemes S_n over $\mathbf{W}_n(k)$ for $n \in \mathbb{N}$ so that $S_n \cong S_{n+1} \otimes_{\mathbf{W}_{n+1}(k)} \mathbf{W}_n(k)$. A formal 1-motive $\mathbb{M}$ over S is the datum of a 1-motive $\mathbb{M}_n$ over S_n with an isomorphism $\mathbb{M}_n \cong \mathbb{M}_{n+1} \otimes_{\mathbf{W}_{n+1}(k)} \mathbf{W}_n(k)$ for every $n \in \mathbb{N}$. We define $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ as the projective limit $\lim\limits_{\infty \leftarrow n} \mathbf{T}_{\mathrm{dR}}(\mathbb{M}_n)$. Then, $\mathbf{T}_{\mathrm{crys}}(\mathbb{M}_k)$ coincides with $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ so that it is endowed not only with the weight filtration, but also with the Hodge filtration defined by the inverse limit of the Hodge filtrations on each $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}_n)$. In particular, the target category of $\mathcal{M}_{\mathbb{M},\mathrm{crys}}$ is the category of extensions of crystals endowed with two filtrations.

Comparison with Manin's map. Suppose we are given an abelian scheme $\mathbb{A}$ over a smooth, connected affine curve S over a field k of characteristic 0. Assume we are given a point $x \in \mathbb{A}(S)$. Let $\mathbb{A}(x) := \big[\mathbf{Z} \to \mathbb{A}\big]$ be the 1–motive sending $1 \mapsto x$. Consider $\mathbf{T}_{\mathrm{dR}}\big(\mathbb{A}(x)^\vee\big)$, where $\mathbb{A}(x)^\vee$ is the Cartier dual of $\mathbb{A}(x)$, with its Gauss-Manin connection. It is an extension of $\mathbf{T}_{\mathrm{dR}}\big(\mathbb{A}^\vee\big)$ by $\mathbf{T}_{\mathrm{dR}}([\mathbf{Z} \to 0]^\vee)$. Note that $\mathbf{T}_{\mathrm{dR}}\big(\mathbb{A}^\vee\big)$ is identified with $\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S)$, with its Gauss-Manin connection by Lemma 4.5, while $\mathbf{T}_{\mathrm{dR}}([\mathbf{Z} \to 0]^\vee) = \mathbf{T}_{\mathrm{dR}}\big(\mathbf{G}_{m,S}\big)$ coincides with $\mathcal{O}_S$ with its derivation by Example 4.2. In particular, we get an element

$$\mathcal{M}_{\mathbb{A},\mathrm{dR}}\big(x\big) \in \mathrm{Ext}^1_S\left(\big(\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S), \nabla_{\mathbb{A}}\big), \big(\mathcal{O}_S, d\big)\right).$$

We then prove (see §7)

Proposition 1.1. *For every $x \in \mathbb{A}(S)$ we have $\mathcal{M}_{\mathbb{A},\mathrm{Man}}(x) = \mathcal{M}_{\mathbb{A},\mathrm{dR}}\big(x\big)$.*

Here, $\mathcal{M}_{\mathbb{A},\mathrm{Man}}$ is Manin's original map (1). To prove the comparison with Coleman's construction in [12, §4], we need a compatibility with dualities. Indeed, given a 1–motive $\mathbb{M}$ over S one has a natural perfect pairing between $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ and $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}^\vee)$ (or more generally for their crystalline variants). One needs to show that such duality preserves the Gauss-Manin connections (resp. provides a pairing of crystals). This follows from the fact that the duality too is crystalline in nature. See Theorem 2.5, Corollary 2.8 for the implication regarding the connections and Corollary 2.6 for its crystalline counterpart.

We remark that the Hodge realization appears already in [12, I §6] for the 1–motive $\mathbb{A}(x)$.

2. The results

2.1. Universal extension crystals

Let S_0 be a scheme and let $S_0 \hookrightarrow S$ be a locally nilpotent PD thickening of S_0: it is a closed immersion defined by an ideal sheaf $\mathcal{I}$ endowed with a locally nilpotent divided powers structure $\{\gamma_n\colon \mathcal{I} \to \mathcal{I}\}_n$. We will denote it by $(S_0 \subset S)$ in the future. Given a morphism of abelian schemes $\beta_0\colon \mathbb{A}_0 \to \mathbb{B}_0$ over S_0 and given deformations $\mathbb{A}$ and $\mathbb{B}$ of $\mathbb{A}_0$ and $\mathbb{B}_0$ respectively as abelian schemes over S, in [19, §II.1] a canonical morphism of S-group schemes

$$\mathbb{E}_S(\beta_0)\colon \mathbb{E}\,(\mathbb{A}) \longrightarrow \mathbb{E}\,(\mathbb{B}) \tag{3}$$

is constructed such that (i) it is functorial in β_0, (ii) it is additive in β_0, (iii) it is functorial w.r.t. divided powers morphisms $(S_0 \subset S) \to (S_0 \subset T)$, (iv) if there exists a morphism of abelian schemes $\beta\colon \mathbb{A} \to \mathbb{B}$

deforming β_0, then $\mathbb{E}_S(\beta_0)$ is the morphism $\mathbb{E}(\mathbb{A}) \to \mathbb{E}(\mathbb{B})$ induced by β. See Theorem 2.1 for a precise formulation of these properties.

Let now $s_0 \colon \mathbb{M}_0 \to \mathbb{N}_0$ be a morphism of 1-motives over S_0. Let $\mathbb{M} = [\mathbb{X} \to \mathbb{G}]$ and $\mathbb{N} := [\mathbb{Y} \to \mathbb{H}]$ be two 1-motives over S deforming $\mathbb{M}_0$ and $\mathbb{N}_0$ respectively. Let $\mathbb{G}$ be an extension of the abelian scheme $\mathbb{A}$ by the torus $\mathbb{T}$ and let $\mathbb{H}$ be the extension of the abelian scheme $\mathbb{B}$ by the torus $\mathbb{T}'$. Then, s_0 induces maps $\alpha_0 \colon \mathbb{X}_0 \to \mathbb{Y}_0$, $\gamma_0 \colon \mathbb{T}_0 \to \mathbb{T}'_0$ and $\beta_0 \colon \mathbb{A}_0 \to \mathbb{B}_0$. Note that the category of étale group schemes over S is equivalent to the category of étale group schemes over S_0. Since a torus is determined by its character group, this also shows that the category of tori over S is equivalent to the category of tori over S_0. Hence, we get unique homomorphisms of S-group schemes

$$\alpha \colon \mathbb{X} \longrightarrow \mathbb{Y}, \qquad \gamma \colon \mathbb{T} \longrightarrow \mathbb{T}', \tag{4}$$

whose base change to S_0 induce the morphisms α_0 and γ_0 respectively.

Theorem 2.1. *Let notations be as above. There is a canonical morphism of complexes of group-schemes over S*

$$\mathbb{E}_S(s_0) \colon \mathbb{E}(\mathbb{M}) \longrightarrow \mathbb{E}(\mathbb{N})$$

which coincides with the morphism defined in (3) *when $\mathbb{X}_0 = \mathbb{Y}_0 = 0$ and $\mathbb{T}_0 = \mathbb{T}'_0 = 0$, i.e., for morphisms of abelian schemes, and furthermore,*

(i) $\mathbb{E}_S(s_0)$ is functorial in s_0, i.e., given morphisms of 1–motives $s_0 \colon \mathbb{M}_0 \to \mathbb{M}'_0$ and $q_0 \colon \mathbb{M}'_0 \to \mathbb{M}''_0$ we have $\mathbb{E}_S(q_0 \circ s_0) = \mathbb{E}_S(q_0) \circ \mathbb{E}_S(s_0)$;

(ii) $\mathbb{E}_S(s_0)$ is additive in s_0, i.e., given morphisms of 1–motives $s_0 \colon \mathbb{M}_0 \to \mathbb{M}'_0$ and $q_0 \colon \mathbb{M}_0 \to \mathbb{M}'_0$, we have $\mathbb{E}_S(s_0 + q_0) = \mathbb{E}_S(s_0) + \mathbb{E}_S(q_0)$;

(iii) it is functorial w.r.t. divided powers morphisms $(S_0 \subset S) \to (S_0 \subset T)$, i.e., given any such morphism ϕ and given deformations $\mathbb{M}$ and $\mathbb{N}$ of $\mathbb{M}_0$ and $\mathbb{N}_0$ over T, the map of S-schemes

$$\mathbb{E}_S(s_0) \colon \mathbb{E}(\mathbb{M} \times_T S) \to \mathbb{E}(\mathbb{N} \times_T S)$$

is the base change via ϕ of the map of T-schemes $\mathbb{E}_T(s_0) \colon \mathbb{E}(\mathbb{M}) \to \mathbb{E}(\mathbb{N})$;

(iv) if there exists a morphism of 1–motives $s \colon \mathbb{M} \to \mathbb{N}$ deforming s_0 then $\mathbb{E}_S(s_0)$ is the morphism of universal extensions $\mathbb{E}(\mathbb{M}) \to \mathbb{E}(\mathbb{N})$ induced by s.

This theorem answers a question raised by Buium; see Proposition 4.7. The crystalline nature of the universal extension of a 1–motive was already established in [1, §3]. Unfortunately, despite the fact that in loc. cit. the statements are given for any nilpotent PD thickening $(S_0 \subset S)$, the proofs in loc. cit. do not work in general but only work under the assumption that S_0 is of characteristic 0 or that we work in a p-adic context as in §6(∗). This second case is sufficient for the applications in [1]. See Remarks 3.4 and 5.5 for explanations of where the problem lies and Proposition 6.2 for a proof that no problem appears if we have the stronger assumptions imposed above.

Remark 2.2. *(a) It follows from (i) that $\mathbb{E}_S(s_0)$ respects the weight filtrations, i.e., for every $i \in \mathbf{N}$, one has $\mathbb{E}_S(s_0)\big(W_{-i}(\mathbb{E}(\mathbb{M}))\big) \subseteq W_{-i}(\mathbb{E}(\mathbb{N}))$. It follows from (iv) that the morphism $\mathbb{E}[\mathbb{X} \to 0] \to \mathbb{E}[\mathbb{Y} \to 0]$ induced by $\mathbb{E}_S(s_0)$ is the one associated to the map α defined in* (4). *Since the category of tori over S is equivalent to the category of tori over S_0, the morphism $\mathbb{T} \to \mathbb{T}'$ induced by $\mathbb{E}_S(s_0)$ is the morphism γ defined in* (4). *Notice that we require that the morphism $\mathbb{E}(\mathbb{A}) \to \mathbb{E}(\mathbb{B})$, induced by $\mathbb{E}_S(s_0)$, coincides with that of* (3). *We conclude that the behavior of $\mathbb{E}_S(s_0)$ on the graded pieces is uniquely determined.*

(b) It follows from (iii) applied to the morphism $(S_0 \subset S_0) \to (S_0 \subset S)$ and from (iv) applied to $s_0 \colon \mathbb{M}_0 \to \mathbb{N}_0$ that the base change via $S_0 \to S$ of the morphism $\mathbb{E}_S(s_0) \colon \mathbb{E}(\mathbb{M}) \to \mathbb{E}(\mathbb{N})$, as in the statement of the Theorem, is the morphism $\mathbb{E}_{S_0}(s_0) \colon \mathbb{E}(\mathbb{M}_0) \to \mathbb{E}(\mathbb{N}_0)$ over S_0 induced by s_0 and the functoriality of the universal extension.

(c) *There is a converse to (iv). In fact, $\mathbb{E}_S(s_0)$ arises from a morphism of 1–motives $s\colon \mathbb{M}\to\mathbb{N}$ if and only if $\mathbb{E}_S(s_0)\big(\mathbb{W}(\mathbb{M})\big)\subseteq\mathbb{W}(\mathbb{N})$. As in the case of abelian schemes, this provides a way of studying the deformation theory of 1–motives via the deformations of the Hodge filtration.*

As a first consequence of Theorem 2.1 we get the following corollary, which will be proved in §4.1:

Corollary 2.3. *Let S_0 be a scheme and denote by S_0^{crys} the locally nilpotent crystalline site of S_0. Then, there exists an additive exact functor $\mathbf{T}_{\mathrm{crys}}$ from the category of 1-motives over S_0 to the category of filtered crystals of $\mathcal{O}_{S_0}^{\mathrm{crys}}$-modules such that*

(i) *given $(U\subset Z)\in S_0^{\mathrm{crys}}$, a 1-motive $\mathbb{M}_0$ over S_0 and a 1–motive $\mathbb{M}$ over Z deforming (the restriction to U of) $\mathbb{M}_0$, we have $\mathbf{T}_{\mathrm{crys},S_0}\big(\mathbb{M}_0\big)(U\subset Z)=\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$;*

(ii) *given $(U\subset Z)\in S_0^{\mathrm{crys}}$, a morphism of 1–motives $s_0\colon \mathbb{M}_0\to\mathbb{N}_0$ over S_0 and a morphism of 1–motives $s\colon\mathbb{M}\to\mathbb{N}$ over Z deforming s_0 (restricted to U), the morphism $\mathbf{T}_{\mathrm{crys}}\big(s_0\big)(U\subset Z)$ is induced by the morphism of Lie algebras $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})\to\mathbf{T}_{\mathrm{dR}}(\mathbb{N})$ defined by s.*

2.2. Gauss-Manin connection of 1–motives

Let $q\colon S\to T$ be a morphism of schemes and let $\mathbb{M}$ be a 1–motive over S. As a corollary of Theorem 2.1 we will construct in §4.2 a connection, called the *Gauss-Manin connection*, associated to a 1–motive $\mathbb{M}$ over S:

$$\nabla_{\mathbb{M}}\colon \mathbf{T}_{\mathrm{dR}}(\mathbb{M})\longrightarrow\mathbf{T}_{\mathrm{dR}}(\mathbb{M})\otimes_{\mathcal{O}_S}\Omega^1_{S/T}.$$

We prove that it is functorial in $\mathbb{M}$ and it preserves the weight filtration on $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$. Furthermore, it is integrable if q is a smooth morphism. Its existence was claimed, without proof, in [10, (2.2.2.1)].

2.3. Hodge realization as variation of MHS

Assume next that S is a scheme smooth over Spec $(\mathbf{C})$. Let S^{an} be the associated complex analytic space. Let $\mathbb{M}$ be a 1–motive over S. Consider a point $x\in S^{\mathrm{an}}$. In [14, §10.1.3] the Hodge realization $\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_x)$ of the 1–motive $\mathbb{M}_x$, defined over $\mathbf{C}$, is constructed. It is a mixed Hodge structure without torsion, of type $\{(0,0),(-1,0),(0,-1),(-1,-1)\}$. Its graded pieces for the weight filtration are

$$\mathrm{Gr}_{-2}\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_x)=\mathrm{H}_1(\mathbb{T}_x,\mathbf{Z})=\mathbb{Y}_x,\quad \mathrm{Gr}_{-1}\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_x)=\mathrm{H}_1(\mathbb{A}_x,\mathbf{Z}),\mathrm{Gr}_{0}\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_x)=\mathbb{X}_x.$$

As in loc. cit. put $\mathbf{T}_{\mathbf{C}}(\mathbb{M}_x):=\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_x)\otimes_{\mathbf{Z}}\mathbf{C}$. By [14, §10.1.8] we have a comparison isomorphism $\eta_{\mathbb{M}_x}\colon \mathbf{T}_{\mathbf{C}}(\mathbb{M}_x)\cong\mathbf{T}_{\mathrm{dR}}(\mathbb{M}_x)$ compatible with the weight and the Hodge filtrations. In §4.3 we prove the following:

Proposition 2.4. *There is a local system of torsion free $\mathbf{Z}$-modules $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})$ over S^{an}, endowed with a weight filtration defined by locally constant $\mathbf{Z}$-submodules, and an isomorphism $\eta_{\mathbb{M}}\colon\mathbf{T}_{\mathbf{Z}}(\mathbb{M})\otimes_{\mathbf{Z}}\mathcal{O}_{S^{\mathrm{an}}}\cong\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ of $\mathcal{O}_{S^{\mathrm{an}}}$-modules, compatible with the weight filtration, such that*

(i) *for every $x\in S^{\mathrm{an}}$ we have an isomorphism $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})_x\cong\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_x)$, compatible with the weight filtrations, and, via this identification, we have $\big(\eta_{\mathbb{M}}\big)_x=\eta_{\mathbb{M}_x}$;*

(ii) *$\mathbf{T}_{\mathbf{Z}}(\mathbb{M})$ and the isomorphism $\eta_{\mathbb{M}}$ are functorial in $\mathbb{M}$;*

(iii) *$\mathbf{T}_{\mathbf{Z}}(\mathbb{M})\subset\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ is horizontal for the Gauss-Manin connection $\nabla_{\mathbb{M}}$;*

(iv) *the functor $\mathbf{T}_{\mathbf{Z}}$ is exact.*

In particular, $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})$ is a *variation of mixed Hodge structures*, called the *Hodge realization* of $\mathbb{M}$. In fact, due to (i) every fiber is endowed with the structure of a MHS in such a way that the weight filtration is locally constant and the Hodge filtration, defined using (iii) and the Hodge filtration on $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$, varies holomorphically over S^{an}. The fact that the connection on $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})\otimes_{\mathbf{Z}}\mathcal{O}_{S^{\mathrm{an}}}$, having $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})$ as horizontal sections, satisfies Griffiths' transversality for the Hodge filtration is clear since the Hodge filtration has only two steps; see the introduction.

Claim (iii) proves [10, (2.2.2.2)].

Claim (iv) allows one to define the Hodge-Manin map $\mathcal{M}_{\mathbb{M},\mathbf{Z}}$ by passing to Hodge realizations in $\mathcal{M}_{\mathbb{M}}$.

2.4. Duality results

We come to the second kind of results regarding dualities. Let $[\mathrm{u}\colon \mathbb{X} \to \mathbb{G}] = \mathbb{M}$ be a 1-motive over S. Its dual 1-motive $\mathbb{M}^\vee = [\mathrm{u}'\colon \mathbb{X}' \to \mathbb{G}']$ is defined as follows (see also [14]): $\mathbb{G}'$ is the smooth group scheme over S that represents the fppf sheaf $\underline{\mathrm{Ext}}^1([\mathbb{X} \to \mathbb{A}], \mathbf{G}_{m,S})$, where the homomorphism $\mathbb{X} \to \mathbb{A}$ is the composition of u with $\mathbb{G} \to \mathbb{A}$; the group $\mathbb{G}'$ is an extension of the dual abelian scheme $\mathbb{A}^\vee$ by the torus $\mathbb{T}'$ whose character group is $\mathbb{X}$. The étale group $\mathbb{X}' = \underline{\mathrm{Hom}}(\mathbb{T}, \mathbf{G}_{m,S})$ is the group of characters of the maximal subtorus $\mathbb{T}$ of $\mathbb{G}$; the homomorphism u' is the first boundary map of the long $\underline{\mathrm{Ext}}(-, \mathbf{G}_{m,S})$ sequence of $0 \to \mathbb{T} \to \mathbb{M} \to [\mathbb{X} \to \mathbb{A}] \to 0$.

Assume as before that $S_0 \hookrightarrow S$ is a closed immersion defined by locally nilpotent divided powers. The main theorem, to be proved in §5.4, is the following:

Theorem 2.5. *Let $s_0\colon \mathbb{M}_0 \to \mathbb{N}_0$ be a morphism of 1–motives over S_0 and denote by $s_0^\vee\colon \mathbb{N}_0^\vee \to \mathbb{M}_0^\vee$ its dual morphism. Let $\mathbb{M}$ and $\mathbb{N}$ be 1–motives over S deforming $\mathbb{M}_0$ and $\mathbb{N}_0$ respectively. Then, there is an isomorphism*

$$\rho_S(s_0)\colon \left(\mathrm{Id} \times \mathbb{E}_S(s_0^\vee)\right)^* \left(\mathcal{P}_{\mathbb{E}(\mathbb{M})}\right) \xrightarrow{\sim} \left(\mathbb{E}_S(s_0) \times \mathrm{Id}\right)^* \left(\mathcal{P}_{\mathbb{E}(\mathbb{N})}\right)$$

as $\mathbf{G}_{m,S}$-biextensions of $\mathbb{E}(\mathbb{M})$ and $\mathbb{E}(\mathbb{N}^\vee)$ such that

(i) $\rho_S(s_0)$ is functorial in s_0, i.e., given morphisms of 1–motives $s_0\colon \mathbb{M}_0 \to \mathbb{M}_0'$, $q_0\colon \mathbb{M}_0' \to \mathbb{M}_0''$ and corresponding deformations $\mathbb{M}$, $\mathbb{M}'$ and $\mathbb{M}''$ as 1–motives over S we have

$$\rho_S(q_0 \circ s_0) = \left(\mathbb{E}_S(s_0) \times \mathrm{Id}\right)^* \left(\rho_S(q_0)\right) \circ \left(\mathrm{Id} \times \mathbb{E}_S(q_0^\vee)\right)^* \left(\rho_S(s_0)\right);$$

(ii) it is additive in s_0, i.e., given morphisms of 1–motives s_0, $q_0\colon \mathbb{M}_0 \to \mathbb{M}_0'$, we have $\rho_S(s_0 + q_0) = \rho_S(s_0) \otimes \rho_S(q_0)$;

(iii) it is functorial w.r.t. divided powers morphisms $(S_0 \subset S) \to (S_0 \subset T)$, i.e., given any such morphism ϕ and given deformations $\mathbb{M}$ and $\mathbb{N}$ of $\mathbb{M}_0$ and $\mathbb{N}_0$ over T the map of S-schemes $\rho_S(s_0)$ is the base change via ϕ of the map of T-schemes $\rho_T(s_0)$;

(iv) if there exists a morphism of 1-motives $s\colon \mathbb{M} \to \mathbb{N}$ over S deforming s_0 then $\rho_S(s_0)$ is the morphism $\rho(s)$ induced by s (see §5.1); in particular $\rho_S(\mathrm{Id}_{\mathbb{M}}) = \mathrm{Id}_{\mathcal{P}_{\mathbb{E}(\mathbb{M})}}$.

To make sense of Properties (i)–(iv) one needs to use the properties analogous to those stated in Theorem 2.1 for $\mathbb{E}_S(s_0)$ and $\mathbb{E}_S(s_0^\vee)$. In (ii) we use that given a biextension P of group schemes A and B by G and given group homomorphisms $f, g\colon C \to B$, we then have a canonical isomorphism

$$\left(\mathrm{Id}_A \times (f+g)\right)^*(P) \cong \left(\mathrm{Id}_A \times f\right)^*(P) + \left(\mathrm{Id}_A \times g\right)^*(P);$$

see Lemma 5.9. As a first corollary in §5.5 we obtain:

Corollary 2.6. *Given a morphism of 1–motives $s_0\colon \mathbb{M}_0 \to \mathbb{N}_0$ over S_0, there exists a bilinear pairing of filtered crystals in $\mathcal{O}_{S_0}^{\mathrm{crys}}$-modules*

$$\langle\, ,\, \rangle_{s_0}\colon \mathbf{T}_{\mathrm{crys}}\left(\mathbb{M}_0\right) \otimes_{\mathcal{O}_{S_0}^{\mathrm{crys}}} \mathbf{T}_{\mathrm{crys}}\left(\mathbb{N}_0^\vee\right) \to \mathbf{T}_{\mathrm{crys}}\left(\mathbf{G}_{m,S_0}\right) = \mathcal{O}_{S_0}^{\mathrm{crys}}$$

such that

(i) it is functorial in s_0, i.e., given a morphism of 1–motives $q_0\colon \mathbb{N}_0 \to \mathbb{N}_0'$ over S_0 then $\langle\, ,\, \rangle_{q_0 \circ s_0}$ is the morphism $\langle\, ,\, \rangle_{s_0} \circ (\mathrm{Id} \otimes \mathbf{T}_{\mathrm{crys}}(q_0^\vee))$ (resp. given $q_0\colon \mathbb{M}_0' \to \mathbb{M}_0$ then $\langle\, ,\, \rangle_{s_0 \circ q_0}$ is the morphism $\langle\, ,\, \rangle_{s_0} \circ (\mathbf{T}_{\mathrm{crys}}(q_0) \otimes \mathrm{Id})$);

(ii) given an object $(U \subset Z)$ in S_0^{crys} and if s_0 restricted to U deforms to a morphism of 1–motives $s\colon \mathbb{M} \to \mathbb{N}$ over Z, then $\langle\, ,\, \rangle_{s_0}(U \subset Z)$ is the canonical pairing (see (14) in §5.5) via the identification

$$\mathbf{T}_{\mathrm{crys}}\left(\mathbb{M}_0\right) \otimes_{\mathcal{O}_{S_0}^{\mathrm{crys}}} \mathbf{T}_{\mathrm{crys}}\left(\mathbb{N}_0^\vee\right)(U \subset Z) \cong \mathbf{T}_{\mathrm{dR}}\left(\mathbb{M}\right) \otimes_{\mathcal{O}_Z} \mathbf{T}_{\mathrm{dR}}\left(\mathbb{N}^\vee\right)(Z)$$

of Corollary 2.3.

Remark 2.7. *(a) To say that $\langle\, ,\, \rangle_{s_0}$ is a bilinear map of filtered crystals amounts to affirming that*

a.1) for every object $(U \subset Z)$ in S_0^{crys}, then $\langle\, ,\, \rangle_{s_0}(U \subset Z)$ is $\Gamma(Z, \mathcal{O}_Z)$-bilinear;

a.2) given a morphism of objects $(U \subset Z) \to (U' \subset Z')$ in S_0^{crys}, then the pairing $\langle\, ,\, \rangle_{s_0}(U \subset Z)$ is the base change of $\langle\, ,\, \rangle_{s_0}(U' \subset Z')$ to Z;

a.3) the pairing $\langle\, ,\, \rangle_{s_0}$ respects filtrations, i.e., $\langle W_i \mathbf{T}_{\mathrm{crys}}(\mathbb{M}_0) \otimes_{\mathcal{O}_{S_0}^{\mathrm{crys}}} W_j \mathbf{T}_{\mathrm{crys}}(\mathbb{N}_0^\vee)\rangle_{s_0}$ is contained in $W_{i+j}\mathbf{T}_{\mathrm{crys}}(\mathbf{G}_m)$; hence it is 0 if $i+j<-2$ and it induces a pairing on graded pieces

$$\mathrm{Gr}_i \mathbf{T}_{\mathrm{crys}}(\mathbb{M}_0) \otimes \mathrm{Gr}_j \mathbf{T}_{\mathrm{crys}}(\mathbb{N}_0^\vee) \longrightarrow \mathrm{Gr}_{-2}\mathbf{T}_{\mathrm{crys}}(\mathbf{G}_{m,0}) = \mathcal{O}_{S_0}^{\mathrm{crys}}$$

if $i+j=-2$.

(b) It follows from (i), taking q_0 to be the identity, that $\mathbf{T}_{\mathrm{crys}}(s_0)$ and $\mathbf{T}_{\mathrm{crys}}(s_0^\vee)$ are transpose one of the other w.r.t. $\langle\, ,\, \rangle_{\mathrm{Id}_{\mathbb{M}_0}}$ and $\langle\, ,\, \rangle_{\mathrm{Id}_{\mathbb{N}_0}}$.

(c) It follows from (ii) applied to $U = S_0$ and $Z = S_0$ and from Section 5.5 that the pairing $\langle\, ,\, \rangle_{s_0}(S_0 = S_0)$ coincides with Deligne's pairing (14) *on $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}_0) \otimes_{\mathcal{O}_{S_0}} \mathbf{T}_{\mathrm{dR}}(\mathbb{N}_0^\vee)$; see [5, §4]. In particular, if $s_0 = \mathrm{Id}_{\mathbb{M}_0}$, it is perfect.*

(d) One deduces from (c) and (a.2) that $\langle\, ,\, \rangle_{\mathrm{Id}_{\mathbb{M}_0}}$ is perfect on the Zariski site S_0^{Zar} of S_0. Since a morphism of crystals over S_0^{crys}, which is an isomorphism when restricted to S_0^{Zar}, is an isomorphism, we conclude that $\langle\, ,\, \rangle_{\mathrm{Id}_{\mathbb{M}_0}}$ is a perfect pairing.

Let $q\colon S \to T$ be a morphism of schemes and let $s\colon \mathbb{M} \to \mathbb{N}$ be a morphism of 1–motives over S. Then, s induces Deligne's pairing $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}) \otimes_{\mathcal{O}_S} \mathbf{T}_{\mathrm{dR}}(\mathbb{N}^\vee) \to \mathcal{O}_S$, see [5, §4], and, hence, a morphism of $\mathcal{O}_S$-modules

$$\nu_s\colon \mathbf{T}_{\mathrm{dR}}(\mathbb{N}^\vee) \to \mathbf{T}_{\mathrm{dR}}(\mathbb{M})^\vee := \underline{\mathrm{Hom}}_{\mathcal{O}_S}\big(\mathbf{T}_{\mathrm{dR}}(\mathbb{M}), \mathcal{O}_S\big).$$

As a corollary of Theorem 2.5 we also prove in §5.6:

Corollary 2.8. *The morphism of $\mathcal{O}_S$-modules ν_s is horizontal considering the Gauss-Manin connection $\nabla_{\mathbb{N}^\vee}$ on $\mathbf{T}_{\mathrm{dR}}(\mathbb{N}^\vee)$ and the connection dual to the Gauss-Manin connection $\nabla_{\mathbb{M}}$ on $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})^\vee$.*

This gives a positive answer to the question of D. Bertrand on whether the Gauss-Manin connection on $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}^\vee)$ is the adjoint of the Gauss-Manin connection on $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ via Deligne's pairing (14). This fact is used in [9].

3. The crystalline nature of $\mathbb{E}(\mathbb{M})$

Let $s_0 := \mathbb{M}_0 \longrightarrow \mathbb{N}_0$ be a homomorphism of 1–motives over S_0. Write $\mathbb{M}_0 := [\mathbb{X}_0 \to \mathbb{G}_0]$ and $\mathbb{N}_0 := [\mathbb{Y}_0 \to \mathbb{H}_0]$ where $\mathbb{G}_0$ and $\mathbb{H}_0$ are semiabelian schemes over S_0 extensions of an abelian scheme by a torus. Let $\mathrm{u}\colon \mathbb{X} \to \mathbb{G}$ and $\mathrm{u}'\colon \mathbb{Y} \to \mathbb{H}$ be 1–motives over S deforming $\mathrm{u}_0\colon \mathbb{X}_0 \to \mathbb{G}_0$ and $\mathrm{u}'_0\colon \mathbb{Y}_0 \to \mathbb{H}_0$ respectively. Let $\mathbb{M}^\vee = [\mathrm{u}^\vee\colon \mathbb{X}' \to \mathbb{G}']$, $\mathbb{N}^\vee = [\mathrm{u}'^\vee\colon \mathbb{Y}' \to \mathbb{H}']$ be the Cartier duals of $\mathbb{M}$ and $\mathbb{N}$ respectively.

Let $(S_0 \subset S)$ be a locally nilpotent PD thickening and G_0 a commutative group scheme over S_0. For $\mathcal{M}$ and $\mathcal{L}$ invertible sheaves on G_0 (resp. $\mathbf{G}_m$-torsors on G_0, resp. invertible modules on the nilpotent crystalline site of G_0 relative to S), denote by $\mathrm{Isom}\big(\mathcal{M}, \mathcal{L}\big)$ the group of isomorphisms between $\mathcal{M}$ and $\mathcal{L}$.

3.1. Crystals and sheaves with connections

Let the notations be as in §2.1. In order to construct the morphism $\mathbb{E}_S(s_0)$ in Theorem 2.1, we need to interpret $\natural$-extensions as invertible modules on the nilpotent crystalline site. We recall that a $\natural$-extension of $\mathbb{M}^\vee$ by $\mathbf{G}_{m,S}$ is a $\mathbf{G}_m$-extension $[\mathbb{X}' \to E]$ of $\mathbb{M}^\vee$ where E is endowed with an integrable connection as $\mathbf{G}_m$-torsor over $\mathbb{G}'$ and the canonical isomorphism $m^*(E) \cong p_1^*(E)p_2^*(E)$ is horizontal (cf. [14, 10.2.7.1], [5, 3.7]). As usual m denotes the multiplication on $\mathbb{G}'$. We then have the following equivalence of commutative Picard categories, i.e., categories $\mathcal{C}$ such that every morphism is an isomorphism (groupoid) with a functor $+\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the usual constrains of associativity, commutativity, existence of unity and opposite elements:

($\mathcal{C}_1$) The category $\mathrm{EXT}^\natural(\mathbb{M}^\vee, \mathbf{G}_{m,S})$ of $\natural$-extensions $(\mathbb{L}, \nabla)$ of $\mathbb{M}^\vee$ by $\mathbf{G}_{m,S}$;

($\mathcal{C}_2$) The category of quadruples $(\mathbb{L}, \nabla_\mathbb{L}, \iota_\mathbb{L}, j_\mathbb{L})$ where $\mathbb{L}$ is a $\mathbf{G}_m$-torsors over $\mathbb{G}'$, $\nabla_\mathbb{L}$ is an integrable connection on $\mathbb{L}$, $\iota_\mathbb{L}\colon m^*(\mathbb{L}) \cong p_1^*(\mathbb{L})p_2^*(\mathbb{L})$ is a horizontal isomorphism of torsors over $\mathbb{G}' \times_S \mathbb{G}'$ and

$$j_\mathbb{L}\colon \mathbb{L} \times_{\mathbb{G}'} \mathbb{X}' := \mathbb{L}_{\mathbb{X}'} \longrightarrow \mathbf{G}_{m,\mathbb{X}'} = \mathbf{G}_{m,S} \times_S \mathbb{X}'$$

is an isomorphism of $\mathbf{G}_m$-torsors over $\mathbb{X}'$ such that its image via the morphism

$$\mathrm{Isom}\big(\mathbb{L}_{\mathbb{X}'}, \mathbf{G}_{m,\mathbb{X}'}\big) \to \mathrm{Isom}\Big(m^*\big(\mathrm{u}^{\vee *}\mathbb{L}\big)p_1^*\big(\mathrm{u}^{\vee *}\mathbb{L}\big)^{-1}p_2^*\big(\mathrm{u}^{\vee *}\mathbb{L}\big)^{-1}, \mathbf{G}_{m,\mathbb{X}'\times\mathbb{X}'}\Big)$$

is induced by the pull-back along $\mathrm{u}^\vee$ of $\iota_\mathbb{L}$;

($\mathcal{C}_3$) The category of quadruples $\big(\mathcal{L}, \nabla_\mathcal{L}, \iota_\mathcal{L}, j_\mathcal{L}\big)$ where $\mathcal{L}$ is an invertible sheaf on $\mathbb{G}'$, $\nabla_\mathcal{L}$ is an integrable connection on $\mathcal{L}$,

$$\iota_\mathcal{L}\colon m^*\big(\mathcal{L}\big) \otimes p_1^*\big(\mathcal{L}\big)^{-1} \otimes p_2^*\big(\mathcal{L}\big)^{-1} \longrightarrow \mathcal{O}_{\mathbb{G}'\times_S\mathbb{G}'}$$

is a horizontal isomorphism and $j_\mathcal{L}\colon \mathrm{u}^{\vee *}\mathcal{L} \to \mathcal{O}_{\mathbb{X}'}$ is an isomorphism such that its image via the morphism

$$\mathrm{Isom}\big(\mathrm{u}^{\vee *}\mathcal{L}, \mathcal{O}_{\mathbb{X}'}\big) \to \mathrm{Isom}\Big(m^*\big(\mathrm{u}^{\vee *}\mathcal{L}\big) \otimes p_1^*\big(\mathrm{u}^{\vee *}\mathcal{L}\big)^{-1} \otimes p_2^*\big(\mathrm{u}^{\vee *}\mathcal{L}\big)^{-1}, \mathcal{O}_{\mathbb{X}'\times_S\mathbb{X}'}\Big),$$

is induced by the pull-back along $\mathrm{u}^\vee$ of $\iota_\mathcal{L}$;

($\mathcal{C}_4$) The category of triples $\big(\mathcal{L}, \iota_\mathcal{L}, j_\mathcal{L}\big)$ where $\mathcal{L}$ is an invertible module on the nilpotent crystalline site of $\mathbb{G}_0'$ relative to S,

$$\iota_\mathcal{L}\colon m^*\big(\mathcal{L}\big) \otimes p_1^*\big(\mathcal{L}\big)^{-1} \otimes p_2^*\big(\mathcal{L}\big)^{-1} \longrightarrow \mathcal{O}_{(\mathbb{G}_0'\times_{S_0}\mathbb{G}_0')/S,\mathrm{crys}}$$

is an isomorphism of crystals and $j_\mathcal{L}\colon \mathrm{u}_0^{\vee *}\mathcal{L} \to \mathcal{O}_{\mathbb{X}_0/S,\mathrm{crys}}$ is an isomorphism such that its image via the morphism

$$\mathrm{Isom}\big(\mathrm{u}_0^{\vee *}\mathcal{L}, \mathcal{O}_{\mathbb{X}_0/S,\mathrm{crys}}\big) \to \mathrm{Isom}\Big(m^*\big(\mathrm{u}_0^{\vee *}\mathcal{L}\big) \otimes p_1^*\big(\mathrm{u}_0^{\vee *}\mathcal{L}\big)^{-1} \otimes p_2^*\big(\mathrm{u}_0^{\vee *}\mathcal{L}\big)^{-1}, \mathcal{O}_{(\mathbb{X}_0\times_{S_0}\mathbb{X}_0)/S,\mathrm{crys}}\Big),$$

is induced by the pull-back along $\mathrm{u}_0^\vee$ of $\iota_\mathcal{L}$. Observe that by [7, 6.7] we could write $\mathbb{G}'$, $\mathbb{X}'$ in place of $\mathbb{G}_0'$, $\mathbb{X}_0'$.

The category ($\mathcal{C}_4$) is equivalent to the category $\mathrm{EXT}^{\mathrm{crys}/S}(\mathbb{M}_0^\vee, \mathbf{G}_{m,S_0})$ by [19, 6.8 i)] and then equivalent to the category ($\mathcal{C}_1$) by [19, 6.12]. The equivalence of ($\mathcal{C}_1$) and ($\mathcal{C}_2$), where the group structure is given by the Baer sum, holds by definition of $\natural$-extensions; see [14, 10.2.7.1]. For the equivalence of ($\mathcal{C}_2$) and ($\mathcal{C}_3$) as Picard categories, considering on the latter the usual product, see for example the more general [19, II.6.8(i)]. The equivalence of ($\mathcal{C}_3$) and ($\mathcal{C}_4$) holds by [7, 6.8] or [6, IV, 1.6.5]. In particular, passing to isomorphism classes, we get isomorphisms between the groups of isomorphism classes of $\natural$-extensions as in ($\mathcal{C}_1$), of torsors with connection as in ($\mathcal{C}_2$), of invertible sheaves with connection as in ($\mathcal{C}_3$) and of invertible modules as in ($\mathcal{C}_4$). Observe furthermore that the equivalences ($\mathcal{C}_1$), ($\mathcal{C}_2$) and ($\mathcal{C}_3$) hold also forgetting connections.

3.2. Universal extension crystals of semiabelian schemes

We briefly recall the proof of Theorem 2.1 for abelian schemes given in [19, §II.1].

Let $\mathbb{A}_0 \to S_0$ be an abelian scheme. Let $\mathrm{H}^1\big(\mathbb{A}_0, \mathcal{O}^*_{\mathbb{A}_0/S,\mathrm{crys}}\big)$ be the group of isomorphism classes of invertible modules on the nilpotent crystalline site of $\mathbb{A}_0$ relative to S. Define $\mathrm{Prim}\left(\mathrm{H}^1\big(\mathbb{A}_0, \mathcal{O}^*_{\mathbb{A}_0/S,\mathrm{crys}}\big)\right)$ to be the kernel of the the "primitivity map"

$$m^* - p_1^* - p_2^* \colon \mathrm{H}^1\big(\mathbb{A}_0, \mathcal{O}^*_{\mathbb{A}_0/S,\mathrm{crys}}\big) \longrightarrow \mathrm{H}^1\big(\mathbb{A}_0 \times_{S_0} \mathbb{A}_0, \mathcal{O}^*_{\mathbb{A}_0\times_{S_0}\mathbb{A}_0/S,\mathrm{crys}}\big)$$

where $m\colon \mathbb{A}_0 \times_{S_0} \mathbb{A}_0 \to \mathbb{A}_0$ is the multiplication map and $p_1, p_2\colon \mathbb{A}_0 \times_{S_0} \mathbb{A}_0 \to \mathbb{A}_0$ are the first and the second projection respectively. Define $\mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{A}_0/S}$ to be the sheaf for the fppf topology of S associated to the presheaf whose value on an fppf morphism $T \to S$ is the abelian group $\mathrm{Prim}\left(\mathrm{H}^1\big(\mathbb{A}_0 \times_S T, \mathcal{O}^*_{\mathbb{A}_0\times_S T/T,\mathrm{crys}}\big)\right)$. More precisely, $\mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{A}_0/S}$ is the sheaf associated to the presheaf that associates to an S-scheme T the isomorphism classes of invertible modules $\mathcal{L}$ on the nilpotent crystalline site of $\mathbb{A}_0 \times_S T$ relative to T for which there exists an isomorphism of crystals

$$\iota_{\mathcal{L}}\colon m^*(\mathcal{L}) \otimes p_1^*(\mathcal{L})^{-1} \otimes p_2^*(\mathcal{L})^{-1} \cong \mathcal{O}_{(\mathbb{A}_0\times_{S_0}\mathbb{A}_0\times_S T)/T,\mathrm{crys}}.$$

Since $\Gamma\big(T, \mathcal{O}_{(\mathbb{A}_0\times_{S_0}\mathbb{A}_0\times_S T)/T,\mathrm{crys}}\big)^* = \Gamma\big(T, \mathcal{O}_T\big)^*$, any two such isomorphisms differ by a unit in $\Gamma\big(T, \mathcal{O}_T\big)$ and, hence, by an isomorphism of $\mathcal{L}$. In particular, if $\iota_{\mathcal{L}}$ exists, it is unique up to isomorphism of $\mathcal{L}$. Furthermore $\mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{A}_0/S}(T) \cong \mathrm{Ext}^{\mathrm{crys}/T}(\mathbb{A}_0, \mathbf{G}_m)$. Then:

Proposition 3.1. *Let $\mathbb{A} \to S$ be an abelian scheme deforming $\mathbb{A}_0 \to S_0$. The S-group scheme $\mathbb{E}\big(\mathbb{A}^\vee\big)$ represents the sheaf $\mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{A}_0/S}$.*

PROOF. See [19, §II.1.4]. The proof goes as follows. Let $T \to S$ be an fppf morphism. The T-valued points of $\mathbb{E}\big(\mathbb{A}^\vee\big)$ correspond to invertible sheaves $\mathcal{L}$ of $\mathcal{O}_{\mathbb{A}\times_S T}$-modules, endowed with an integrable connection, for which there exists a horizontal isomorphism $m^*(\mathcal{L}) \cong p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L})$ ([19, I, §4.2]). Using the equivalence between $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ in §3.1, we get that the category of invertible sheaves with integrable connection on $\mathbb{A} \times_S T$ is equivalent to the category of invertible modules on the nilpotent crystalline site of $\mathbb{A}_0 \times_S T$ relative to T. Via this equivalence the elements in Prim correspond to isomorphism classes of invertible sheaves $\mathcal{L}$ on $\mathbb{A} \times_S T$ endowed with integrable connection and for which there is a horizontal isomorphism $m^*(\mathcal{L}) \cong p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L})$ on $\mathbb{A} \times_S \mathbb{A}$. Hence, we are done. □

Consider now a 1–motive $\mathbb{M}_0$ over S_0 given by $\mathrm{u}_0\colon \mathbb{X}_0 \to \mathbb{A}_0$ with $\mathbb{A}_0$ an abelian scheme over S_0. Define $\mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{M}_0/S}$ to be the sheaf for the fppf topology of S associated to the presheaf whose value on an fppf morphism $T \to S$ is given by the isomorphism classes of triples $\big(\mathcal{L}, \iota_{\mathcal{L}}, j_{\mathcal{L}}\big)$ where $\mathcal{L}$ is an invertible module on the nilpotent crystalline site of $\mathbb{A}_0 \times_S T$ relative to T,

$$\iota_{\mathcal{L}}\colon m^*\big(\mathcal{L}\big) \otimes p_1^*\big(\mathcal{L}\big)^{-1} \otimes p_2^*\big(\mathcal{L}\big)^{-1} \to \mathcal{O}_{(\mathbb{A}_0\times_{S_0}\mathbb{A}_0)\times_S T/T,\mathrm{crys}}$$

is an isomorphism and $j_{\mathcal{L}}\colon \mathrm{u}_0^*\mathcal{L} \to \mathcal{O}_{\mathbb{X}_0\times_S T/T,\mathrm{crys}}$ is an isomorphism such that its image via the morphism

$$\mathrm{Isom}\big(\mathrm{u}_0^*\mathcal{L}, \mathcal{O}_{\mathbb{X}_0/S,\mathrm{crys}}\big) \longrightarrow \mathrm{Isom}\Big(m^*\big(\mathrm{u}_0^*\mathcal{L}\big) \otimes p_1^*\big(\mathrm{u}_0^*\mathcal{L}\big)^{-1} \otimes p_2^*\big(\mathrm{u}_0^*\mathcal{L}\big)^{-1}, \mathcal{O}_{(\mathbb{X}_0\times_{S_0}\mathbb{X}_0)/S,\mathrm{crys}}\Big),$$

is induced by the pull-back along u_0 of $\iota_{\mathcal{L}}$. In other words

$$\mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{M}_0/S}(T) \cong \mathrm{Ext}^{\mathrm{crys}/T}(\mathbb{M}_0, \mathbf{G}_m).$$

Proposition 3.2. *Let $\mathbb{M} := [\mathrm{u}\colon \mathbb{X} \to \mathbb{A}]$ be a 1–motive deforming $\mathbb{M}_0$ to S. The S-group scheme $\mathbb{E}\big(\mathbb{M}^\vee\big)$ represents the sheaf $\mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{M}_0/S}$.*

Proof. Let $T \to S$ be an fppf morphism. Recall that $\mathbb{E}(\mathbb{M}^\vee)$ is the fiber product of $\mathbb{E}(\mathbb{A}^\vee)$ and $\mathbb{M}^\vee$ over $\mathbb{A}^\vee$. Using this description, we get that it represents the sheaf for the fppf topology on S that associates to $T \to S$ the isomorphism classes of triples $(\mathbb{L}, \nabla_{\mathbb{L}}, \sigma_{\mathbb{L}})$ where $\mathbb{L}$ is a $\mathbf{G}_m$-extension of $\mathbb{A}$ over T, $\nabla_{\mathbb{L}}$ is an integrable connection on the underlying invertible sheaf $\mathcal{L}$ and $\sigma_{\mathbb{L}}$ is a trivialization (as extension) of the pull-back of $\mathbb{L}$ along u. The group law on $\mathbb{L}$ determines and is determined by an isomorphism $\iota_{\mathbb{L}} \colon m^*(\mathcal{L}) \cong p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L})$ of invertible sheaves over $\mathbb{A} \times_S T$ which is unique up to multiplication by a unit of $\Gamma(T, \mathcal{O}_T)$ and is horizontal by [19, I, Prop 4.2.1]. In particular, $\mathbb{E}(\mathbb{M}^\vee)$ represents the sheaf for the fppf topology on S that associates to $T \to S$ the isomorphism classes of $\natural$-extensions of $\mathbb{M}$ by $\mathbf{G}_m$ over T. Via the equivalence between $(\mathcal{C}_1)$ and $(\mathcal{C}_4)$ in §3.1 any such element corresponds to a T-section of $\mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{M}_0/S}$ and vice versa. The conclusion follows. $\square$

Corollary 3.3. *Theorem 2.1 holds true for* 1*–motives* $\mathbb{M}_0$ *of the form* $[0 \to \mathbb{G}_0]$.

Proof. Let $s_0 \colon \mathbb{G}_0 \to \mathbb{H}_0$ be a morphism of semiabelian schemes, extensions of an abelian scheme by a torus. Write the Cartier duals $\mathbb{G}_0^\vee := [\mathbb{X}_0' \to \mathbb{A}_0^\vee]$ and $\mathbb{H}_0^\vee := [\mathbb{Y}_0' \to \mathbb{B}_0^\vee]$. Let $s_0^\vee := \mathbb{H}_0^\vee \to \mathbb{G}_0^\vee$ be the morphism dual to s_0.

Let $\mathbb{G}$ and $\mathbb{H}$ be deformations of $\mathbb{G}_0$ and $\mathbb{H}_0$ to semiabelian schemes over S. Then, $\mathbb{G}^\vee = [\mathbb{X}' \to \mathbb{A}^\vee]$ and $\mathbb{H}^\vee := [\mathbb{Y}' \to \mathbb{B}^\vee]$ deform $\mathbb{G}_0^\vee$ and $\mathbb{H}_0^\vee$ respectively. The morphism $s_0^\vee$ defines by pull-back a morphism $\mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{G}_0^\vee/S} \longrightarrow \mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{H}_0^\vee/S}$. Using Proposition 3.2 we get a morphism

$$\mathbb{E}_S(s_0) \colon \mathbb{E}(\mathbb{G}) \cong \mathbb{E}\left((\mathbb{G}^\vee)^\vee\right) \cong \mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{G}_0^\vee/S} \longrightarrow \mathrm{Pic}^{\mathrm{crys},0}_{\mathbb{H}_0^\vee/S} \cong \mathbb{E}\left((\mathbb{H}^\vee)^\vee\right) \cong \mathbb{E}(\mathbb{H}).$$

By construction it coincides with the morphism defined in [19] if $\mathbb{G}_0$ and $\mathbb{H}_0$ are abelian schemes. It is clearly functorial and additive in s_0 and it satisfies properties (iii) and (iv) of Theorem 2.1. The claim follows. $\square$

3.3. Proof of Theorem 2.1

To construct the homomorphism $\mathbb{E}_S(s_0)$ we first construct a functor

$$\mathbb{E}_S^\natural(s_0) \colon \mathrm{EXT}^\natural(\mathbb{M}^\vee, \mathbf{G}_{m,S}) \to \mathrm{EXT}^\natural(\mathbb{N}^\vee, \mathbf{G}_{m,S}) \tag{5}$$

from the category of $\natural$-extensions of $\mathbb{M}^\vee$ by $\mathbf{G}_{m,S}$ to the category of $\natural$-extensions of $\mathbb{N}^\vee$ by $\mathbf{G}_{m,S}$. Starting with a $\natural$-extension (E, ∇) of $\mathbb{M}^\vee$ via the equivalence between $(\mathcal{C}_1)$ and $(\mathcal{C}_4)$ in §3.1, we get a triple $\left(\mathcal{L}, \iota_{\mathcal{L}}, j_{\mathcal{L}}\right)$ where $\mathcal{L}$ is an invertible module on the nilpotent crystalline site of $T_0 \times_{S_0} \mathbb{G}_0'$. We then apply the pull-back via $s_0^\vee \colon \mathbb{N}_0^\vee \to \mathbb{M}_0^\vee$ getting a triple $\left(\mathcal{M}, \iota_{\mathcal{M}}, j_{\mathcal{M}}\right)$ with $\mathcal{M}$ now an invertible module on the nilpotent crystalline site of $T_0 \times_{S_0} \mathbb{H}_0'$ relative to T. Again the equivalence between $(\mathcal{C}_1)$ and $(\mathcal{C}_4)$ provides a $\natural$-extension $\mathbb{E}_S^\natural(s_0)(E, \nabla)$ of $\mathbb{N}^\vee$ by $\mathbf{G}_{m,S}$. This construction works over any S-scheme T. Since both the equivalence between $(\mathcal{C}_1)$ and $(\mathcal{C}_4)$ and the pull-back along $s_0^\vee$ pass to isomorphism classes, we get a homomorphism of fppf sheaves, denoted in the same way,

$$\mathbb{E}_S^\natural(s_0) \colon \underline{\mathrm{Ext}}^\natural(\mathbb{M}^\vee, \mathbf{G}_m) \to \underline{\mathrm{Ext}}^\natural(\mathbb{N}^\vee, \mathbf{G}_m).$$

Denote by $\rho \colon \mathbb{M}^\vee \to [\mathbb{X}' \to \mathbb{A}^\vee]$ the homomorphism coming from $\mathbb{G}' \to \mathbb{A}^\vee$ and by $\mathcal{P}_{\mathbb{G}}$ the Poincaré biextension of $\mathbb{G}$ and $[\mathbb{X}' \to \mathbb{A}^\vee]$. For T an S-scheme and g a T-rational point of $\mathbb{G}$, the "fibre" $\mathcal{P}_g$ of $\mathcal{P}_{\mathbb{G}}$ at g will always be viewed as a $\mathbf{G}_m$-extension of $[\mathbb{X}' \to \mathbb{A}^\vee]$ over T. Recall now that by [5, 3.8] the group scheme $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$ represents the functor that associates to a scheme T over S the pairs (g, ∇) where g is a T-valued point of $\mathbb{G}$ and ∇ is an integrable connection on the pull-back $\rho^*\left(\mathcal{P}_g\right)$ to $\mathbb{M}^\vee$ of the fibre $\mathcal{P}_g$, i.e., ∇ makes $\rho^*\mathcal{P}_g$ a $\natural$-extension of $\mathbb{M}^\vee$ by $\mathbf{G}_m$. Using this interpretation of $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$, we have a natural homomorphism $\mathbb{E}(\mathbb{M})_{\mathbb{G}} \to \underline{\mathrm{Ext}}^\natural(\mathbb{M}^\vee, \mathbf{G}_m)$ associating to a T-valued point (g, ∇) of $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$ the $\natural$-extension $\left(\rho^*\left(\mathcal{P}_g\right), \nabla\right)$ of $\mathbb{M}^\vee$ by $\mathbf{G}_m$. This morphism is functorial in $\mathbb{M}$. It follows from [5, 5.12] that this map is surjective with kernel $\tilde{\mathbb{X}}$ equal to the image of $\mathbb{X}$ in $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$, i.e., we have exact sequences

$$0 \to \tilde{\mathbb{X}} \to \mathbb{E}(\mathbb{M})_{\mathbb{G}} \to \underline{\mathrm{Ext}}^\natural(\mathbb{M}^\vee, \mathbf{G}_m) \to 0,$$
$$0 \to \tilde{\mathbb{Y}} \to \mathbb{E}(\mathbb{N})_{\mathbb{H}} \to \underline{\mathrm{Ext}}^\natural(\mathbb{N}^\vee, \mathbf{G}_m) \to 0,$$

where, being a quotient of $\mathbb{X}$ (resp. of $\mathbb{Y}$), the group scheme $\tilde{\mathbb{X}}$ (resp. $\tilde{\mathbb{Y}}$), étale locally on T, is a constant group scheme.

We now consider the problem of lifting the map $\mathbb{E}(\mathbb{M})_{\mathbb{G}} \to \underline{\mathrm{Ext}}^{\natural}(\mathbb{N}^{\vee}, \mathbf{G}_m)$, obtained by composing $\mathbb{E}(\mathbb{M})_{\mathbb{G}} \to \underline{\mathrm{Ext}}^{\natural}(\mathbb{M}^{\vee}, \mathbf{G}_m)$ with $\mathbb{E}^{\natural}_S(s_0)$, to a morphism $\mathbb{E}(\mathbb{M})_{\mathbb{G}} \to \mathbb{E}(\mathbb{N})_{\mathbb{H}}$. Such a lift exists if we prove that $\mathrm{Ext}^1(\mathbb{E}(\mathbb{M})_{\mathbb{G}}, \tilde{\mathbb{Y}}) = 0$. Take an element in that Ext group. It corresponds to an extension as fppf sheaves which is represented by a group scheme F extension of $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$ by $\tilde{\mathbb{Y}}$ due to [22, Prop. 17.4]. Any such extension splits where the splitting is defined by taking the connected component F^0 of F containing the 0-section. Thus, $\mathrm{Ext}^1(\mathbb{E}(\mathbb{M})_{\mathbb{G}}, \tilde{\mathbb{Y}}) = 0$. Two lifts differ by a homomorphism $\mathbb{E}(\mathbb{M})_{\mathbb{G}} \to \tilde{\mathbb{Y}}$. Since $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$ is connected any such homomorphism is zero. We then conclude that there exists a *unique* homomorphism $\mathbb{E}(\mathbb{M})_{\mathbb{G}} \to \mathbb{E}(\mathbb{N})_{\mathbb{H}}$ compatible with $\mathbb{E}^{\natural}(s_0)$. In particular, it maps $\tilde{\mathbb{X}}$ to $\tilde{\mathbb{Y}}$. The uniqueness over S_0 implies that it lifts the morphism $\mathbb{E}(\mathbb{M}_0)_{\mathbb{G}_0} \to \mathbb{E}(\mathbb{N}_0)_{\mathbb{H}_0}$ induced by s_0 by the functoriality of the universal vector extensions. Then, it extends to a unique morphism of complexes $\mathbb{E}_S(s_0)\colon \mathbb{E}(\mathbb{M}) \to \mathbb{E}(\mathbb{N})$ compatible with the map $\alpha_0\colon \mathbb{X} \to \mathbb{Y}$.

Due to the functoriality of $\mathbb{E}^{\natural}_S(s_0)$ and the uniqueness of $\mathbb{E}_S(s_0)$, the association $s_0 \mapsto \mathbb{E}_S(s_0)$ is functorial in s_0, i.e., Property (i) of Theorem 2.1 holds. In particular, $\mathbb{E}(\mathbb{G}) \to \mathbb{E}(\mathbb{M}) \to \mathbb{E}(\mathbb{N})$ factors via $\mathbb{E}(\mathbb{H})$ and the induced map is determined by $\mathbb{E}^{\natural}_S(\beta_0)$. By Proposition 3.3 this coincides with $\mathbb{E}_S(\beta_0)$. In particular, the construction of $\mathbb{E}_S(s_0)$ is compatible with Mazur-Messing's construction if $\mathbb{M}_0$ and $\mathbb{N}_0$ are abelian schemes as requested in Theorem 2.1.

If s_0 lifts to a morphism $s\colon \mathbb{M} \to \mathbb{N}$ over S, then the induced map on universal vector extensions $\mathbb{E}(s)\colon \mathbb{E}(\mathbb{M}) \to \mathbb{E}(\mathbb{N})$ is compatible with the map $\mathbb{E}^{\natural}_S(s_0)$. In particular, $\mathbb{E}(s) = \mathbb{E}_S(s_0)$ since $\mathbb{E}_S(s_0)$ is the unique morphism with this property. Thus, Property (iv) of Theorem 2.1 holds.

Due to the uniqueness of the construction of $\mathbb{E}_S(s_0)$ from $\mathbb{E}^{\natural}_S(s_0)$, in order to check properties (ii) and (iii) of Theorem 2.1 for $\mathbb{E}_S(s_0)$, it suffices to check them for $\mathbb{E}^{\natural}_S(s_0)$. The result is clear for Property (iii). We come next to Property (ii). Let s_0 and $q_0\colon \mathbb{M}_0 \to \mathbb{N}_0$ be morphisms of 1–motives over S_0. The morphism

$$\mu^{\natural}_{\mathbb{N}}\colon \underline{\mathrm{Ext}}^{\natural}(\mathbb{N}^{\vee} \times \mathbb{N}^{\vee}, \mathbf{G}_m) \xrightarrow{\sim} \underline{\mathrm{Ext}}^{\natural}(\mathbb{N}^{\vee}, \mathbf{G}_m) \times \underline{\mathrm{Ext}}^{\natural}(\mathbb{N}^{\vee}, \mathbf{G}_m) \longrightarrow \underline{\mathrm{Ext}}^{\natural}(\mathbb{N}^{\vee}, \mathbf{G}_m)$$

induced by the Baer sum of $\natural$-extensions is compatible with the sum

$$\mathbb{E}(\mathbb{N} \times \mathbb{N})_{\mathbb{H}\times\mathbb{H}} \cong \mathbb{E}(\mathbb{N})_{\mathbb{H}} \times \mathbb{E}(\mathbb{N})_{\mathbb{H}} \to \mathbb{E}(\mathbb{N})_{\mathbb{H}}.$$

In particular, if $m_0\colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ denotes the sum we have $\mathbb{E}^{\natural}_S(m_0) = \mu^{\natural}_{\mathbb{N}}$. Similarly, the diagonal map

$$\Delta^{\natural}\colon \underline{\mathrm{Ext}}^{\natural}(\mathbb{M}^{\vee}, \mathbf{G}_m) \to \underline{\mathrm{Ext}}^{\natural}(\mathbb{M}^{\vee}, \mathbf{G}_m) \times \underline{\mathrm{Ext}}^{\natural}(\mathbb{M}^{\vee}, \mathbf{G}_m) \xrightarrow{\sim} \underline{\mathrm{Ext}}^{\natural}(\mathbb{M}^{\vee} \times \mathbb{M}^{\vee}, \mathbf{G}_m)$$

is compatible with the diagonal map $\mathbb{E}(\mathbb{M})_{\mathbb{G}} \to \mathbb{E}(\mathbb{M})_{\mathbb{G}} \times \mathbb{E}(\mathbb{M})_{\mathbb{G}} \cong \mathbb{E}(\mathbb{M} \times \mathbb{M})_{\mathbb{G}\times\mathbb{G}}$. In particular, if Δ_0 denotes the diagonal embedding $\mathbb{M}_0 \to \mathbb{M}_0 \times \mathbb{M}_0$, we have $\Delta^{\natural} = \mathbb{E}^{\natural}_S(\Delta_0)$. Thus

$$\mathbb{E}^{\natural}_S(s_0) + \mathbb{E}^{\natural}_S(q_0) = \mathbb{E}^{\natural}_S(m_0) \circ \big(\mathbb{E}^{\natural}_S(s_0) \times \mathbb{E}^{\natural}_S(q_0)\big) \circ \Delta^{\natural} = \mathbb{E}^{\natural}_S\big(m_0 \circ (s_0, q_0) \circ \Delta_0\big)$$

by functoriality and the latter coincides with $\mathbb{E}^{\natural}_S(s_0 + q_0)$. This proves Property (ii) for $\mathbb{E}^{\natural}_S$ as desired.

Remark 3.4. *Theorem 2.1 asserts the existence of a canonical morphism $\mathbb{E}_S(s_0)$ satisfying certain constrains, not its uniqueness. More precisely $\mathbb{E}_S(s_0)$ is the unique one compatible with $\mathbb{E}^{\natural}_S(s_0)$. The uniqueness stated in [1, Lemma 3.2.1] is wrong unless S_0 is flat over $\mathbf{Z}$ or one works with a p-adic basis as in §6. For the first case, the proofs in loc. cit. work. In the last case, which is needed in [1], the uniqueness will be proven in Proposition 6.2.*

4. Consequences of theorem 2.1

4.1. Crystalline realization of 1*–motives. Proof of Corollary 2.3*

We use Theorem 2.1 to produce a covariant additive functor $\mathbf{T}_{\mathrm{crys}}$ from the category of 1–motives over S_0 to the category of filtered crystals as follows. Suppose we are given a 1–motive $\mathbb{M}_0 := [\mathbb{X}_0 \to \mathbb{G}_0]$ over S_0

and an object $(U \subset Z)$ of S_0^{crys} defined by an ideal $\mathcal{I}$ with locally nilpotent divided powers. Étale locally on U the 1–motive $\mathbb{M}_0$ can be deformed to a 1–motive $\mathbb{M}$ over Z. Indeed, let $\{U_{i,0}\}_i$ be an étale cover of U, with $U_{i,0}$ affine, such that $\mathbb{X}_0$ and the character group $\mathbb{X}_0'$ of the toric part of $\mathbb{G}_0$ are constant on each $U_{i,0}$. Since $\mathcal{I}$ is locally nilpotent, the étale sites of U and Z are identified and each $U_{i,0}$ defines an étale open subset U_i of Z. Furthermore, $\mathbb{X}_0$ and $\mathbb{X}_0'$ admit unique deformations $\mathbb{X}_i$ and $\mathbb{X}_i'$ to constant group schemes over U_i. Since U_i is affine, the abelian part $\mathbb{A}_0$ of $\mathbb{G}_0$ can be lifted over U_i to an abelian scheme $\mathbb{A}_i$. The semiabelian part $\mathbb{G}_0$ is defined by a homomorphism $\mathbb{X}_0' \to \mathbb{A}_0^\vee$ and, since $\mathbb{X}_i'$ is split and $\mathbb{A}_i^\vee \to U_i$ is smooth, it can be deformed to a group scheme homomorphism $\mathbb{X}_i' \to \mathbb{A}_i^\vee$. This defines a semiabelian scheme $\mathbb{G}_i$ over U_i deforming $\mathbb{G}_0$. Since $\mathbb{G}_i \to U_i$ is smooth, also the map $\mathbb{X}_0 \to \mathbb{G}_0$ can be lifted to a homomorphism $\mathbb{X}_i \to \mathbb{G}_i$. This defines a 1–motive $\mathbb{M}_i$ over U_i deforming $\mathbb{M}_0$. Due to Theorem 2.1(iii) the group schemes $\{\mathbb{E}(\mathbb{M}_i)\}_i$ satisfy a descent datum and, hence, $\{\mathbf{T}_{\mathrm{dR}}\mathbb{E}(\mathbb{M}_i)\}_i$ as well. In particular, they descend to a locally free $\mathcal{O}_Z$-module, whose sections over Z define

$$\mathbf{T}_{\mathrm{crys}}(\mathbb{M}_0)(U \subset Z).$$

It is endowed with a filtration by locally free $\mathcal{O}_Z$-modules obtained from the one introduced in Remark 2.2(a), i.e.,

$$W_i\left(\mathbf{T}_{\mathrm{crys}}(\mathbb{M}_0)(U \subset Z)\right) := \begin{cases} \mathbf{T}_{\mathrm{crys}}(\mathbb{M}_0)(U \subset Z) & \text{if } i \geq 0 \\ \mathbf{T}_{\mathrm{crys}}(\mathbb{G}_0)(U \subset Z) & \text{if } i = -1 \\ \mathbf{T}_{\mathrm{crys}}(\mathbb{T}_0)(U \subset Z) & \text{if } i = -2 \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mathbf{T}_{\mathrm{crys}}(\mathbb{M}_0)$ defines a filtered crystal, due to Theorem 2.1(iii), which is functorial in $\mathbb{M}_0$ by Theorem 2.1(i) and it is additive by Theorem 2.1(ii). By construction if $\mathbb{M}$ is a 1–motive over Z deforming the restriction of $\mathbb{M}_0$ to U we have a canonical identification

$$\mathbf{T}_{\mathrm{crys}}(\mathbb{M}_0)(U \subset Z) \cong \mathbf{T}_{\mathrm{dR}}\big(\mathbb{E}(\mathbb{M})\big). \tag{6}$$

Assume that $\mathbb{M}_0 = \mathbb{A}_0$ is an abelian scheme with structural morphism $f\colon \mathbb{A}_0 \to S_0$. Let $g\colon \mathbb{A}_0^\vee \to S_0$ be the dual abelian scheme. Then, as remarked in [19, §I.4 & II.1.5], we have an isomorphism of crystals on S_0^{crys}:

$$\mathbf{T}_{\mathrm{crys}}(\mathbb{A}_0)\big(U \subset Z\big) \cong \mathrm{R}^1 g_{\mathrm{crys},*}\big(\mathcal{O}_{\mathbb{A}_0^\vee/S}\big)\big(U \subset Z\big).$$

Indeed, let $(U \subset Z)$ be an object of S_0^{crys} with Z-affine and let $\mathbb{A}$ be an abelian scheme over Z deforming $\mathbb{A}_0$ (restricted to U). Then, $\mathbf{T}_{\mathrm{crys}}(\mathbb{A}_0)\big(U \subset Z\big)$ coincides with $\mathbf{T}_{\mathrm{dR}}(\mathbb{A}) = \mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}^\vee/Z)$ while $\mathrm{R}^1 g_{\mathrm{crys},*}\big(\mathcal{O}_{\mathbb{A}_0^\vee/S}\big)\big(U \subset Z\big)$ coincides with $\mathrm{H}^1\big(\mathbb{A}_0, \mathcal{O}_{\mathbb{A}_0^\vee/Z,\mathrm{crys}}\big)$ which is $\mathbb{H}^1_{\mathrm{dR}}\big(\mathbb{A}^\vee/Z\big)$ by [7, Cor. 7.4].

Lemma 4.1. *The universal extension functor $\mathbb{E}$ is exact w.r.t. short exact sequences of 1–motives over S_0. In particular the functors $\mathbf{T}_{\mathrm{dR}}$ and $\mathbf{T}_{\mathrm{crys}}$ are exact.*

Proof. The exactness of $\mathbb{E}$ is trivially true for 1–motives concentrated in degree -1 and for exact sequences of the kind $0 \to \mathbb{G} \to \mathbb{M} \to [\mathbb{X} \to 0]$. One is then reduced to proving that $\mathbb{E}$ is exact for exact sequences of the form

$$0 \to \mathbb{G}_1 \to \mathbb{G} \to \mathbb{G}_2 \to 0$$

with $\mathbb{G}$, $\mathbb{G}_1$ and $\mathbb{G}_2$ semiabelian schemes. This follows since the sequence of vector groups $\mathbb{W}(\mathbb{G}_i) = \underline{\mathrm{Ext}}^1_{\mathrm{Zar}}(\mathbb{G}_i, \mathbf{G}_a)^\vee$ is exact.

The exactness of $\mathbf{T}_{\mathrm{dR}}$ follows from the exactness of $\mathbb{E}$. To prove the exactness of $\mathbf{T}_{\mathrm{crys}}$ it suffices to show that it is exact when restricted to the Zariski site of S_0, where it coincides with $\mathbf{T}_{\mathrm{dR}}$, and the result then follows from the exactness of the latter. □

4.2. Construction of the Gauss-Manin connection

Let $q\colon S \to T$ be a morphism of schemes and let $\mathbb{M}$ be a 1–motive over S. In this section we construct the Gauss-Manin connection associated to $\mathbb{M}$:

$$\nabla_{\mathbb{M}}\colon \mathbf{T}_{\mathrm{dR}}(\mathbb{M}) \longrightarrow \mathbf{T}_{\mathrm{dR}}(\mathbb{M}) \otimes_{\mathcal{O}_S} \Omega^1_{S/T}$$

and we prove the following properties:

(a) it is functorial in $\mathbb{M}$;

(b) $\nabla_{\mathbb{M}}$ preserves the weight filtration on $\mathbf{T}_{\rm dR}(\mathbb{M})$ described in (2);

(c) if $\mathbb{M}$ arises by pull-back from a 1–motive $\mathbb{N}$ over T then $\nabla_{\mathbb{M}}$ is the trivial connection vanishing on $q^{-1}(\mathbf{T}_{\rm dR}(\mathbb{N}))$;

(d) it is integrable if q is a smooth morphism.

Consider the diagonal map $\Delta\colon S \to S \times_T S$. It is a locally closed T-immersion. Let $\mathcal{J} \subset \mathcal{O}_{S\times_T S}$ be the ideal defining the schematic closure of Δ. Define $S' \subset S \times_T S$ to be the closed subscheme defined by $\mathcal{J}^2$. Let $p_i\colon S' \to S$, for $i = 1$ and 2, be the two projections onto S. Then, Δ factors via a closed immersion $\iota\colon S \to S'$ defined by the ideal $\mathcal{I} := \mathcal{J}/\mathcal{J}^2$ whose square is 0. In particular, $\mathcal{I}$ has a unique divided power structure given by $\gamma_1 = \mathrm{Id}$ and $\gamma_n = 0$ for $n \geq 2$. We recall that a connection on an $\mathcal{O}_S$-module $\mathcal{L}$ is an isomorphism $\nabla'\colon p_2^*\mathcal{L} \to p_1^*\mathcal{L}$ of $\mathcal{O}_{S'}$-modules restricting to the identity on S, or, equivalently, an $\mathcal{O}_S$-linear homomorphism $\nabla\colon \mathcal{L} \to \mathcal{L} \otimes \Omega^1_{S/T}$ (cf. [19, I.3.1]). The relation between the two descriptions is that $\nabla = \nabla' \circ p_2^* - p_1^*$.

Let $\mathbb{M}_i$ be the base-change of $\mathbb{M}$ to S' via p_i. Then, the base-change of $\mathbb{E}(\mathbb{M}_i)$ via ι is $\mathbb{E}(\mathbb{M})$. Thanks to Theorem 2.1, there exists an isomorphism of group schemes over S'

$$\mathbb{E}_{S'}\big(\mathrm{id}_{\mathbb{M}}\big)\colon \mathbb{E}(\mathbb{M}_2)_{\mathbb{G}_2} \longrightarrow \mathbb{E}(\mathbb{M}_1)_{\mathbb{G}_1} \tag{7}$$

reducing to the identity map over S. At the level of de Rham realizations we have

$$\mathbf{T}_{\rm dR}(\mathbb{M}_1) := \mathbf{Lie}\left(\mathbb{E}(\mathbb{M}_1)_{\mathbb{G}_1}\right) = \mathbf{Lie}\left(\mathbb{E}(\mathbb{M})_{\mathbb{G}}\right) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} =: \mathbf{T}_{\rm dR}(\mathbb{M}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$$

and similarly $\mathbf{T}_{\rm dR}(\mathbb{M}_2) := \mathbf{Lie}\left(\mathbb{E}(\mathbb{M}_2)_{\mathbb{G}_2}\right) = \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathbf{T}_{\rm dR}(\mathbb{M})$. Hence on passing to Lie algebras in (7) we get an $\mathcal{O}_{S'}$-linear map

$$\nabla'_{\mathbb{M}} := \mathbf{Lie}\ \left(\mathbb{E}_{S'}\big(\mathrm{id}_{\mathbb{M}}\big)\right) : \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathbf{T}_{\rm dR}(\mathbb{M}) \longrightarrow \mathbf{T}_{\rm dR}(\mathbb{M}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \tag{8}$$

inducing the identity on $\mathbf{T}_{\rm dR}(\mathbb{M})$ modulo $\mathcal{I}$, i.e., we obtain the so-called *Gauss-Manin connection*

$$\nabla_{\mathbb{M}}\colon \mathbf{T}_{\rm dR}(\mathbb{M}) \longrightarrow \mathbf{T}_{\rm dR}(\mathbb{M}) \otimes_{\mathcal{O}_S} \Omega^1_{S/T}.$$

It follows from Theorem 2.1(i) that $\nabla_{\mathbb{M}}$ is functorial in $\mathbb{M}$ and, in particular, it preserves the weight filtration on $\mathbf{T}_{\rm dR}(\mathbb{M})$. This proves Properties (a) and (b). Property (c) follows from Theorem 2.1(iv).

There remains only to prove claim (d). Let $\mathcal{D}^{(n)}_{S/T}(1)$ be the n-th order divided power neighborhood of $S \subset S \times_T S$ diagonally embedded; see [7, Def. 4.1]. Let p_1^n and $p_2^n\colon \mathrm{Spec}\big(\mathcal{D}^{(n)}_{S/T}\big) \to S$ be the projection onto the first (resp. second) factor. Due to Theorem 2.1 we have an isomorphism $a\colon (p_2^n)^*\big(\mathbb{E}(\mathbb{M})\big) \cong (p_1^n)^*\big(\mathbb{E}(\mathbb{M})\big)$. Taking Lie algebras we get a collection of isomorphisms

$$\epsilon_n\colon \mathcal{D}^{(n)}_{S/T}(1) \otimes_{\mathcal{O}_S} \mathbf{T}_{\rm dR}(\mathbb{M}) \xrightarrow{\sim} \mathbf{T}_{\rm dR}(\mathbb{M}) \otimes_{\mathcal{O}_S} \mathcal{D}^{(n)}_{S/T}(1)$$

for $n \in \mathbb{N}$ such that $\epsilon_0 = \mathrm{Id}$, they are compatible for varying n by Theorem 2.1(iii) and they satisfy the cocycle condition of [7, Def. 4.3] due to Theorem 2.1(i). Thus, the ϵ_n's define a PD stratification. Since $S \to T$ is smooth, $\mathrm{Spec}\big(\mathcal{D}^{(1)}_{S/T}(1)\big)$ coincides with S' (*cf.* [7, Rmk. 4.2]) and ϵ_1 is the isomorphism $\mathbf{Lie}\left(\mathbb{E}_{S'}(\mathrm{Id}_{\mathbb{M}})\right)$ constructed above. Thus, the connection associated to the PD stratification $(\epsilon_n)_{n\in\mathbf{N}}$ is $\nabla_{\mathbb{M}}$ and it is integrable by [7, Thm. 4.8].

Example 4.2. *Assume that* $\mathbb{M} = [\mathbb{X} \to 0]$. *Then,* $\mathbf{T}_{\rm dR}(\mathbb{M}) = \mathbb{X} \otimes_{\mathbf{Z}} \mathcal{O}_S$. *The isomorphism* $\mathbf{Lie}\,\mathbb{E}\big(\mathrm{id}_{\mathbb{M}}\big)\colon \mathbb{X} \otimes \mathcal{O}_{S'} \to \mathbb{X} \otimes \mathcal{O}_{S'}$ *is the identity by Remark 2.2(a). In particular, the connection* $\nabla_{\mathbb{M}}\colon \mathbb{X}\otimes\mathcal{O}_S \to \mathbb{X}\otimes\Omega^1_{S/T}$ *vanishes on* $\mathbb{X}$ *and by the Leibniz rule it is characterized by this property.*

Example 4.3. *Assume that $\mathbb{M} = [0 \to \mathbb{T}]$ where $\mathbb{T}$ is a torus with cocharacter group $\mathbb{Y}$. Then, $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}) = \mathbb{Y} \otimes \mathcal{O}_S$ and $\mathbf{Lie}\,\mathbb{E}\big(\mathrm{id}_{\mathbb{M}}\big)\colon \mathbb{Y} \otimes \mathcal{O}_{S'} \to \mathbb{Y} \otimes \mathcal{O}_{S'}$ is the identity by Remark 2.2(a). As before we conclude that $\nabla_{\mathbb{M}}\colon \mathbb{Y} \otimes \mathcal{O}_S \to \mathbb{Y} \otimes \Omega^1_{S/T}$ vanishes on $\mathbb{Y}$.*

Example 4.4. *Assume that $\mathbb{M} = [0 \to \mathbb{A}]$ where $g\colon \mathbb{A} \to S$ is an abelian scheme. Then $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ is identified with the first de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{A}^\vee/S\big)$ of the dual abelian scheme $\mathbb{A}^\vee$; see [19, §I.4]. Then:*

Lemma 4.5. *Assume that $S \to T$ is a smooth morphism. The connection $\nabla_{\mathbb{A}}$ on $\mathbf{Lie}\,\mathbb{E}(\mathbb{A})$, identified with $\mathbb{H}^1_{\mathrm{dR}}\big(\mathbb{A}^\vee/S\big)$, coincides with the Gauss-Manin connection on $\mathbb{H}^1_{\mathrm{dR}}\big(\mathbb{A}^\vee/S\big)$.*

Proof. We have already remarked that there is an isomorphism of crystals

$$\mathbf{T}_{\mathrm{crys}}\,(\mathbb{A})\,(U \subset Z) \cong \mathrm{R}^1 g_{\mathrm{crys},*}\big(\mathcal{O}_{\mathbb{A}^\vee/S}\big)(U \subset Z)$$

for $(U \subset Z)$ in S^{crys}. In particular, we have a diagram with commutative squares

$$\begin{array}{ccccc}
\mathbf{Lie}\,\mathbb{E}(\mathbb{A}_2) & \xrightarrow{\sim} & \mathbf{T}_{\mathrm{crys}}\,(\mathbb{A})\,(S \subset S') & \xrightarrow{\sim} & \mathbf{Lie}\,\mathbb{E}(\mathbb{A}_1) \\
\| & & \| & & \| \\
\mathrm{H}^1\big(\mathbb{A}_2^\vee, \mathcal{O}_{\mathbb{A}_2^\vee/S',\mathrm{crys}}\big) & \xrightarrow{\sim} & \mathrm{H}^1\big(\mathbb{A}^\vee, \mathcal{O}_{\mathbb{A}^\vee/S',\mathrm{crys}}\big) & \xrightarrow{\sim} & \mathrm{H}^1\big(\mathbb{A}_1^\vee, \mathcal{O}_{\mathbb{A}_1^\vee/S,\mathrm{crys}}\big) \\
\| & & & & \| \\
p_2^*\big(\mathbb{H}^1_{\mathrm{dR}}\big(\mathbb{A}^\vee/S\big)\big) & & & & p_1^*\mathbb{H}^1_{\mathrm{dR}}\big(\mathbb{A}^\vee/S\big);
\end{array}$$

The lower horizontal isomorphisms come from [7, 5.17]. The existence of the upper horizontal isomorphisms and the commutativity of the squares follow by definition of $\mathbf{T}_{\mathrm{crys}}$. The lower vertical identifications follow from the identifications $\mathrm{H}^1\big(\mathbb{A}^\vee, \mathcal{O}_{\mathbb{A}^\vee/S',\mathrm{crys}}\big) \cong \mathbb{H}^1_{\mathrm{dR}}\big(\mathbb{A}_i^\vee/S'\big)$ see [7, Cor. 7.4]. It follows that $\nabla_{\mathbb{A}}$ induces on $\mathbf{Lie}\,\mathbb{E}(\mathbb{A}) \cong \mathbb{H}^1_{\mathrm{dR}}\big(\mathbb{A}^\vee/S\big)$ the connection defined by the lower horizontal arrows. Due to [6, Cor. V.3.6.3 & Prop. V.3.6.4] this is the Gauss-Manin connection on $\mathbb{H}^1_{\mathrm{dR}}\big(\mathbb{A}^\vee/S\big)$. □

4.3. The Hodge realization of 1–motives. Proof of Proposition 2.4

We use the notations of Proposition 2.4. Assume that S is a scheme smooth over $\mathrm{Spec}\,(\mathbf{C})$. Let S^{an} be the associated complex analytic space. Let $\mathbb{M} := [\mathbb{X} \to \mathbb{G}]$ be a 1–motive over S with $\mathbb{G}$ an extension of the abelian scheme $\mathbb{A}$ by the torus $\mathbb{T}$ with cocharacter group $\mathbb{Y}$. For brevity we write $\mathbb{G}^\natural$ to denote the group scheme $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$.

It follows from §4.2 that the Gauss-Manin connection $\nabla_{\mathbb{M}}$ is integrable. In particular, the horizontal sections of $\nabla_{\mathbb{M}}^{\mathrm{an}}\colon \mathbf{Lie}\,\mathbb{G}^{\natural,\mathrm{an}} \to \mathbf{Lie}\,\mathbb{G}^{\natural,\mathrm{an}} \otimes \Omega^1_{S^{\mathrm{an}}}$ form a local system of $\mathbf{C}$-vector spaces of dimension equal to the dimension of $\mathbb{G}^\natural$. We denote it by $\mathbf{T}_{\mathbf{C}}(\mathbb{M})$. It has the property that $\mathbf{T}_{\mathbf{C}}(\mathbb{M}) \otimes_{\mathbf{C}} \mathcal{O}_{S^{\mathrm{an}}} \cong \mathbf{Lie}\,\mathbb{G}^{\natural,\mathrm{an}}$ as $\mathcal{O}_{S^{\mathrm{an}}}$-modules. It is endowed with a weight filtration $W_\bullet \mathbf{T}_{\mathbf{C}}(\mathbb{M}) := \mathbf{T}_{\mathbf{C}}(\mathbb{M}) \cap W_\bullet \mathbf{Lie}\,\mathbb{G}^{\natural,\mathrm{an}}$ and $\mathbf{T}_{\mathbf{C}}(\mathbb{M}) \otimes_{\mathbf{C}} \mathcal{O}_{S^{\mathrm{an}}}$ is endowed with the Hodge filtration $F^\bullet\big(\mathbf{T}_{\mathbf{C}}(\mathbb{M}) \otimes_{\mathbf{C}} \mathcal{O}_{S^{\mathrm{an}}}\big)$ defined by the Hodge filtration on $\mathbf{Lie}\,\mathbb{G}^{\natural,\mathrm{an}}$. To prove Proposition 2.4 we need only construct a $\mathbf{Z}$-structure on $\mathbf{T}_{\mathbf{C}}(\mathbb{M})$. We proceed as follows.

Consider a point $x \in S(\mathbf{C})$. Let $S_{x,m} := \mathrm{Spec}\big(\mathcal{O}_{S,x}/\mathrm{max}_x^m\big)$ for $m \in \mathbf{N}$. Then, the formal completion $\widehat{S}_x$ of S at x is $\lim_m S_{x,m}$. Since S is a $\mathbf{Q}$-scheme, $S_{x,m}$ admits a unique structure of PD thickening of x. In particular, by Theorem 2.1 we have an isomorphism $\mu_m\colon \mathbb{G}^\natural|_{S_{x,m}} \cong \mathbb{G}^\natural_x \times_{\mathbf{C}} S_{x,m}$ of group schemes for every $m \in \mathbf{N}$, compatibly for varying $m \in \mathbf{N}$. It follows from [2, Prop. 2.3 & Cor 2.4] that the isomorphisms μ_m can be approximated by an isomorphism between $\mathbb{G}^\natural$ and $\mathbb{G}^\natural_x \times_{\mathbf{C}} S$ in an étale neighborhood of x in S and, hence, in a small enough analytic neighborhood $x \in U \subset S^{\mathrm{an}}$. The exponential map of $\mathbb{G}^\natural_x$ provides an isomorphism of complexes

$$\big[\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_x)/\mathbf{T}_{\mathbf{Z}}(\mathbb{G}_x) \to \mathbf{T}_{\mathrm{dR}}(\mathbb{M}_x)/\mathbf{T}_{\mathbf{Z}}(\mathbb{G}_x)\big] \xrightarrow{\sim} \mathbb{E}(\mathbb{M}_x)^{\mathrm{an}} = [\mathbb{X} \to \mathbb{G}_x^{\natural,\mathrm{an}}];$$

cf. [14, §10.1.8]. We then get an isomorphism of complexes:

$$\mu^{\mathrm{an}}\colon \Big(\big[\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_x)/\mathbf{T}_{\mathbf{Z}}(\mathbb{G}_x) \to \mathbf{T}_{\mathrm{dR}}(\mathbb{M}_x)/\mathbf{T}_{\mathbf{Z}}(\mathbb{G}_x)\big]\Big) \times_{\mathbf{C}} U \xrightarrow{\sim} \mathbb{E}(\mathbb{M})^{\mathrm{an}}|_U.$$

Passing to Lie algebras the isomorphism μ^{an} gives an isomorphism

$$\mathbf{Lie}\,\mu^{\mathrm{an}}\colon \mathbf{T}_{\mathrm{dR}}(\mathbb{M}_x)\otimes_{\mathbf{C}}\mathcal{O}_U \cong \mathbf{Lie}\,\mathbb{G}^{\natural,\mathrm{an}}|_U$$

and the induced map $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}_x)\otimes_{\mathbf{C}}\mathcal{O}_U \to \mathbb{G}^{\natural,\mathrm{an}}_{|U}$ is the exponential map at each fibre since it is a surjective homomorphism of analytic groups, inducing an isomorphism on Lie algebras. Possibly composing $\mathbf{Lie}\,\mu^{\mathrm{an}}$ with an automorphism of $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}_x)\otimes_{\mathbf{C}}\mathcal{O}_U$ we may assume that $\mathbf{Lie}\,\mu^{\mathrm{an}}$ coincides with the restriction to U of the isomorphism $\mathbf{T}_{\mathbf{C}}(\mathbb{M})\otimes_{\mathbf{C}}\mathcal{O}_{S^{\mathrm{an}}}\cong \mathbf{Lie}\,\mathbb{G}^{\natural,\mathrm{an}}$. In particular, $\mathbf{T}_{\mathbf{C}}(\mathbb{M})|_U$ is endowed with the $\mathbf{Z}$-structure $\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_x)\times_{\mathbf{C}}U \to \mathbf{T}_{\mathrm{dR}}(\mathbb{M}_x)\otimes_{\mathbf{C}}\mathcal{O}_U\cong \mathbf{T}_{\mathbf{C}}(\mathbb{M})|_U$. Denote it by $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})|_U$. To prove the Proposition 2.4 there remains only to show that for every $y\in U$ we have $\left(\mathbf{T}_{\mathbf{Z}}(\mathbb{M})|_U\right)_y = \mathbf{T}_{\mathbf{Z}}(\mathbb{M}_y)$ (as submodules of $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})_y$) compatibly with the weight filtration. Recall that $\mathbf{T}_{\mathbf{Z}}(\mathbb{M}_y)$ is the fibre product of $\mathbf{Lie}\,\mathbb{G}_y$ and $\mathbb{X}$ over $\mathbb{G}_y$ via the exponential map, or equivalently the fibre product of $\mathbf{Lie}\,\mathbb{G}^{\natural,\mathrm{an}}_y$ and $\mathbb{X}$ over $\mathbb{G}^{\natural,\mathrm{an}}_y$. Taking the fibre at y of the isomorphism μ^{an} introduced above we get the result.

Notice that the functor $\mathbf{T}_{\mathbf{Z}}$ is exact on 1–motives over $\mathbf{C}$. Hence, it is exact also on 1–motives over S being exact on each fibre.

Example 4.6. *Consider the cases $\mathbb{M}=[\mathbb{X}\to 0]$, or $\mathbb{M}=[0\to\mathbb{T}]$ with $\mathbb{T}$ a torus, or $\mathbb{M}=[0\to\mathbb{A}]$ with $f\colon\mathbb{A}\to S$ abelian scheme. In the first case, due to Example 4.2, we have $\mathbf{T}_{\mathbf{C}}(\mathbb{M})=\mathbb{X}\otimes_{\mathbf{Z}}\mathbf{C}$ and $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})=\mathbb{X}$. Using Example 4.3, in the second case we get that $\mathbf{T}_{\mathbf{C}}(\mathbb{M})=\mathbb{Y}\otimes_{\mathbf{Z}}\mathbf{C}$ and $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})=\mathbb{Y}$. In the third case, $\mathbf{T}_{\mathbf{C}}(\mathbb{A})\subset\mathbf{T}_{\mathrm{dR}}(\mathbb{A})$ is identified with $\mathrm{R}^1f^{\mathrm{an}}_*\mathbf{C}\subset\mathrm{R}^1f_*(\Omega^\bullet_{\mathbb{A}^{\mathrm{an}}/S^{\mathrm{an}}})$; see [13, §3.5–3.7], cf. [8, Thm 2.1] for more details. Then, $\mathbf{T}_{\mathbf{Z}}(\mathbb{M})=\mathrm{R}^1f^{\mathrm{an}}_*(\mathbf{Z})$ via this identification.*

4.4. Buium's conjecture on projective hulls

Let k be a field of characteristic 0 and $\mathbb{A}$ an abelian variety over k. In [11, Ch. 3,2.1], Buium defines a *projective hull* of $\mathbb{A}$ as a connected algebraic k-group $\mathbb{G}$, extension of $\mathbb{A}$ by a linear k-group $\mathbb{T}\times\mathbb{W}$ ($\mathbb{T}$ a torus, $\mathbb{W}$ a vector group), such that the push-out homomorphism $m\colon\mathrm{Hom}(\mathbb{T}\times\mathbb{W},\mathbf{G}_m)\to\mathrm{Ext}(\mathbb{A},\mathbf{G}_m)$ is injective and the analogous $a\colon\mathrm{Hom}(\mathbb{T}\times\mathbb{W},\mathbf{G}_a)\to\mathrm{Ext}(\mathbb{A},\mathbf{G}_a)$ is an isomorphism. As an example, $\mathbb{E}(\mathbb{A})$ is a projective hull of $\mathbb{A}$. He then conjectured that any projective hull of an abelian variety has a crystalline nature. In view of Proposition 4.7 below, this follows from Theorem 2.1 and, as a matter of fact, the hypothesis on m is superfluous.

Proposition 4.7. *Let $\mathbb{G}$ be an algebraic k-group, extension of $\mathbb{A}$ by $\mathbb{T}\times\mathbb{W}$ as above. Suppose that the push-out homomorphism*

$$a\colon\mathrm{Hom}(\mathbb{T}\times\mathbb{W},\mathbf{G}_a)\to\mathrm{Ext}(\mathbb{A},\mathbf{G}_a)$$

is an isomorphism. Then, $\mathbb{G}$ is the universal extension of $\mathbb{G}_\times := \mathbb{G}/\mathbb{W}$. In particular, $\mathbb{G}$ has crystalline nature.

Proof. Observe that $\mathbb{G}$ is an extension of $\mathbb{G}_\times$ by $\mathbb{W}$ and $\mathbb{G}_\times$ is an extension of $\mathbb{A}$ by $\mathbb{T}$. Clearly we have a commutative square

$$\begin{array}{ccc}\mathrm{Hom}(\mathbb{T}\times\mathbb{W},\mathbf{G}_a) & \xrightarrow{a} & \mathrm{Ext}(\mathbb{A},\mathbf{G}_a)\\ \downarrow & & \downarrow\\ \mathrm{Hom}(\mathbb{W},\mathbf{G}_a) & \longrightarrow & \mathrm{Ext}(\mathbb{G}_\times,\mathbf{G}_a)\end{array}$$

where the vertical arrows are isomorphisms. By hypothesis a is an isomorphism, hence the lower push-out map is an isomorphism too. As k has characteristic 0, the group of homomorphisms $\mathbb{W}\to\mathbf{G}_a$ as group schemes coincides with the group of homomorphisms as vector groups. Hence $\mathbb{G}$ is a universal extension of $\mathbb{G}_\times$. Thus by Theorem 2.1, it has crystalline nature. □

Remark 4.8. *Let k be a field of characteristic zero. We recall that a* Laumon k-1–motive *(also called generalized 1–motive) $\mathbb{M}=[\mathrm{u}\colon\mathbb{X}\to\mathbb{G}]$ is a two term complex (in degree -1, 0) where $\mathbb{X}$ is a formal k-group without torsion and $\mathbb{G}$ is a connected algebraic k-group. The morphism u is a morphism as fppf sheaves* *on the category of affine k-schemes. In [3] a sharp universal extension $\mathbb{E}^\sharp(\mathbb{M})$ of $\mathbb{M}$ was introduced as well as its sharp de Rham realization $\mathbf{T}_\sharp(\mathbb{M})$. These constructions generalize $\mathbb{E}(\mathbb{M})$ and $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ for Deligne 1–motives. Although it is possible to work with "linearized" Laumon 1-motives over any base (cf. [3, 1.5] we can not expect a crystalline nature for $\mathbb{E}^\sharp(\mathbb{M})$. Indeed, let $(S_0 \subset S)$ be a locally nilpotent PD thickening of S_0 and $\mathbb{M}_0 = [0 \to \mathbb{W}_0]$ a linearized 1–motive with $\mathbb{W}_0$ a vector group over S_0. One has $\mathbb{E}^\sharp(\mathbb{M}_0) = \mathbb{M}_0$. Let now $\mathbb{M}_i = [0 \to \mathbb{W}_i]$, $i = 1, 2$, be two deformations of $\mathbb{M}_0$ to S. Since $\mathbb{W}_0$ is locally free the vector groups $\mathbb{W}_i$ are isomorphic, but there is no canonical isomorphism in general. For vector groups of the type $\mathbb{X}_i \otimes \mathbf{G}_a$, $i = 0, 1, 2$, with $\mathbb{X}_i$ étale locally constant and $\mathbb{X}_i$, $i = 1, 2$, deformations of $\mathbb{X}_0$, a canonical isomorphism is deduced from the canonical isomorphism $\mathbb{X}_1 \to \mathbb{X}_2$ that deforms the identity on $\mathbb{X}_0$. But this is a very special case. The previous observation does not imply that a Gauss-Manin connection on $\mathbb{T}_\sharp(\mathbb{M})$ does not exist. For example, one can hope to define it via a "differential structure" on the sharp de Rham extension of $\mathbb{M}$ in the style of [11].*

5. Duality theory

5.1. Duality of 1–motives and biextensions.

Let $\mathbb{A} \to S$ be an abelian scheme. Let $\mathbb{A}^\vee$ be the abelian scheme dual to $\mathbb{A}$; it is the abelian scheme representing the functor $\mathrm{Pic}^0_{\mathbb{A}/S}$ of line bundles, rigidified over the zero section of $\mathbb{A}$ and fiberwise over the base algebraically equivalent to 0. Its existence is guaranteed by [15, Thm 1.9]. The representability implies that $\mathbb{A} \times_S \mathbb{A}^\vee$ is endowed with a tautological $\mathbf{G}_m$-torsor $\mathcal{P}_\mathbb{A}$ which, by construction, is trivial over $\mathbb{A} \times \{0\}$ and over $\{0\} \times \mathbb{A}^\vee$. Choose rigidifications over $\mathbb{A} \times \{0\}$ and $\{0\} \times \mathbb{A}^\vee$. Then, $\mathcal{P}_\mathbb{A}$ is a birigidified $\mathbf{G}_m$-torsor. Note that there is a unique morphism $\mathbb{A} \to \big(\mathbb{A}^\vee\big)^\vee$ such that $\mathcal{P}_\mathbb{A}$ is the pull-back of $\mathcal{P}_{\mathbb{A}^\vee}$ as birigidified $\mathbf{G}_m$-torsor. Such morphism has trivial kernel (cf. [21, Last Cor.§13]) and, hence, it is an isomorphism and allows one to identify $\mathcal{P}_\mathbb{A}$ and $\mathcal{P}_{\mathbb{A}^\vee}$. Furthermore, one has a natural morphism

$$j \colon \underline{\mathrm{Ext}}^{\,1}(\mathbb{A}, \mathbf{G}_m) \longrightarrow \mathbb{A}^\vee$$

of fppf sheaves on S defined as follows. Consider an fppf morphism $T \to S$ and an extension G_T of $\mathbb{A} \times_S T$ by $\mathbf{G}_{m,T}$. Then, G_T is represented by a T-group scheme by [22, Prop. 17.4]. The underlying $\mathbf{G}_{m,T}$-bundle defines a T-valued point of $\mathbb{A}^\vee$, i.e., there is a unique morphism $x \colon T \to \mathbb{A}^\vee$ such that G_T is uniquely isomorphic to the pull-back of $\mathcal{P}_\mathbb{A}$ via $\mathrm{Id} \times x \colon \mathbb{A} \times_S T \to \mathbb{A} \times_S \mathbb{A}^\vee$ as $\mathbf{G}_{m,T}$-bundles rigidified over the zero section of $\mathbb{A} \times_S T$. The functor j is an isomorphism. The inverse associates to a line bundle L over $A \times_S T$ the theta group $\mathcal{G}(L)$, extension of $\mathbb{A} \times_S T$ by $\mathbf{G}_{m,T}$ (cf. [21, Thm. 1, §23]).

Suppose we are given a 1–motive $\mathbb{M}_0$ over S_0 and a deformation $\mathbb{M} = \big[\mathbb{X} \to \mathbb{G}\big]$ over S where $\mathbb{G}$ is a semiabelian scheme, extension of $\mathbb{A}$ by a torus $\mathbb{T}$. Let $\mathbb{X}'$ be the character group of $\mathbb{T}$. The previous discussion shows that $\mathbb{G}$, as extension of $\mathbb{A}$ by $\mathbb{T}$, is defined by a homomorphism $\mathbb{X}' \to \mathbb{A}^\vee$. It also follows that $\mathcal{P}_\mathbb{A}$ inherits a unique structure of biextension, called the Poincaré $\mathbf{G}_m$-biextension on $\mathbb{A} \times_S \mathbb{A}^\vee$. In particular, $\mathbb{M}$ is defined by a trivialization, as $\mathbf{G}_m$-biextension, of the bull-back of $\mathcal{P}_\mathbb{A}$ to $\mathbb{X} \times \mathbb{X}'$. Switching the roles of $\mathbb{A}$ and $\mathbb{A}^\vee$ and of $\mathbb{X}$ and $\mathbb{X}'$, this also provides the definition of the Cartier dual $\mathbb{M}^\vee := [\mathbb{X}' \to \mathbb{G}']$ of $\mathbb{M}$ where $\mathbb{G}'$ is the extension of $\mathbb{A}^\vee$ by the torus with character group $\mathbb{X}$ defined by the morphism $\mathbb{X} \to \mathbb{A} \cong \big(\mathbb{A}^\vee\big)^\vee$. The construction of $\mathbb{M}^\vee$ goes together with the construction of the Poincaré biextension $\mathcal{P}_\mathbb{M}$ of $\mathbb{M}$ and $\mathbb{M}^\vee$ over $\mathbf{G}_m$, whose underlying biextension is the pull-back of $\mathcal{P}_\mathbb{A}$ to $\mathbb{G} \times \mathbb{G}'$ ([14, 10.2.11]). Denote furthermore by $\mathcal{P}_{\mathbb{E}(\mathbb{M})}$ the pull-back of the biextension $\mathcal{P}_\mathbb{M}$ to $\mathbb{E}\big(\mathbb{M}\big) \times_S \mathbb{E}\big(\mathbb{M}^\vee\big)$.

Given abelian schemes $\mathbb{A}, \mathbb{B}$ over S, we have a morphism

$$\rho \colon \mathrm{Hom}\Big(\mathbb{A}, \underline{\mathrm{Ext}}^{\,1}\big(\mathbb{B}^\vee, \mathbf{G}_m\big)\Big) \longrightarrow \mathrm{Biext}^1\Big(\mathbb{A}, \mathbb{B}^\vee; \mathbf{G}_m\Big)$$

associating to a morphism $\mathbb{A} \to \mathbb{B}$ the pull-back of the Poincaré biextension $\mathcal{P}_\mathbb{B}$ on $\mathbb{A} \times \mathbb{B}^\vee$. We also have a morphism in the other direction

$$\tau \colon \mathrm{Biext}^1\Big(\mathbb{A}, \mathbb{B}^\vee; \mathbf{G}_m\Big) \longrightarrow \mathrm{Hom}\Big(\mathbb{A}, \underline{\mathrm{Ext}}^{\,1}\big(\mathbb{B}^\vee, \mathbf{G}_m\big)\Big)$$

associating to a biextension Q of $\mathbb{A}\times_S\mathbb{B}^\vee$ by $\mathbf{G}_m$, to an fppf morphism $T\to S$ and a T-valued point $x\colon T\to\mathbb{A}$ the extension of $\mathbb{B}^\vee\times_S T$ by $\mathbf{G}_{m,T}$ defined by pull-back of Q via $x\times\mathrm{Id}$. In particular, if $\tau(Q)=0$ then Q is trivial as $\mathbf{G}_m$-extension of $\mathbb{B}^\vee_{\mathbb{A}}$. In particular, it is trivial as $\mathbf{G}_m$-bundle over $\mathbb{B}^\vee_{\mathbb{A}}=\mathbb{A}\times\mathbb{B}^\vee=\mathbb{A}_{\mathbb{B}^\vee}$ and, since the map j above is an isomorphism, also as $\mathbf{G}_m$-extension of $\mathbb{A}_{\mathbb{B}^\vee}$. Thus, Q is the trivial biextension, i.e., τ is injective. By construction we have that $\tau\circ\rho=\mathrm{Id}$ so that τ and ρ are isomorphisms, one being the inverse of the other producing the isomorphism $\mathrm{Hom}\Big(\mathbb{A},\mathbb{B}\Big)\cong\mathrm{Biext}^1\Big(\mathbb{A},\mathbb{B}^\vee;\mathbf{G}_m\Big)$. Consider the following diagram

$$\begin{array}{ccc}\mathrm{Hom}(\mathbb{A},\mathbb{B}) & \xrightarrow{\sim} & \mathrm{Biext}^1(\mathbb{A},\mathbb{B}^\vee;\mathbf{G}_m)\\ \wr\downarrow & & \wr\downarrow\\ \mathrm{Hom}(\mathbb{B}^\vee,\mathbb{A}^\vee) & \xrightarrow{\sim} & \mathrm{Biext}^1(\mathbb{B}^\vee,\mathbb{A};\mathbf{G}_m)\end{array}$$

where the first vertical map associates to β its dual $\beta^\vee$, the second vertical map associates to a biextension its symmetric one, while the upper and lower horizontal isomorphisms are those just seen above. Note that $\beta^\vee:\mathbb{B}^\vee\to\mathbb{A}^\vee$ is, by definition, the unique morphism such that $(\mathrm{Id}\times\beta^\vee)^*(\mathcal{P}_{\mathbb{A}})$ is isomorphic to $(\beta\times\mathrm{Id})^*(\mathcal{P}_{\mathbb{B}})$ as birigidified $\mathbf{G}_m$-torsors on $\mathbb{A}\times\mathbb{B}^\vee$, and hence as $\mathbf{G}_m$-biextensions. Thus, the diagram commutes.

Suppose we are given a morphism of 1–motives $s\colon\mathbb{M}\to\mathbb{N}$ with $\mathbb{N}:=[\mathbb{Y}\to\mathbb{H}]$ and $\mathbb{H}$ an extension of $\mathbb{B}$ by the torus $\mathbb{T}'$. Let $\mathbb{Y}'$ be the character group of $\mathbb{T}'$. Then, s defines a morphism of abelian schemes $\beta\colon\mathbb{A}\to\mathbb{B}$ and of group schemes $\alpha\colon\mathbb{X}\to\mathbb{Y}$ and $\alpha'\colon\mathbb{Y}'\to\mathbb{X}'$ and morphisms of 1–motives

$$t:=(\alpha,\beta)\colon[\mathbb{X}\to\mathbb{A}]\to[\mathbb{Y}\to\mathbb{B}]\qquad\text{and}\qquad t':=(\alpha',\beta^\vee)\colon[\mathbb{Y}'\to\mathbb{B}^\vee]\to[\mathbb{X}'\to\mathbb{A}^\vee].$$

In fact, s is defined by morphisms of 1–motives (t,t') such that the isomorphism as $\mathbf{G}_{m,S}$-biextensions of $\mathbb{A}$ and $\mathbb{B}^\vee$ between $\big(\mathrm{Id}\times\beta^\vee\big)^*\big(\mathcal{P}_{\mathbb{A}}\big)$ and $\big(\beta\times\mathrm{Id}\big)^*\big(\mathcal{P}_{\mathbb{B}}\big)$ is compatible with the induced trivializations on $\mathbb{X}\times\mathbb{Y}'$. This fact follows from the commutativity of the diagram above. In particular, any given s provides a canonical isomorphism as $\mathbf{G}_{m,S}$-biextensions of the 1–motives $\mathbb{M}$ and $\mathbb{N}^\vee$ between $\big(\mathrm{Id}\times s^\vee\big)^*\big(\mathcal{P}_{\mathbb{M}}\big)$ and $\big(s\times\mathrm{Id}\big)^*\big(\mathcal{P}_{\mathbb{N}}\big)$. Pulling-back it gives an isomorphism

$$\rho(s)\colon\big(\mathrm{Id}\times\mathbb{E}(s^\vee)\big)^*\big(\mathcal{P}_{\mathbb{E}(\mathbb{M})}\big)\xrightarrow{\sim}\big(\mathbb{E}(s)\times\mathrm{Id}\big)^*\big(\mathcal{P}_{\mathbb{E}(\mathbb{N})}\big)\tag{9}$$

as $\mathbf{G}_{m,S}$-biextensions of the 1–motives $\mathbb{E}\big(\mathbb{M}\big)$ and $\mathbb{E}\big(\mathbb{N}^\vee\big)$. This is the morphism used in Theorem 2.5(iv). By construction we have $\rho(\mathrm{Id}_{\mathbb{M}})=\mathrm{Id}_{\mathcal{P}_{\mathbb{E}(\mathbb{M})}}$.

5.2. Exp and Log

Let $S_0\hookrightarrow S$ be a locally nilpotent PD thickening defined by an ideal $\mathcal{I}$. Let $\mathcal{D}$ be a quasi-coherent and flat $\mathcal{O}_S$-algebra. The divided power structure on $\mathcal{I}$ extends uniquely to a divided power structure on $\mathcal{I}\mathcal{D}$ by the flatness of $\mathcal{D}$; see [7, Cor 3.22]. We get a homomorphism

$$\exp:\Big\{a\in\mathcal{D}|a\equiv 0\bmod\mathcal{I}\mathcal{D}\Big\}\to\Big\{m\in\mathcal{D}|m\equiv 1\bmod\mathcal{I}\mathcal{D}\Big\}$$

defined by $a\mapsto\sum_n\gamma_n(a)$. The inverse is the logarithm

$$m\mapsto\log(m):=\sum_n(n-1)!\gamma_n(m).$$

Thus, exp and log are isomorphisms.

Let G be a smooth, commutative group scheme over S. Let $\mathcal{J}$ be the $\mathcal{O}_G$-ideal defining the 0-section of G. Due to the smoothness assumption, $\mathcal{O}_G/\mathcal{J}^n$ is a locally free $\mathcal{O}_S$-module of finite rank for every $n\in\mathbf{N}$. Let $\mathcal{D}_n$ be the coherent and locally free $\mathcal{O}_S$-module defined by $\underline{\mathrm{Hom}}_{\mathcal{O}_S}\big(\mathcal{O}_G/\mathcal{J}^n,\mathcal{O}_S\big)$. Let

$$\Delta_n\colon\mathcal{D}_n\to\mathcal{D}_n\otimes_{\mathcal{O}_S}\mathcal{D}_n$$

be the dual of the multiplication on the $\mathcal{O}_S$-algebra $\mathcal{O}_G/\mathcal{J}^n$ and let $\mathrm{cu}\colon\mathcal{D}_n\to\mathcal{O}_S$ be the dual of the structural morphism $\mathcal{O}_S\to\mathcal{O}_G/\mathcal{J}^n$. The group law on G defines for every n a map $\mathcal{O}_G/\mathcal{J}^{2n}\to\big(\mathcal{O}_G/\mathcal{J}^n\big)\otimes_{\mathcal{O}_S}\big(\mathcal{O}_G/\mathcal{J}^n\big)$

and, dualizing, a map $m_n\colon \mathcal{D}_n\otimes_{\mathcal{O}_S}\mathcal{D}_n\to\mathcal{D}_{2n}$. Similarly the 0-section of G defines a morphism $\varepsilon_n\colon \mathcal{O}_S\to\mathcal{D}_n$. Write $\mathcal{D}$ for the direct limit $\mathcal{D}:=\lim_n\mathcal{D}_n$. It is locally free $\mathcal{O}_S$-module (of infinite rank) and $m:=\lim_n m_n$ and $\varepsilon:=\lim_n\varepsilon_n$ define the structure of $\mathcal{O}_S$-algebra on $\mathcal{D}$. Put $\Delta:=\lim_n\Delta_n$ and $\mathrm{cu}:=\lim_n\mathrm{cu}_n$.

Fix an fppf morphism $q\colon T\to S$. Since $\mathcal{I}$ is a nilpotent ideal, the T-valued points of G, reducing to the identity section over $T_0:=T\times_S S_0$, factor via $q^*\big(\mathcal{O}_G/\mathcal{J}^N\big)$ for some $N\in\mathbf{N}$ and correspond to the subgroup of $\Gamma\big(T,q^*(\mathcal{D})\big)$ consisting of homomorphisms $q^*\big(\mathcal{O}_G/\mathcal{J}^N\big)\to\mathcal{O}_T$ respecting the structure of $\mathcal{O}_T$-algebras and congruent to 1 modulo $\mathcal{I}\mathcal{O}_T$. Thus,

$$\mathrm{Ker}\,\big(G(T)\to G(T_0)\big)=\big\{x\in\Gamma\big(T,1+\mathcal{I}q^*(\mathcal{D})\big)|\Delta(x)=x\otimes x,\,\mathrm{cu}(x)=1\big\}\,.$$

On the other hand, $\mathbf{Lie}\,(G)$ is the vector group scheme over S whose values for every fppf morphism $q\colon T\to S$ consist of the morphisms of T-schemes $\tau\colon T[\varepsilon]\to G\times_S T$ such that $\tau\equiv 0 \mod \varepsilon$. Here, $T[\varepsilon]$ are the relative dual numbers over T. Hence,

$$\mathbf{Lie}\,G(T)=\big\{d\in\Gamma\big(T,q^*(\mathcal{D})\big)|\Delta(d)=d\otimes 1+1\otimes d,\,\mathrm{cu}(d)=0\big\}$$

and

$$\mathrm{Ker}\,\left(\mathbf{Lie}\,G(T)\to\mathbf{Lie}\,G(T_0)\right)=\big\{d\in\Gamma\big(T,\mathcal{I}q^*(\mathcal{D})\big)|\Delta(d)=d\otimes 1+1\otimes d,\,\mathrm{cu}(d)=0\big\}\,.$$

Since $\mathcal{D}$ is a flat $\mathcal{O}_S$-module and $q\colon T\to S$ is also flat, the exponential and the logarithm on $\mathcal{I}q^*(\mathcal{D})$ define a group isomorphism

$$\exp_G\colon \mathrm{Ker}\,\left(\mathbf{Lie}\,G(T)\to\mathbf{Lie}\,G(T_0)\right)\longrightarrow\mathrm{Ker}\,\big(G(T)\to G(T_0)\big);$$

see [20, Rmk III.2.2.6]. This is an isomorphism of sheaves for the fppf topology on S. If $\rho\colon H\to G$ is a morphism of commutative and smooth S-group schemes, we get a commutative diagram, functorial in T:

$$\begin{array}{ccc}\mathrm{Ker}\,\left(\mathbf{Lie}\,H(T)\to\mathbf{Lie}\,H(T_0)\right) & \xrightarrow{\exp_H} & \mathrm{Ker}\,\big(H(T)\to H(T_0)\big)\\ {\scriptstyle\mathbf{Lie}\,\rho}\big\downarrow & & {\scriptstyle\rho}\big\downarrow\\ \mathrm{Ker}\,\left(\mathbf{Lie}\,G(T)\to\mathbf{Lie}\,G(T_0)\right) & \xrightarrow{\exp_G} & \mathrm{Ker}\,\big(G(T)\to G(T_0)\big)\end{array}$$

5.3. Deformations of biextensions

Let S_0 be a scheme. Let $S_0\hookrightarrow S$ be a thickening defined by an ideal $\mathcal{I}$ with nilpotent divided powers.

Lemma 5.1. *Let X be a flat and separated group scheme over S. The group of isomorphism classes of $\mathbf{G}_{m,S}$-torsors over X, deforming the trivial torsor over $X_0:=X\times_S S_0$, is naturally isomorphic to the group of isomorphism classes of $\mathbf{G}_{a,S}$-torsors over X, deforming the trivial torsor over X_0.*

The group of isomorphism classes of extensions of X by $\mathbf{G}_{m,S}$, deforming the trivial extension over S_0, is naturally isomorphic to the group of isomorphism classes of extensions of X by $\mathbf{G}_{a,S}$, deforming the trivial extension over S_0.

Given two representatives P_m and P_a of corresponding isomorphism classes of torsors (resp. extensions), there is a natural isomorphism between the group of automorphisms of P_m, preserving the trivialization over X_0, and the group of automorphisms of P_a, preserving the trivialization over X_0.

Proof. Let $\mathbf{G}=\mathbf{G}_{m,S}$ or $\mathbf{G}_{a,S}$. Let P be a $\mathbf{G}$-torsor over X and let

$$\alpha\colon P\times_S S_0\xrightarrow{\sim}(X\times_S\mathbf{G})\times_S S_0$$

be a trivialization as a $\mathbf{G}$-torsor over X_0. Choose an open affine covering $\{U_i\}_i$ of X and, for each i, a trivialization

$$\beta_i\colon P\times_X U_i\xrightarrow{\sim}\mathbf{G}\times_S U_i$$

compatible with α. For $i\neq j$, let $\mathrm{Spec}\,(\mathcal{D}_{i,j}):=U_{i,j}$. The trivializations β_i and β_j restricted to $U_{i,j}:=U_i\times_X U_j$ differ by

(i) a multiplicative cocycle $m_{i,j} \in \mathcal{D}^*_{i,j}$ such that $m_{i,j}$ is 1 mod $\mathcal{I}$ if $\mathbf{G} = \mathbf{G}_{m,S}$;

(ii) an additive cocycle $a_{i,j} \in \mathcal{D}_{i,j}$ such that $a_{i,j}$ is 0 mod $\mathcal{I}$ if $\mathbf{G} = \mathbf{G}_{a,S}$.

In case (i) define the $\mathbf{G}_{a,S}$-torsor P_a by the additive cocycle $a_{i,j} := \log\bigl(m_{i,j}\bigr)$ in $\mathcal{D}_{i,j}$; see §5.2 for log. In case (ii) define the $\mathbf{G}_{m,S}$-torsor P_m by the multiplicative cocycle $m_{i,j} := \exp\bigl(a_{i,j}\bigr) \in \mathcal{D}_{i,j}$. It is easily checked that these maps define an equivalence between $\mathbf{G}_{m,S}$-torsors and $\mathbf{G}_{a,S}$-torsors with a trivialization over X_0.

The group of automorphisms of P as $\mathbf{G}$-torsor over X, reducing to the identity over X_0, coincides with the kernel of $\mathrm{H}^0(X,\mathbf{G}) \to \mathrm{H}^0(X_0,\mathbf{G})$, i.e., with the group of morphisms as S-schemes $X \to \mathbf{G}$ reducing to the trivial one over X_0. For $\mathbf{G} = \mathbf{G}_m$ the latter group coincides with the elements $x \in \Gamma\bigl(X, \mathcal{O}_X\bigr)$ such that $x \equiv 1$ modulo $\mathcal{I}$. For $\mathbf{G} = \mathbf{G}_a$ it coincides with the elements $x \in \Gamma\bigl(X, \mathcal{O}_X\bigr)$ such that $x \equiv 0$ modulo $\mathcal{I}$. Since X is flat over S, the exponential and logarithm maps are defined over $\mathcal{I}\mathcal{O}_X$ and define a bijection between those two sets.

Let m, p_1 and p_2 be the maps from $X \times_S X$ to X defined by the multiplication, the first and the second projection respectively. Let P be a $\mathbf{G}$-torsor over X which is trivial over S_0. Giving a multiplication law on P, compatible with the one on X and with the action of $\mathbf{G}$ and inducing the standard group law on $P \times_S S_0$, is equivalent to giving an isomorphism

$$X \times_S X \times_S \mathbf{G} \longrightarrow m^*(P)p_1^*(P)^{-1}p_2^*(P)^{-1}$$

as $\mathbf{G}$-torsors reducing to the identity after base change to S_0. Using the identifications given above between $\mathbf{G}_{m,S}$-torsors and $\mathbf{G}_{a,S}$-torsors (with trivialization over X_0) and between their automorphism groups, one passes from the case $\mathbf{G} = \mathbf{G}_{m,S}$ to the case $\mathbf{G} = \mathbf{G}_{a,S}$ and vice versa. Since the commutativity and the associativity of the multiplication are preserved via this correspondence, we get the claimed correspondence for the extensions.

Given an extension P of X by $\mathbf{G}$ with a trivialization over S_0, the group of automorphisms of P as extension coincides with the group of homomorphisms $X \to \mathbf{G}$ reducing to the trivial one over S_0. The correspondence associates to an automorphism f the endomorphism $f - \mathrm{Id}$ which factors via $P \to X$ and induces a homomorphism $X \to \mathbf{G}$, reducing to the trivial one over S_0. Let $\Delta\colon \Gamma\bigl(X, \mathcal{O}_X\bigr) \to \Gamma\bigl(X \times_S X, \mathcal{O}_{X\times_S X}\bigr)$ be the map induced by the multiplication law on X. For $\mathbf{G} = \mathbf{G}_m$ the above group of automorphisms is identified with the elements $x \in \Gamma\bigl(X, \mathcal{O}_X\bigr)$ such that $x \equiv 1$ modulo $\mathcal{I}$ (and $\Delta(x) = x \otimes x$ in the case of extensions). For $\mathbf{G} = \mathbf{G}_a$ it gets identified with the elements $x \in \Gamma\bigl(X, \mathcal{O}_X\bigr)$ such that $x \equiv 0$ modulo $\mathcal{I}$ (and $\Delta(x) = x \otimes 1 + 1 \otimes x$ in the case of extensions). The bijection between $\{x \in \Gamma\bigl(X, \mathcal{O}_X\bigr) | x \equiv 1 \mod \mathcal{I}\}$ and $\{x \in \Gamma\bigl(X, \mathcal{O}_X\bigr) | x \equiv 0 \mod \mathcal{I}\}$, defined using the exponential and the logarithm, preserves the conditions that $\Delta(x) = x \otimes x$ and $\Delta(x) = x \otimes 1 + 1 \otimes x$ respectively. This proves the correspondence between the automorphism groups of extensions. □

Definition 5.2. *Let A and B be flat and separated group schemes over S and let $\mathbf{G}$ be a commutative S-group scheme. Let $\mathrm{Biext}^1_{S/S_0}\Bigl(A, B; \mathbf{G}\Bigr)$ be the set of isomorphism classes of pairs consisting of a biextension P of A and B by $\mathbf{G}$, see [16, Ex. VII], and of an isomorphism ζ_{P_0} of $P \times_S S_0$ with the trivial biextension, called a* trivialization.

One verifies as in [16, Ex. VII, §2.5] that $\mathrm{Biext}^1_{S/S_0}\Bigl(A, B; \mathbf{G}\Bigr)$ has a group structure. More generally,

Definition 5.3. *Let $\mathbf{G}$ be as above and let $M = [X \to A]$ and $N = [Y \to B]$ be complexes of flat and separated group schemes over S. Let $\mathrm{Biext}^1_{S/S_0}\Bigl(M, N; \mathbf{G}\Bigr)$ be the set of isomorphism classes of pairs $(P,\ \zeta_{P_0})$ consisting of a biextension P of M and N by $\mathbf{G}$, (i.e., a biextension of A and B by $\mathbf{G}$ together with trivializations ψ_1 on $X \times B$ and ψ_2 on $A \times Y$ that coincide on $X \times Y$, see [14, 10.2.1]), and of an isomorphism ζ_{P_0} of $P \times_S S_0$ with the trivial biextension, called a* trivialization.

Let $\mathrm{Biext}^0_{S/S_0}\Bigl(M, N; \mathbf{G}\Bigr) = \ker\Bigl(\mathrm{Biext}^0\left(M, N; \mathbf{G}\right) \longrightarrow \mathrm{Biext}^0\left(M_0, N_0, \mathbf{G}\right)\Bigr)$, *where* $\mathrm{Biext}^0\left(M, N; \mathbf{G}\right) = \mathrm{Hom}\left(H^0(M) \otimes H^0(N), \mathbf{G}\right)$ *stands for pairs of bilinear homomorphisms $A \times Y \to \mathbf{G}$ and $X \times B \to \mathbf{G}$ that coincide on $X \times Y$ ([14, 10.2]).*

Then,

Corollary 5.4. *There is an isomorphism*

$$\mathrm{Biext}^1_{S/S_0}\Big(M,N;\mathbf{G}_{m,S}\Big) \xrightarrow{\sim} \mathrm{Biext}^1_{S/S_0}\Big(M,N;\mathbf{G}_{a,S}\Big).$$

Given two representatives P_m and P_a of corresponding isomorphism classes there is a natural isomorphism between the group $\mathrm{Biext}^0_{S/S_0}\Big(M,N;\mathbf{G}_{m,S}\Big)$ of automorphisms of P_m, preserving the trivialization over S_0, and the group $\mathrm{Biext}^0_{S/S_0}\Big(M,N;\mathbf{G}_{a,S}\Big)$ of automorphisms of P_a, preserving the trivialization over S_0.

Proof. A biextension of M and N by $\mathbf{G}$ is given by a $\mathbf{G}$-torsor P over $A\times_S B$ and two structures of extensions on P. The first is an extension of the A-group scheme $A\times_S B$ by $\mathbf{G}$ with a trivialization ψ_1 over $A\times Y$ and the second is an extension of the B-group scheme $A\times_S B$ by $\mathbf{G}$ with a trivialization ψ_2 over $X\times B$ that coincides with ψ_1 over $X\times Y$. The two structures are supposed to be compatible [16, Ex. VII, Def. 2.1] and the compatibility is expressed by imposing that two trivializations of a given $\mathbf{G}$-torsor are the same. The first claim follows then from Lemma 5.1.

An automorphism of a biextension P is an automorphism as $\mathbf{G}$-torsor, inducing automorphisms on the two structures of extension given on P and preserving the trivializations ψ_i. The second claim follows again from Lemma 5.1. $\square$

Before applying the previous results to universal extensions of 1–motives, we need to remark the following fact:

Remark 5.5. *Let $\mathbb{N}=[\mathbb{Y}\to\mathbb{H}]$ be a 1–motive over S. We have a long exact sequence of groups*

$$0\longrightarrow \mathrm{Hom}\big(\mathbb{N},\mathbf{G}_{a,S}\big)\longrightarrow \mathrm{Hom}\big(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)\longrightarrow \mathrm{Hom}\big(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)$$
$$\xrightarrow{\delta} \mathrm{Ext}^1\big(\mathbb{N},\mathbf{G}_{a,S}\big)\longrightarrow \mathrm{Ext}^1\big(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)\longrightarrow \mathrm{Ext}^1\big(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big).$$

where

a) $\mathrm{Hom}\big(\mathbb{N},\mathbf{G}_{a,S}\big)=\{0\}$ *since $\mathbb{H}\to S$ is semi-abelian;*

b) the homomorphism δ is defined by push-out and is split surjective. Indeed, let $\mathrm{Hom}_{\mathcal{O}_S}\big(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)$ *be the group of homomorphisms as vector groups. It is a subgroup of* $\mathrm{Hom}\big(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)$ *and it maps isomorphically to* $\mathrm{Ext}^1\big(\mathbb{N},\mathbf{G}_{a,S}\big)$ *via δ by definition of universal extension.*

Nevertheless, the map δ is not an isomorphism in general as mistakenly assumed in [1, Prop 3.3.3]. If S is flat over $\mathbf{Z}$, then $\mathrm{Hom}(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S})=0$ and δ is an isomorphism.

Proposition 5.6. *Let $\mathbb{N}=[\mathbb{Y}\to\mathbb{H}]$ be a 1–motive over S. Let $M=[X\to A]$ be a complex of flat and separated group schemes over S. Let $\mathbf{G}$ be $\mathbf{G}_{a,S}$ or $\mathbf{G}_{m,S}$. The morphism*

$$\mathrm{Biext}^1_{S/S_0}\Big(M,\mathbb{E}(\mathbb{N});\mathbf{G}\Big)\to \mathrm{Biext}^1_{S/S_0}\Big(M,\mathbb{W}(\mathbb{N});\mathbf{G}\Big),$$

defined by pull-back via $\mathbb{W}(\mathbb{N})\to\mathbb{E}(\mathbb{N})$, is injective. Such identification is compatible with the isomorphisms in Lemma 5.1 and Corollary 5.4.

Proof. Put $N=\mathbb{E}(\mathbb{N})$ or $\mathbb{W}(\mathbb{N})$. The groups $\mathrm{Biext}^1_{S/S_0}\Big(M,N;\mathbf{G}\Big)$ are identified for $\mathbf{G}=\mathbf{G}_{m,S}$ and $\mathbf{G}=\mathbf{G}_{a,S}$ due to Corollary 5.4. Thus, we may assume that $\mathbf{G}=\mathbf{G}_{a,S}$. Consider the natural map

$$\mathrm{Biext}^1_{S/S_0}\Big(M,N;\mathbf{G}_{a,S}\Big)\to \mathrm{Biext}^1\Big(M,N;\mathbf{G}_{a,S}\Big),$$

to the group of isomorphism classes of biextensions of M and N by $\mathbf{G}_{a,S}$, defined by forgetting the trivialization over S_0. Let $\mathcal{K}\big(M,N,\mathbf{G}_{a,S}\big)$ be its kernel. It suffices to prove that the maps

$$\upsilon\colon \mathrm{Biext}^1\Big(M,\mathbb{E}(\mathbb{N});\mathbf{G}_{a,S}\Big)\longrightarrow \mathrm{Biext}^1\Big(M,\mathbb{W}(\mathbb{N});\mathbf{G}_{a,S}\Big),\quad \lambda\colon \mathcal{K}\big(M,\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)\longrightarrow \mathcal{K}\big(M,\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)$$

induced by the map $\mathbb{W}(\mathbb{N})\to\mathbb{E}(\mathbb{N})$ are injective.

Step I: Injectivity of υ: We have the long exact sequence of sheaves for the flat topology

$$\begin{aligned}0\longrightarrow\underline{\mathrm{Hom}}\,\big(\mathbb{N},\mathbf{G}_{a,S}\big)\longrightarrow\underline{\mathrm{Hom}}\,\big(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)\longrightarrow\underline{\mathrm{Hom}}\,\big(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)&\\ \xrightarrow{\delta}\underline{\mathrm{Ext}}^{\,1}\big(\mathbb{N},\mathbf{G}_{a,S}\big)\longrightarrow\underline{\mathrm{Ext}}^{\,1}\big(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)\longrightarrow\underline{\mathrm{Ext}}^{\,1}\big(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)&.\end{aligned}$$

By Remark 5.5 a), one has $\underline{\mathrm{Hom}}\,\big(\mathbb{N},\mathbf{G}_{a,S}\big)=\{0\}$ and the map δ defined by push forward is split surjective. Furthermore, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1\Big(M,\underline{\mathrm{Hom}}\big(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)\Big) \arrow[r]\arrow[d] & \mathrm{Biext}^1\Big(M,\mathbb{E}(\mathbb{N});\mathbf{G}_{a,S}\Big) \arrow[r]\arrow[d,"\upsilon"'] & \mathrm{Hom}\Big(M,\underline{\mathrm{Ext}}^{\,1}\big(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)\Big)\arrow[d]\\
\mathrm{Ext}^1\Big(M,\underline{\mathrm{Hom}}\big(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)\Big) \arrow[r] & \mathrm{Biext}^1\Big(M,\mathbb{W}(\mathbb{N});\mathbf{G}_{a,S}\Big) \arrow[r] & \mathrm{Hom}\Big(M,\underline{\mathrm{Ext}}^{\,1}\big(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)\Big).
\end{tikzcd}$$

where the horizontal sequences come from [16, Ex. VIII, 1.1.4] and are exact on the left. The vertical maps are induced by the morphism $\mathbb{W}(\mathbb{N})\to\mathbb{E}(\mathbb{N})$. Since the vertical maps on the right and on the left are injective by Remark 5.5 a) & b), we conclude that υ is injective.

Step II: Injectivity of λ: Since

$$\mathrm{Biext}^0_{S/S_0}\Big(X,\mathbb{E}(\mathbb{N});\mathbf{G}_{a,S}\Big)\cong\ker\Big(\mathrm{Biext}^1_{S/S_0}(M,\mathbb{E}(\mathbb{N});\mathbf{G}_{a,S})\xrightarrow{f}\mathrm{Biext}^1_{S/S_0}(A,\mathbb{E}(\mathbb{N});\mathbf{G}_{a,S})\Big),$$

where the map f is obtained by forgetting the trivializations ψ_2 over $X\times\mathbb{E}(\mathbb{N})_{\mathbb{H}}$, such kernel maps injectively into

$$\mathrm{Biext}^0\Big(X,\mathbb{E}(\mathbb{N});\mathbf{G}_{a,S}\Big)\cong\ker\Big(\mathrm{Biext}^1(M,\mathbb{E}(\mathbb{N});\mathbf{G}_{a,S})\to\mathrm{Biext}^1(A,\mathbb{E}(\mathbb{N});\mathbf{G}_{a,S})\Big).$$

Hence we have a canonical injection $\mathcal{K}\big(M,\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)\to\mathcal{K}\big(A,\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)$. Similarly we have an injection $\mathcal{K}\big(M,\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)\to\mathcal{K}\big(A,\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)$. Hence we are reduced to proving that

$$\lambda'\colon\mathcal{K}\big(A,\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)\to\mathcal{K}\big(A,\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}\big)$$

is injective. Let $P\in\mathrm{Ker}\,(\lambda')$. It corresponds to a biextension of A and $\mathbb{E}(\mathbb{N})$ by $\mathbf{G}_{a,S}$ endowed with (i) a trivialization ζ_{S_0} over $\big(A\times_S\mathbb{E}(\mathbb{N})\big)\times_S S_0$, (ii) an isomorphism ϕ with the trivial biextension since it lies in $\mathcal{K}\big(A,\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S}\big)$ and (iii) with a trivialization η over $A\times\mathbb{W}(\mathbb{N})$ such that $\eta\times_S S_0$ and ζ_{S_0} define the same trivialization over $\big(A\times\mathbb{W}(\mathbb{N})\big)\times_S S_0$ since $\lambda'(P)=0$.

The trivialization $\big(\phi\times_S S_0\big)\zeta_{S_0}^{-1}$ defines an element in $\mathrm{Biext}^0\big(A_{S_0},\mathbb{E}(\mathbb{N})\times_S S_0;\mathbf{G}_{a,S_0}\big)$. Similarly consider $\phi|_{A\times\mathbb{W}(\mathbb{N})}\eta^{-1}$ as element of $\mathrm{Biext}^0\big(A,\mathbb{W}(\mathbb{N});\mathbf{G}_{a,S}\big)=\mathrm{Bil}\big(A,\mathbb{W}(\mathbb{N});\mathbf{G}_{a,S}\big)$, i.e., it is a map $A\times_S\mathbb{W}(\mathbb{N})\to\mathbf{G}_{a,S}$ that is a homomorphism in each component. Its image in $\mathrm{Bil}\big(A_{S_0},\mathbb{W}(\mathbb{N})\times_S S_0;\mathbf{G}_{a,S_0}\big)$ is the same as the image of $\big(\phi\times_S S_0\big)\zeta_{S_0}^{-1}$.

Note that $\mathrm{Biext}^0\big(A,N;\mathbf{G}_{a,S}\big)\cong\mathrm{Hom}\,(A,\underline{\mathrm{Hom}}\,(N,\mathbf{G}_{a,S}))$ for $N=\mathbb{W}(\mathbb{N})$ and $N=\mathbb{E}(\mathbb{N})$. Furthermore, we have

$$\underline{\mathrm{Hom}}\,(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S})\cong\underline{\mathrm{Hom}}\,(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S})\oplus\underline{\mathrm{Hom}}_{\mathcal{O}_S}\,(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S})\tag{10}$$

(cf. Remark 5.5) and the same over S_0. We then get a direct sum decomposition

$$\mathrm{Biext}^0\big(A,\mathbb{W}(\mathbb{N});\mathbf{G}_{a,S}\big)=\mathrm{Biext}^0\,(A,\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,S})\oplus\mathrm{Hom}\,(X,\underline{\mathrm{Hom}}_{\mathcal{O}_S}\,(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,S}))$$

and similarly over S_0. Write $\phi|_{A\times\mathbb{W}(\mathbb{N})}\eta^{-1} =: (\alpha,\beta)$ in this decomposition. Its image in the group $\mathrm{Biext}^0\big((A\times\mathbb{W}(\mathbb{N}))\times_S S_0;\mathbf{G}_{a,S_0}\big)$, using the analogous direct sum decomposition, is $\big((\phi\times_S S_0)\zeta_{S_0}^{-1},0\big)$. Thus, replacing ϕ with the trivialization $\phi' := \phi\alpha^{-1}$ we get a trivialization of the biextension P such that $\phi'\times_S S_0 = \zeta_{S_0}$, i.e., P is trivial in $\mathrm{Biext}^1_{S/S_0}\Big(A,\mathbb{E}(\mathbb{N});\mathbf{G}_{a,S}\Big)$ which proves the injectivity of λ'. □

Proposition 5.7. *Let $\mathbb{M}$ and $\mathbb{N}$ be 1–motives over T with $T=S$ or S_0. We have a direct sum decomposition*

$$\begin{aligned}\mathrm{Biext}^0\left(\mathbb{W}(\mathbb{N}),\mathbb{W}(\mathbb{M});\mathbf{G}_{a,T}\right) = \\ \mathrm{Biext}^0\left(\mathbb{E}(\mathbb{N}),\mathbb{E}(\mathbb{M}),\mathbf{G}_{a,T}\right)\oplus \mathrm{Hom}\left(\mathbb{E}(\mathbb{N}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{M}),\mathbf{G}_{a,T}\right)\right)\oplus \\ \oplus\,\mathrm{Hom}\big(\mathbb{E}(\mathbb{M}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}\right)\big)\oplus \mathrm{Bil}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{N}),\mathbb{W}(\mathbb{M}),\mathbf{G}_{a,T}\right). \quad (11)\end{aligned}$$

Here, $\underline{\mathrm{Hom}}_{\mathcal{O}_T}(-,\mathbf{G}_{a,T})$ stands for the homomorphisms as vector groups and $\mathrm{Bil}_{\mathcal{O}_T}(-,-,\mathbf{G}_{a,T})$ stands for $\mathbf{G}_{a,T}$-bilinear maps. The morphisms $\mathbb{W}(\mathbb{N})\to\mathbb{E}(\mathbb{N})$ and $\mathbb{W}(\mathbb{M})\to\mathbb{E}(\mathbb{M})$ identify

$$\mathrm{Biext}^0(\mathbb{E}(\mathbb{N}),\mathbb{W}(\mathbb{M});\mathbf{G}_{a,T})\cong \mathrm{Biext}^0\left(\mathbb{E}(\mathbb{N}),\mathbb{E}(\mathbb{M}),\mathbf{G}_{a,T}\right)\oplus\mathrm{Hom}\left(\mathbb{E}(\mathbb{N}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{M}),\mathbf{G}_{a,T}\right)\right)$$

and

$$\mathrm{Biext}^0(\mathbb{W}(\mathbb{N}),\mathbb{E}(\mathbb{M});\mathbf{G}_{a,T})\cong \mathrm{Biext}^0\left(\mathbb{E}(\mathbb{N}),\mathbb{E}(\mathbb{M}),\mathbf{G}_{a,T}\right)\oplus\mathrm{Hom}\left(\mathbb{E}(\mathbb{M}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}\right)\right)$$

in this decomposition.

These decompositions and identifications are functorial in $\mathbb{M}$ and $\mathbb{N}$ and in T. In particular, the group $\mathrm{Biext}^0_{S/S_0}\left(\mathbb{W}(\mathbb{N}),\mathbb{W}(\mathbb{M});\mathbf{G}\right)$ is a direct summand of $\mathrm{Biext}^0_{S/S_0}\big(\mathbb{M},\mathbb{N};\mathbf{G}\big)$ for $\mathbf{G}=\mathbf{G}_{a,S}$ or $\mathbf{G}_{m,S}$ functorially in $\mathbb{M}$ and $\mathbb{N}$ and in $(S_0\subset S)$.

Proof. We have $\mathrm{Biext}^0\left(\mathbb{W}(\mathbb{N}),\mathbb{W}(\mathbb{M});\mathbf{G}_{a,T}\right)=\mathrm{Hom}\left(\mathbb{W}(\mathbb{M}),\underline{\mathrm{Hom}}\left(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}\right)\right)$ by adjunction. By (10) we then get

$$\mathrm{Biext}^0\left(\mathbb{W}(\mathbb{N}),\mathbb{W}(\mathbb{M});\mathbf{G}_{a,T}\right)=\mathrm{Hom}\left(\mathbb{W}(\mathbb{M}),\underline{\mathrm{Hom}}\left(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,T}\right)\right)\oplus\mathrm{Hom}\left(\mathbb{W}(\mathbb{M}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}\right)\right).$$

Analogously,

$$\begin{aligned}&\mathrm{Hom}\big(\mathbb{W}(\mathbb{M}),\underline{\mathrm{Hom}}\left(\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,T}\right)\big)=\\ &\quad=\mathrm{Biext}^0\left(\mathbb{W}(\mathbb{M}),\mathbb{E}(\mathbb{N}),\mathbf{G}_{a,T}\right)=\mathrm{Hom}\left(\mathbb{E}(\mathbb{N}),\underline{\mathrm{Hom}}\left(\mathbb{W}(\mathbb{M}),\mathbf{G}_{a,T}\right)\right)\\ &\quad=\mathrm{Hom}\Big(\mathbb{E}(\mathbb{N}),\underline{\mathrm{Hom}}\left(\mathbb{E}(\mathbb{M}),\mathbf{G}_{a,T}\right)\oplus\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{M}),\mathbf{G}_{a,T}\right)\Big)\\ &\quad=\mathrm{Biext}^0\left(\mathbb{E}(\mathbb{N}),\mathbb{E}(\mathbb{M}),\mathbf{G}_{a,T}\right)\oplus\mathrm{Hom}\left(\mathbb{E}(\mathbb{N}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{M}),\mathbf{G}_{a,T}\right)\right).\end{aligned}$$

Since $\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}\right)$ is a vector group scheme, we have

$$\begin{aligned}&\mathrm{Hom}\big(\mathbb{W}(\mathbb{M}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}\right)\big)\\ &\quad=\mathrm{Hom}_{\mathcal{O}_T}(\mathbb{W}(\mathbb{M}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}))\oplus\mathrm{Hom}\big(\mathbb{E}(\mathbb{M}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}\big)\\ &\quad=\mathrm{Bil}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{N}),\mathbb{W}(\mathbb{M}),\mathbf{G}_{a,T}\right)\oplus\mathrm{Hom}\big(\mathbb{E}(\mathbb{M}),\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}\right)\big)\end{aligned}$$

where $\mathrm{Bil}_{\mathcal{O}_T}\left(\mathbb{W}(\mathbb{M}),\mathbb{W}(\mathbb{N}),\mathbf{G}_{a,T}\right)$ is the group of $\mathbf{G}_{a,T}$-bilinear maps. Putting everything together we get the claimed direct sum decomposition which is functorial in $\mathbb{M}$ and $\mathbb{N}$ and in T. The last statement for $\mathbf{G}=\mathbf{G}_{a,S}$ follows from the definition of $\mathrm{Biext}^0_{S/S_0}\big(\mathbb{M},\mathbb{N};\mathbf{G}_{a,S}\big)$. The statement for $\mathbf{G}=\mathbf{G}_{m,S}$ follows from this and Corollary 5.4. □

5.4. The crystalline nature of the Poincaré biextension. Proof of Theorem 2.5

Let $(S_0 \subset S)$ be a locally nilpotent PD thickening. Let $s_0\colon \mathbb{M}_0 \to \mathbb{N}_0$ be a morphism of 1–motives over S_0. Let $\mathbb{M} := [\mathbb{X} \to \mathbb{G}]$ and $\mathbb{N} := [\mathbb{Y} \to \mathbb{H}]$ be 1-motives over S deforming $\mathbb{M}_0$ and $\mathbb{N}_0$ respectively. Consider the morphisms of motives

$$\begin{aligned} \mathrm{Id} \times \mathbb{E}_S(s_0^\vee)\colon\ \mathbb{E}\big(\mathbb{M}\big) \times_S \mathbb{E}\big(\mathbb{N}^\vee\big) &\longrightarrow \mathbb{E}\big(\mathbb{M}\big) \times_S \mathbb{E}\big(\mathbb{M}^\vee\big) \\ \mathbb{E}_S(s_0) \times \mathrm{Id}\colon\ \mathbb{E}\big(\mathbb{M}\big) \times_S \mathbb{E}\big(\mathbb{N}^\vee\big) &\longrightarrow \mathbb{E}\big(\mathbb{N}\big) \times_S \mathbb{E}\big(\mathbb{N}^\vee\big) \end{aligned} \tag{12}$$

where $\mathbb{E}_S(s_0)$, $\mathbb{E}_S(s_0^\vee)$ are those in Theorem 2.1. Let $\mathcal{P}_{\mathbb{E}(\mathbb{M})}$ be the pull-back of the Poincaré biextension $\mathcal{P}_{\mathbb{M}}$ on $\mathbb{M}\times_S\mathbb{M}^\vee$ to $\mathbb{E}(\mathbb{M})\times_S\mathbb{E}(\mathbb{M}^\vee)$ and let $\mathcal{P}_{\mathbb{E}(\mathbb{N})}$ be the pull-back of the Poincaré biextension $\mathcal{P}_{\mathbb{N}}$ on $\mathbb{N}\times_S\mathbb{N}^\vee$ to $\mathbb{E}(\mathbb{N})\times_S\mathbb{E}(\mathbb{N}^\vee)$. Call $\mathcal{P}'_{\mathbb{M}}$ (resp. $\mathcal{P}'_{\mathbb{N}}$) the pull-back of $\mathcal{P}_{\mathbb{E}(\mathbb{M})}$ (resp. $\mathcal{P}_{\mathbb{E}(\mathbb{N})}$) to $\mathbb{E}\big(\mathbb{M}\big)\times_S\mathbb{E}\big(\mathbb{N}^\vee\big)$ via the maps in (12). The pull-back of $\mathcal{P}'_{\mathbb{M}}$ to $\mathbb{W}\big(\mathbb{M}\big)\times_S\mathbb{W}\big(\mathbb{N}^\vee\big)$ admits a canonical trivialization $\xi_{\mathbb{M}}$ since it is obtained from the map $\mathbb{W}\big(\mathbb{M}\big)\to\mathbb{M}$ which is the zero map by construction and since $\mathcal{P}_{\mathbb{M}}|_{\{0\}\times\mathbb{M}^\vee}$ is canonically trivialized. Analogously, the pull-back of $\mathcal{P}'_{\mathbb{N}}$ to $\mathbb{W}\big(\mathbb{M}\big)\times_S\mathbb{W}\big(\mathbb{N}^\vee\big)$ admits a canonical trivialization $\xi_{\mathbb{N}}$ since the map $\mathbb{W}\big(\mathbb{N}^\vee\big)\to\mathbb{N}^\vee$ is the zero map. Moreover, by (9) we have a canonical isomorphism $\rho_{S_0}(s_0)$ between $\mathcal{P}'_{\mathbb{M}}$ and $\mathcal{P}'_{\mathbb{N}}$ over S_0. Thus, $\mathcal{P}'_{\mathbb{M}}\big(\mathcal{P}'_{\mathbb{N}}\big)^{-1}$ is a $\mathbf{G}_{m,S}$-biextension of $\mathbb{E}\big(\mathbb{M}\big)$ and $\mathbb{E}\big(\mathbb{N}^\vee\big)$ endowed with a trivialization ζ_0 over S_0 (induced by $\rho_{S_0}(s_0)$) and a trivialization $\xi=\xi_{\mathbb{M}}\xi_{\mathbb{N}}^{-1}$ over $\mathbb{W}\big(\mathbb{M}\big)\times_S\mathbb{W}\big(\mathbb{N}^\vee\big)$, defining the same trivialization as ζ_0 over $\mathbb{W}\big(\mathbb{M}_0\big)\times\mathbb{W}\big(\mathbb{N}_0^\vee\big)$. It follows from Corollary 5.4, Proposition 5.6 and Proposition 5.7 that there is a unique trivialization $\zeta_S(s_0)$ of the biextension $\mathcal{P}'_{\mathbb{M}}\big(\mathcal{P}'_{\mathbb{N}}\big)^{-1}$ deforming ζ_0 and such that the component of

$$\zeta_S(s_0)|_{\mathbb{W}\left(\mathbb{M}\right)\times_S\mathbb{W}\left(\mathbb{N}^\vee\right)}\xi^{-1}\in \mathrm{Biext}^0_{S/S_0}\left(\mathbb{W}\big(\mathbb{M}\big),\mathbb{W}\big(\mathbb{N}^\vee\big);\mathbf{G}_{m,S}\right)$$

in the factor $\mathrm{Biext}^0_{S/S_0}\left(\mathbb{E}\big(\mathbb{M}\big),\mathbb{E}\big(\mathbb{N}^\vee\big);\mathbf{G}_{m,S}\right)$ of the decomposition analogous to (11) is 0. This produces an isomorphism of biextensions of $\mathbb{E}(\mathbb{M})$ and $\mathbb{E}(\mathbb{N}^\vee)$ by $\mathbf{G}_{m,S}$

$$\rho := \rho_S(s_0)\colon \mathcal{P}'_{\mathbb{M}} \xrightarrow{\sim} \mathcal{P}'_{\mathbb{N}}. \tag{13}$$

It is characterized by the property that

$$\rho^*(\xi_{\mathbb{N}})\xi_{\mathbb{M}}^{-1}\in \mathrm{Biext}^0_{S/S_0}\left(\mathbb{W}\big(\mathbb{M}\big),\mathbb{W}\big(\mathbb{N}^\vee\big);\mathbf{G}_{m,S}\right)$$

projects to 0 in the factor $\mathrm{Biext}^0_{S/S_0}\left(\mathbb{E}\big(\mathbb{M}\big),\mathbb{E}\big(\mathbb{N}^\vee\big);\mathbf{G}_{m,S}\right)$.

Lemma 5.8. *The isomorphism $\rho=\rho_S(s_0)$ satisfies the requirements of Theorem 2.5.*

Proof. Suppose we have morphisms of 1–motives over S_0, $s_0\colon \mathbb{M}_0\to\mathbb{M}'_0$ and $q_0\colon\mathbb{M}'_0\to\mathbb{M}''_0$. Fix deformations $\mathbb{M}$, $\mathbb{M}'$ and $\mathbb{M}''$ of $\mathbb{M}_0$, $\mathbb{M}'_0$ and $\mathbb{M}''_0$ respectively. The pull-back of $\rho_S(s_0)$ along $\big(\mathrm{Id}\times\mathbb{E}_S(q_0^\vee)\big)$ defines an isomorphism

$$\rho'\colon \big(\mathrm{Id}\times\mathbb{E}_S(s_0^\vee\circ q_0^\vee)\big)^*(\mathcal{P}_{\mathbb{E}(\mathbb{M})})\xrightarrow{\sim}\big(\mathbb{E}_S(s_0)\times\mathbb{E}_S(q_0^\vee)\big)^*(\mathcal{P}_{\mathbb{E}(\mathbb{M}')}).$$

Similarly the pull-back $\big(\mathbb{E}_S(s_0)\times\mathrm{Id}\big)^*\big(\rho_S(q_0)\big)$ defines an isomorphism

$$\rho''\colon \big(\mathbb{E}_S(s_0)\times\mathbb{E}_S(q_0^\vee)\big)^*(\mathcal{P}_{\mathbb{E}(\mathbb{M}')})\xrightarrow{\sim}\big(\mathbb{E}_S(q_0\circ s_0)\times\mathrm{Id}\big)^*(\mathcal{P}_{\mathbb{E}(\mathbb{M}'')}).$$

Here, we use Theorem 2.1(i). Composing we get an isomorphism

$$\rho\colon \big(\mathrm{Id}\times\mathbb{E}_S(s_0^\vee\circ q_0^\vee)\big)^*(\mathcal{P}_{\mathbb{E}(\mathbb{M})})\xrightarrow{\sim}\big(\mathbb{E}_S(q_0\circ s_0)\times\mathrm{Id}\big)^*(\mathcal{P}_{\mathbb{E}(\mathbb{M}'')}).$$

The functoriality of the decomposition in Proposition 5.7 implies that the projection of $\rho^*\big(\xi_{\mathbb{M}''}\big)\xi_{\mathbb{M}}^{-1}$ in $\mathrm{Biext}^0\left(\mathbb{E}\big(\mathbb{M}\big),\mathbb{E}\big((\mathbb{M}'')^\vee\big);\mathbf{G}_{m,S}\right)$ is the sum of the images of $(\rho')^*\big(\xi_{\mathbb{M}'}\big)\xi_{\mathbb{M}}^{-1}$ and of $(\rho'')^*\big(\xi_{\mathbb{M}''}\big)\xi_{\mathbb{M}'}^{-1}$. In particular, it is 0. Hence, $\rho=\rho_S(q_0\circ s_0)$ proving Property (i).

Property (ii) follows similarly using Theorem 2.1(ii), Lemma 5.9 and the additivity of the decomposition in Proposition 5.7.

Since (iii) holds for $\mathbb{E}_S(s_0)$ and $\mathbb{E}_S(s_0^\vee)$ by Theorem 2.1 and the decomposition in Proposition 5.7 is functorial on the base, the isomorphism $\zeta_S(s_0)$ also satisfies (iii). Since $\rho_S(s_0)$ is obtained from $\zeta_S(s_0)$, then $\rho_S(s_0)$ satisfies (iii).

Assume that s_0 deforms to a morphism $s\colon \mathbb{M} \to \mathbb{N}$. Then, s induces an isomorphism of biextensions $\big(\mathrm{Id} \times s^\vee)^*(\mathcal{P}_\mathbb{M}) \xrightarrow{\sim} \big(s \times \mathrm{Id})^*(\mathcal{P}_\mathbb{N})$; see §5.1. Pulling it back to $\mathbb{E}(\mathbb{M}) \times_S \mathbb{E}(\mathbb{N}^\vee)$ and due to Theorem 2.1(iv) it provides an isomorphism of biextensions $\rho(s)\colon \mathcal{P}'_\mathbb{M} \xrightarrow{\sim} \mathcal{P}'_\mathbb{N}$. These biextensions have trivializations $\xi_\mathbb{M}$ and $\xi_\mathbb{N}$ over $\mathbb{W}(\mathbb{M}) \times_S \mathbb{W}(\mathbb{N})$ and $\rho(s)$ preserves these trivializations being obtained by pull-back i.e., $\xi_\mathbb{N} = \rho(s)\xi_\mathbb{M}$. This implies that $\rho(s) \circ \rho_S(s_0)^{-1} \in \mathrm{Biext}^0_{S/S_0}\big(\mathbb{E}(\mathbb{M}), \mathbb{E}(\mathbb{N}^\vee); \mathbf{G}_{m,S}\big)$ is trivial, i.e., $\rho(s) = \rho_S(s_0)$. This proves Property (iv). $\square$

Lemma 5.9. *Let P be a biextension of group schemes A and B by G over S. Let f and $g\colon C \to B$ be group homomorphisms. Then, we have a canonical isomorphism*

$$\big(\mathrm{Id}_A \times (f+g)\big)^*(P) \cong \big(\mathrm{Id}_A \times f\big)^*(P) + \big(\mathrm{Id}_A \times g\big)^*(P),$$

where the latter is the composite biextension as in [16, Ex. VII, §2.5].

Proof. Let μ_B (resp. $\mu_{P,B}$) be the multiplication on B (resp. on P as G-extension of B_A) and, for $i = 1$, 2, let $p_i\colon A \times_S B \times_S B \to A \times_S B$ be the projection on the i-th component of B. Then, $\mu_{P,B}$ defines an isomorphism of G-biextensions of A and $B \times_S B$ from $p_1^*(P) + p_2^*(P)$ to $\big(\mathrm{Id}_A \times \mu_B\big)^*(P)$ ([16, VII, 1.1.3.2 & 1.2]). Pulling-back via the morphism $(f, g)\colon C \to B \times_S B$ we get the claimed isomorphism. $\square$

5.5. The canonical pairing. Proof of Corollary 2.6

Let G, G' be two S-group schemes and let $\mathcal{P}$ be a biextension of G, G' by $\mathbf{G}_{m,S}$. A $\natural$-structure on $\mathcal{P}$ (cf. [14, 10.2.7.2] is a connection on the $\mathbf{G}_m$-torsor $\mathcal{P}$ over $G \times_S G'$, such that the canonical morphisms

$$\begin{aligned} \nu_1\colon p_{13}^*\mathcal{P} + p_{23}^*\mathcal{P} &\to (\mu \times \mathrm{Id})^*\mathcal{P}, \quad \text{on} \quad G \times_S G \times_S G', \\ \nu_2\colon p_{12}^*\mathcal{P} + p_{13}^*\mathcal{P} &\to (\mathrm{Id} \times \mu')^*\mathcal{P} \quad \text{on} \quad G \times_S G' \times_S G' \end{aligned}$$

are horizontal; here p_{ij} denotes the obvious projection morphism and μ, (resp. μ') the group law on G (resp. G'). The curvature ∇ can be seen as the sum of a $\natural$-1–structure on $\mathcal{P}$ (i.e., a connection on the $\mathbf{G}_m$-torsor $\mathcal{P}$ relative to $G \times_S G' \to G'$ such that ν_1, ν_2 are horizontal) and a $\natural$-2–structure (analogous definition with the roles of G and G' interchanged). For $\mathbb{M} = [\mathbb{X} \to \mathbb{G}]$ and $\mathbb{N} = [\mathbb{Y} \to \mathbb{H}]$ two 1–motives over S and $\mathcal{P}$ a $\mathbf{G}_m$-biextension of the 1–motives $\mathbb{M}$ and $\mathbb{N}$, a $\natural$-structure is a $\natural$-structure as $\mathbf{G}_m$-biextension of $\mathbb{G}$ and $\mathbb{H}$ together with trivializations (as $\natural$-biextensions) of the pull-backs to $\mathbb{X} \times \mathbb{H}$ and $\mathbb{G} \times \mathbb{Y}$.

Recall now that, if $\mathcal{P}$ denotes the Poincaré biextension of $\mathbb{M}$ and $\mathcal{P}_{\mathbb{E}(\mathbb{M})}$ its pull-back to the universal extensions, the biextension $\mathcal{P}_{\mathbb{E}(\mathbb{M})}$ is endowed with a canonical $\natural$-structure ([14, 10.2.7.2], [5, §4]). More generally, given any $\mathbf{G}_m$-biextension $\mathcal{P}$ of 1–motives $\mathbb{M}$ and $\mathbb{N}$ there is a canonical $\natural$-structure on the pull-back $\mathcal{P}_\mathbb{E}$ of $\mathcal{P}$ to $\mathbb{E}(\mathbb{M})$ and $\mathbb{E}(\mathbb{N})$. This $\natural$-structure allows one to define a curvature form (cf. [14, 10.2.7.3])

$$\Phi\colon \mathbf{T}_{\mathrm{dR}}(\mathbb{M}) \otimes \mathbf{T}_{\mathrm{dR}}(\mathbb{N}) \longrightarrow \mathbf{T}_{\mathrm{dR}}(\mathbf{G}_m) = \mathbf{Lie}\,\mathbf{G}_m. \tag{14}$$

It is proven in [14, 10.2.7.4] that over a field of characteristic 0, $\mathcal{P}_\mathbb{E}$ admits a unique $\natural$-structure. This is no longer true over a general base ([5, 3.11]). However, we will prove below that all $\natural$-structures on $\mathcal{P}_\mathbb{E}$ define the same curvature form.

Proposition 5.10. *Let $\mathbb{M}$ and $\mathbb{N}$ be 1–motives over S. Let $\mathcal{P}_\mathbb{E}$ be a $\mathbf{G}_m$-biextension of $\mathbb{E}(\mathbb{M})$ and $\mathbb{E}(\mathbb{N})$. If ∇ and ∇' are two $\natural$-structures on $\mathcal{P}_\mathbb{E}$, then their curvature forms coincide.*

Proof. We may suppose that $\mathcal{P}_{\mathbb{E}}$ is the trivial biextension and $\nabla' = 0$. Furthermore, working Zariski locally, we may assume that all locally free sheaves that we meet are indeed free. Let $\nabla = \nabla_1 + \nabla_2$ be the decomposition of ∇ in its $\natural$-1–structure and its $\natural$-2–structure. Then ∇_1 is associated to an invariant differential $\omega_1 = \sum_j F_j\omega_{1,j}$ of the underlying $\mathbf{G}_m$-extension of $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$ over $\mathbb{E}(\mathbb{N})_{\mathbb{H}}$, where $\{\omega_{1,j}\}_j$ is a free basis of invariant differentials of $\mathbb{E}(\mathbb{M})_{\mathbb{G}}$ over S and F_j is (the pull-back of) a global section of $\mathbb{E}(\mathbb{N})_{\mathbb{H}}$. The horizontality of ν_2 in the definition of $\natural$-structures, requires that F_j is "additive", i.e., it corresponds to a homomorphism $F_j\colon \mathbb{E}(\mathbb{N})_{\mathbb{H}} \to \mathbf{G}_a$. The triviality of its pull-back to $\mathbb{E}(\mathbb{M})_{\mathbb{G}} \times \mathbb{Y}$ implies that F_j maps $\mathbb{Y}$ to 0, i.e., it is indeed a morphism $F_j\colon \mathbb{E}(\mathbb{N}) \to \mathbf{G}_a$. Recall now the following diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{E}(\mathbb{N}), \mathbf{G}_a) \arrow[r] \arrow[d, "\iota"'] & \mathrm{Hom}(\mathbb{W}(\mathbb{N}), \mathbf{G}_a) \arrow[r] \arrow[d, equal] & \mathrm{Ext}(\mathbb{N}, \mathbf{G}_a) \arrow[d, "j"'] \\
\mathrm{Hom}(\mathbb{E}(\mathbb{N})_{G'}, \mathbf{G}_a) \arrow[r] & \mathrm{Hom}(\mathbb{W}(\mathbb{N}), \mathbf{G}_a) \arrow[r] & \mathrm{Ext}(\mathbb{H}, \mathbf{G}_a)
\end{tikzcd}$$

where the Coker $(\iota) = $ Ker $(j) = \mathrm{Hom}(\mathbb{Y}, \mathbf{G}_a)$, j is Zariski locally surjective and ι is injective. Recall also the splitting of the upper horizontal sequence (cf. Remark 5.5) where the group on the right is isomorphic to $\mathrm{Hom}_{\mathcal{O}_S}(\mathbb{W}(\mathbb{N}), \mathbf{G}_a)$. After identifying, Zariski locally on S, the vector group $\mathbb{W}(\mathbb{N})$ with $\mathbf{G}_a^r$ for some integer r, the group $\mathrm{Hom}_{\mathcal{O}_S}(\mathbb{W}(\mathbb{N}), \mathbf{G}_a)$ is the additive group of linear polynomials in the r variables $x_1, \ldots, x_r$. The group $\mathrm{Hom}(\mathbb{W}(\mathbb{N}), \mathbf{G}_a)$ consists of the so-called "additive polynomials", i.e., to those polynomials $\sum_I a_I x^I$ such that

$$\sum_I a_I x^I \otimes 1 + \sum_I 1 \otimes a_I x^I = \sum_I a_I (x \otimes 1 + 1 \otimes x)^I,$$

with the obvious multi-index notation, $I \in \mathbb{N}^r$. The image of $\mathrm{Hom}(\mathbb{E}(\mathbb{M}'), \mathbf{G}_a)$ in $\mathrm{Hom}(\mathbb{W}(\mathbb{M}'), \mathbf{G}_a)$ corresponds then to those additive polynomials where the indices $I = (i_1, \ldots i_r)$ satisfy $|I| := \sum_{h=1}^r i_h \geq 2$. By a direct computation one checks that the coefficients a_I satisfy $i_h a_I = 0$ for all $h = 1, \ldots r$. Hence taking the derivative, we get

$$d \sum_{|I|>1} a_I x^I = \sum_{|I|>1} a_I dx^I = \sum_{|I|>1} a_I \sum_{h=1}^r i_h x^{I-e_h} dx_h = \sum_{|I|>1} \sum_{h=1}^r i_h a_I x^{I-e_h} dx_h = 0,$$

where e_h is the multi-index with 1 at place h and 0 elsewhere. We conclude that $dF_j = 0$ so that $dF_j \wedge \omega_{1,j} = 0$ and hence the curvature of ω_1 is 0. Similarly one proves that the curvature of ω_2 is trivial. ☐

Corollary 5.11. *Let $\mathbb{M}$ be a 1–motive over S and $\mathcal{P}$ its Poincaré biextension. Deligne's pairing in [5, §4] constructed via the canonical $\natural$-structure $\nabla_{\mathbb{M}}$ on $\mathcal{P}_{\mathbb{E}(\mathbb{M})}$ could be constructed using any $\natural$-structure on $\mathcal{P}_{\mathbb{E}(\mathbb{M})}$.*

Proof of Corollary 2.6. Let now $s_0\colon \mathbb{M}_0 \to \mathbb{N}_0$ be a morphism of 1–motives over S_0. Let $(U \subset Z)$ be in S_0^{crys}. Let $\mathbb{M}$ and $\mathbb{N}$ be deformations of $\mathbb{M}_0$ and $\mathbb{N}_0$ over Z. The pull-back via $(\mathrm{Id} \times \mathbb{E}_Z(s_0^\vee))$ of the canonical $\natural$-biextension on $\mathbb{E}(\mathbb{M}) \times \mathbb{E}(\mathbb{M}^\vee)$ by $\mathbf{G}_m$ provides a $\natural$-biextension on $\mathbb{E}(\mathbb{M}) \times \mathbb{E}(\mathbb{N}^\vee)$ by $\mathbf{G}_{m,Z}$. This defines a pairing $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}) \otimes \mathbf{T}_{\mathrm{dR}}(\mathbb{N}^\vee) \to \mathbf{Lie}\, \mathbf{G}_{m,Z}$ over Z, i.e., a pairing

$$\langle\, ,\, \rangle_{s_0}\colon \mathbf{T}_{\mathrm{crys}}(\mathbb{M}_0)(U \subset Z) \otimes \mathbf{T}_{\mathrm{crys}}(\mathbb{N}_0^\vee)(U \subset Z) \to \mathbf{T}_{\mathrm{crys}}(\mathbf{G}_m)(U \subset Z).$$

It follows from Theorem 2.5 that the construction of the biextension above is crystalline in nature. Thanks to Proposition 5.10 the same holds for the associated pairing. Hence the above construction provides a pairing of crystals. It follows from Theorem 2.5(i) and Corollary 5.11 that over $(U \subset Z)$ the pairing given above is the same as the one defined by the $\natural$-biextension of $\mathbb{E}(\mathbb{M}) \times \mathbb{E}(\mathbb{N}^\vee)$ by $\mathbf{G}_m$ given by the pull-back via $\mathbb{E}_Z(s_0) \times \mathrm{Id}$ of the canonical $\natural$-biextension on $\mathbb{E}(\mathbb{N}) \times \mathbb{E}(\mathbb{N}^\vee)$. Thus, property (i) follows. Property (ii) holds by construction. Due to (i) and Remark 2.2(a) to show that $\langle\, ,\, \rangle_{s_0}$ is a pairing of filtered crystals, we may assume that $s_0 = \mathrm{Id}_{\mathbb{M}_0}$. In this case, the fact that $\langle\, ,\, \rangle_{\mathrm{Id}_{\mathbb{M}_0}}$ respects weight filtrations follows because Deligne's pairing does by [5, §4.5].

5.6. Proof of Corollary 2.8

We use the notations and constructions of §4.2. Let $s\colon \mathbb{M} \to \mathbb{N}$ be a morphism of 1–motives over S. In (8) we have defined the Gauss-Manin connection on $\mathbf{T}_{\mathrm{dR}}(\mathbb{M})$ and $\mathbf{T}_{\mathrm{dR}}(\mathbb{N}^\vee)$ via isomorphisms of $\mathcal{O}_{S'}$-modules

$$\nabla'_{\mathbb{M}}\colon \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathbf{T}_{\mathrm{dR}}(\mathbb{M}) = \mathbf{T}_{\mathrm{dR}}(\mathbb{M}_2) \longrightarrow \mathbf{T}_{\mathrm{dR}}(\mathbb{M}_1) = \mathbf{T}_{\mathrm{dR}}(\mathbb{M}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'},$$
$$\nabla'_{\mathbb{N}^\vee}\colon \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathbf{T}_{\mathrm{dR}}(\mathbb{N}^\vee) = \mathbf{T}_{\mathrm{dR}}(\mathbb{N}_2^\vee) \longrightarrow \mathbf{T}_{\mathrm{dR}}(\mathbb{N}_1^\vee) = \mathbf{T}_{\mathrm{dR}}(\mathbb{N}^\vee) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}.$$

Let $\langle\ ,\ \rangle_2\colon \mathbf{T}_{\mathrm{dR}}(\mathbb{M}_2) \otimes_{\mathcal{O}_{S'}} \mathbf{T}_{\mathrm{dR}}(\mathbb{N}_2^\vee) \longrightarrow p_2^*\mathbf{Lie}\,\mathbf{G}_m$ be the pairing in (14) associated to the curvature form of any $\natural$-structure on $(\mathrm{Id}_{\mathbb{E}(\mathbb{M}_2)} \times \mathbb{E}_{S'}(s^\vee))^*\mathcal{P}_{\mathbb{E}(\mathbb{M}_2)}$; cf. Proposition 5.10. Analogously, let $\langle\ ,\ \rangle_1\colon \mathbf{T}_{\mathrm{dR}}(\mathbb{M}_1) \otimes_{\mathcal{O}_{S'}} \mathbf{T}_{\mathrm{dR}}(\mathbb{N}_1^\vee) \longrightarrow p_1^*\mathbf{Lie}\,\mathbf{G}_m$ be the pairing associated to the curvature form of any $\natural$-structure on the biextension $(\mathrm{Id}_{\mathbb{E}(\mathbb{M}_1)} \times \mathbb{E}_{S'}(s^\vee))^*\mathcal{P}_{\mathbb{E}(\mathbb{M}_1)}$. In order to prove Corollary 2.8, it suffices to show that

$$\langle\ ,\ \rangle_1 \circ (\nabla'_{\mathbb{M}} \otimes \nabla'_{\mathbb{N}^\vee}) = \langle\ ,\ \rangle_2. \tag{15}$$

By Theorem 2.5 we have a canonical isomorphism

$$\mathcal{P}_{\mathbb{E}(\mathbb{M}_2)} \to (\mathbb{E}_{S'}(\mathrm{Id}_{\mathbb{M}}) \times \mathbb{E}_{S'}(\mathrm{Id}_{\mathbb{M}^\vee}))^*\mathcal{P}_{\mathbb{E}(\mathbb{M}_1)}$$

of $\mathbf{G}_m$-biextensions of $\mathbb{E}(\mathbb{M}_2)$ and $\mathbb{E}(\mathbb{M}_2^\vee)$ induced by $\rho_{S'}(\mathrm{Id}_{\mathbb{M}})$. We then obtain isomorphisms

$$\begin{aligned}(\mathrm{Id}_{\mathbb{E}(\mathbb{M}_2)} \times \mathbb{E}_{S'}(s^\vee))^*\mathcal{P}_{\mathbb{E}(\mathbb{M}_2)} &\cong (\mathrm{Id}_{\mathbb{E}(\mathbb{M}_2)} \times \mathbb{E}_{S'}(s^\vee))^*(\mathbb{E}_{S'}(\mathrm{Id}_{\mathbb{M}}) \times \mathbb{E}_{S'}(\mathrm{Id}_{\mathbb{M}^\vee}))^*\mathcal{P}_{\mathbb{E}(\mathbb{M}_1)}\\ &\cong (\mathbb{E}_{S'}(\mathrm{Id}_{\mathbb{M}}) \times \mathbb{E}_{S'}(\mathrm{Id}_{\mathbb{N}^\vee})^*(\mathrm{Id}_{\mathbb{E}(\mathbb{M}_1)} \times \mathbb{E}_{S'}(s^\vee))^*\mathcal{P}_{\mathbb{E}(\mathbb{M}_1)}\\ &\cong (\mathbb{E}_{S'}(\mathrm{Id}_{\mathbb{M}}) \times \mathbb{E}_{S'}(s^\vee))^*\mathcal{P}_{\mathbb{E}(\mathbb{M}_1)}\end{aligned}$$

of $\mathbf{G}_m$-biextensions of $\mathbb{E}(\mathbb{M}_2)$ and $\mathbb{E}(\mathbb{N}_2^\vee)$. Due to Proposition 5.10, any $\natural$-structure on the first biextension induces the pairing $\langle\ ,\ \rangle_2$. Choosing on the last biextension the $\natural$-structure given by pull-back of any $\natural$-structure on $\mathcal{P}_{\mathbb{E}(\mathbb{M}_1)}$ and taking the associated curvature form, we obtain the pairing $\langle\ ,\ \rangle_1 \circ (\nabla'_{\mathbb{M}} \otimes \nabla'_{\mathbb{N}^\vee})$. As the curvature forms associated to isomorphic $\natural$-biextensions coincide, we conclude that the pairings in (15) coincide. This proves the corollary.

6. Uniqueness results

As already stressed in Remark 3.4 the homomorphism $\mathbb{E}_S(s_0)$ in Theorem 2.1 is canonical but not unique in general. The problem disappears when working with schemes S flat over $\mathbf{Z}$. Indeed, any two such morphisms $\mathbb{E}(\mathbb{M}) \to \mathbb{E}(\mathbb{N})$ will differ by a homomorphism $\mathbb{E}(\mathbb{M}) \to \mathcal{I}\mathbf{Lie}\,(\mathbb{E}(\mathbb{N})) \to \mathbf{Lie}\,(\mathbb{E}(\mathbb{N}))$ that is trivial (cf. Remark 5.5). Another case when the canonical morphism is also the unique one is the case of adic rings which are flat over $\mathbf{Z}$. More precisely:

(∗) Let $(A, \mathcal{I}, \gamma)$ be a ring with an ideal $\mathcal{I}$ endowed with divided powers. Assume that A is complete and separated with respect to a sub-PD ideal $P \subseteq \mathcal{I}$ and that the induced divided powers structure on the image of $\mathcal{I}A_n$, where $A_n := A/P^n$ for $n \geq 1$, is locally nilpotent. Assume that A is flat as $\mathbf{Z}$-module.

Put $A_0 := A/\mathcal{I}$ and $S_0 := \mathrm{Spec}\,(A_0)$. Assume we are given compatible systems of deformations $\mathbb{M}_n$ and $\mathbb{N}_n$ over $S_n := \mathrm{Spec}\,(A_n)$. Write $\mathbb{M}_n := [\mathbb{X}_n \to \mathbb{G}_n]$ with $\mathbb{G}_n$ extension of the abelian scheme $\mathbb{A}_n$ by the torus $\mathbb{T}_n$ and $\mathbb{N}_n := [\mathbb{Y}_n \to \mathbb{H}_n]$ with $\mathbb{H}_n$ extension of the abelian scheme $\mathbb{B}_n$ by the torus $\mathbb{T}'_n$. Let $s_0\colon \mathbb{M}_0 \to \mathbb{N}_0$ be a morphism of 1–motives over S_0 inducing morphisms $\alpha_0\colon \mathbb{X}_0 \to \mathbb{Y}_0$, $\beta_0\colon \mathbb{A}_0 \to \mathbb{B}_0$ and $\gamma_0\colon \mathbb{T}_0 \to \mathbb{T}'_0$.

Lemma 6.1. *Let $\{W_n \to S_n\}_n$ and $\{U_n \to S_n\}_n$ be a compatible system of vector groups. For every $n \in \mathbf{N}$ let $\mathrm{Hom}_{\mathcal{O}_{S_n}}(W_n, U_n)$ be the subgroup of the group of homomorphisms $\mathrm{Hom}(W_n, U_n)$ as S_n-group schemes consisting of morphisms as vector groups. Then,*

$$\lim_{\infty \leftarrow n} \mathrm{Hom}_{\mathcal{O}_{S_n}}(W_n, U_n) \cong \lim_{\infty \leftarrow n} \mathrm{Hom}(W_n, U_n).$$

In particular, with the notations above, we have

$$\lim_{\infty \leftarrow n} \mathrm{Hom}(\mathbb{E}(\mathbb{G}_n), U_n) = \{0\}.$$

Proof. Consider the second statement. Recall that $\mathbb{G}_n$ is an extension of an abelian scheme $\mathbb{A}_n$ by a torus $\mathbb{T}_n$. Since $\mathbb{A}_n$ is proper, we have $\mathrm{Hom}_{S_n}(\mathbb{A}_n, U_n) = 0$. Since $\mathrm{Hom}(\mathbb{T}_n, U_n) = 0$, we have $\mathrm{Hom}(\mathbb{G}_n, U_n)$ for every $n \in \mathbf{N}$. Thus, we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}\big(\mathbb{E}(\mathbb{G}_n), U_n\big) \longrightarrow \mathrm{Hom}\big(\mathbb{W}(\mathbb{G}_n), U_n\big) \xrightarrow{\delta_n} \mathrm{Ext}^1\big(\mathbb{G}_n, U_n\big),$$

where δ_n is defined by push-forward. As explained in Remark 5.5 the latter identifies the subgroup $\mathrm{Hom}_{\mathcal{O}_{S_n}}\big(\mathbb{W}(\mathbb{G}_n), U_n\big)$ of $\mathrm{Hom}\big(\mathbb{W}(\mathbb{G}_n), U_n\big)$, consisting of homomorphisms of vector groups, with $\mathrm{Ext}^1\big(\mathbb{G}_n, U_n\big)$ by definition of universal extension, so that we have a direct sum decomposition

$$\mathrm{Hom}\big(\mathbb{W}(\mathbb{G}_n), U_n\big) \cong \mathrm{Hom}\big(\mathbb{E}(\mathbb{G}_n), U_n\big) \oplus \mathrm{Hom}_{\mathcal{O}_{S_n}}\big(\mathbb{W}(\mathbb{G}_n), U_n\big).$$

Thus, the claim follows from the first statement.

We now prove the first statement. Since it can be checked Zariski locally over S_0, we may assume that $W_n \cong \mathbf{G}_{a,S_n}^s$ and $U_n \cong \mathbf{G}_{a,S_n}^r$. Moreover, working componentwise, we may also assume that $s = r = 1$. Write $\mathbf{G}_{a,S_n} = \mathrm{Spec}\big(A_n[T]\big)$ with group law $T \mapsto T \otimes 1 + 1 \otimes T$ and zero section $T \mapsto 0$. A compatible system of endomorphisms $\xi_n : \mathbf{G}_{a,S_n} \to \mathbf{G}_{a,S_n}$ induces an endomorphism $\widehat{\xi}$ of the additive formal group $\mathrm{Spf}(A\langle T\rangle)$ where $A\langle T\rangle$ is the ring of converging power series with respect to the P-adic topology. The endomorphisms of the latter are of the form $T \mapsto aT$ for some $a \in A$. Indeed, they are defined by converging power series $f(T) = \sum_{n\geq 1} a_n T^n$ such that $\sum_{n\geq 1} a_n\big((T\otimes 1 + 1\otimes T)^n - T^n \otimes 1 - 1 \otimes T^n\big) = 0$ which implies that $\binom{n}{i} a_n = 0$ for every $1 \leq i \leq n-1$, i.e., since A is $\mathbf{Z}$-flat, $a_n = 0$ for every $n \geq 2$. Since ζ_n is obtained from ζ by reduction modulo P^n, also $\zeta_n(T) = \bar{a}T$ and hence ζ_n is a morphism of vector group schemes. □

Proposition 6.2. *There is a unique compatible system of morphisms $\{\zeta_n(s_0) : \mathbb{E}(\mathbb{M}_n) \to \mathbb{E}(\mathbb{N}_n)\}_{n\in\mathbf{N}}$ of group schemes over S_n such that $\zeta_{n+1}(s_0) \equiv \zeta_n(s_0)$ over S_n for every $n \in \mathbf{N}$ and $\zeta_0(s_0)$ is the morphism of universal extensions induced by s_0.*

In particular, for any $n \in \mathbf{N}$, $\zeta_n(s_0)$ is the morphism $\mathbb{E}_{S_n}(s_0)$ defined in Theorem 2.1.

Proof. The existence of such systems follows from Theorem 2.1. If we prove that it is unique, the last statement follows from Theorem 2.1(iii). Suppose that we are given two compatible systems of morphisms $\zeta_n : \mathbb{E}\big(\mathbb{M}_n\big) \to \mathbb{E}\big(\mathbb{N}_n\big)$ and $\zeta_n' : \mathbb{E}\big(\mathbb{M}_n\big) \to \mathbb{E}\big(\mathbb{N}_n\big)$ extending $\mathbb{E}(s_0)$.

STEP 1: We claim that any such system preserves the weight filtrations, i.e., $\zeta_n\big(\mathbb{T}_n\big) \subseteq \mathbb{T}_n'$ and $\zeta_n\big(\mathbb{E}(\mathbb{G}_n)\big) \subseteq \mathbb{E}(\mathbb{H}_n)$, and similarly for ζ_n', for every $n \in \mathbf{N}$ and that the morphisms $\mathbb{T}_n \to \mathbb{T}_n'$, $\mathbb{E}(\mathbb{A}_n) \to \mathbb{E}(\mathbb{B}_n)$ and $\mathbb{E}\big[\mathbb{X}_n \to 0\big] \to \mathbb{E}\big[\mathbb{Y}_n \to 0\big]$, induced by ζ_n and ζ_n' on the graded pieces are the same.

W_{-2}-part and induced map on Gr_{-2}*:* The composite homomorphism

$$\mathbb{T}_n \to \mathbb{E}(\mathbb{M}_n)_{\mathbb{G}_n} \to \mathbb{E}(\mathbb{N}_n)_{\mathbb{H}_n} \to \mathbb{E}\big([\mathbb{Y}_n \to \mathbb{B}_n]\big)_{\mathbb{B}_n},$$

where the map $\mathbb{E}(\mathbb{M}_n)_{\mathbb{G}_n} \to \mathbb{E}(\mathbb{N}_n)_{\mathbb{H}_n}$ is induced by ζ_n, is a morphism of group schemes which is zero map over S_0. Due to §5.2 it provides a morphism of group schemes $\mathbb{T}_n \to \mathbf{Lie}\,\big(\mathbb{E}\big([\mathbb{Y}_n \to \mathbb{B}_n]\big)\big)_{\mathbb{B}_n}$ that factors through $\mathcal{I}\mathbf{Lie}\,\big(\mathbb{E}\big([\mathbb{Y}_n \to \mathbb{B}_n]\big)\big)_{\mathbb{B}_n}$. The group scheme $\mathbf{Lie}\,\big(\mathbb{E}\big([\mathbb{Y}_n \to \mathbb{B}_n]\big)\big)_{\mathbb{B}_n}$ is Zariski locally over S_n isomorphic to the sum of copies of $\mathbf{G}_{a,S_n}$. Since the only morphism from a torus to $\mathbf{G}_{a,S_n}$ is the trivial one, it follows that the map $\mathbb{T}_n \to \mathbb{E}\big([\mathbb{Y}_n \to \mathbb{B}_n]\big)$ is trivial so that $\zeta_n(\mathbb{T}_n) \subseteq \mathbb{T}_n'$. Since the morphisms of tori correspond to morphisms of their character groups and those are determined by their behavior over S_0, the induced map $\zeta_n : \mathbb{T}_n \to \mathbb{T}_n'$ is the unique one extending $\gamma_0 : \mathbb{T}_0 \to \mathbb{T}_0'$.

W_{-1}-part and induced map on Gr_0*:* Consider the map $\mathbb{E}(\mathbb{G}_n) \to \mathbb{E}\big([\mathbb{Y}_n \to 0]\big)$ induced by ζ_n. Since it is zero over S_0 and it induces a compatible system of morphisms $j_n : \mathbb{E}(\mathbb{G}_n) \to \mathbb{Y}_n \otimes \mathbf{G}_{a,S_n}$, it follows from Lemma 6.1 that such system is the trivial one, i.e., $j_n = 0$ for every n and, hence, ζ_n induces a map $\mathbb{E}(\mathbb{G}) \to \mathbb{E}(\mathbb{H})$. In particular, $\{\zeta_n\}_n$ induces compatible maps $\mathbb{E}\big([\mathbb{X}_n \to 0]\big) \to \mathbb{E}\big([\mathbb{Y}_n \to 0]\big)$. By Lemma 6.1 they are morphisms of vector extensions $i_n : \mathbb{E}\big([\mathbb{X}_n \to 0]\big)_0 \to \mathbb{E}\big([\mathbb{Y}_n \to 0]\big)_0$ in degree 0. Since $\mathbb{E}\big([\mathbb{X}_n \to 0]\big)_0 = \mathbb{X}_n \otimes \mathbf{G}_{a,S_n}$ and $\mathbb{E}\big([\mathbb{Y}_n \to 0]\big)_0 = \mathbb{Y}_n \otimes \mathbf{G}_{a,S_n}$, it follows that i_n is determined by its values

on $\mathbb{X}_n$, i.e., by the map of lattices $\mathbb{X}_n \to \mathbb{Y}_n$. The latter is uniquely determined by its behavior over S_0. We conclude that i_n is unique.

Induced map on Gr_{-1}*:* Suppose we have two compatible systems of morphisms of group schemes a_n and $a'_n\colon \mathbb{E}(\mathbb{A}_n) \to \mathbb{E}(\mathbb{B}_n)$ such that $a_0 = a'_0$. Then, $a'_n - a_n$ defines a compatible system of morphisms $\mathbb{E}(\mathbb{A}_n) \to \mathcal{I}\mathbf{Lie}\,\mathbb{E}(\mathbb{B}_n) \subseteq \mathbb{E}(\mathbb{B}_n)$ and $a'_0 - a_0 = 0$. Any such system is trivial by Lemma 6.1 so that $a_n = a'_n$ for every $n \in \mathbf{N}$. Thus, the map $\mathbb{E}(\mathbb{A}_n) \to \mathbb{E}(\mathbb{B}_n)$ induced by ζ_n does not depend on the system of morphisms $\{\zeta_n\}_n$.

STEP 2: Suppose that we are given compatible systems of morphisms ζ_n and $\zeta'_n\colon \mathbb{E}(\mathbb{M}_n) \to \mathbb{E}(\mathbb{N}_n)$ deforming $\mathbb{E}_{S_0}(s_0)$, preserving the weight filtrations and inducing the same systems of morphisms on the graded pieces $\mathbb{T}_n \to \mathbb{T}'_n$, $\mathbb{E}(\mathbb{A}_n) \to \mathbb{E}(\mathbb{B}_n)$ and $\mathbb{E}\big[\mathbb{X}_n \to 0\big] \to \mathbb{E}\big[\mathbb{Y}_n \to 0\big]$.

In particular, the maps $\zeta'_n - \zeta_n$ on $\mathbb{E}(\mathbb{G}_n) \to \mathbb{E}(\mathbb{H}_n)$ factor via a compatible system of morphisms $\mathbb{E}(\mathbb{A}_n) \to \mathbb{T}'_n$ reducing to 0 modulo S_0. It follows from §5.2 that this is defined by a compatible system of morphisms $\mathbb{E}(\mathbb{A}_n) \to \mathcal{I}\mathbf{Lie}\,\mathbb{T}'_n \subset \mathbf{Lie}\,\mathbb{T}'_n$ which is 0 over S_0. Any such system is trivial by Lemma 6.1 so that $\zeta'_n = \zeta_n$ on $\mathbb{E}(\mathbb{G}_n)$. Thus, $\zeta'_n - \zeta_n$ factors via a compatible system of morphisms $\mathbb{E}\big([\mathbb{X}_n \to 0]\big)_0 \to \mathbb{E}(\mathbb{H}_n)$ reducing to 0 over S_0 and mapping $\mathbb{X}_n$ to 0. Thanks to results in §5.2 this is defined by a compatible system of morphisms $\mathbb{E}\big([\mathbb{X}_n \to 0]\big)_0 \to \mathcal{I}\mathbf{Lie}\,\mathbb{E}(\mathbb{H}_n) \subset \mathbf{Lie}\,(\mathbb{H}_n)$ which are 0 on $\mathbb{X}_n$. It follows from Lemma 6.1 that there is a homomorphism of vector groups $\mathbb{E}\big([\mathbb{X}_n \to 0]\big)_0 \to \mathbf{Lie}\,(\mathbb{H}_n)$ mapping $\mathbb{X}_n$ to 0. Since $\mathbb{E}\big([\mathbb{X}_n \to 0]\big)_0 = \mathbb{X}_n \otimes \mathbf{G}_{a,S_n}$ any such map must be 0. We conclude that $\zeta_n = \zeta'_n$ for every $n \in \mathbf{N}$. □

This completes the proof of [1, 4.2.1]. In particular there exists a canonical isomorphism $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}) \cong \mathbf{T}_{\mathrm{crys}}(\mathbb{M}_k)$ for $\mathbb{M}$ a 1-motive over the Witt vectors of a perfect field k of positive characteristic $p > 2$.

7. Motivic definition of Manin's map. Proof of Proposition 1.1

We briefly recall the definition of Manin's map following [12, §4]. Let S be a smooth irreducible curve over a field k of characteristic 0, let $\mathbb{A}$ be an abelian scheme over S and fix a section $x \in \mathbb{A}(S)$, distinct from the zero section 0 of $\mathbb{A}$. Let $Z_x \subset \mathbb{A}$ be the closed subscheme $\{x\} \cup \{0\}$. We let $\Omega^\bullet_{\mathbb{A}/S,Z_x}$ be the complex defined as the kernel of the homomorphism $\Omega^\bullet_{\mathbb{A}/S} \to \Omega^\bullet_{Z_x/S}$. Denote by $\mathbb{H}^i_{\mathrm{dR}}(\mathbb{A}/S, Z_x)$ the i-th hypercohomology group of the complex $\Omega^\bullet_{\mathbb{A}/S,Z_x}$. It is endowed with a connection, the Gauss-Manin connection. One proves that the exact sequence of complexes

$$0 \to \Omega^\bullet_{\mathbb{A}/S,Z_x} \to \Omega^\bullet_{\mathbb{A}/S} \to \Omega^\bullet_{Z_x/S} \to 0$$

provides a short exact sequence of sheaves

$$0 \to \mathcal{O}_S \to \mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S, Z_x) \to \mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S) \to 0$$

and that the morphisms in the sequence are compatible with the connections. In this way one gets a map, the *Manin map*,

$$\mathcal{M}_{\mathbb{A},\mathrm{Man}}\colon \mathbb{A}(S) \longrightarrow \mathrm{Ext}^1_S\left(\big(\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S), \nabla_{\mathbb{A}}\big), \big(\mathcal{O}_S, d\big)\right),$$

defined as above for sections different from 0 and sending 0 to the trivial extension. One proves that it is a group homomorphism (cf. [12, Thm. 1.4.1]). Its kernel coincides with the subgroup of $\mathbb{A}(S)$ of the so-called constant sections (cf. [12, Thm. 1.4.3]). This result plays a fundamental role in Manin's proof of Mordell's conjecture over function fields. In the following sections we will generalize this construction to 1-motives.

7.1. The motivic Manin map and its realizations

Let $\mathbb{M} = [\mathrm{u}\colon \mathbb{X} \to \mathbb{G}]$ be a 1–motive over any base scheme S. Define an S-valued point of $\mathbb{M}$ to be a homomorphism $\mathbf{Z} \to \mathbb{M}$ in the bounded derived category of sheaves on the fppf site of S. We have isomorphisms

$$\mathbb{M}(S) = \mathrm{Hom}_{D^b(fppf)}(\mathbf{Z}, \mathbb{M}) \cong \mathrm{Ext}^1_{D^b(fppf)}(\mathbf{Z}[1], \mathbb{M}) \cong \mathrm{Ext}_{1-\mathrm{Mot}}(\mathbf{Z}[1], \mathbb{M}) \cong \mathrm{Ext}^1_{1-\mathrm{Mot}}(\mathbb{M}^\vee, \mathbf{G}_m), \qquad (16)$$

whose composite

$$\mathcal{M}_{\mathbb{M}} \colon \mathbb{M}(S) \to \mathrm{Ext}^1_{1-\mathrm{Mot}}(\mathbb{M}^\vee, \mathbf{G}_m)$$

will be called the *motivic Manin map*. The last isomorphism in (16) is defined by Cartier duality. The isomorphism between $\mathrm{Ext}^1_{D^b(fppf)}(\mathbf{Z}[1], \mathbb{M})$ and the group of extensions $\mathrm{Ext}_{1-\mathrm{Mot}}(\mathbf{Z}[1], \mathbb{M})$, in the category of 1-motives, follows by remarking that the $\mathrm{Ext}^n_{D^b(fppf)}$ groups of fppf sheaves correspond to Yoneda n-fold extension classes and that any Yoneda extension of the complex $\mathbf{Z}[1]$ by the complex $\mathbb{M}$ is represented by a 1–motive.

If we tensor the group $\mathbb{M}(S)$ of S-valued points of $\mathbb{M}$ with $\mathbf{Q}$, we have a more explicit description. Indeed,

Lemma 7.1. *We have* $\mathbb{M}(S) \otimes_{\mathbf{Z}} \mathbf{Q} = \big(\mathbb{G}(S)/\mathbb{X}(S)\big) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Proof. Consider the exact sequence

$$\mathbb{X}(S) \longrightarrow \mathbb{G}(S) \longrightarrow \mathbb{M}(S) \longrightarrow \mathrm{Hom}\big(\mathbf{Z}, \mathbb{X}[1]\big) = \mathrm{Ext}^1\big(\mathbf{Z}, \mathbb{X}\big) = \mathrm{H}^1(S_{\mathrm{fppf}}, \mathbb{X}).$$

It suffices to show that $\mathrm{H}^1(S_{\mathrm{fppf}}, \mathbb{X})$ is a torsion group. This group coincides with $\mathrm{H}^1(S_{\mathrm{et}}, \mathbb{X})$ since $\mathbb{X}$ is étale over S so that $\mathbb{X}$-torsors are themselves étale. If $S \subset S'$ is a finite, étale, Galois extension with group G where $\mathbb{X}$ becomes constant, we have $\mathrm{H}^1(S'_{\mathrm{et}}, \mathbb{X}) = 0$ by [16, Prop. VIII.5.1]. Thus, $\mathrm{H}^1(S_{\mathrm{et}}, \mathbb{X}) = \mathrm{H}^1(G, \mathbb{X})$ (Galois cohomology) which is torsion since G is a finite group. □

Thus, the isomorphism $\mathcal{M}_{\mathbb{M}} \otimes_{\mathbf{Z}} \mathbf{Q}$ has the following description: suppose we are given an S-valued point y of $\mathbb{M}$. Thanks to Lemma 7.1, possibly after taking a multiple of y, we may assume that it comes from a point $x \in \mathbb{G}(S)$ and $\mathcal{M}_{\mathbb{M}}(y)$ is the pull-back of the extension $\mathcal{M}_{\mathbb{G}}(x)$ via the map $\mathbb{M}^\vee \to \mathbb{G}^\vee$. More concretely, let $\mathbb{M}(x)$ be the 1–motive defined by $\mathbb{X} \oplus \mathbf{Z} \to \mathbb{G}$ which is u on $\mathbb{X}$ and sends $\mathbf{Z} \ni 1 \mapsto x$. Note that $\mathbb{M}(x)$ is an extension of $[\mathbf{Z} \to 0]$ by $\mathbb{M}$ which depends, up to isomorphism, only on the class of x in $\mathbb{G}(S)/\mathbb{X}(S)$. Passing to Cartier duals we get $\mathcal{M}_{\mathbb{G}}(x) = \mathbb{M}(x)^\vee$ as extension of $\mathbb{M}^\vee$ by $\mathbf{G}_{m,S}$.

By Proposition 2.4 the Hodge realization functor $\mathbf{T}_{\mathbf{Z}}$ is exact and functorial. Hence we get immediately from $\mathcal{M}_{\mathbb{M}}$ its Hodge realization, i.e., the *Hodge-Manin map*

$$\mathcal{M}_{\mathbb{M},\mathbf{Z}} \colon \mathbb{M}(S) \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHS}/S^{\mathrm{an}}}\left(\mathbf{T}_{\mathbf{Z}}(\mathbb{M}^\vee), \mathbf{Z}\right).$$

Similarly one gets the *crystalline Manin map*

$$\mathcal{M}_{\mathbb{M},\mathrm{crys}} \colon \mathbb{M}(S) \longrightarrow \mathrm{Ext}^1_{S^{\mathrm{crys}}}\left(\mathbf{T}_{\mathrm{crys}}(\mathbb{M}^\vee), \mathcal{O}_S^{\mathrm{crys}}\right)$$

because the crystalline realization functor $\mathbf{T}_{\mathrm{crys}}$ is exact by Lemma 4.1 and functorial.

Analogously, suppose that S is a smooth scheme over a scheme T. Taking the de Rham realizations with Gauss-Manin connection, which is functorial by §4.2 and exact by Lemma 4.1, we get a map

$$\mathcal{M}_{\mathbb{M},\mathrm{dR}} \colon \mathbb{M}(S) \longrightarrow \mathrm{Ext}^1_S\left(\big(\mathbf{T}_{\mathrm{dR}}(\mathbb{M}^\vee), \nabla_{\mathbb{M}^\vee}\big), \big(\mathcal{O}_S, d\big)\right)$$

called *de Rham Manin's map* for the 1–motive $\mathbb{M}$.

Proposition 7.2. *The homomorphism $\mathcal{M}_{\mathbb{M}}$ is functorial in $\mathbb{M}$ and in S. The map $\mathcal{M}_{\mathbb{M},\mathrm{dR}}$ is a group homomorphism functorial in $\mathbb{M}$ and if S varies in the category of schemes flat over $\mathbf{Z}$, it is also functorial in S.*

Proof. We have already remarked that $\mathcal{M}_{\mathbb{M}}$ is functorial in $\mathbb{M}$ and in S. The functoriality of $\mathcal{M}_{\mathbb{M},\mathrm{dR}}$ in $\mathbb{M}$ follows from the fact that $\mathbf{T}_{\mathrm{dR}}$ with the Gauss-Manin connection is functorial in $\mathbb{M}$ by §4.2(a). There remains only to check that $\mathcal{M}_{\mathbb{M},\mathrm{dR}}$ is a group homomorphism and that it is functorial in S for S flat over $\mathbf{Z}$. Let $\mathbb{N}$ and $\mathbb{N}'$ be two $\mathbf{G}_m$-extensions of $\mathbb{M}^\vee$ and denote by $\mathbb{N}''$ their sum. It is defined by $\mathbb{N}'' = \Delta^*_{\mathbb{M}^\vee} \circ \mu_{\mathbf{G}_m *}(\mathbb{N} \oplus \mathbb{N}')$ where $\Delta_{\mathbb{M}^\vee}$ is the diagonal morphism $\mathbb{M}^\vee \to \mathbb{M}^\vee \times \mathbb{M}^\vee$ and $\mu_{\mathbf{G}_m}$ is the multiplication map on $\mathbf{G}_m$. By the functoriality of $\mathbf{T}_{\mathrm{dR}}$ and the Gauss-Manin connection, we get that the Gauss-Manin connection on the de Rham realization of $\mathbb{N}''$ is obtained via pull-back along $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}^\vee) \to \mathbf{T}_{\mathrm{dR}}(\mathbb{M}^\vee) \oplus \mathbf{T}_{\mathrm{dR}}(\mathbb{M}^\vee)$ and push-out along the product $\mathcal{O}_S \oplus \mathcal{O}_S \to \mathcal{O}_S$ of the sum of $\mathbf{T}_{\mathrm{dR}}(\mathbb{N}) \oplus \mathbf{T}_{\mathrm{dR}}(\mathbb{N}')$, each term being endowed with its Gauss-Manin connection. Hence, $\mathcal{M}_{\mathbb{M},\mathrm{dR}}$ is a group homomorphism. If S flat over $\mathbf{Z}$, then [1, 3.4.2] holds (see §6) and $\mathbb{E}(\mathbb{M}^\vee)$ is functorial in S. In particular, $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}^\vee)$ is functorial in S and we are done. □

7.2. Comparison with the classical Manin map

Before proving Proposition 1.1, we need the following

Lemma 7.3. *Let $S' \to S$ be an étale morphism with S' affine and (L, ∇_L) a locally free module with integrable connection on S. The base-change homomorphism*

$$\mathrm{Ext}^1_S\left((L,\nabla_L),(\mathcal{O}_S,d)\right) \to \mathrm{Ext}^1_{S'}\left((L,\nabla_L)\otimes_{\mathcal{O}_S}\mathcal{O}_{S'},(\mathcal{O}_{S'},d)\right)$$

is injective.

Proof. The map is injective when $S' \to S$ is an open immersion with S' non empty by [12, Cor. 1.1.3]. Thus, on replacing S with an open dense subscheme we may assume that $S' \to S$ is finite and étale of degree n. Note that an extension class (H, ∇_H) over S (resp. $H' := H \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ over S') is trivial if and only if the same holds for its dual $(H^\vee, \nabla_H^\vee)$ ([12, p.397]), i.e., if and only if there exists a section $a \in H^\vee$ (resp. in $H'^\vee$) which is horizontal with respect to $\nabla_H^\vee$ and maps to $1 \in \mathcal{O}_S$ (resp. $1 \in \mathcal{O}'_S$). Let $(H^\vee, \nabla_H^\vee) \in \mathrm{Ext}^1_S\left((\mathcal{O}_S,d),(L^\vee,\nabla_L^\vee)\right)$ such that $H'^\vee$ is trivial, i.e., it admits a section $a' \in \mathbb{H}'^\vee$ horizontal w.r.t. $\nabla_H^\vee \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ mapping to $1 \in \mathcal{O}_{S'}$. Let a be $\frac{1}{n}$ times the trace of a' with respect to $\mathcal{O}'_S/\mathcal{O}_S$. It is an element of $H^\vee$ such that $\nabla_H^\vee(a) = 0$ and it maps to $1 \in \mathcal{O}_S$. Hence, $(H^\vee, \nabla_H^\vee)$ is the trivial class as claimed. □

Lemma 7.4. *Let S be a smooth connected affine curve over a field k of characteristic 0. Proposition 1.1 holds true if for any $\mathbb{A} = \mathrm{Jac}\,(C/S)$ with $C \to S$ a smooth, proper and geometrically connected curve and any $x \in \mathbb{A}(S)$ that is the difference of two sections s and $t\colon S \to C$ such that $s \cap t = \emptyset$ one has $\mathcal{M}_{\mathbb{A},Man}(x) = \mathcal{M}_{\mathbb{A},\mathrm{dR}}(x)$.*

Proof. Let $\mathbb{A}$ be an abelian scheme over S. Note that $\mathbb{A}$ is the quotient of a Jacobian $\mathrm{Jac}\,(C/S)$ of a smooth proper curve $C \to S$. Furthermore, every S-valued point of $\mathbb{A}$ lifts, after a possible étale extension $S' \to S$, to an S'-valued point of $\mathrm{Jac}\,(C/S)$. Using the functoriality of $\mathcal{M}_{\mathbb{A},\mathrm{dR}}$ and $\mathcal{M}_{\mathbb{A},\mathrm{Man}}$ in $\mathbb{A}$ and in S and Lemma 7.3, it suffices to prove that Proposition 1.1 holds assuming that $\mathbb{A} = \mathrm{Jac}\,(C/S)$.

Passing to an étale extension $S' \to S$ we may also assume that $C \to S$ is geometrically connected and admits a section $s\colon S \to C$. Let g be the genus of C. Since the map $C^g \to \mathrm{Jac}\,(C/S)$ given by $(P_1, \ldots, P_g) \mapsto (P_1 - s) + \cdots + (P_g - s)$ is surjective, passing to an étale extension $S' \to S$ and using the fact that $\mathcal{M}_{\mathbb{A},\mathrm{dR}}$ and $\mathcal{M}_{\mathbb{A},\mathrm{Man}}$ are group homomorphisms, we may assume that $\mathbb{A} = \mathrm{Jac}\,(C/S)$ and x is the difference $x = t - s$ with $t\colon S \to C$ an S-section.

The map

$$\mathrm{Ext}^1_S\left((\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S),\nabla_{\mathbb{A}}),(\mathcal{O}_S,d)\right) \to \mathrm{Ext}^1_{S'}\left((\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S'),\nabla_{\mathbb{A}}),(\mathcal{O}_{S'},d)\right)$$

is injective for $S' \subset S$ open non-empty by [12, Cor. 1.1.3]. Passing to an open non-empty subset S', we may then assume that either $s = t$ or $s \cap t = \emptyset$. In the first case $x = 0$ and there is nothing to prove since $\mathcal{M}_{\mathbb{A},\mathrm{dR}}$ and $\mathcal{M}_{\mathbb{A},\mathrm{Man}}$ are group homomorphisms. □

Proof of Proposition 1.1. In view of Lemma 7.4 we may assume that $\mathbb{A} = \mathrm{Jac}\,(C/S)$ with $C \to S$ a smooth, proper and geometrically connected curve and that $x \in \mathbb{A}(S)$ is the difference of two sections s and t of C over S such that $s \cap t = \emptyset$. Using the point s we get a closed immersion $C \to \mathbb{A}$ which provides an isomorphism $\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S) \cong \mathbb{H}^1_{\mathrm{dR}}(C/S)$ compatible with the Gauss-Manin connections on the two sides. Via this identification $\mathcal{M}_{\mathbb{A},\mathrm{Man}}(x)$ is an extension of $\mathbb{H}^1_{\mathrm{dR}}(C/S)$ by $\left(\mathcal{O}_S, d\right)$. Let $Z \subset C$ be the closed subscheme defined by $s \cup t$ and put $W := C \backslash Z$. Let $\mathbb{H}^1_{\mathrm{dR}}(W/S)$ be the hypercohomology of the complex $\Omega^\bullet_{C/S}\left(\log Z\right)$. Thus, $\mathbb{H}^1_{\mathrm{dR}}(W/S)$ is endowed with a Gauss-Manin connection, *cf.* [12, p. 404], which is an extension of $(\mathcal{O}_S, d)$ by $\mathbb{H}^1_{\mathrm{dR}}(C/S)$. It follows from [12, Prop. 1.5.5] that such extension is dual to $\mathcal{M}_{\mathbb{A},\mathrm{Man}}(x)$ (with the dual connection) on identifying $\mathbb{H}^1_{\mathrm{dR}}(C/S) \cong \mathbb{H}^1_{\mathrm{dR}}(C/S)^\vee$ by Poincaré duality. Let $\mathbb{A}(x) = [x\colon \mathbf{Z} \to \mathbb{A}]$. Note that $\mathbf{T}_{\mathrm{dR}}\left(\mathbb{A}(x)\right)$ is the dual of $\mathbf{T}_{\mathrm{dR}}\left(\mathbb{A}(x)^\vee\right)$ (with the dual connection) by Corollary 2.8. It is an extension of $(\mathcal{O}_S, d) \cong \mathbf{T}_{\mathrm{dR}}([\mathbf{Z} \to 0])$ by $\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}^\vee/S)$. The principal polarization on $\mathbb{A}$ allows one to identify $\mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}^\vee/S) \cong \mathbb{H}^1_{\mathrm{dR}}(\mathbb{A}/S) \cong \mathbb{H}^1_{\mathrm{dR}}(C/S)$ (compatibly with the connections). Thus, to conclude the proof of Proposition 1.1 it suffices to prove Lemma 7.5 below. □

Lemma 7.5. *We have $\mathbf{T}_{\mathrm{dR}}\big(\mathbb{A}(x)\big) \cong \mathbb{H}^1_{\mathrm{dR}}(W/S)$ compatibly with Gauss-Manin connections and as extensions of $(\mathcal{O}_S, d)$ by $\mathbb{H}^1_{\mathrm{dR}}(C/S)$.*

Proof. By construction $\mathbf{T}_{\mathrm{dR}}\big(\mathbb{A}(x)\big)$ is the $\mathcal{O}_S$-module with connection associated to the crystal $\mathbf{T}_{\mathrm{crys}}\big(\mathbb{A}(x)\big)$. We proceed as follows:

Step I. Let $(U \subset U') \in S^{\mathrm{crys}}$. In §7.3 we construct a sheaf $\mathrm{Pic}\,^{\mathrm{logcrys},0}_{C/U'}$ for the fppf topology of U' with a 1-step filtration given by $\mathrm{Pic}\,^{\mathrm{crys},0}_{C/U'} \subset \mathrm{Pic}\,^{\mathrm{logcrys},0}_{C/U'}$, functorial in $(U \subset U')$. We then prove that $\mathbf{T}_{\mathrm{crys}}\big(\mathbb{A}(x)\big)(U \subset U')$ coincides with $\mathbf{Lie}\,\mathrm{Pic}\,^{\mathrm{logcrys},0}_{C/U'}$ as a filtered crystal over S^{crys}; see (18).

Step II. Let C^{logcrys} be the locally nilpotent log-crystalline site of C with logarithmic structure associated to $Z \subset C$; *cf.* [17, §5]. We prove that we have an isomorphism of filtered crystals $\mathbf{Lie}\,\mathrm{Pic}\,^{\mathrm{logcrys},0}_{C/U'} \cong \mathrm{H}^1_{\mathrm{logcrys}}\big(C \times_S U/U'\big)$. Here, $\mathrm{H}^1_{\mathrm{logcrys}}\big(C \times_S U/U'\big)$ is endowed with the filtration

$$\mathrm{H}^1_{\mathrm{crys}}\big(C \times_S U/U'\big) \subset \mathrm{H}^1_{\mathrm{logcrys}}\big(C \times_S U/U'\big).$$

Note that $\mathrm{H}^1_{\mathrm{logcrys}}\big(C \times_S U/U\big)$ is $\mathbb{H}^1_{\mathrm{dR}}(W \times_S U/U)$ compatibly with the filtration given by $\mathbb{H}^1_{\mathrm{dR}}(C \times_S U/U) \subset \mathbb{H}^1_{\mathrm{dR}}(W \times_S U/U)$. We are then left with the task of proving that the integrable connection on $\mathbb{H}^1_{\mathrm{dR}}(W \times_S U/U)$, arising from the crystal

$$(U \subset U') \mapsto \mathrm{H}^1_{\mathrm{logcrys}}\big(C \times_S U/U'\big),$$

is the Gauss-Manin connection. This is the word by word transposition from the crystalline context to the log-crystalline context of [6, V.3.6.3 & V.3.6.4]. $\square$

7.3. $\mathrm{Pic}\,^{\mathrm{logcrys},0}$.

Let $(U \subset U') \in S^{\mathrm{crys}}$. For $X \to U'$ a homomorphism of schemes, let $\mathrm{H}^1_{\mathrm{crys}}(X/U') := \mathrm{H}^1(X, \mathcal{O}_{X/U',\mathrm{crys}})$. Define $\mathrm{Pic}\,^{\mathrm{logcrys}}_{C/U'}$ to be the sheaf for the fppf topology on U' associated to the presheaf whose value on an fppf morphism $T \to U'$ is the group $\mathrm{H}^1\big(C \times_S T, \mathcal{O}^*_{C \times_S T/T,\mathrm{logcrys}}\big)$. Assume that C admits a deformation as a smooth projective curve $\pi'\colon C' \to U'$ over U' and that s', $t' \in C'(U')$ deform s and t; this holds for U affine. Put $Z' := s' \cup t'$.

Because of the equivalence between the category of crystals on the locally nilpotent log crystalline site C^{logcrys} and the category of invertible sheaves endowed with an integral connection in [17, 6.2],[6, IV 1.6.5], we have a canonical isomorphism between $\mathrm{Pic}\,^{\mathrm{logcrys}}_{C/U'}$ and the sheaf $\mathrm{Pic}\,^{\natural-\log}_{C'/U'}$ for the fppf topology on U' associated to the presheaf whose value on an fppf morphism $T \to U'$ is the group of invertible sheaves of $\mathcal{O}_{C' \times_{U'} T}$-modules endowed with an integrable connection relative to T and with log-poles along $Z' \times_{U'} T$ *cf.* [17, Thm. 6.2]. The group of such connections on the trivial sheaf $\mathcal{O}_{C' \times_{U'} T}$ is in bijection with $\mathrm{H}^0\big(C' \times_{U'} T, \Omega^1_{C' \times_{U'} T/T}(\log Z' \times_{U'} T)\big)$. In particular, we have an exact sequence

$$0 \to \pi'_*\big(\Omega^1_{C'/U'}(\log Z')\big) \longrightarrow \mathrm{Pic}\,^{\natural-\log}_{C'/U'} \longrightarrow \mathrm{Pic}\,_{C'/U'},$$

where the sheaf on the left associates to an fppf morphism $T \to U'$ the group

$$\pi'_*\big(\Omega^1_{C'/U'}(\log Z')\big)(T) \cong \big(\pi' \times_{U'} T\big)_*\big(\Omega^1_{C' \times_{U'} T/T}(\log Z' \times_{U'} T)\big)(T).$$

Since C' is a smooth, proper and geometrically connected curve over U' and since Z' consists of two disjoint sections of C', the sheaf of $\mathcal{O}_{U'}$-modules $\pi'_*\big(\Omega^1_{C'/U'}(\log Z')\big)$ is locally free so that it defines a vector group. Let $\mathrm{Pic}\,^{\natural-\log,0}_{C'/U'}$ be the inverse image of $\mathrm{Pic}\,^0_{C'/U'}$. Let furthermore $\mathrm{Pic}\,^{\mathrm{logcrys},0}_{C/U'} \subset \mathrm{Pic}\,^{\mathrm{logcrys}}_{C/U'}$ be the inverse image of $\mathrm{Pic}\,^0_{C/U} \subset \mathrm{Pic}\,_{C/U}$ via the composition of maps $\mathrm{Pic}\,^{\mathrm{logcrys}}_{C/U'} \to \mathrm{Pic}\,^{\mathrm{logcrys}}_{C/U} \to \mathrm{Pic}\,_{C/U}$. Then, for a fixed deformation $\pi'\colon C' \to U'$ as above, we have $\mathrm{Pic}\,^{\natural-\log,0}_{C'/U'} \cong \mathrm{Pic}\,^{\mathrm{logcrys},0}_{C/U'}$ and we have an exact sequence

$$0 \longrightarrow \pi'_*\big(\Omega^1_{C'/U'}(\log Z')\big) \longrightarrow \mathrm{Pic}\,^{\natural-\log,0}_{C'/U'} \longrightarrow \mathrm{Pic}\,^0_{C'/U'} \longrightarrow 0.$$

The surjectivity amounts to saying that every invertible sheaf on $C'\times_{U'}T$, algebraically equivalent to 0, can be endowed with a connection. The obstruction for this defines a map $\operatorname{Pic}^{0}_{C'/U'}\to \mathrm{R}^1\pi'_*\big(\Omega^1_{C'/U'}(\log Z')\big)$. Since $\Omega^1_{C'/U'}(\log Z')$ is a relative Cartier divisor on a smooth, proper and geometrically connected curve $C'\to U'$, it follows that the sheaf $\mathrm{R}^1\pi'_*\big(\Omega^1_{C'/U'}(\log Z')\big)$ is an invertible $\mathcal{O}_{U'}$-module and so it defines a vector group. In particular, the obstruction map defines a morphism from $\operatorname{Pic}^{0}_{C'/U'}$, which is an abelian scheme over U', to a vector group which is an affine scheme relative to U'. We conclude that the obstruction map vanishes. Arguing as in [19, § II.1.5] one gets an isomorphism of crystals

$$\mathbf{Lie}\operatorname{Pic}^{\mathrm{logcrys},0}_{C/U'}=\mathbf{Lie}\operatorname{Pic}^{\mathrm{logcrys}}_{C/U'}\cong \mathrm{H}^1_{\mathrm{logcrys}}\big(C\times_S U/U'\big).$$

Taking $Z=\emptyset$ in the foregoing construction, we get the definition of $\operatorname{Pic}^{\mathrm{crys},0}_{C/U'}$. The pull-back from the crystalline site of C/U' to the log-crystalline site of C/U' induces maps $\operatorname{Pic}^{\mathrm{crys},0}_{C/U'}\to\operatorname{Pic}^{\mathrm{logcrys},0}_{C/U'}$, and $\mathrm{H}^1_{\mathrm{crys}}\big(C\times_S U/U'\big)\to \mathrm{H}^1_{\mathrm{logcrys}}\big(C\times_S U/U'\big)$, and gives the commutative diagram

$$\begin{array}{ccc}\mathbf{Lie}\operatorname{Pic}^{\mathrm{crys},0}_{C/U'} & \stackrel{\sim}{\longrightarrow} & \mathrm{H}^1_{\mathrm{crys}}\big(C\times_S U/U'\big)\\ \downarrow & & \downarrow\\ \mathbf{Lie}\operatorname{Pic}^{\mathrm{logcrys},0}_{C/U'} & \stackrel{\sim}{\longrightarrow} & \mathrm{H}^1_{\mathrm{logcrys}}\big(C\times_S U/U'\big).\end{array}\tag{17}$$

If one fixes a deformation $\pi'\colon C'\to U'$ and sections $s',t'\in C'(S')$ as above, the map $\operatorname{Pic}^{\mathrm{crys},0}_{C/U'}\to\operatorname{Pic}^{\mathrm{logcrys},0}_{C/U'}$ is the natural map $\operatorname{Pic}^{\natural,0}_{C'/U'}\to\operatorname{Pic}^{\natural-\log,0}_{C'/U'}$. Both group schemes map surjectively onto $\operatorname{Pic}^{0}_{C'/U'}$ and the induced map on kernels coincide with the map of vector groups associated to the injective morphism of $\mathcal{O}_{U'}$-modules

$$\pi'_*\big(\Omega^1_{C'/U'}\big)\longrightarrow \pi'_*\big(\Omega^1_{C'/U'}(\log Z')\big).$$

In particular, the map $\operatorname{Pic}^{\natural,0}_{C'/U'}\to\operatorname{Pic}^{\natural-\log,0}_{C'/U'}$ is a closed immersion. Thus, the vertical arrows in (17) are injective as well. Let $\mathcal{I}_s$ be the kernel of the morphism $\mathcal{O}_{C/U',\mathrm{logcrys}}\to s_*(\mathcal{O}_{U/U',\mathrm{logcrys}})$ and similarly for $\mathcal{I}_t$. They are locally free sheaves of $\mathcal{O}^{\mathrm{logcrys}}_{C/U'}$-modules. Consider the natural map $u\colon\mathbf{Z}\to\operatorname{Pic}^{\mathrm{logcrys}}_{C/U'}$ sending 1 to $\mathcal{I}_t^{-1}\mathcal{I}_s$. If we fix a deformation, via the identification $\operatorname{Pic}^{\natural-\log,0}_{C'/U'}\cong\operatorname{Pic}^{\mathrm{logcrys},0}_{C/U'}$, then $u(1)$ belongs to $\operatorname{Pic}^{\mathrm{logcrys},0}_{C/U'}$ and corresponds to the invertible sheaf $\mathcal{O}_{C'}(t'-s')$ with its standard derivation. In particular, u factors via $u\colon\mathbf{Z}\to\operatorname{Pic}^{\mathrm{logcrys},0}_{C/U'}$. The map $\mathbf{Z}\to\operatorname{Pic}^{\natural-\log,0}_{C'/U'}$ identifies $\operatorname{Pic}^{\natural-\log,0}_{C'/U'}$ as a vectorial extension of $\mathbb{M}':=\big[\mathbf{Z}\to\operatorname{Pic}^{0}_{C'/U'}\big]$ sending $1\mapsto t'-s'$. Analogously, $\operatorname{Pic}^{\natural,0}_{C'/U'}$ is a vectorial extension of $\operatorname{Pic}^{0}_{C'/U'}$. By universality we get a map of complexes $v\colon\mathbb{E}(\mathbb{M}')\to\big[\mathbf{Z}\to\operatorname{Pic}^{\natural-\log,0}_{C'/U'}\big]$, sending $\mathbb{E}\big(\operatorname{Pic}^{0}_{C'/U'}\big)$ to $\operatorname{Pic}^{\natural,0}_{C'/U'}$. Since it is an isomorphism fiberwise over the closed points of U' by [4, Lemma 4.5.2], it is an isomorphism. Note that $\mathbb{M}'$ is a deformation of $\mathbb{A}(x)=[x\colon\mathbf{Z}\to\mathbb{A}]$, identifying $\operatorname{Pic}^{0}_{C/S}\cong\mathbb{A}=\operatorname{Jac}(C/S)$ via the principal polarization on $\mathbb{A}$. Hence, $\mathbf{T}_{\mathrm{dR}}(\mathbb{M}')=\mathbf{T}_{\mathrm{crys}}\big(\mathbb{A}(x)\big)(U\subset U')$. In view of Remark 3.4 the isomorphism v is compatible for different choices of deformations with the isomorphisms of Theorem 2.1. We thus get an isomorphism of filtered crystals

$$\mathbf{T}_{\mathrm{crys}}\big(\mathbb{A}(x)\big)(U\subset U')\cong\mathbf{Lie}\operatorname{Pic}^{\mathrm{logcrys},0}_{C/U'}(U').\tag{18}$$

By construction it is functorial in $U\subset U'$.

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