HEIGHT PAIRINGS ON ORTHOGONAL SHIMURA VARIETIES

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Abstract. Let M be the Shimura variety associated to the group of spinor similitudes of a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$. We prove a conjecture of Bruinier and Yang, relating the arithmetic intersection multiplicities of special divisors and CM points on M to the central derivatives of certain L-functions. Each such L-function is the Rankin-Selberg convolution associated with a cusp form of half-integral weight $n/2 + 1$, and the weight $n/2$ theta series of a positive definite quadratic space of rank n.

When $n = 2$ the Shimura variety M is a classical quaternionic Shimura curve, and our result is a variant of the Gross-Zagier theorem on heights of Heegner points.

Contents

1. Introduction

In this paper we prove a generalization, conjectured by Bruinier-Yang [BY09], of the Gross-Zagier theorem on heights of Heegner points on modular curves.

Instead of a modular curve we work on the Shimura variety defined by a reductive group over $\mathbb{Q}$ of type $\mathrm{GSpin}(n, 2)$. The Heegner points are replaced by 0-cycles arising from embeddings of rank two tori into $\mathrm{GSpin}(n, 2)$, and by divisors arising from embeddings of $\mathrm{GSpin}(n-1, 2)$ into $\mathrm{GSpin}(n, 2)$. We

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prove that the arithmetic intersections of these special cycles are related to the central derivatives of certain Rankin-Selberg convolution L-functions.

1.1. **The Shimura variety.** Fix an integer $n \geq 1$ and a quadratic space (V, Q) over $\mathbb{Q}$ of signature $(n, 2)$. From V one can construct a Shimura datum $(G, \mathcal{D})$, in which $G = \mathrm{GSpin}(V)$ is a reductive group over $\mathbb{Q}$ sitting in an exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{GSpin}(V) \to \mathrm{SO}(V) \to 1,$$

and the hermitian domain $\mathcal{D}$ is an open subset of the space of isotropic lines in $\mathbb{P}^1(V_{\mathbb{C}})$.

Any choice of lattice $L \subset V$ on which Q is $\mathbb{Z}$-valued determines a compact open subgroup $K \subset G(\mathbb{A}_f)$. We fix such a lattice, and assume that L is maximal, in the sense that it admits no superlattice on which Q is $\mathbb{Z}$-valued. The data $L \subset V$ now determines a complex orbifold

$$M(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K$$

which, by the theory of canonical models of Shimura varieties, is the space of complex points of an algebraic stack $M \to \mathrm{Spec}(\mathbb{Q})$. (Throughout this article *stack* means *Deligne-Mumford stack*.)

Except for small values of n, the Shimura variety M is not of PEL type; that is, it is not naturally a moduli space of abelian varieties with polarization, endomorphisms, and level structures. It is, however, of Hodge type, and so the recent work of Kisin [Kis10] (and its extension in [MPar]) provides us with a regular and flat integral model

$$\mathcal{M} \to \mathrm{Spec}(\mathbb{Z}[1/2]).$$

1.2. **Special divisors.** The integral model carries over it a canonical family of abelian varieties: the *Kuga-Satake abelian scheme*

$$\mathcal{A} \to \mathcal{M}.$$

The Kuga-Satake abelian scheme is endowed with a right action of the Clifford algebra $C(L)$, and with a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathcal{A} = \mathcal{A}^+ \times \mathcal{A}^-$.

For any scheme $S \to \mathcal{M}$, the pullback $\mathcal{A}_S$ comes with a distinguished $\mathbb{Z}$-module $V(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S)$ of *special endomorphisms*, and there is a positive definite quadratic form $Q : V(\mathcal{A}_S) \to \mathbb{Z}$ characterized by $x \circ x = Q(x) \cdot \mathrm{Id}$. Slightly more generally, there is a distinguished subset

$$V_\mu(\mathcal{A}_S) \subset V(\mathcal{A}_S) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for each coset $\mu \in L^\vee/L$, where $L^\vee$ is the dual lattice of L. Taking $\mu = 0$ recovers $V(\mathcal{A}_S)$.

The special endomorphisms allow us to define a family of *special divisors* on $\mathcal{M}$. For $m \in \mathbb{Q}_{>0}$ and $\mu \in L^\vee/L$, let $\mathcal{Z}(m, \mu) \to \mathcal{M}$ be the moduli stack that assigns to any $\mathcal{M}$-scheme $S \to \mathcal{M}$ the set

$$\mathcal{Z}(m, \mu)(S) = \{x \in V_\mu(\mathcal{A}_S) : Q(x) = m\}.$$

The morphism $\mathcal{Z}(m,\mu) \to \mathcal{M}$ is relatively representable, finite, and unramified, and allows us to view $\mathcal{Z}(m,\mu)$ as a Cartier divisor on $\mathcal{M}$. These are the *special divisors* of the title.

In the generic fiber of $\mathcal{M}$ the special divisors agree with the divisors appearing in [Bor98], [Bru02], [BY09], and [Kud04]. In special cases where V has signature $(1,2)$, $(2,2)$, or $(3,2)$, the Shimura variety $\mathcal{M}$ is a (quaternionic version of a) modular curve, Hilbert modular surface, or Siegel threefold, respectively. In these cases the special divisors are traditionally known as *complex multiplication points*, *Hirzebruch-Zagier divisors*, and *Humbert surfaces*. See, for example, [vdG88], [KR99], [KR00], and [KRY06].

1.3. **CM points.** Let $V_0 \subset V$ be a negative definite plane in V, and set $L_0 = V_0 \cap L$. The Clifford algebra $C(L_0)$ is an order in a quaternion algebra over $\mathbb{Q}$, and its even part $C^+(L_0) \subset C(L_0)$ is an order in a quadratic imaginary field $\boldsymbol{k}$. Fix an embedding $\boldsymbol{k} \to \mathbb{C}$, and assume from now on that $C^+(L_0) = \mathcal{O}_{\boldsymbol{k}}$ is the maximal order in $\boldsymbol{k}$.

The group $\mathrm{GSpin}(V_0) \cong \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m$ is a rank two torus, and has associated to it a 0-dimensional Shimura variety $Y \to \mathrm{Spec}(\boldsymbol{k})$, which can be reinterpreted as the moduli stack of elliptic curves with CM by $\mathcal{O}_{\boldsymbol{k}}$. This allows us to construct a smooth integral model $\mathcal{Y} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$. The inclusion $V_0 \to V$ induces an embedding $\mathrm{GSpin}(V_0) \to G$, which induces a relatively representable, finite, and unramified morphism

$$i : \mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]} \to \mathcal{M}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}.$$

The stack $\mathcal{Y}$ has its own Kuga-Satake abelian scheme $\mathcal{A}_0 \to \mathcal{Y}$, endowed with a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathcal{A}_0 \cong \mathcal{A}_0^+ \times \mathcal{A}_0^-$, and a right action of $C(L_0)$. Here $\mathcal{A}_0^+$ is the universal elliptic curve with CM by $\mathcal{O}_{\boldsymbol{k}}$ and $\mathcal{A}_0 \cong \mathcal{A}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L_0)$. It is related to the Kuga-Satake abelian scheme $\mathcal{A} \to \mathcal{M}$ by a $C(L)$-linear isomorphism

$$\mathcal{A}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \cong \mathcal{A}_0|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \otimes_{C(L_0)} C(L).$$

Let $\widehat{\mathrm{Pic}}(\mathcal{M})$ denote the group of metrized line bundles on $\mathcal{M}$, and similarly with $\mathcal{M}$ replaced by $\mathcal{M}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$ or $\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$. The composition

$$\widehat{\mathrm{Pic}}(\mathcal{M}) \to \widehat{\mathrm{Pic}}(\mathcal{M}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}) \xrightarrow{i^*} \widehat{\mathrm{Pic}}(\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}) \xrightarrow{\widehat{\deg}} \mathbb{R}/\mathbb{Q}\log(2)$$

is denoted $\widehat{\mathcal{Z}} \mapsto [\widehat{\mathcal{Z}} : \mathcal{Y}]$. We call it *arithmetic degree along* $\mathcal{Y}$.

1.4. **Harmonic modular forms.** Denote by $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$, and by

$$\omega_L : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(\mathfrak{S}_L)$$

the Weil representation on the space $\mathfrak{S}_L$ of $\mathbb{C}$-valued functions on $L^\vee/L$. There is a complex conjugate representation $\overline{\omega}_L$ on the same space, and a contragredient action $\omega_L^\vee$ on the dual space $\mathfrak{S}_L^\vee$.

Bruinier and Funke [BF04] have defined a surjective conjugate-linear differential operator

$$\xi : H_{1-\frac{n}{2}}(\omega_L) \to S_{1+\frac{n}{2}}(\overline{\omega}_L).$$

Here $S_{1+\frac{n}{2}}(\overline{\omega}_L)$ is the space of weight $1+\frac{n}{2}$ cusp forms valued in $\mathfrak{S}_L$, and transforming according to the representation $\overline{\omega}_L$. The space $H_{1-\frac{n}{2}}(\omega_L)$ is defined similarly, but the forms are *harmonic weak Mass forms* in the sense of [BY09, §3.1].

To any $f \in H_{1-\frac{n}{2}}(\omega_L)$ there is associated a formal q-expansion

$$f^+(\tau) = \sum_{\substack{m\in\mathbb{Q} \\ m\gg-\infty}} c_f^+(m)q^m \in \mathfrak{S}_L[[q]],$$

and each coefficient is uniquely a linear combination $\sum_\mu c_f^+(m,\mu)\varphi_\mu$ of the characteristic functions φ_μ of cosets $\mu \in L^\vee/L$. Assuming that f^+ has *integral principal part*, in the sense that $c_f^+(m,\mu) \in \mathbb{Z}$ whenever $m < 0$, we may form the Cartier divisor

$$\mathcal{Z}(f) = \sum_{m>0} \sum_{\mu\in L^\vee/L} c_f^+(-m,\mu)\cdot \mathcal{Z}(m,\mu)$$

(the sum is finite) on the integral model $\mathcal{M}$, following the work of Borcherds [Bor98], Bruinier [Bru02], and Bruinier-Yang [BY09].

Using the formalism of regularized theta lifts developed in [Bor98], Bruinier [Bru02] defined a Green function $\Phi(f)$ for $\mathcal{Z}(f)$. This Green function determines a metric on the corresponding line bundle, yielding a metrized line bundle

$$\widehat{\mathcal{Z}}(f) \in \widehat{\mathrm{Pic}}(\mathcal{M}).$$

1.5. **The main result.** The central problem is to compute the arithmetic degree $[\widehat{\mathcal{Z}}(f) : \mathcal{Y}]$. Assuming that the divisor $\mathcal{Z}(f)_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$ intersects the cycle $\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$ properly, the arithmetic degree decomposes as a sum of local contributions. The calculation of the archimedean contribution, which is essentially the sum of the values of $\Phi(f)$ at all $y \in \mathcal{Y}(\mathbb{C})$, is the main result of [BY09].

Based on their calculation of the archimedean contribution, Bruinier and Yang formulated a conjecture [BY09, Conjecture 1.1] relating $[\widehat{\mathcal{Z}}(f) : \mathcal{Y}]$ to the derivative of an L-function. Our main result is a proof of this conjecture, up to a rational multiple of $\log(2)$.

The statement of the result requires two more ingredients. The first is the *metrized cotautological bundle*

$$\widehat{\boldsymbol{T}} \in \widehat{\mathrm{Pic}}(\mathcal{M}).$$

By definition, the hermitian domain $\mathcal{D}$ is an open subset of the space of isotropic lines in $\mathbb{P}^1(V_{\mathbb{C}})$. Restricting the tautological bundle on $\mathbb{P}^1(V_{\mathbb{C}})$ yields a line bundle $\omega_{\mathcal{D}}$ on $\mathcal{D}$, which descends first to the orbifold $M(\mathbb{C})$, and then to the canonical model M. The resulting line bundle, ω, is the *tautological bundle* on M, also called the *line bundle of weight one modular forms*. There is a natural metric (5.8) on ω, and our $\widehat{\boldsymbol{T}}$ is an integral model of $\widehat{\omega}^{-1}$.

The second ingredient is a classical vector valued theta series. From the $\mathbb{Z}$-quadratic space

$$\Lambda = \{x \in L : x \perp L_0\}$$

of signature $(n,0)$, one can construct a half-integral weight theta series

$$\Theta_\Lambda(\tau) \in M_{\frac{n}{2}}(\omega_\Lambda^\vee).$$

Here ω_Λ is the Weil representation of $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ on the space $\mathfrak{S}_\Lambda$ of functions on $\Lambda^\vee/\Lambda$, and $\omega_\Lambda^\vee$ is its contragredient. See §3.3 for the precise definition.

The following appears in the text as Theorem 5.7.3.

Theorem A. *Assume that the discriminant of $\boldsymbol{k}$ is odd, and let $h_{\boldsymbol{k}}$ and $w_{\boldsymbol{k}}$ be the class number and number of roots of unity in $\boldsymbol{k}$, respectively. Every weak harmonic Maass form $f \in H_{1-n/2}(\omega_L)$ with integral principal part satisfies*

$$[\widehat{\mathcal{Z}}(f) : \mathcal{Y}] + c_f^+(0,0) \cdot [\widehat{\boldsymbol{T}} : \mathcal{Y}] = -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \cdot L'(\xi(f), \Theta_\Lambda, 0),$$

where $c_f^+(0,0)$ is the value of $c_f^+(0) \in \mathfrak{S}_L$ at the trivial coset in $L^\vee/L$, and $L(\xi(f), \Theta_\Lambda, s)$ is the Rankin-Selberg convolution L-function of §3.3.

Remark 1.5.1. The theorem asserts an equality in $\mathbb{R}/\mathbb{Q}\log(2)$. Indeed, because we lack a good extension of $\mathcal{M} \to \mathrm{Spec}(\mathbb{Z}[1/2])$ to a stack over $\mathrm{Spec}(\mathbb{Z})$, the left hand side of the identity is only defined modulo rational multiples of $\log(2)$.

Remark 1.5.2. The analogous theorem for Shimura varieties associated with unitary similitude groups was proved in [BHY15].

Remark 1.5.3. The convolution L-function appearing in the theorem can be realized by integrating $\xi(f)$ and Θ_Λ against a certain Eisenstein series $E_{L_0}(\tau, s)$. This Eisenstein series vanishes at $s = 0$, and hence the same is true of $L(\xi(f), \Theta_\Lambda, s)$.

Remark 1.5.4. As noted earlier, our cycles $\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]} \to \mathcal{M}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$ arise from embeddings of rank two tori into $\mathrm{GSpin}(n,2)$. One could instead look at the cycles defined by embeddings of maximal tori; these are the *big CM cycles* of [BKY12]. The methods used to prove Theorem A are expected to yield a similar result for big CM cycles, and such a result will have applications to Colmez's conjecture on the Faltings heights of CM abelian varieties (via the method of Yang [Yan10]), and to the André-Oort conjecture on the Zariski closures of special points on Shimura varieties (via recent results of Tsimerman [private communication]). These generalizations and applications will be explored in future work of the authors.

Remark 1.5.5. As the equality of the theorem is only up to rational multiples of $\log(2)$, in principle it is possible to drop the assumption that $d_{\boldsymbol{k}}$ is odd. It should also be possible to drop the assumption that $C^+(L_0)$ is maximal and replace it by the weaker assumption that $C^+(L_0)$ is maximal away from 2. Such improvements come at a cost: under these weaker hypotheses one does not know anything like the explicit formulas appearing in Proposition 3.3.1 and Theorem 4.5.1.

Instead, one should be able (we have not checked the details) to prove the main result by using the Siegel-Weil arguments of [BKY12] and [KY13]. In essence, the Siegel-Weil formula allows one to compare the point counting part of Theorem 4.5.1 with the values of certain Whittaker functions appearing in the Eisenstein series coefficients of Proposition 3.3.1, without deriving explicit formulas for either one.

A rough outline of the contents is as follows:

In §2 we establish the basic properties of the Shimura variety $\mathcal{M}$ on which we will be performing intersection theory, and of the special cycles $\mathcal{Z}(m,\mu)$. We prove that the $\mathcal{Z}(m,\mu)$ define Cartier divisors on $\mathcal{M}$, and explore the the functoriality of the formation of $\mathcal{M}$ and its divisors with respect to the quadratic space L. More precisely, given an isometric embedding $L_0 \hookrightarrow L$ of maximal $\mathbb{Z}$-quadratic spaces of signatures $(n_0,2)$ and $(n,2)$, with $1 \le n_0 \le n$, we prove the existence of a canonical morphism $\mathcal{M}_0 \to \mathcal{M}$ between the associated integral models, and show that the pullback of a special divisor on $\mathcal{M}$ is a prescribed linear combination of special divisors on $\mathcal{M}_0$.

Section 3 contains little original material, and consists primarily of a quick reminder of some of the analytic theory used in [BY09]. In particular, we recall the essential properties of harmonic weak Maass forms, the divisors $Z(f)$ on M associated to such forms, and the Green functions $\Phi(f)$ for these divisors, which are defined as regularized theta lifts. We also recall the theta series and convolution L-functions that appear in the main theorem, and Schofer's [Sch09] calculation of the Fourier coefficients of the Eisenstein series of Remark 1.5.3. The only thing new here is that, thanks to the constructions of §2, we are able to define an extension of the divisor $Z(f)$ on M to a divisor $\mathcal{Z}(f)$ on the integral model $\mathcal{M}$.

In §4 we fix a quadratic space L_0 over $\mathbb{Z}$ of signature $(0,2)$, whose even Clifford algebra is isomorphic to the maximal order in a quadratic imaginary field $\boldsymbol{k}$. The Clifford algebra of L_0 then has the form $C(L_0) \xrightarrow{\simeq} \mathcal{O}_{\boldsymbol{k}} \oplus L_0$. The Shimura variety associated to the rank two torus $\mathrm{GSpin}(L_{0\mathbb{Q}})$ has dimension 0, and can be realized as the moduli stack of elliptic curves with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$. Using this interpretation we define an integral model $\mathcal{Y} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$, and define the Kuga-Satake abelian scheme as $\mathcal{A}_0 = \mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L_0)$, where $\mathcal{E} \to \mathcal{Y}$ is the universal CM elliptic curve. All of the results of §2 are then extended to this new setting.

In particular, we define a family of divisors $\mathcal{Z}_0(m,\mu)$ on the arithmetic curve $\mathcal{Y}$, each of which can be characterized as the locus of points where $\mathcal{A}_0$

has an extra quasi-endomorphism with prescribed properties. This can also be expressed in terms of endomorphisms of the universal CM elliptic curve: along each special divisor the action of $\mathcal{O}_{\boldsymbol{k}}$ on $\mathcal{E}$ extends to an action of an order in a definite quaternion algebra. Theorem 4.5.1 gives explicit formulas for the degrees of the divisors $\mathcal{Z}_0(m,\mu)$, and these degrees match up with Schofer's formulas for the coefficients of Eisenstein series. Many cases of Theorem 4.5.1 already appear in work of Kudla-Rapoport-Yang [KRY99] and Kudla-Yang [KY13], and our proof, like theirs, makes essential use of Gross's calculation [Gro86] of the endomorphism rings of canonical lifts of CM elliptic curves.

In §5 we prove Theorem A. The greatest difficulty comes from the fact that the 1-cycle $\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$ on $\mathcal{M}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$ may have irreducible components contained entirely within $\mathcal{Z}(m,\mu)_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$. Thus we must compute improper intersections. Our techniques for doing this are a blend of the methods used in [BHY15] on unitary Shimura varieties, and the methods of Hu's thesis [Hu99], which reconstructs the arithmetic intersection theory of Gillet-Soulé [GS90] using Fulton's method of deformation to the normal cone.

The cycle $\mathcal{Y}$ has a normal bundle $N_{\mathcal{Y}}\mathcal{M} \to \mathcal{Y}$, and the arithmetic divisor $\widehat{\mathcal{Z}}(f)$ on $\mathcal{M}$ has a *specialization to the normal bundle* $\sigma(\widehat{\mathcal{Z}}(f))$, which is an arithmetic divisor on $N_{\mathcal{Y}}\mathcal{M}$. The specialization is defined in such a way that the arithmetic degree of $\widehat{\mathcal{Z}}(f)$ along $\mathcal{Y} \to \mathcal{M}$ is equal to the arithmetic degree of $\sigma(\widehat{\mathcal{Z}}(f))$ along the zero section $\mathcal{Y} \to N_{\mathcal{Y}}\mathcal{M}$. What we are able to show, essentially, is that every component of the arithmetic divisor $\sigma(\widehat{\mathcal{Z}}(f))$ that meets the zero section improperly has $\sigma(\widehat{\boldsymbol{T}})$, the specialization of the cotautological bundle, as its associated line bundle. This implies that every component of $\widehat{\mathcal{Z}}(f)$ that meets $\mathcal{Y}$ improperly contributes the same quantity, $[\widehat{\boldsymbol{T}} : \mathcal{Y}]$, to the degree $[\widehat{\mathcal{Z}}(f) : \mathcal{Y}]$. The contribution to the intersection of the remaining components can be read off from the degrees of the divisors $\widehat{\mathcal{Z}}_0(m,\mu)$ computed in §4, and the calculation of $[\widehat{\boldsymbol{T}} : \mathcal{Y}]$ quickly reduces to the Chowla-Selberg formula.

Finally, we remark that the proof of Theorem A makes essential use of the Bruinier-Yang calculation (following Schofer [Sch09]) of the values of $\Phi(f)$ at the points of $\mathcal{Y}(\mathbb{C})$. Because of the particular way in which the divergent integral defining $\Phi(f)$ is regularized, the Green function $\Phi(f)$ has a well-defined value even at points of divisor $\mathcal{Z}(f)$ along which $\Phi(f)$ has a logarithmic singularity. In other words, the Green function comes, by construction, with a discontinuous extension to all points of $\mathcal{M}(\mathbb{C})$. Hu's thesis sheds some light on this phenomenon, and the *over-regularized* values of $\Phi(f)$ at the points of $\mathcal{Z}(f)$ will play an essential role in our calculation of improper intersections.

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2. The GSpin Shimura variety and its special divisors

2.1. **Preliminaries.** Let R be a commutative ring. A *quadratic space* over R is a projective R-module V of finite rank, equipped with a homogenous function $Q\colon V \to R$ of degree 2 for which the symmetric pairing

$$[x, y] = Q(x + y) - Q(x) - Q(y)$$

is R-bilinear. We say that V is *self-dual* if this pairing induces an isomorphism $V \xrightarrow{\simeq} \mathrm{Hom}_R(V, R)$.

The *Clifford algebra* $C(V)$ is the quotient of the tensor algebra $\bigotimes V$ by the ideal generated by elements of the form $x \otimes x - Q(x)$. The R-algebra $C(V)$ is generated by the image of the natural injection $V \to C(V)$, and the grading on $\bigotimes V$ induces a $\mathbb{Z}/2\mathbb{Z}$ grading

$$C(V) = C^+(V) \oplus C^-(V).$$

The Clifford algebra $C(V)$ has the following universal property: given an associative algebra B over R and a map of R-modules $f : V \to B$ satisfying $f(x)^2 = Q(x)$ for all $x \in V$, there is a unique map of R-algebras $\tilde{f} : C(V) \to B$ satisfying $\tilde{f}|_V = f$. In particular, applying this property with $B = C(V)^{\mathrm{op}}$ and f the natural inclusion of V in $C(V)^{\mathrm{op}}$, we obtain a canonical anti-involution $*$ on $C(V)$ satisfying:

$$(x_1 x_2 \cdots x_{r-1} x_r)^* = x_r x_{r-1} \cdots x_2 x_1$$

for any $x_1, x_2, \ldots, x_{r-1}, x_r \in V$.

To a self-dual quadratic space V, we can attach the reductive group scheme $\mathrm{GSpin}(V)$ over R with functor of points

$$\mathrm{GSpin}(V)(S) = \{g \in C^+(V)_S^\times : gV_S g^{-1} = V_S\}$$

for any R-algebra S. Denote by $g \bullet v = gvg^{-1}$ the natural action of $\mathrm{GSpin}(V)$ on V. For any R-algebra S, the map $g \mapsto g^* g$ on $\mathrm{GSpin}(V)(S)$ takes values in $S^\times$ [Bas74, 3.2.1], and so defines a canonical homomorphism of R-group schemes

$$\nu : \mathrm{GSpin}(V) \to \mathbb{G}_{m,R},$$

which we call the *spinor norm.*

Choose an element $\delta \in C^+(V)^\times$ satisfying $\delta^* = -\delta$. We can use this element to define a symplectic pairing on $C(V)$ as follows (see [MPar, (1.6)]). The self-duality hypothesis on V implies that $C(V)$ is a graded Azumaya algebra over R in the sense of [Bas74, §2.3, Theorem]. In particular, there is a canonical reduced trace form $\mathrm{Trd} : C(V) \to R$, and the pairing $(z_1, z_2) \mapsto \mathrm{Trd}(z_1 z_2)$ is a perfect symmetric bilinear form on $C(V)$. The desired symplectic pairing on $C(V)$ is $\psi_\delta(z_1, z_2) = \mathrm{Trd}(z_1 \delta z_2^*)$.

It is easily checked that

$$\psi_\delta(gz_1, gz_2) = \nu(g)\psi_\delta(z_1, z_2).$$

for any R-algebra S, $z_1, z_2 \in C(V)_S$, and $g \in \mathrm{GSpin}(V)(S)$. In other words, the action of $\mathrm{GSpin}(V)$ on $C(V)$ by left multiplication realizes $\mathrm{GSpin}(V)$

as a subgroup of the R-group scheme GSp_δ of symplectic similitudes of $(C(V), \psi_\delta)$, and the symplectic similitude character restricts to the spinor norm on $\mathrm{GSpin}(V)$.

2.2. **The complex Shimura variety.** Let $n \geq 1$ be an integer. Fix a quadratic space (L, Q) over $\mathbb{Z}$ of signature $(n, 2)$, and set $V = L_\mathbb{Q}$. We assume throughout that L is *maximal* in the sense that there is no lattice $L \subsetneq L' \subset V$ satisfying $Q(L') \subset \mathbb{Z}$. Set $G = \mathrm{GSpin}(V)$; this is a reductive group over $\mathbb{Q}$.

Let $\mathcal{D}$ be the space of oriented negative 2-planes $\boldsymbol{h} \subset V_\mathbb{R}$. If we extend the $\mathbb{Q}$-bilinear form on V to a $\mathbb{C}$-bilinear form on $V_\mathbb{C}$, the real manifold $\mathcal{D}$ is isomorphic to the Grassmannian

$$\mathcal{D} \xrightarrow{\simeq} \{z \in V_\mathbb{C} \smallsetminus \{0\} : [z, z] = 0 \text{ and } [z, \overline{z}] < 0\} / \mathbb{C}^\times, \tag{2.1}$$

and in particular is naturally a complex manifold. The isomorphism is as follows: for each $\boldsymbol{h}$ pick an oriented basis $\{x, y\}$ so that $Q(x) = Q(y)$ and $[x, y] = 0$, and let $z = x + iy$. The inverse construction is obvious.

There are canonical inclusions of $\mathbb{R}$-algebras $\mathbb{C} \hookrightarrow C(\boldsymbol{h}) \hookrightarrow C(V_\mathbb{R})$, where the first is determined by

$$i \mapsto \frac{xy}{\sqrt{Q(x)Q(y)}}.$$

The induced map $\mathbb{C}^\times \to C(V_\mathbb{R})^\times$ takes values in $G(\mathbb{R})$, and arises from a morphism of real algebraic groups

$$\alpha_{\boldsymbol{h}} : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_\mathbb{R}. \tag{2.2}$$

In this way we identify $\mathcal{D}$ with a $G(\mathbb{R})$-conjugacy class in $\mathrm{Hom}(\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, G_\mathbb{R})$.

Set $\widehat{L} = L_{\widehat{\mathbb{Z}}}$, and define a compact open subgroup

$$K = G(\mathbb{A}_f) \cap C\big(\widehat{L}\big)^\times$$

of $G(\mathbb{A}_f)$. Note that $g \bullet \widehat{L} = \widehat{L}$ for all $g \in K$. Denote by

$$L^\vee = \{x \in V : [x, L] \subset \mathbb{Z}\}$$

the dual lattice of L. As in the proof of [MPar, (2.6)], the action of K on $\widehat{L}$ defines a homomorphism $K \to \mathrm{SO}(\widehat{L})$ whose image is precisely the subgroup of elements acting trivially on the discriminant group $L^\vee/L \simeq \widehat{L}^\vee/\widehat{L}$.

The pair $(G, \mathcal{D})$ is a Shimura datum, and

$$M(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K \xrightarrow{\simeq} \coprod_{g \in G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K} \Gamma_g \backslash \mathcal{D}, \tag{2.3}$$

is the complex analytic orbifold of $\mathbb{C}$-points of the Shimura variety attached to this datum and level subgroup K. Here we have set $\Gamma_g = G(\mathbb{Q}) \cap gKg^{-1}$. We will refer to $M(\mathbb{C})$ as the *GSpin Shimura variety* associated with L.

Given an algebraic representation $G \to \underline{\mathrm{Aut}}(N)$ on a $\mathbb{Q}$-vector space N, and a K-stable lattice $N_{\widehat{\mathbb{Z}}} \subset N_{\mathbb{A}_f}$, we obtain a $\mathbb{Z}$-local system $\boldsymbol{N}_{\mathrm{Be}}$ on $M(\mathbb{C})$ whose fiber at a point $[(\boldsymbol{h}, g)] \in M(\mathbb{C})$ is identified with $N \cap gN_{\widehat{\mathbb{Z}}}$. The corresponding vector bundle $\boldsymbol{N}_{\mathrm{dR},M(\mathbb{C})} = \mathcal{O}_{M(\mathbb{C})} \otimes \boldsymbol{N}_{\mathrm{Be}}$ is equipped with a filtration $\mathrm{Fil}^\bullet \boldsymbol{N}_{\mathrm{dR},M(\mathbb{C})}$, which at any point $[(\boldsymbol{h}, g)]$ equips the fiber of $\boldsymbol{N}_{\mathrm{Be}}$ with the Hodge structure determined by (2.2). This gives us a functorial assignment from pairs $(N, N_{\widehat{\mathbb{Z}}})$ as above to variations of $\mathbb{Z}$-Hodge structures over $M(\mathbb{C})$.

Applying this to V and the lattice $\widehat{L} \subset V_{\mathbb{A}_f}$, we obtain a canonical variation of polarized $\mathbb{Z}$-Hodge structures $(\boldsymbol{V}_{\mathrm{Be}}, \mathrm{Fil}^\bullet \boldsymbol{V}_{\mathrm{dR},M(\mathbb{C})})$. For each point z of (2.1) the induced Hodge decomposition of $V_{\mathbb{C}}$ has

$$V_{\mathbb{C}}^{(1,-1)} = \mathbb{C}z, \qquad V_{\mathbb{C}}^{(-1,1)} = \mathbb{C}\overline{z}, \qquad V_{\mathbb{C}}^{(0,0)} = (\mathbb{C}z + \mathbb{C}\overline{z})^{\perp}.$$

It follows that $\mathrm{Fil}^1 \boldsymbol{V}_{\mathrm{dR},M(\mathbb{C})}$ is an isotropic line and $\mathrm{Fil}^0 \boldsymbol{V}_{\mathrm{dR},M(\mathbb{C})}$ is its annihilator with respect to the pairing on $\boldsymbol{V}_{\mathrm{dR},M(\mathbb{C})}$ induced from that on L.

Similarly, viewing $C(V)$ as a representation of G under left multiplication, with its lattice $C(L)_{\widehat{\mathbb{Z}}} \subset C(V)_{\mathbb{A}_f}$, we obtain a variation of $\mathbb{Z}$-Hodge structures $(\boldsymbol{H}_{\mathrm{Be}}, \mathrm{Fil}^\bullet \boldsymbol{H}_{\mathrm{dR},M(\mathbb{C})})$. This variation has type $(-1,0),(0,-1)$ and is therefore the homology of a family of complex tori over $M(\mathbb{C})$.

This variation of $\mathbb{Z}$-Hodge structures is *polarizable*, and so the family of complex tori in fact arises from an *abelian scheme* over $M(\mathbb{C})$. To see this, first consider the representation of G on $\mathbb{Q}$ afforded by the spinor norm ν, along with the obvious K-stable lattice $\widehat{\mathbb{Z}} \subset \mathbb{A}_f$. For any $\boldsymbol{h} \in \mathcal{D}$, the composition $\nu \circ \alpha_{\boldsymbol{h}}$ is the homomorphism

$$\begin{aligned} \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m &\to \mathbb{G}_{m,\mathbb{R}} \\ z &\mapsto z\overline{z}. \end{aligned}$$

From this, we see that the associated variation of $\mathbb{Z}$-Hodge structures is the Tate twist $\underline{\mathbb{Z}}(1)$, whose underlying $\mathbb{Z}$-local system is just the constant sheaf $\underline{\mathbb{Z}}$, but whose (constant) Hodge filtration is concentrated in degree -1.

Now, choose $\delta \in C^+(L) \cap C^+(V)^\times$ satisfying $\delta^* = -\delta$. As explained in (2.1), this defines a G-equivariant symplectic pairing

$$\begin{aligned} \psi_\delta : C(V) \times C(V) &\to \mathbb{Q}(\nu) \\ (x,y) &\mapsto \mathrm{Trd}(x\delta y^*). \end{aligned}$$

Since δ lies in $C^+(L)$, the restriction of this form to $C(L)$ is $\mathbb{Z}$-valued, and in particular induces a K-invariant alternating form on $C(L)_{\widehat{\mathbb{Z}}}$ with values in $\widehat{\mathbb{Z}}(\nu)$.

We can arrange our choice of δ to have the following positivity property: For every $\boldsymbol{h} \in \mathcal{D}$, the Hermitian form $\psi_\delta(\alpha_{\boldsymbol{h}}(i)x, \overline{y})$ on $C(V)_{\mathbb{C}}$ is (positive or negative) definite. To produce a concrete such choice, choose a negative definite plane $\boldsymbol{h}_0 \subset V$ defined over $\mathbb{Q}$, and an oriented basis $x, y \in \boldsymbol{h}_0 \cap L$ for $\boldsymbol{h}_0$ satisfying $[x,y] = 0$. Then it can be checked that $\delta = xy \in C(L)$ has the desired positivity property.

Any such choice of δ produces an alternating pairing

$$\boldsymbol{H}_{\mathrm{Be}} \times \boldsymbol{H}_{\mathrm{Be}} \to \underline{\mathbb{Z}}(1)$$

of variations of Hodge structures, and the positivity property implies that this is actually a polarization of variations of Hodge structures.

In sum, we have produced an abelian scheme $\mathcal{A}_{M(\mathbb{C})} \to M(\mathbb{C})$, whose homology is canonically identified with the polarizable variation of Hodge structures $(\boldsymbol{H}_{\mathrm{Be}}, \boldsymbol{H}_{\mathrm{dR},M(\mathbb{C})})$. We call this the *Kuga-Satake abelian scheme.* It has relative dimension 2^{n+1}, and is endowed with a right action of the Clifford algebra $C(L)$ as well as a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathcal{A} = \mathcal{A}^+ \times \mathcal{A}^-$. Elements of $C^+(L)$ and $C^-(L)$ act as homogeneous endomorphisms of graded degrees zero and one, respectively.

Remark 2.2.1. By our construction, any choice of $\delta \in C^+(L) \cap C(V)^\times$ with the required positivity property produces a polarization on $\mathcal{A}_{M(\mathbb{C})}$. While the polarization on $\mathcal{A}_{M(\mathbb{C})}$ depends on the choice of δ, the abelian scheme itself does not.

By construction, the left multiplication action of L on $C(L)$ gives us an embedding

$$(\boldsymbol{V}_{\mathrm{Be}}, \mathrm{Fil}^\bullet \boldsymbol{V}_{\mathrm{dR},M(\mathbb{C})}) \subset \big(\underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{Be}}), \mathrm{Fil}^\bullet \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR},M(\mathbb{C})})\big)$$

of variations of Hodge structures.

For $x \in V$ of positive length, define a divisor on $\mathcal{D}$ by

$$\mathcal{D}(x) = \{z \in \mathcal{D} : z \perp x\}.$$

As in the work of Borcherds [Bor98], Bruinier [Bru02], and Kudla [Kud04], for every $m \in \mathbb{Q}_{>0}$ and $\mu \in L^\vee/L$ we define a complex orbifold

$$Z(m,\mu)(\mathbb{C}) = \coprod_{g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K} \Gamma_g \backslash \Big(\coprod_{\substack{x \in \mu_g + L_g \\ Q(x)=m}} \mathcal{D}(x) \Big). \tag{2.4}$$

Here $L_g \subset V$ is the $\mathbb{Z}$-lattice determined by $\widehat{L}_g = g \bullet \widehat{L}$, and $\mu_g = g \bullet \mu \in L_g^\vee/L_g$. Comparing with (2.3), the natural finite and unramified morphism

$$Z(m,\mu)(\mathbb{C}) \to M(\mathbb{C})$$

allows us to view (2.4) as an effective Cartier divisor on $M(\mathbb{C})$. We will see below that (2.4) admits a moduli interpretation in terms of the Kuga-Satake abelian scheme.

2.3. **Canonical models over $\mathbb{Q}$.** The reflex field of the Shimura datum $(G, \mathcal{D})$ is $\mathbb{Q}$, and hence $M(\mathbb{C})$ is the complex fiber of an algebraic stack M over $\mathbb{Q}$. By [MPar, §3] the Kuga-Satake abelian scheme, along with all its additional structures, has a canonical descent $\pi : \mathcal{A}_M \to M$.

We denote by[1]

$$\boldsymbol{H}_{\mathrm{dR},M} = \underline{\mathrm{Hom}}\big(\mathbb{R}^1\pi_{\mathrm{Zar},*}\Omega^\bullet_{\mathcal{A}_M/M}, \mathcal{O}_M\big)$$

[1]In [MPar], one works consistently with cohomology rather than homology, and so the conventions here will be dual to the conventions there.

the first de Rham homology of $\mathcal{A}$ relative to M. It is a locally free $\mathcal{O}_M$-module of rank 2^{n+2}, equipped with its Hodge filtration $\mathrm{Fil}^0\boldsymbol{H}_{\mathrm{dR},M} \subset \boldsymbol{H}_{\mathrm{dR},M}$ and Gauss–Manin connection. It also has a $\mathbb{Z}/2\mathbb{Z}$-grading $\boldsymbol{H}_{\mathrm{dR},M} = \boldsymbol{H}^+_{\mathrm{dR},M} \oplus \boldsymbol{H}^-_{\mathrm{dR},M}$ arising from the grading $\mathcal{A} = \mathcal{A}^+ \times \mathcal{A}^-$ on the Kuga-Satake abelian scheme, and similarly a right action of $C(L)$ induced from that on $\mathcal{A}$. Over $M(\mathbb{C})$, it is canonically identified with $\boldsymbol{H}_{\mathrm{dR},M(\mathbb{C})}$ with its filtration and integrable connection.

From [MPar, (3.12)], we find that there is a canonical descent $\boldsymbol{V}_{\mathrm{dR},M}$ of $\boldsymbol{V}_{\mathrm{dR},M(\mathbb{C})}$, along with its filtration and integrable connection, so that we have an embedding

$$\boldsymbol{V}_{\mathrm{dR},M} \subset \underline{\mathrm{End}}_{C(L)}(\boldsymbol{H}_{\mathrm{dR},M})$$

of filtered vector bundles over M with integrable connection. This is a descent of the one already available over $M(\mathbb{C})$. There is a canonical quadratic form $\boldsymbol{Q} : \boldsymbol{V}_{\mathrm{dR},M} \to \mathcal{O}_M$ given on sections by $x \circ x = \boldsymbol{Q}(x) \cdot \mathrm{Id}$, where the composition takes place in $\underline{\mathrm{End}}_{C(L)}(\boldsymbol{H}_{\mathrm{dR},M})$.

In the same way we define

$$\boldsymbol{H}_{\ell,\mathrm{et}} = \underline{\mathrm{Hom}}\big(R^1\pi_{\mathrm{et},*}\underline{\mathbb{Z}}_p, \underline{\mathbb{Z}}_p\big)$$

to be the first ℓ-adic étale homology of $\mathcal{A}_M$ relative to M. Over $M(\mathbb{C})$, it is canonically identified with $\mathbb{Z}_\ell \otimes \boldsymbol{H}_{\mathrm{Be}}$. The local system $\mathbb{Z}_\ell \otimes \boldsymbol{V}_{\mathrm{Be}}$ descends to an ℓ-adic sheaf $\boldsymbol{V}_{\ell,\mathrm{et}}$ over M, and there is a canonical embedding as a local direct summand

$$\boldsymbol{V}_{\ell,\mathrm{et}} \subset \underline{\mathrm{End}}_{C(L)}(\boldsymbol{H}_{\ell,\mathrm{et}}).$$

Again, composition on the right hand side induces a canonical quadratic form on $\boldsymbol{V}_{\ell,\mathrm{et}}$ with values in the constant sheaf $\underline{\mathbb{Z}}_\ell$.

2.4. **Integral models.** The stack M admits a regular integral model $\mathcal{M}$ over $\mathbb{Z}[1/2]$, which we now describe, following [MPar, §7].

It follows from [MPar, (6.8)] that we may fix an isometric embedding $L \hookrightarrow \widetilde{L}$, where $\widetilde{L}$ is a maximal quadratic space over $\mathbb{Z}$, self-dual at p and of signature $(\tilde{n}, 2)$. Set

$$\Lambda = L^\perp = \{x \in \widetilde{L} : x \perp L\}.$$

There is a canonical finite and unramified map of the associated Shimura varieties $M \to \widetilde{M}$ over $\mathbb{Q}$. Over $\widetilde{M}$ we have the de Rham realization $\widetilde{\boldsymbol{V}}_{\mathrm{dR},\widetilde{M}}$ associated with $\widetilde{L}$. Let $\widetilde{\boldsymbol{V}}_{\mathrm{dR},M}$ be the restriction of this realization to M. By construction, we have an isometric embedding

$$\boldsymbol{V}_{\mathrm{dR},M} \hookrightarrow \widetilde{\boldsymbol{V}}_{\mathrm{dR},M}$$

of filtered vector bundles with integrable connections over M. Setting $\boldsymbol{\Lambda}_{\mathrm{dR},M} = \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_M$, we obtain an injection of vector bundles

$$\boldsymbol{\Lambda}_{\mathrm{dR},M} \hookrightarrow \widetilde{\boldsymbol{V}}_{\mathrm{dR},M} \tag{2.5}$$

identifying $\boldsymbol{\Lambda}_{\mathrm{dR},M}$ with a local direct summand of $\widetilde{\boldsymbol{V}}_{\mathrm{dR},M}$.

By the main theorem of [Kis10], the Shimura variety $\widetilde{M}$ associated with $\widetilde{L}$ admits a smooth canonical model $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$. We now recall the construction.

Choose $\delta \in C^+(\widetilde{L}) \cap C(\widetilde{V})^\times$ as in §2.2, so that the associated symplectic pairing ψ_δ on $C(\widetilde{L})$ produces a polarization on the Kuga-Satake abelian scheme

$$\widetilde{A}_{\widetilde{M}(\mathbb{C})} \to \widetilde{M}(\mathbb{C}).$$

This polarization has degree $\tilde{d}$, where $\tilde{d}$ is the discriminant of the restriction of ψ_δ to $C(\widetilde{L})$. The polarized Kuga-Satake abelian scheme defines a morphism of complex algebraic stacks

$$i_\delta : \widetilde{M}_{\mathbb{C}} \to \mathcal{X}^{\mathrm{Siegel}}_{2^{\tilde{n}+1},\tilde{d},\mathbb{C}},$$

where $\mathcal{X}^{\mathrm{Siegel}}_{2^{\tilde{n}+1},\tilde{d}}$ is the Siegel moduli stack over $\mathbb{Z}$ parameterizing polarized abelian schemes of dimension $2^{\tilde{n}+1}$ of degree $\tilde{d}$. It can be checked using complex uniformizations that this map is finite and unramified. The theory of canonical models of Shimura varieties implies that i_δ descends to a finite and unramified map of $\mathbb{Q}$-stacks

$$i_\delta : \widetilde{M} \to \mathcal{X}^{\mathrm{Siegel}}_{2^{\tilde{n}+1},\tilde{d},\mathbb{Q}}.$$

The integral model $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ is defined as the normalization of $\mathcal{X}^{\mathrm{Siegel}}_{2^{\tilde{n}+1},\tilde{d},\mathbb{Z}_{(p)}}$ in $\widetilde{M}$. Here, given an algebraic stack $\mathcal{X}$ over $\mathbb{Z}_{(p)}$, and a normal algebraic stack Y over $\mathbb{Q}$ equipped with a finite map $j_{\mathbb{Q}} : Y \to \mathcal{X}_{\mathbb{Q}}$, the *normalization* of $\mathcal{X}$ in Y is the finite $\mathcal{X}$-stack $j : \mathcal{Y} \to \mathcal{X}$, characterized by the property that $j_*\mathcal{O}_{\mathcal{Y}}$ is the integral closure of $\mathcal{O}_{\mathcal{X}}$ in $(j_{\mathbb{Q}})_*\mathcal{O}_Y$. It is characterized by the following universal property: given a finite morphism $\mathcal{Z} \to \mathcal{X}$ with $\mathcal{Z}$ a normal algebraic stack over $\mathbb{Z}_{(p)}$, any map of $\mathcal{X}_{\mathbb{Q}}$-stacks $\mathcal{Z}_{\mathbb{Q}} \to Y$ extends uniquely to a map of $\mathcal{X}$-stacks $\mathcal{Z} \to \mathcal{Y}$.

Proposition 2.4.1. *The stack $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ is smooth over $\mathbb{Z}_{(p)}$. It does not depend on the choice of $\delta \in C^+(\widetilde{L}) \cap C(\widetilde{V})^\times$.*

Proof. This essentially follows from [Kis10, (2.3.8)]. We provide some of the details.

Abbreviate

$$\mathcal{X}^{\mathrm{Siegel}} = \mathcal{X}^{\mathrm{Siegel}}_{2^{\tilde{n}+1},\tilde{d},\mathbb{Z}_{(p)}},$$

and let $(\mathcal{A}^{\mathrm{Siegel}}, \lambda^{\mathrm{Siegel}})$ be the universal polarized abelian scheme over $\mathcal{X}^{\mathrm{Siegel}}$. For any $m \in \mathbb{Z}_{>0}$ coprime to $p\tilde{d}$, let $\mathcal{X}^{\mathrm{Siegel}}_m \to \mathcal{X}^{\mathrm{Siegel}}$ be the finite étale cover parameterizing full level-m structures on $\mathcal{A}^{\mathrm{Siegel}}$; that is, parametrizing isomorphisms of group schemes

$$C(\widetilde{L}) \otimes \underline{\mathbb{Z}/m\mathbb{Z}} \xrightarrow{\simeq} \mathcal{A}^{\mathrm{Siegel}}[m]$$

that carry the symplectic form $\psi_\delta \otimes 1$ on the left hand side to a $(\mathbb{Z}/m\mathbb{Z})^\times$-multiple of the Weil pairing on $\mathcal{A}^{\mathrm{Siegel}}[m]$ induced by $\lambda^{\mathrm{Siegel}}$. For $m \geq 3$ the stack $\mathcal{X}_m^{\mathrm{Siegel}}$ is in fact a scheme over $\mathbb{Z}_{(p)}$.

Write $\widetilde{G} = \mathrm{GSpin}(\widetilde{V})$, and set

$$\widetilde{K} = \widetilde{G}(\mathbb{A}_f) \cap C(\widetilde{L})^\times_{\widehat{\mathbb{Z}}}.$$

Let $\widetilde{K}(m) \subset \widetilde{K}$ be the largest subgroup acting trivially on $C(\widetilde{L}) \otimes (\mathbb{Z}/m\mathbb{Z})$. We obtain a finite étale cover $\widetilde{M}_m \to \widetilde{M}$ of Shimura varieties with Galois group $\widetilde{K}/\widetilde{K}(m)$, where

$$\widetilde{M}_m(\mathbb{C}) = \widetilde{G}(\mathbb{Q}) \backslash \widetilde{\mathcal{D}} \times \widetilde{G}(\mathbb{A}_f)/\widetilde{K}(m)$$

and $\widetilde{\mathcal{D}}$ is the space of oriented negative definite planes in $\widetilde{V}_{\mathbb{R}}$.

The map $i_\delta : \widetilde{M} \to \mathcal{X}_{\mathbb{Q}}^{\mathrm{Siegel}}$ lifts to a map $i_{\delta,m} : \widetilde{M}_m \to \mathcal{X}_{m,\mathbb{Q}}^{\mathrm{Siegel}}$. From [Kis10, (2.1.2)] we find that $i_{\delta,m}$ is a closed immersion of $\mathbb{Q}$-schemes for sufficiently large m. Fix such an m, and let $\widetilde{\mathcal{M}}_{m,\mathbb{Z}_{(p)}}$ be the normalization of $\mathcal{X}_m^{\mathrm{Siegel}}$ in $\widetilde{M}_m$. Since $i_{\delta,m}$ is a closed immersion, this is simply the normalization of the Zariski closure of $\widetilde{M}_m$ in $\mathcal{X}_m^{\mathrm{Siegel}}$. It now follows from [Kis10, (2.3.8)] that $\widetilde{\mathcal{M}}_{m,\mathbb{Z}_{(p)}}$ is smooth over $\mathbb{Z}_{(p)}$, and moreover, that it is, up to unique isomorphism, *independent* of the choice of δ.

The stack $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ is the étale quotient of $\widetilde{\mathcal{M}}_{m,\mathbb{Z}_{(p)}}$ by the action of $\widetilde{K}/\widetilde{K}(m)$, and so the proposition follows. □

By the construction of the integral model $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$, the Kuga-Satake abelian scheme $\widetilde{A} \to \widetilde{M}$ extends to $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$. A non-trivial consequence of the construction is that the vector bundle $\widetilde{\boldsymbol{V}}_{\mathrm{dR},\widetilde{M}}$, with its integrable connection and filtration, extends to a filtered vector bundle with integrable connection $(\widetilde{\boldsymbol{V}}_{\mathrm{dR},\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}}, \mathrm{Fil}^\bullet \widetilde{\boldsymbol{V}}_{\mathrm{dR},\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}})$ over $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ (see [MPar, §4] for details).

Let $\widetilde{\boldsymbol{H}}_{\mathrm{dR},\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}}$ be the de Rham homology of $\widetilde{A}$. Then we have an embedding of filtered vector bundles

$$\widetilde{\boldsymbol{V}}_{\mathrm{dR},\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}} \subset \underline{\mathrm{End}}_{C(L)}\big(\widetilde{\boldsymbol{H}}_{\mathrm{dR},\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}}\big) \tag{2.6}$$

extending that in the generic fiber and exhibiting its source as a local direct summand of the target. Composition in the endomorphism ring gives a canonical non-degenerate quadratic form on

$$\widetilde{\boldsymbol{Q}} : \widetilde{\boldsymbol{V}}_{\mathrm{dR},\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}} \to \mathcal{O}_{\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}}$$

again extending that on its generic fiber.

Similarly, let

$$\widetilde{\boldsymbol{H}}_{\mathrm{crys}} = \underline{\mathrm{Hom}}\big(R^1\pi_{\mathrm{crys},*}\mathcal{O}^{\mathrm{crys}}_{\widetilde{\mathcal{A}}_{\mathbb{F}_p}/\mathbb{Z}_p}, \mathcal{O}^{\mathrm{crys}}_{\widetilde{\mathcal{M}}_{\mathbb{F}_p}/\mathbb{Z}_p}\big)$$

be the first crystalline homology of $\widetilde{\mathcal{A}}_{\mathbb{F}_p}$ relative to $\widetilde{\mathcal{M}}_{\mathbb{F}_p}$. As $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ is smooth over $\mathbb{Z}_{(p)}$, the vector bundle with integrable connection (2.6) determines a crystal $\widetilde{\boldsymbol{V}}_{\mathrm{crys}}$ over $\widetilde{\mathcal{M}}_{\mathbb{F}_p}$ equipped with an embedding

$$\widetilde{\boldsymbol{V}}_{\mathrm{crys}} \subset \underline{\mathrm{End}}_{C(L)}\big(\widetilde{\boldsymbol{H}}_{\mathrm{crys}}\big)$$

as a local direct summand; see [MPar, (4.14)] for details.

We are now ready to give the construction of the desired integral model $\mathcal{M}_{\mathbb{Z}_{(p)}}$ for M. First, observe that the direct summands $\Lambda_{\mathbb{Q}} \subset \widetilde{V}$ and $\Lambda_{\widehat{\mathbb{Z}}} \subset \widetilde{L}_{\widehat{\mathbb{Z}}}$ are pointwise stabilized by G and K, respectively. From this, arguing as in [MPar, (6.5)], we find that there is a canonical embedding

$$\Lambda \subset \mathrm{End}_{C(\widetilde{L})}(\widetilde{\mathcal{A}}|_M) \tag{2.7}$$

whose de Rham realization is (2.5).

Let $\check{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ be the normalization of $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ in M. Then, by [FC90, Prop. 2.7], the embedding (2.7) extends to

$$\Lambda \subset \mathrm{End}_{C(\widetilde{L})}\big(\widetilde{\mathcal{A}}|_{\check{\mathcal{M}}_{\mathbb{Z}_{(p)}}}\big). \tag{2.8}$$

Since $\widetilde{\boldsymbol{V}}_{\mathrm{dR},\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}}$ is a direct summand of $\underline{\mathrm{End}}_{C(L)}\big(\widetilde{\boldsymbol{H}}_{\mathrm{dR},\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}}\big)$, the de Rham realization of (2.8) provides us with an extension

$$\boldsymbol{\Lambda}_{\mathrm{dR}} \to \widetilde{\boldsymbol{V}}_{\mathrm{dR},\check{\mathcal{M}}_{\mathbb{Z}_{(p)}}} \tag{2.9}$$

of (2.5), where $\boldsymbol{\Lambda}_{\mathrm{dR}} = \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{\check{\mathcal{M}}_{\mathbb{Z}_{(p)}}}$.

Let $\check{\mathcal{M}}^{\mathrm{pr}}_{\mathbb{Z}_{(p)}} \subset \check{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ be the open locus where the cokernel of (2.9) is a vector bundle of rank n (see [MPar, 6.16(i)]). This is also the locus over which $\boldsymbol{\Lambda}_{\mathrm{dR}}$ embeds as a local direct summand of $\widetilde{\boldsymbol{V}}_{\mathrm{dR},\check{\mathcal{M}}_{\mathbb{Z}_{(p)}}}$ via (2.9). In particular, it contains the generic fiber M of $\check{\mathcal{M}}_{\mathbb{Z}_{(p)}}$.

Let $\mathrm{M}^{\mathrm{loc}}(L)$ be the projective scheme over $\mathbb{Z}[1/2]$ parameterizing isotropic lines $L^1 \subset L$. The following can be deduced from [MPar, (6.16),(6.20),(2.16)].

Proposition 2.4.2.

(i) Around any point of $\check{\mathcal{M}}^{\mathrm{pr}}_{\mathbb{Z}_{(p)}}$, we can find an étale neighborhood T such that there exists an isometry

$$\xi : \mathcal{O}_T \otimes \widetilde{L} \xrightarrow{\simeq} \widetilde{\boldsymbol{V}}_{\mathrm{dR},T}$$

satisfying $\xi(\mathcal{O}_T \otimes \Lambda) = \boldsymbol{\Lambda}_{\mathrm{dR},T}$, and with the following additional property: the map $T \to \mathrm{M}^{\mathrm{loc}}(L)$ corresponding to the isotropic line

$$\xi^{-1}(\mathrm{Fil}^1 \widetilde{\boldsymbol{V}}_{\mathrm{dR},T}) \subset \mathcal{O}_T \otimes \widetilde{L}$$

is étale.

(ii) If p^2 does not divide the discriminant of L, then $\check{\mathcal{M}}^{\mathrm{pr}}_{\mathbb{Z}_{(p)}} = \check{\mathcal{M}}_{\mathbb{Z}_{(p)}}$.

(iii) If p^2 divides the discriminant of L, then $\check{\mathcal{M}}^{\mathrm{pr}}_{\mathbb{Z}_{(p)}}$ is regular outside of a 0 dimensional substack.

We now set

- $\mathcal{M}_{\mathbb{Z}_{(p)}} = \check{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ if p^2 does not divide the discriminant of L; and
- $\mathcal{M}_{\mathbb{Z}_{(p)}} =$ the regular locus of $\check{\mathcal{M}}^{\mathrm{pr}}_{\mathbb{Z}_{(p)}}$, otherwise.

Then $\mathcal{M}_{\mathbb{Z}_{(p)}}$ is the desired regular integral model for M over $\mathbb{Z}_{(p)}$.

Remark 2.4.3. In the case where p^2 divides the discriminant of L, this is certainly not the optimal definition of the regular model $\mathcal{M}_{\mathbb{Z}_{(p)}}$. For a discussion of this, see [MPar, (6.27)].

Equip $\boldsymbol{\Lambda}_{\mathrm{dR}} = \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}_{\mathbb{Z}_{(p)}}}$ with the integrable connection $1 \otimes \mathrm{d}$, and the trivial filtration concentrated in degree 0. By construction, over $\mathcal{M}_{\mathbb{Z}_{(p)}}$ we have a canonical isometric embedding

$$\boldsymbol{\Lambda}_{\mathrm{dR}} \subset \widetilde{\boldsymbol{V}}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$$

of filtered vector bundles with integrable connections. Therefore, since the quadratic form on $\widetilde{\boldsymbol{V}}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$ is self-dual, the orthogonal complement

$$\boldsymbol{V}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}} = \boldsymbol{\Lambda}_{\mathrm{dR}}^{\perp} \subset \widetilde{\boldsymbol{V}}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$$

is again a vector sub-bundle with integrable connection. Indeed, it is precisely the kernel of the surjection of vector bundles

$$\widetilde{\boldsymbol{V}}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}} \xrightarrow{\simeq} \widetilde{\boldsymbol{V}}^{\vee}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}} \to \widetilde{\boldsymbol{V}}^{\vee}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}/\mathrm{ann}(\boldsymbol{\Lambda}_{\mathrm{dR}}).$$

Moreover, the isotropic line $\mathrm{Fil}^1\widetilde{\boldsymbol{V}}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$ is actually contained in $\boldsymbol{V}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$. For convenience, we abbreviate

$$\boldsymbol{V}^1_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}} = \mathrm{Fil}^1\widetilde{\boldsymbol{V}}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}.$$

Proposition 2.4.4. *The integral models $\mathcal{M}_{\mathbb{Z}_{(p)}}$ and $\check{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ are, up to unique isomorphism, independent of the auxiliary choice of $\widetilde{L}$. Furthermore, the vector bundle $\boldsymbol{V}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$ and the isotropic line*

$$\boldsymbol{V}^1_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}} \subset \boldsymbol{V}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$$

are also independent of this auxiliary choice.

Proof. Suppose that we have a different embedding $L \hookrightarrow \widetilde{L}_1$ with $\widetilde{L}_1$ maximal, self-dual at p, and of signature $(\tilde{n}_1, 2)$.

First, assume that the embedding $L \hookrightarrow \widetilde{L}_1$ factors as $L \hookrightarrow \widetilde{L} \hookrightarrow \widetilde{L}_1$. The construction of Kisin in [Kis10] shows that the map of integral models $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}} \to \widetilde{\mathcal{M}}_{1,\mathbb{Z}_{(p)}}$ associated with $\widetilde{L} \hookrightarrow \widetilde{L}_1$ is finite and unramified. Moreover, Proposition 2.4.2 shows that, if $\Lambda_1 = \widetilde{L}^{\perp} \subset \widetilde{L}_1$, then the corresponding map

$$\boldsymbol{\Lambda}_{1,\mathrm{dR},\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}} \to \widetilde{\boldsymbol{V}}_{1,\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$$

is an embedding of vector bundles, and that its orthogonal complement is exactly $\widetilde{\boldsymbol{V}}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$. From this, we easily deduce that the constructions of both $\check{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ and $\check{\mathcal{M}}^{\mathrm{pr}}_{\mathbb{Z}_{(p)}}$ do not depend on the choice of isometric embedding. The proof also shows that the vector bundle $\boldsymbol{V}_{\mathrm{dR},\mathcal{M}_{\mathbb{Z}_{(p)}}}$, along with its isotropic line, is also independent of this choice.

For the general case, consider the embedding

$$\begin{aligned}\Delta : L &\hookrightarrow \widetilde{L} \oplus \widetilde{L}_1\\ v &\mapsto (v, -v)\end{aligned}$$

The quadratic form on $\widetilde{L} \oplus \widetilde{L}_1$ given by the orthogonal sum of those on its individual summands induces a quadratic form on the quotient

$$\widetilde{L}_2 = (\widetilde{L} \oplus \widetilde{L}_1)/\Delta(L)$$

of signature $(\widetilde{n} + \widetilde{n}_1 - n, 2)$. Moreover, the natural maps $\widetilde{L} \to \widetilde{L}_2$ and $\widetilde{L}_1 \to \widetilde{L}_2$ are isometric inclusions of direct summands, whose restrictions to L coincide.

Now, $\widetilde{L}_2$ need not be self-dual at p, but, via the argument in [MPar, (6.8)], we can embed it isometrically as a direct summand in a maximal quadratic space $\widetilde{L}_3$ of signature $(\tilde{n}_3, 2)$, which is self-dual at p. In particular, replacing $\widetilde{L}_1$ with $\widetilde{L}_3$, we are reduced to the already considered case where $\widetilde{L}$ is an isometric direct summand of $\widetilde{L}_1$. □

We can now give the construction of the stack $\mathcal{M}$ over $\mathbb{Z}[1/2]$. It is simply obtained by patching together the spaces $\mathcal{M}_{\mathbb{Z}_{(p)}}$. To do this rigorously, we choose a finite collection of maximal quadratic spaces $\widetilde{L}_1, \widetilde{L}_2, \ldots, \widetilde{L}_r$ with the following properties:

- For each $i = 1, 2, \ldots, r$, $\widetilde{L}_i$ has signature $(\tilde{n}_i, 2)$, for $\tilde{n}_i \in \mathbb{Z}_{>0}$;
- For each i, there is an isometric embedding $L \hookrightarrow \widetilde{L}_i$;
- If, for each i, we denote by S_i the set of primes containing all the odd prime divisors of the discriminant of $\widetilde{L}_i$, then $\bigcap_{i=1}^r S_i = \emptyset$.

It is always possible to find such a collection. For instance, begin by choosing any maximal quadratic lattice of signature $(\tilde{n}_1, 2)$, admitting L as an isometric direct summand. Let $\{p_2, \ldots, p_r\}$ be the set of odd primes dividing the discriminant of $\widetilde{L}_1$. Now, for each $i = 2, 3, \ldots, r$, let $\widetilde{L}_i$ be a maximal quadratic lattice, self-dual at p_i, of signature $(\tilde{n}_i, 2)$, admitting L as an isometric direct summand. Then the collection $\{\widetilde{L}_1, \ldots, \widetilde{L}_r\}$ does the job.

For $i = 1, 2, \ldots, r$, let $\widetilde{M}_i$ be the GSpin Shimura variety over $\mathbb{Q}$ attached to $\widetilde{L}_i$. Given an appropriate choice of $\delta_i \in C^+(\widetilde{L}_i) \cap C(\widetilde{L}_i)^\times_{\mathbb{Q}}$, with associated alternating form ψ_{δ_i} on $C(\widetilde{L}_i)$, we obtain a sequence of finite and unramified map of $\mathbb{Q}$-stacks

$$M \to \widetilde{M}_i \xrightarrow{i_{\delta_i}} \mathcal{X}^{\mathrm{Siegel}}_{2^{\tilde{n}_i+1},\tilde{d}_i,\mathbb{Q}}.$$

Here, $\tilde{d}_i$ is the discriminant of ψ_{δ_i}.

Let $\mathbb{Z}[(2S_i)^{-1}] \subset \mathbb{Q}$ be the localization of $\mathbb{Z}$ obtained by inverting 2 and the primes in S_i. Let $\mathcal{M}_{\mathbb{Z}[(2S_i)^{-1}]}$ be the stack over $\mathbb{Z}[(2S_i)^{-1}]$ obtained by normalizing $\mathcal{X}^{\mathrm{Siegel}}_{2^{\tilde{n}_i+1},\tilde{d}_i,\mathbb{Z}[(2S_i)^{-1}]}$ in M.

Proposition 2.4.5. *There is a regular, flat algebraic $\mathbb{Z}[1/2]$-stack $\mathcal{M}$ such that, for each i, the restriction of $\mathcal{M}$ over $\mathbb{Z}[(2S_i)^{-1}]$ is isomorphic to $\mathcal{M}_{\mathbb{Z}[(2S_i)^{-1}]}$. Moreover, the vector bundle $\boldsymbol{V}_{\mathrm{dR},M}$, along with its integrable connection and the isotropic line $\boldsymbol{V}^1_{\mathrm{dR},M}$, has a canonical extension $(\boldsymbol{V}_{\mathrm{dR}}, \boldsymbol{V}^1_{\mathrm{dR}})$ over $\mathcal{M}$.*

Proof. This is immediate from Proposition 2.4.4. □

Definition 2.4.6. The line bundle $\boldsymbol{V}^1_{\mathrm{dR}} \subset \boldsymbol{V}_{\mathrm{dR}}$ is the *tautological bundle* on $\mathcal{M}$. Its dual is the *cotautological bundle*.

Remark 2.4.7. Under the uniformization

$$\mathcal{D} \to \Gamma_g \backslash \mathcal{D} \subset \mathcal{M}(\mathbb{C}),$$

of (2.3), and using the notation of (2.1), the tautological bundle and cotautological bundle pull-back to the line bundles $z \mapsto \mathbb{C}z$ and $z \mapsto V_{\mathbb{C}}/(\mathbb{C}z)^{\perp}$, respectively.

By (7.12) of [MPar], the Kuga-Satake abelian scheme over M extends (necessarily uniquely) to an abelian scheme

$$\pi \colon \mathcal{A} \to \mathcal{M}$$

endowed with a right $C(L)$-action and $\mathbb{Z}/2\mathbb{Z}$-grading. The de Rham realization of $\mathcal{A}$ provides us with an extension $\boldsymbol{H}_{\mathrm{dR}}$ over $\mathcal{M}$ of the filtered vector bundle with connection $\boldsymbol{H}_{\mathrm{dR},M}$, equipped with its $\mathbb{Z}/2\mathbb{Z}$-grading and right $C(L)$-action. Let

$$\boldsymbol{H}_{\mathrm{crys}} = \underline{\mathrm{Hom}}\big(R^1\pi_{\mathrm{crys},*}\mathcal{O}^{\mathrm{crys}}_{\mathcal{A}_{\mathbb{F}_p}/\mathbb{Z}_p}, \mathcal{O}^{\mathrm{crys}}_{\mathcal{M}_{\mathbb{F}_p}/\mathbb{Z}_p}\big)$$

be the first crystalline homology of $\mathcal{A}_{\mathbb{F}_p}$ relative to $\mathcal{M}_{\mathbb{F}_p}$.

Denote by $\mathbf{1}$ the structure sheaf $\mathcal{O}_{\mathcal{M}}$ with its standard connection and trivial filtration in the de Rham case; the constant sheaf $\underline{\mathbb{Z}_p}$ on $\mathcal{M}[1/p]$ in the étale case; the constant crystal $\mathcal{O}_{\mathcal{M}_{\mathbb{F}_p},\mathrm{crys}}$ in the crystalline case; and the trivial variation of $\mathbb{Z}$-Hodge structures over $M^{\mathrm{an}}_{\mathbb{C}}$ in the Betti case. If $\boldsymbol{H}$ is any one of $\boldsymbol{H}_{\mathrm{dR}}$, $\boldsymbol{H}_{\ell,\mathrm{et}}$, $\boldsymbol{H}_{\mathrm{crys}}$, or $\boldsymbol{H}_{\mathrm{Be}}$, then $\boldsymbol{H}$ is a $\mathbf{1}$-module in the appropriate category.

It follows from [MPar, (3.12),(7.13)] that there is a canonical local direct summand

$$\boldsymbol{V} \subset \underline{\mathrm{End}}_{C(L)}(\boldsymbol{H}) \tag{2.10}$$

of grade shifting endomorphisms. In fact, as we will see below in Proposition 2.5.1, the homological realizations of the canonical isometric embedding $\Lambda \hookrightarrow \mathrm{End}(\widetilde{\mathcal{A}}|_{\mathcal{M}})$ exhibit $\Lambda \otimes \mathbf{1}$ as an isometric local direct summand of $\widetilde{\boldsymbol{V}}|_{\mathcal{M}}$, and that

$$\boldsymbol{V} = (\Lambda \otimes \mathbf{1})^{\perp} \subset \widetilde{\boldsymbol{V}}|_{\mathcal{M}}.$$

In the de Rham or Betti case, $\boldsymbol{V}$ is identified with the realizations already constructed above. Moreover, for any prime p, the étale realization $\boldsymbol{V}_{p,\mathrm{et}} \subset \underline{\mathrm{End}}_{C(L)}(\boldsymbol{H}_{p,\mathrm{et}})$ is the unique extension of that over M. Here, we are using the following fact: Given a normal flat Noetherian algebraic $\mathbb{Z}[1/p]$-stack S, and a lisse p-adic sheaf F over S, any lisse subsheaf of $F|_{S_{\mathbb{Q}}}$ extends uniquely to a lisse subsheaf of F. To see this, we can reduce to the case where S is a normal, connected, flat Noetherian $\mathbb{Z}[1/p]$-scheme, where the assertion amounts to saying that the map of étale fundamental groups $\pi_1(S_{\mathbb{Q}}) \to \pi_1(S)$ is *surjective*. This is shown in [Gro03, Exp. V, Prop. 8.2].

For any section x of $\boldsymbol{V}$ the element $x \circ x$ is a scalar endomorphism of $\boldsymbol{H}$. This defines a quadratic form

$$\boldsymbol{Q} : \boldsymbol{V} \to \mathbf{1} \tag{2.11}$$

with $x \circ x = \boldsymbol{Q}(x) \cdot \mathrm{Id}$.

Remark 2.4.8. Although we will not require this in what follows, we note that there is a certain compatibility between the crystalline realization $\boldsymbol{V}_{\mathrm{crys}}$ over $\mathcal{M}_{\mathbb{F}_p}$ and the p-adic realization $\boldsymbol{V}_{p,\mathrm{et}}$ over $\mathcal{M}[1/p]$. Suppose that we are given a finite extension $F/\mathbb{Q}_p$, contained in an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, with residue field k. Write $F_0 \subset F$ for the maximal unramified subextension of F. Suppose that we are given a point $s \in \mathcal{M}(F)$ specializing to a point $s_0 \in \mathcal{M}(k)$. Write $\overline{s}$ for the corresponding $\overline{\mathbb{Q}}_p$-point of $\mathcal{M}$. Then the comparison isomorphism between the crystalline cohomology of $\mathcal{A}_{s_0}$ with respect to $W(k) = \mathcal{O}_{F_0}$ and the p-adic étale cohomology of $\mathcal{A}_{\overline{s}}$ induces, by [MPar, (7.11)], an isometry

$$\boldsymbol{V}_{\mathrm{crys},s_0} \otimes_{F_0} B_{\mathrm{crys}} \xrightarrow{\simeq} \boldsymbol{V}_{p,\mathrm{et},\overline{s}} \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}.$$

2.5. **Functoriality of integral models.** We consider the question of functoriality of the previous constructions with respect to the maximal quadratic space L. Let $L_0 \subset L$ be an isometric embedding of maximal quadratic spaces over $\mathbb{Z}$, of signatures $(n_0, 2)$ and $(n, 2)$, respectively. This implies that L_0 is a $\mathbb{Z}$-module direct summand of L. We will maintain our assumption that $n_0 \geq 1$.

The quadratic spaces L_0 and L have associated Shimura varieties M_0 and M, with level structure defined by the compact open subgroups

$$K_0 = \mathrm{GSpin}(L_0 \otimes \mathbb{A}_f) \cap C(\widehat{L}_0)^{\times}$$
$$K = \mathrm{GSpin}(L \otimes \mathbb{A}_f) \cap C(\widehat{L})^{\times}.$$

The inclusion $L_0 \hookrightarrow L$ induces a homomorphism of Clifford algebras, which then restricts to an inclusion of algebraic groups $\mathrm{GSpin}(V_0) \to \mathrm{GSpin}(V)$

satisfying $K_0 = K \cap \mathrm{GSpin}(V_0)(\mathbb{A}_f)$. By the theory of canonical models of Shimura varieties, there is an induced finite and unramified morphism of $\mathbb{Q}$-stacks

$$M_0 \to M.$$

Let $\mathcal{M}_0$ and $\mathcal{M}$ be the integral models over $\mathbb{Z}[1/2]$, and let $\mathcal{A}_0 \to \mathcal{M}_0$ and $\mathcal{A} \to \mathcal{M}$ be their Kuga-Satake abelian schemes. They are equipped with right actions of $C(L_0)$ and $C(L)$, respectively. Let $\boldsymbol{H}_0$ and $\boldsymbol{H}$ stand for the various homological realizations of $\mathcal{A}_0$ and $\mathcal{A}$, respectively. We have the canonical sub-objects $\boldsymbol{V}_0 \subset \underline{\mathrm{End}}_{C(L_0)}(\boldsymbol{H}_0)$ and $\boldsymbol{V} \subset \underline{\mathrm{End}}_{C(L)}(\boldsymbol{H})$ introduced in §2.4.

The Kuga-Satake abelian schemes over M_0 and M are determined by the lattices $C(L_0) \subset C(V_0)$ and $C(L) \subset C(V)$ (in the sense of [Del72, §4.12]), and the argument in [MPar, (6.4)] shows that there is a $C(L)$-linear isomorphism of abelian schemes

$$\big(\mathcal{A}_0 \otimes_{C(L_0)} C(L)\big)|_{M_0} \xrightarrow{\simeq} \mathcal{A}|_{M_0} \tag{2.12}$$

over the generic fiber M_0 of $\mathcal{M}_0$. As L_0 is a direct summand of L, one easily checks that $C(L)$ is a free $C(L_0)$-module, and so the tensor construction $\mathcal{A}_0 \otimes_{C(L_0)} C(L)$ in (2.12) is defined.

Proposition 2.5.1. *The morphism $M_0 \to M$ extends uniquely to an unramified morphism $\mathcal{M}_0 \to \mathcal{M}$, and (2.12) extends to a $C(L)$-linear isomorphism of abelian schemes*

$$\mathcal{A}_0 \otimes_{C(L_0)} C(L) \xrightarrow{\simeq} \mathcal{A}|_{\mathcal{M}_0} \tag{2.13}$$

over $\mathcal{M}_0$. In particular,

$$\boldsymbol{H}_0 \otimes_{C(L_0)} C(L) \xrightarrow{\simeq} \boldsymbol{H}|_{\mathcal{M}_0}. \tag{2.14}$$

Set $\Lambda = \{\lambda \in L : \lambda \perp L_0\}$, and define a sheaf $\boldsymbol{\Lambda} = \Lambda \otimes \mathbf{1}$ on $\mathcal{M}_0$. There is a canonical embedding $\Lambda \subset \mathrm{End}\big(\mathcal{A}|_{\mathcal{M}_0}\big)$ with the following properties:

(i) Its homological realization $\boldsymbol{\Lambda} \hookrightarrow \underline{\mathrm{End}}(\boldsymbol{H}|_{\mathcal{M}_0})$ arises from an isometric map $\boldsymbol{\Lambda} \hookrightarrow \boldsymbol{V}|_{\mathcal{M}_0}$, and exhibits $\boldsymbol{\Lambda}$ as a local direct summand of $\boldsymbol{V}|_{\mathcal{M}_0}$.

(ii) The injection

$$\underline{\mathrm{End}}_{C(L_0)}(\boldsymbol{H}_0) \to \underline{\mathrm{End}}_{C(L)}(\boldsymbol{H}|_{\mathcal{M}_0})$$

induced by (2.14) identifies $\boldsymbol{V}_0$ with the submodule of all elements of $\boldsymbol{V}|_{\mathcal{M}_0}$ anticommuting with all elements of $\boldsymbol{\Lambda}$. Furthermore, $\boldsymbol{V}_0 \subset \boldsymbol{V}|_{\mathcal{M}_0}$ is locally a direct summand.

(iii) In the de Rham case, the inclusion $\boldsymbol{V}_{\mathrm{dR}} \subset \widetilde{\boldsymbol{V}}_{\mathrm{dR}}|_{\mathcal{M}}$ identifies

$$\boldsymbol{V}^1_{0,\mathrm{dR}} \xrightarrow{\simeq} \boldsymbol{V}^1_{\mathrm{dR}}|_{\mathcal{M}_0}.$$

Proof. It suffices to prove the claim after localizing at a prime $p \neq 2$. So we can assume that we have a sequence of isometric embeddings $L_0 \hookrightarrow L \hookrightarrow \widetilde{L}$, where $\widetilde{L}$ is maximal and self-dual at p.

By taking the normalizations of $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ in M_0 and M, respectively, we see that the finite morphism $M_0 \to M$ extends to a map

$$\check{\mathcal{M}}_{0,\mathbb{Z}_{(p)}} \to \check{\mathcal{M}}_{\mathbb{Z}_{(p)}}. \tag{2.15}$$

Set $\Lambda_1 = L^\perp \subset \widetilde{L}$ and $\widetilde{\Lambda} = L_0^\perp \subset \widetilde{L}$, so that Λ_1 is a direct summand in $\widetilde{\Lambda}$ with orthogonal complement Λ. From this, it is clear that (2.15) restricts to a map $\check{\mathcal{M}}^{\mathrm{pr}}_{0,\mathbb{Z}_{(p)}} \to \check{\mathcal{M}}^{\mathrm{pr}}_{\mathbb{Z}_{(p)}}$. We can deduce from Proposition 2.4.2 that, étale locally on the source, this map is isomorphic to an étale neighborhood of the map of quadrics $\mathrm{M}^{\mathrm{loc}}(L_0) \to \mathrm{M}^{\mathrm{loc}}(L)$. In particular, it is unramified.

To check that this carries $\mathcal{M}_{0,\mathbb{Z}_{(p)}}$ to $\mathcal{M}_{\mathbb{Z}_{(p)}}$, it suffices now to show that the map $\mathrm{M}^{\mathrm{loc}}(L_0)_{\mathbb{Z}_{(p)}} \to \mathrm{M}^{\mathrm{loc}}(L)_{\mathbb{Z}_{(p)}}$ of quadrics over $\mathbb{Z}_{(p)}$ carries the regular locus of the source into that of the target. If the discriminant of $L_{\mathbb{Z}_{(p)}}$ is not divisible by p^2, the entire target is regular (see [MPar, (2.16)]), and there is nothing to check. Otherwise, the locus where $\mathrm{M}^{\mathrm{loc}}(L)_{\mathbb{Z}_{(p)}}$ is not regular consists exactly of the two $\mathbb{F}_{p^2}$-valued points where the corresponding isotropic line $L^1 \subset L_{\mathbb{F}_{p^2}}$ lies in the radical $\mathrm{rad}(L_{\mathbb{F}_{p^2}})$ and is also isotropic for the non-degenerate quadratic form on this radical (see *loc. cit.*).

Suppose therefore that we have an isotropic line $L^1_{0,\overline{\mathbb{F}}_p} \subset L_{0,\overline{\mathbb{F}}_p}$. We need to show: If, as an isotropic line in $L_{\overline{\mathbb{F}}_p}$, it corresponds to one of the two singular points above, then the discriminant of L_0 is also divisible by p^2, and $L^1_{0,\overline{\mathbb{F}}_p}$ corresponds to a singular point of $\mathrm{M}^{\mathrm{loc}}(L_0)$.

It is easy to see that any *smooth* point of $\mathrm{M}^{\mathrm{loc}}(L_0)$ (that is, a point at which the space is smooth over $\mathbb{Z}[1/2]$) maps to a smooth point of $\mathrm{M}^{\mathrm{loc}}(L)$. So the only remaining possibility is that the discriminant of L_0 is divisible exactly by p, and that $L^1_{0,\overline{\mathbb{F}}_p}$ is just the radical of $L_{0,\overline{\mathbb{F}}_p}$, corresponding to the unique non-smooth point of $\mathrm{M}^{\mathrm{loc}}(L_0)_{\mathbb{Z}_{(p)}}$. However, this point is defined over $\mathbb{F}_p$, whereas both the singular points of $\mathrm{M}^{\mathrm{loc}}(L)_{\mathbb{Z}_{(p)}}$ are only defined over $\mathbb{F}_{p^2}$. So this possibility can also be excluded.

Now, the fact that the isomorphism of abelian schemes also extends over $\mathcal{M}_0$ is a consequence of [FC90, Proposition I.2.7].

The canonical embedding $\Lambda \subset \mathrm{End}\big(\mathcal{A}|_{\mathcal{M}_{0,\mathbb{Z}_{(p)}}}\big)$ is constructed as follows: First, we will define an action of Λ on

$$\mathcal{A}_0 \otimes_{\mathbb{Z}} C(L) = \big(\mathcal{A}_0^+ \otimes_{\mathbb{Z}} C(L)\big) \times \big(\mathcal{A}_0^- \otimes_{\mathbb{Z}} C(L)\big).$$

Given a section of the form $(a^+ \otimes c, a^- \otimes c')$ of this product and an element $\lambda \in \Lambda$, we set:

$$\lambda \cdot (a^+ \otimes c, a^- \otimes c') = (a^+ \otimes \lambda c, -a^- \otimes \lambda c').$$

As L_0 anti-commutes with Λ within $C(L)$, it is easy to check that this action of Λ descends to one on

$$\mathcal{A}_0 \otimes_{C(L_0)} C(L) \xrightarrow[\simeq]{(2.13)} \mathcal{A}|_{\mathcal{M}_{0,\mathbb{Z}_{(p)}}}.$$

Claims (i), (ii) and (iii) are now shown in [MPar, (7.13)] for the crystalline realizations over $\mathcal{M}_{0,\mathbb{F}_p}$, but in the case where L is self-dual at p. From this, we can easily deduce the general case, just as above, by embedding everything in a quadratic space that is self-dual at p. □

The discriminant modules $L_0^\vee/L_0$ and $L^\vee/L$ are equipped with canonical $\mathbb{Q}/\mathbb{Z}$-valued quadratic forms. We have inclusions of $\mathbb{Z}$-modules

$$L_0 \oplus \Lambda \subset L \subset L^\vee \subset L_0^\vee \oplus \Lambda^\vee.$$

This identifies $L^\vee/L$ as a submodule of a quotient of $(L_0^\vee \oplus \Lambda^\vee)/(L_0 \oplus \Lambda)$.

Recall the canonical quadratic forms $\boldsymbol{Q}_0 : \boldsymbol{V}_0 \to \mathbf{1}$ and $\boldsymbol{Q} : \boldsymbol{V} \to \mathbf{1}$ from (2.11). The bilinear forms associated with these quadratic forms give rise to monomorphisms $\boldsymbol{V}_0 \to \boldsymbol{V}_0^\vee$ and $\boldsymbol{V} \to \boldsymbol{V}^\vee$, respectively. The quotient objects $\boldsymbol{V}_0^\vee/\boldsymbol{V}_0$ and $\boldsymbol{V}^\vee/\boldsymbol{V}$ are equipped with canonical $\mathbb{Q}/\mathbb{Z}\otimes\mathbf{1}$-valued quadratic forms. Arguing as in the previous paragraph and using Proposition 2.5.1, we find that $(\boldsymbol{V}^\vee/\boldsymbol{V})|_{\mathcal{M}_0}$ is canonically a submodule of a quotient of $\boldsymbol{V}_0^\vee/\boldsymbol{V}_0 \oplus \big((\Lambda^\vee/\Lambda)\otimes\mathbf{1}\big)$.

Corollary 2.5.2. *There is a canonical isometry*

$$\alpha_L : \boldsymbol{V}^\vee/\boldsymbol{V} \xrightarrow{\simeq} (L^\vee/L)\otimes_{\mathbb{Z}}\mathbf{1}$$

of $\mathbf{1}$-modules with $\mathbb{Q}/\mathbb{Z}\otimes\mathbf{1}$-valued quadratic forms. It is functorial for embeddings $L_0 \hookrightarrow L$ in the following sense: the isometry

$$\big(\boldsymbol{V}_0^\vee/\boldsymbol{V}_0\big) \oplus \big((\Lambda^\vee/\Lambda)\otimes\mathbf{1}\big) \xrightarrow{\alpha_{L_0}\oplus 1} \big((L_0^\vee/L_0)\otimes\mathbf{1}\big) \oplus \big((\Lambda^\vee/\Lambda)\otimes\mathbf{1}\big)$$

induces an isometry of subquotients $(\boldsymbol{V}^\vee/\boldsymbol{V})|_{\mathcal{M}_0} \xrightarrow{\simeq} (L^\vee/L)\otimes\mathbf{1}$, which agrees with $\alpha_L|_{\mathcal{M}_0}$.

Proof. Over the generic fiber M, the étale realizations of $\boldsymbol{V}^\vee$ and $\boldsymbol{V}$ are attached to the K-representations $L^\vee\otimes\hat{\mathbb{Z}}$ and $L\otimes\hat{\mathbb{Z}}$ via a construction that is essentially described in [MPar, (3.15)]. Therefore, the étale realizations of $\boldsymbol{V}^\vee/\boldsymbol{V}$ over M are attached to the *trivial* representation $L^\vee/L\otimes\hat{\mathbb{Z}}$ of K, which gives us the étale realization of the desired isometry α_L over M. Now, for any prime ℓ, the category of lisse sheaves over a normal scheme over $\mathbb{Z}[(2\ell)^{-1}]$ maps fully faithfully into that over its generic fiber. From this, one can deduce that the ℓ-adic part of α_L extends over $\mathcal{M}_{\mathbb{Z}[(2\ell)^{-1}]}$.

A similar argument works to provide the Betti realization of α_L over the complex fiber of M.

It is clear from the construction that α_L is functorial in L in the desired way, at least for the Betti and étale realizations.

It remains to construct the crystalline realization of α_L over $\mathcal{M}_{\mathbb{F}_p}$, for any prime $p>2$, and to show that it is functorial for embeddings $L_0\hookrightarrow L$.

Assume first that we have chosen an isometric embedding $L\hookrightarrow\widetilde{L}$ with $\widetilde{L}$ self-dual at p of signature $(\tilde{n},2)$. Then we obtain natural isomorphisms:

$$L^\vee/L\otimes\mathbb{Q}_p/\mathbb{Z}_p \xleftarrow{\simeq} \widetilde{L}/(L\oplus\Lambda)\otimes\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\simeq} \Lambda^\vee/\Lambda\otimes\mathbb{Q}_p/\mathbb{Z}_p. \tag{2.16}$$

The induced isomorphism $L^\vee/L \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\simeq} \Lambda^\vee/\Lambda \otimes \mathbb{Q}_p/\mathbb{Z}_p$ preserves the $\mathbb{Q}_p/\mathbb{Z}_p$-valued forms on either side up to sign.

As in the proof of Proposition 2.5.1, the quadratic form on $\widetilde{\boldsymbol{V}}$ is non-degenerate. Thus the induced map $\widetilde{\boldsymbol{V}}_{\mathrm{crys}} \to \widetilde{\boldsymbol{V}}^\vee_{\mathrm{crys}}$ is an isomorphism, and, using *loc. cit.* and arguing as for (2.16), we obtain canonical isomorphisms:

$$\boldsymbol{V}^\vee_{\mathrm{crys}}/\boldsymbol{V}_{\mathrm{crys}} \xrightarrow{\simeq} \widetilde{\boldsymbol{V}}_{\mathrm{crys}}|_{\mathcal{M}_{\mathbb{F}_p}}/(\boldsymbol{V}_{\mathrm{crys}} \oplus \Lambda \otimes \mathbf{1}) \xrightarrow{\simeq} (\Lambda^\vee/\Lambda) \otimes \mathbf{1}.$$

Combining this with (2.16) gives us an isometry

$$\alpha_L : \boldsymbol{V}^\vee_{\mathrm{crys}}/\boldsymbol{V}_{\mathrm{crys}} \xrightarrow{\simeq} (L^\vee/L) \otimes \mathbf{1}$$

of $\mathbb{Q}/\mathbb{Z} \otimes \mathbf{1}$-quadratic spaces.

We claim that α_L is independent of the choice of quadratic space $\widetilde{L}$. Indeed, if we have a different embedding $L \hookrightarrow \widetilde{L}_1$ with $\widetilde{L}_1$ maximal, self-dual at p and of signature $(\tilde{n}_1, 2)$, then as in Proposition 2.4.4, we can assume that the embedding $L \hookrightarrow \widetilde{L}_1$ factors as $L \hookrightarrow \widetilde{L} \hookrightarrow \widetilde{L}_1$.

If we set $\Lambda_1 = L^\perp \subset \widetilde{L}_1$, there is a canonical identification

$$\Lambda_1^\vee/\Lambda_1 \otimes \mathbb{Q}_p/\mathbb{Z}_p = \Lambda^\vee/\Lambda \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

This allows us to easily check that the isometry α_L constructed from the embedding $L \hookrightarrow \widetilde{L}$ has to agree with the one obtained from the embedding $L \hookrightarrow \widetilde{L}_1$.

As for the functoriality of the isometry, given an embedding $L_0 \hookrightarrow L$ of maximal quadratic spaces with L of signature $(\tilde{n}, 2)$, we only have to observe that we can construct both α_{L_0} and α_L using an embedding $L \hookrightarrow \widetilde{L}$ into a maximal quadratic space $\widetilde{L}$ that is self-dual at p. □

2.6. **Special endomorphisms.** For any scheme $S \to \mathcal{M}$ the pull-back $\mathcal{A}_S$ of the Kuga-Satake abelian scheme has a distinguished submodule

$$V(\mathcal{A}_S) \subset \mathrm{End}_{C(L)}(\mathcal{A}_S)$$

of *special endomorphisms*, defined in [MPar, (5.4)]. If S is connected and $s \to S$ is any geometric point, there is a cartesian diagram

$$\begin{array}{ccc} V(\mathcal{A}_S) & \longrightarrow & \mathrm{End}_{C(L)}(\mathcal{A}_S) \\ \downarrow & & \downarrow \\ V(\mathcal{A}_s) & \longrightarrow & \mathrm{End}_{C(L)}(\mathcal{A}_s). \end{array}$$

In other words, if S is connected, an endomorphism is special if and only if it is special at one (equivalently, all) geometric points of S.

At a geometric point s, the property of being special can be characterized homologically, using the subspace

$$\boldsymbol{V}_s \subset \mathrm{End}_{C(L)}(\boldsymbol{H}_s). \tag{2.17}$$

If $s = \mathrm{Spec}(\mathbb{C})$ then $x \in \mathrm{End}_{C(L)}(\mathcal{A}_s)$ is special if and only if its Betti realization lies in the subspace (2.17). This is equivalent to the ℓ-adic étale realization of x lying in (2.17) for some (equivalently, all) primes ℓ. It is also equivalent to requiring that the de Rham realization $t_{\mathrm{dR}}(x)$ lie in the subspace $\boldsymbol{V}_{\mathrm{dR},s}$.

If s is a geometric point of residue characteristic p, then $x \in \mathrm{End}_{C(L)}(\mathcal{A}_s)$ is special if and only if its crystalline realization lies in (2.17). This implies that the ℓ-adic étale realization lies in (2.17) for all ℓ different from the residue characteristic of s; cf. [MPar, (5.22)].

From the very construction of $V(\mathcal{A}_S)$, homological realization induces a map:

$$V(\mathcal{A}_S) \otimes \mathbb{Q} \to V(\mathcal{A}_S) \otimes \mathbb{Q}/\mathbb{Z} \to \boldsymbol{V}|_S \otimes \mathbb{Q}/\mathbb{Z}. \tag{2.18}$$

Within $\boldsymbol{V}|_S \otimes \mathbb{Q}/\mathbb{Z}$ we have the sub-object $(\boldsymbol{V}^\vee/\boldsymbol{V})|_S$, which, using the trivialization α_L from Corollary 2.5.2, we can identify with $(L^\vee/L) \otimes \mathbf{1}|_S$. Given $\mu \in L^\vee/L$, we can view $\alpha_L^{-1}(\mu \otimes 1)$ as a section of $\boldsymbol{V}|_S \otimes \mathbb{Q}/\mathbb{Z}$. We define

$$V_\mu(\mathcal{A}_S) \subset V(\mathcal{A}_S) \otimes \mathbb{Q}$$

to be the subset of elements which map into the preimage of $\alpha_L^{-1}(\mu \otimes 1)$ under the map (2.18), for *every* realization of $\boldsymbol{V}$.

Concretely, this means the following. An endomorphism $x \in V(\mathcal{A}_S) \otimes \mathbb{Q}$ belongs to $V_\mu(\mathcal{A}_S)$ if and only if its restriction to every geometric point $s \to S$ belongs to $V_\mu(\mathcal{A}_s)$. Now, assume that $s = S$ is a geometric point of $\mathcal{M}$. If s is a $\mathbb{C}$-valued point, then $x \in V(\mathcal{A}_s) \otimes \mathbb{Q}$ belongs to $V_\mu(\mathcal{A}_s)$ if and only if its Betti realization $x_{\mathrm{Be}} \in \boldsymbol{V}_{\mathrm{Be},s} \otimes \mathbb{Q}$ belongs to $\boldsymbol{V}^\vee_{\mathrm{Be},s}$, and maps to μ under the maps:

$$\boldsymbol{V}^\vee_{\mathrm{Be},s} \to \boldsymbol{V}^\vee_{\mathrm{Be},s}/\boldsymbol{V}_{\mathrm{Be},s} \xrightarrow{\simeq} L^\vee/L.$$

Here, the last isomorphism is induced by the isometry α_L from Corollary 2.5.2.

If s is valued in a characteristic 0 field, then the condition that x belongs to $V_\mu(\mathcal{A}_s)$ can be checked after embedding the field into $\mathbb{C}$ and using the Betti realization. This can be said more intrinsically. For every prime ℓ, we have the ℓ-adic component $\mu_\ell \in L^\vee/L \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$. The endomorphism x belongs to $V_\mu(\mathcal{A}_s)$ if and only if, for every ℓ, its ℓ-adic realization $x_\ell \in \boldsymbol{V}_{\ell,s} \otimes \mathbb{Q}$ belongs to $\boldsymbol{V}^\vee_{\ell,s}$ and maps to μ_ℓ under the maps

$$\boldsymbol{V}^\vee_{\ell,s} \to \boldsymbol{V}^\vee_{\ell,s}/\boldsymbol{V}_{\ell,s} \xrightarrow{\simeq} L^\vee/L \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

Again, the last isomorphism is the one induced by α_L.

Finally, if s is valued in a field of characteristic $p > 2$, x belongs to $V_\mu(\mathcal{A}_s)$ if and only if two conditions hold: First, its ℓ-adic realization x_ℓ, for $\ell \neq p$, satisfies the criterion in the previous paragraph. Second, its crystalline realization $x_{\mathrm{crys}} \in \boldsymbol{V}_{\mathrm{crys},s} \otimes \mathbb{Q}$ belongs to $\boldsymbol{V}^\vee_{\mathrm{crys},s}$ and maps to μ_p under the maps:

$$\boldsymbol{V}^\vee_{\mathrm{crys},s} \to \boldsymbol{V}^\vee_{\mathrm{crys},s}/\boldsymbol{V}_{\mathrm{crys},s} \xrightarrow{\simeq} L^\vee/L \otimes W(k).$$

Observe that, by definition, $V_0(\mathcal{A}_S) = V(\mathcal{A}_S)$. Moreover, if $V_\mu(\mathcal{A}_S)$ is non-empty, it will be a torsor under translation by $V(\mathcal{A}_S)$.

Proposition 2.6.1. *Each $x \in V_\mu(\mathcal{A}_S)$, viewed as an element of $\mathrm{End}^0(\mathcal{A}_S)$, shifts the grading on $\mathcal{A}_S$ and commutes with the right action of $C(L)$. Moreover, $x \circ x = Q(x) \cdot \mathrm{Id}$ for some non-negative $Q(x) \in \mathbb{Q}$ satisfying*

$$Q(x) \equiv Q(\mu) \pmod{\mathbb{Z}}, \tag{2.19}$$

and $Q(x) = 0$ if and only if $x = 0$.

Proof. Most of this follows from the definitions. The only thing to check is that the quadratic form $Q(x)$ is positive definite. But this follows from the existence of a polarization on $\mathcal{A}_S$ such that every element of $V(\mathcal{A}_S) \otimes \mathbb{Q}$ is fixed under the corresponding Rosati involution; see the proof of [MPar, (5.12)]. □

We will now consider the question of functoriality of special endomorphisms. As before, suppose that we are given an embedding of maximal quadratic spaces $L_0 \hookrightarrow L$ with $\Lambda = L_0^\perp \subset L$. In particular the inclusion $L^\vee \hookrightarrow L_0^\vee \oplus \Lambda^\vee$ induces an injection

$$L^\vee/(L_0 \oplus \Lambda) \hookrightarrow (L_0^\vee/L_0) \oplus (\Lambda^\vee/\Lambda).$$

Fix an $\mathcal{M}_0$-scheme $S \to \mathcal{M}_0$. Then, by construction, the map $\Lambda \hookrightarrow \mathrm{End}(\mathcal{A}_S)$ from Proposition 2.5.1 factors through an isometric embedding $\Lambda \hookrightarrow V(\mathcal{A}_S)$.

Proposition 2.6.2. *Fix an $\mathcal{M}_0$-scheme $S \to \mathcal{M}_0$.*

(i) There is a canonical isometry:

$$V(\mathcal{A}_{0,S}) \xrightarrow{\simeq} \Lambda^\perp \subset V(\mathcal{A}_S). \tag{2.20}$$

(ii) For every $\mu \in L^\vee/L$ and every $(\mu_1, \mu_2) \in \big(\mu + L\big)/\big(L_0 \oplus \Lambda\big)$ the map (2.20), tensored with $\mathbb{Q}$, restricts to an injection

$$V_{\mu_1}(\mathcal{A}_{0,S}) \times (\mu_2 + \Lambda) \hookrightarrow V_\mu(\mathcal{A}_S).$$

(iii) The above injections determine a decomposition

$$V_\mu(\mathcal{A}_S) = \coprod_{(\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)} V_{\mu_1}(\mathcal{A}_{0,S}) \times \big(\mu_2 + \Lambda\big).$$

Proof. Claim (i) follows from the definitions and Proposition 2.5.1; see also [MPar, (8.12)]. In particular, any element $x \in V(\mathcal{A}_S) \otimes \mathbb{Q}$ admits a decomposition:

$$x = (x_0, \nu) \in (V(\mathcal{A}_{0,S}) \otimes \mathbb{Q}) \times (\Lambda \otimes \mathbb{Q}).$$

Using this decomposition, it is an easy exercise to deduce claims (ii) and (iii) from the definitions. The main input is the functoriality of the isometry α_L shown in Corollary 2.5.2. □

2.7. **Special divisors.** For $m \in \mathbb{Q}_{>0}$ and $\mu \in L^\vee/L$, define the *special cycle* $\mathcal{Z}(m,\mu) \to \mathcal{M}$ as the stack over $\mathcal{M}$ with functor of points

$$\mathcal{Z}(m,\mu)(S) = \{x \in V_\mu(\mathcal{A}_S) : Q(x) = m\} \tag{2.21}$$

for any scheme $S \to \mathcal{M}$. Note that, by (2.19), the stack (2.21) is empty unless the image of m in $\mathbb{Q}/\mathbb{Z}$ agrees with $Q(\mu)$.

For later purposes we also define the stacks $\mathcal{Z}(0,\mu)$ in exactly the same way. As the only special endomorphism x with $Q(x) = 0$ is the zero map, we have

$$\mathcal{Z}(0,\mu) = \begin{cases} \emptyset & \text{if } \mu \neq 0 \\ \mathcal{M} & \text{if } \mu = 0. \end{cases}$$

Lemma 2.7.1. *For any postive m and any μ, the complex orbifold $\mathcal{Z}(m,\mu)(\mathbb{C})$ just defined agrees with (2.4).*

Proof. Consider the uniformization $M(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K$. By construction, for every

$$z = (\boldsymbol{h}, g) \in \mathcal{D} \times G(\mathbb{A}_f),$$

the fiber of $\boldsymbol{V}_{\mathrm{Be},\mathbb{Q}}$ at z is the $\mathbb{Q}$-vector space V with Hodge structure determined by the negative plane $\boldsymbol{h} \subset V_\mathbb{R}$. The $(0,0)$ part of $\boldsymbol{V}_{\mathrm{Be},\mathbb{Q},z}$ is $V \cap \boldsymbol{h}^\perp$. The fibers of the subsheaves $\boldsymbol{V}_{\mathrm{Be}}$ and $\boldsymbol{V}_{\mathrm{Be}}^\vee$ of $\boldsymbol{V}_{\mathrm{Be},\mathbb{Q}}$ at z are $\widehat{L}_g \cap V = L_g$ and $\widehat{L}_g^\vee \cap V = L_g^\vee$, respectively. Here, using the notation of § 2.2, $\widehat{L}_g = g \bullet \widehat{L}$ and $\widehat{L}_g^\vee = g \bullet \widehat{L}^\vee$.

By definition, we now have

$$V(\mathcal{A}_z) = \mathrm{End}(\mathcal{A}_z) \cap \boldsymbol{V}_{\mathrm{Be,z}} = (\boldsymbol{V}_{\mathrm{Be},z})^{(0,0)} = L_g \cap \boldsymbol{h}^\perp.$$

This proves the lemma for the trivial class μ.

The isomorphism $\boldsymbol{V}_{\mathrm{Be}}^\vee/\boldsymbol{V}_{\mathrm{Be}} \simeq L^\vee/L \otimes \mathbf{1}$ from (2.5.2) determines, for every $\mu \in L^\vee/L$, a locally constant intermediate sheaf of sets $\boldsymbol{V}_{\mathrm{Be}} \subset \boldsymbol{V}_{\mu,\mathrm{Be}} \subset \boldsymbol{V}_{\mathrm{Be}}^\vee$ whose fiber $\boldsymbol{V}_{\mu,\mathrm{Be},z}$ is $\mu_g + L_g$. It follows that

$$V_\mu(\mathcal{A}_z) = (\mu_g + L_g) \cap \boldsymbol{h}^\perp.$$

From this, the lemma follows for general μ. □

The special cycles behave nicely under pullback by the morphism of Proposition 2.5.1. Indeed, the following is an immediate consequence of Proposition 2.6.2.

Proposition 2.7.2. *Suppose that we are given an embedding of maximal quadratic spaces $L_0 \hookrightarrow L$, and let $\Lambda \subset L$ be the submodule of vectors orthogonal to L_0. Let $\mathcal{M}_0 \to \mathcal{M}$ be the corresponding morphism of Shimura varieties, as in Proposition 2.5.1. Thus, for any $m \in \mathbb{Q}_{\geq 0}$ and $\mu \in L^\vee/L$ there is a special cycle $\mathcal{Z}(m,\mu) \to \mathcal{M}$, and for any $m_1 \in \mathbb{Q}_{\geq 0}$ and $\mu_1 \in L_0^\vee/L_0$ there is a special cycle $\mathcal{Z}_0(m_1,\mu_1) \to \mathcal{M}_0$.*

Then, there is an isomorphism of $\mathcal{M}_0$-stacks

$$\mathcal{Z}(m,\mu) \times_{\mathcal{M}} \mathcal{M}_0 \simeq \coprod_{\substack{m_1+m_2=m \\ (\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)}} \mathcal{Z}_0(m_1,\mu_1) \times \Lambda_{m_2,\mu_2},$$

where

$$\Lambda_{m_2,\mu_2} = \{x \in \mu_2 + \Lambda : Q(x) = m_2\},$$

and $\mathcal{Z}_0(m_1,\mu_1) \times \Lambda_{m_2,\mu_2}$ denotes the disjoint union of $\#\Lambda_{m_2,\mu_2}$ copies of $\mathcal{Z}_0(m_1,\mu_1)$.

We now show that the stacks $\mathcal{Z}(m,\mu)$ with $m > 0$ define (étale local) Cartier divisors on $\mathcal{M}$.

Proposition 2.7.3. *Suppose $m > 0$. The morphism $\mathcal{Z}(m,\mu) \to \mathcal{M}$ is finite and unramified, and its image does not contain any connected component of $\mathcal{M}$. In fact, étale locally on the source, $\mathcal{Z}(m,\mu)$ defines a Cartier divisor on $\mathcal{M}$. More precisely: around every geometric point of $\mathcal{M}$ there is an étale neighborhood U such that for every connected component $Z \subset \mathcal{Z}(m,\mu)_U$, the map $Z \to U$ is a closed immersion defined by a single non-zero equation.*

Before giving the proof, we need to understand the deformation theory of the stacks $\mathcal{Z}(m,\mu)$.

Suppose that we are given a closed point z of $\mathcal{Z}(m,\mu)$ corresponding to an element $x \in V_\mu(\mathcal{A}_z)$. Let R_z be the étale local ring of $\mathcal{M}$ at z and let $J_x \subset R_z$ be the ideal defining the fiber of $\mathcal{Z}(m,\mu)$ over Spec R_z. Set $S = \mathrm{Spec}(R_z/J_x)$, so that $x \in V_\mu(\mathcal{A}_z)$ has a universal deformation to $x_S \in V_\mu(\mathcal{A}_S)$.

Fix an embedding $L \hookrightarrow \widetilde{L}$ so that $\widetilde{L}$ is maximal and self-dual at p of signature $(\tilde{n},2)$, and let $\Lambda \subset \widetilde{L}$ be the subset of vectors orthogonal to L. Let $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{M}}$ be the Kuga-Satake abelian scheme over the integral model of the Shimura variety determined by $\widetilde{L}$.

Proposition 2.6.2 (with $\mathcal{A}_0$ and $\mathcal{A}$ replaced by $\mathcal{A}$ and $\widetilde{\mathcal{A}}$) gives an inclusion

$$V(\mathcal{A}_S) \oplus \Lambda \hookrightarrow V(\widetilde{\mathcal{A}}_S),$$

which becomes an isomorphism after tensoring with $\mathbb{Q}$. There is a $\nu \in \Lambda^\vee$ such that the vector

$$\tilde{x}_S := x_S + \nu \in (V(\mathcal{A}_S) \otimes \mathbb{Q}) \oplus (\Lambda \otimes \mathbb{Q})$$

lies in the $\mathbb{Z}$-lattice $V(\widetilde{\mathcal{A}}_S)$. The crystalline realization $t_{\mathrm{crys}}(\tilde{x})$ allows us to canonically lift the de Rham realization

$$t_{\mathrm{dR}}(\tilde{x}_S) \in \widetilde{\boldsymbol{V}}_{\mathrm{dR},R_z/J_x}$$

to an element

$$t_{\mathrm{crys}}(\tilde{x}_S) \in \widetilde{\boldsymbol{V}}_{\mathrm{dR},R_z/J_x^2}.$$

through the divided power thickening $S \hookrightarrow \mathrm{Spec}\big(R_z/J_x^2\big)$.

It follows from [MPar, (5.15)] that $t_{\mathrm{dR}}(\tilde{x}_S)$ is orthogonal to $\widetilde{\boldsymbol{V}}^1_{\mathrm{dR},S}$. Thus, if

$$\omega \in \boldsymbol{V}^1_{\mathrm{dR},R_z/J_x^2} \xrightarrow{\simeq} \widetilde{\boldsymbol{V}}^1_{\mathrm{dR},R_z/J_x^2}$$

is an R_z-module generator, we obtain an element $\big[\omega, t_{\mathrm{crys}}(\tilde{x}_S)\big] \in J_x/J_x^2$ using the pairing on $\widetilde{\boldsymbol{V}}_{\mathrm{dR}}$. In fact, Grothendieck-Messing theory [MPar, (5.17)] shows that $\big[\omega, t_{\mathrm{crys}}(\tilde{x}_S)\big]$ generates J_x/J_x^2. The element

$$t_{\mathrm{crys}}(x_S) := t_{\mathrm{crys}}(\tilde{x}_S) - \nu$$

lies in $\boldsymbol{V}^\vee_{\mathrm{dR},R_z/J_x^2}$, and the orthogonality relation $[\omega,\nu]=0$ implies

$$[\omega, t_{\mathrm{crys}}(x_S)] = [\omega, t_{\mathrm{crys}}(\widetilde{x}_S)].$$

Lemma 2.7.4. *The element $[\omega, t_{\mathrm{crys}}(x_S)]$ generates J_x/J_x^2 as an R_z-module. In particular, $J_x \subset R_z$ is a principal ideal.*

Proof. The discussion above proves the first assertion. The second follows by Nakayama's lemma. □

Proof of Proposition 2.7.3. Let us first show that the morphism $\mathcal{Z}(m,\mu) \to \mathcal{M}$ is finite. First, we claim that it is quasi-finite. This amounts to seeing that, for any point $z \to \mathcal{M}$, $V_\mu(\mathcal{A}_z)$ has only finitely many elements x satisfying $Q(x)=m$. Indeed, there exists $r \in \mathbb{Z}_{>0}$ such that

$$rV_\mu(\mathcal{A}_z) \subset V(\mathcal{A}_z) \subset V(\mathcal{A}_z)_{\mathbb{Q}}.$$

Therefore, it suffices to show that there are only finitely many $x \in V(\mathcal{A}_z)$ satisfying $Q(x) = r^2 m$. This follows from Proposition 2.6.1, which shows that $V(\mathcal{A}_z)$ is positive definite.

To finish the proof of finiteness, we have to show that the morphism is also proper. Let $\mathscr{O}$ be a discrete valuation ring of residue characteristic $p>2$, and let F be its fraction field. Suppose that we have a map $z : \mathrm{Spec}(\mathscr{O}) \to \mathcal{M}$. We must show:

$$V_\mu(\mathcal{A}_z) = V_\mu\big(\mathcal{A}_z|_{\mathrm{Spec}(F)}\big).$$

For this, using Proposition 2.6.2 and the usual trick of embedding L in a quadratic space that is self-dual at p, we can reduce to the case where L is itself self-dual at p.

In this case, it follows from [MPar, (5.13)], and the fact that $\mathcal{A}_z$ is a Nerón model for its generic fiber, that the natural inclusion

$$V(\mathcal{A}_z) \hookrightarrow V\big(\mathcal{A}_z|_{\mathrm{Spec}(F)}\big)$$

given by restriction is actually a bijection.

Since, in addition, the p-primary part μ_p of μ is necessarily trivial, we see that any element $x \in V_\mu\big(\mathcal{A}_z|_{\mathrm{Spec}(F)}\big)$ lies in $V(\mathcal{A}_z)_{\mathbb{Z}_{(p)}}$. It remains to check that its cohomological realizations over $\mathrm{Spec}(\mathscr{O})$ are sections of $\boldsymbol{V}^\vee$ and that the image in $\boldsymbol{V}^\vee/\boldsymbol{V}$ of these realizations coincide with $\alpha_L(\mu\otimes 1)$. For the ℓ-adic realizations, with $\ell \neq p$, these statements can be checked over $\mathrm{Spec}(F)$, where they hold by hypothesis. Since $\mu_p = 0$, all that is left to show is that the crystalline realization of x is a section of $\boldsymbol{V}_{\mathrm{crys}}$. But this is precisely equivalent to knowing that x lies in $V(\mathcal{A}_z)_{\mathbb{Z}_{(p)}}$.

Now that we know that the morphism is finite, we find, in particular, that $Z(m,\mu)$ is of finite presentation over $\mathcal{M}$. Lemma 2.7.4 shows that $Z(m,\mu)$ is formally unramified over $\mathcal{M}$, and so it is unramified. In fact, the lemma implies the more refined final assertion of the proposition.

It only remains to show that the image of the morphism does not contain any connected component of $\mathcal{M}$. Using the flatness of $\mathcal{M}$ over $\mathbb{Z}[1/2]$, this can be checked over $\mathbb{C}$, where it follows from Lemma 2.7.1. □

Remark 2.7.5. It follows from Proposition 2.7.3 that $\mathcal{Z}(m,\mu)$ is of pure dimension $\dim(\mathcal{M})-1$. The morphism to $\mathcal{M}$ is not itself a closed immersion, but nevertheless $\mathcal{Z}(m,\mu)$ defines a Cartier divisor on $\mathcal{M}$, as in [BHY15, §3.1].

3. Harmonic modular forms and special divisors

Fix a maximal quadratic space L over $\mathbb{Z}$ of signature $(n,2)$ with $n\geq 1$, set $V=L_{\mathbb{Q}}$, and let $\mathcal{M}\to\mathrm{Spec}(\mathbb{Z}[1/2])$ be the corresponding integral model of the Shimura variety (2.3) defined in §2.

This section is a rapid review of some results and constructions of Bruinier [Bru02], Bruinier-Funke [BF04], and Bruinier-Yang [BY09]. In particular, we recall the construction of a divisor $\mathcal{Z}(f)$ and a Green function $\Phi(f)$ on $\mathcal{M}$ from a harmonic weak Maass form f.

3.1. **Vector valued modular forms.**

Let $\mathfrak{S}_L$ be the (finite dimensional) space of $\mathbb{C}$-valued functions on $L^\vee/L$. As in [BY09] there is a Weil representation

$$\omega_L:\widetilde{\mathrm{SL}}_2(\mathbb{Z})\to\mathrm{Aut}_{\mathbb{C}}(\mathfrak{S}_L),$$

where $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ is the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$. Define the conjugate action $\overline{\omega}_L$ by $\overline{\omega}_L(\gamma)\varphi=\overline{\omega_L(\gamma)\overline{\varphi}}$. We denote by $\omega_L^\vee$ the contragredient action of $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ on the complex linear dual $\mathfrak{S}_L^\vee$.

Remark 3.1.1. Our $\overline{\omega}_L$ is the representation denoted ρ_L in [Bor98], [Bru02], [BF04], and [BY09]. If we denote by $-L$ the quadratic space over $\mathbb{Z}$ whose underlying $\mathbb{Z}$-module is L, but endowed with the quadratic form $-Q$, then $\mathfrak{S}_L=\mathfrak{S}_{-L}$ as vector spaces, but $\omega_L=\rho_{-L}$.

For a half-integer k, denote by $H_k(\omega_L)$ the $\mathbb{C}$-vector space of $\mathfrak{S}_L$-valued harmonic weak Maass forms of weight k and representation ω_L, in the sense[2] of [BY09, §3.1]. As in [BY09, §3] there are subspaces

$$S_k(\omega_L)\subset M_k^!(\omega_L)\subset H_k(\omega_L)$$

of cusp forms and weakly modular forms. Define similar spaces of modular forms for the representation $\overline{\omega}_L$. Bruinier and Funke [BF04] have defined a

[2] These satisfy a more restrictive growth condition at the cusp than the *weak Maass forms* of [BF04, §3].

differential operator

$$\xi : H_k(\omega_L) \to S_{2-k}(\overline{\omega}_L) \tag{3.1}$$

by $\xi(f) = 2iv^k\overline{(\partial f/\partial\overline{\tau})}$, where $\tau = u + iv$ is the variable on the upper half-plane, and have proved the exactness of the sequence

$$0 \to M^!_k(\omega_L) \to H_k(\omega_L) \xrightarrow{\xi} S_{2-k}(\overline{\omega}_L) \to 0.$$

Every $f(\tau) \in H_k(\omega_L)$ has an associated formal q-expansion

$$f^+(\tau) = \sum_{\substack{m\in\mathbb{Q} \\ -\infty \ll m}} c_f^+(m) \cdot q^m,$$

called the *holomorphic part of* f, with $c_f^+(m) \in \mathfrak{S}_L$. The delta functions φ_μ at elements $\mu \in L^\vee/L$ form a basis for $\mathfrak{S}_L$, and so each coefficient $c_f^+(m)$ can be decomposed as a sum

$$c_f^+(m) = \sum_{\mu\in L^\vee/L} c_f^+(m,\mu)\cdot\varphi_\mu$$

with $c_f^+(m,\mu) \in \mathbb{C}$. The transformation laws satisfied by f imply that

$$c_f^+(m,\mu) = c_f^+(m,-\mu),$$

and also that $c_f^+(m,\mu) = 0$, unless the image of m in $\mathbb{Q}/\mathbb{Z}$ is equal to $-Q(\mu)$.

3.2. **Divisors and Green functions.** Suppose we are given a harmonic weak Maass form

$$f(\tau) \in H_{1-\frac{n}{2}}(\omega_L)$$

having *integral principal part*, in the sense that $c_f^+(m,\mu) \in \mathbb{Z}$ for all $\mu \in L^\vee/L$ and all $m < 0$. Define a divisor on $\mathcal{M}$ by

$$\mathcal{Z}(f) = \sum_{m\in\mathbb{Q}_{>0}} \sum_{\mu\in L^\vee/L} c_f^+(-m,\mu)\mathcal{Z}(m,\mu).$$

Bruinier [Bru02], following ideas of Borcherds [Bor98] and Harvey-Moore, constructed a Green function $\Phi(f)$ for $\mathcal{Z}(f)$ as follows (see [BY09, §4]). Let $G = \mathrm{GSpin}(V)$ as in §2. For each coset $g \in G(\mathbb{A}_f)/K$ there is a Siegel theta function

$$\theta_L(\tau,z,g) : \mathcal{H}\times\mathcal{D} \to \mathfrak{S}_L^\vee$$

as in [BY09, (2.4)], which is Γ_g-invariant in the variable $z \in \mathcal{D}$, and transforms in the variable $\tau = u + iv \in \mathcal{H}$ like a modular form of representation $\omega_L^\vee$.

The regularized theta integral

$$\Phi(f,z,g) = \int_{\mathcal{F}}^{\mathrm{reg}} \{f(\tau),\theta_L(\tau,z,g)\}\frac{du\,dv}{v^2}$$

of [BY09, (4.7)] defines a Γ_g-invariant function on $\mathcal{D}$, where

$$\{\cdot,\cdot\} : \mathfrak{S}_L \times \mathfrak{S}_L^\vee \to \mathbb{C}$$

is the tautological pairing. Letting g vary and using the uniformization (2.3) yields a function $\Phi(f)$ on the orbifold $\mathcal{M}(\mathbb{C})$. This function is smooth on the complement of $\mathcal{Z}(f)$, and has a logarithmic singularity along $\mathcal{Z}(f)$ in the sense that for any local equation $\Psi(z) = 0$ defining $\mathcal{Z}(f)(\mathbb{C})$, the function

$$\Phi(f, z) + \log|\Psi(z)|^2$$

on $\mathcal{M}(\mathbb{C}) \smallsetminus \mathcal{Z}(f)(\mathbb{C})$ has a smooth extension across $\mathcal{Z}(f)(\mathbb{C})$.

Remark 3.2.1. Amazingly, the regularized integral defining $\Phi(f, z, g)$ converges at every point of $\mathcal{D}$, and so the function $\Phi(f)$ has a well-defined value *even at points of the divisor* $\mathcal{Z}(f)(\mathbb{C})$. The value of the discontinuous function $\Phi(f)$ at the points of $\mathcal{Z}(f)(\mathbb{C})$ will play a key role in our later calculations of improper intersection.

Among the harmonic weak Maass forms of weight $1 - n/2$ are the *Hejhal-Poincarè series*

$$F_{m,\mu}(\tau) \in H_{1-\frac{n}{2}}(\omega_L),$$

defined for all $\mu \in L^\vee/L$ and positive $m \in \mathbb{Z} + Q(\mu)$, which may be rescaled to have holomorphic parts

$$F^+_{m,\mu}(\tau) = \frac{1}{2}(q^{-m}\varphi_\mu + q^{-m}\varphi_{-\mu}) + O(1).$$

In particular $\mathcal{Z}(F_{m,\mu}) = \mathcal{Z}(m, \mu)$. See [Bru02, Def. 1.8 & Prop. 1.10] and [BF04, Rmk 3.10] for details.

3.3. **Eisenstein series and the Rankin-Selberg integral.** Fix a rational negative 2-plane $V_0 \subset V$, and let $L_0 = V_0 \cap L$. We assume that $C^+(L_0)$ is the maximal order in the quadratic imaginary field $C^+(V_0)$. This implies, in particular, that L_0 is a maximal quadratic space. Let

$$\Lambda = \{x \in L : x \perp L_0\},$$

so that $L_0 \oplus \Lambda \subset L$ with finite index, and Λ is also a maximal quadratic space.

Let $\mathfrak{S}_\Lambda$ be the space of complex-valued functions on $\Lambda^\vee/\Lambda$, and let $\omega_\Lambda : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathfrak{S}_\Lambda)$ be the Weil representation. Let $\mathfrak{S}_\Lambda^\vee$ be the complex linear dual of $\mathfrak{S}_\Lambda$, and for each $m \in \mathbb{Q}$ define the representation number $R_\Lambda(m) \in \mathfrak{S}_\Lambda^\vee$ by

$$R_\Lambda(m, \varphi) = \sum_{\substack{x \in \Lambda^\vee \\ Q(x) = m}} \varphi(x)$$

for any $\varphi \in \mathfrak{S}_\Lambda$. As in [BY09, §2], these representation numbers are the Fourier coefficients of a vector-valued modular form

$$\Theta_\Lambda(\tau) = \sum_{m \in \mathbb{Q}} R_\Lambda(m) q^m \in M_{\frac{n}{2}}(\omega_\Lambda^\vee).$$

Abbreviate $R_\Lambda(m,\mu) = R_\Lambda(m,\varphi_\mu)$, where $\varphi_\mu \in \mathfrak{S}_\Lambda$ is the characteristic function of $\mu \in \Lambda^\vee/\Lambda$, so that

$$R_\Lambda(m,\mu) = \#\{x \in \mu + \Lambda : Q(x) = m\}.$$

Let $F(\tau)$ be a cuspidal modular form

$$F(\tau) = \sum_{m \in \mathbb{Q}_{>0}} b_F(m) q^m \in S_{1+\frac{n}{2}}(\overline{\omega}_L).$$

Using the natural inclusion $L^\vee \subset L_0^\vee \oplus \Lambda^\vee$, every function on $L^\vee$ extends to a function on $L_0^\vee \oplus \Lambda^\vee$ identically zero off of $L^\vee$. Thus, there is a natural "extension by zero" map

$$\mathfrak{S}_L \to \mathfrak{S}_{L_0 \oplus \Lambda}, \tag{3.2}$$

which allows us to view $\overline{b_F(m)} \in \mathfrak{S}_{L_0\oplus\Lambda}$. The inclusion $\Lambda^\vee \to L_0^\vee \oplus \Lambda^\vee$ defines a *restriction of functions* homomorphism $\mathfrak{S}_{L_0\oplus\Lambda} \to \mathfrak{S}_\Lambda$. Using this, we can view $R_\Lambda(m) \in \mathfrak{S}^\vee_{L_0\oplus\Lambda}$. This allows us to define the *Rankin-Selberg convolution L-function*

$$L(F,\Theta_\Lambda,s) = \Gamma\left(\frac{s+n}{2}\right) \sum_{m\in\mathbb{Q}^+} \frac{\left\{\overline{b_F(m)}, R_\Lambda(m)\right\}}{(4\pi m)^{(s+n)/2}},$$

in which the pairing on the right is the tautological pairing

$$\{\cdot,\cdot\} : \mathfrak{S}_{L_0\oplus\Lambda} \times \mathfrak{S}^\vee_{L_0\oplus\Lambda} \to \mathbb{C}. \tag{3.3}$$

Define an $\mathfrak{S}^\vee_{L_0}$-valued Eisenstein series $E_{L_0}(\tau,s)$ by

$$E_{L_0}(\tau,s,\varphi) = \sum_{\gamma\in B\backslash \mathrm{SL}_2(\mathbb{Z})} (\omega_{L_0}(\gamma)\varphi)(0) \cdot \frac{\mathrm{Im}(\gamma\tau)^{s/2}}{c\tau+d}$$

for all $\varphi \in \mathfrak{S}_{L_0}$, where $B \subset \mathrm{SL}_2(\mathbb{Z})$ is the subset of upper triangular matrices, and ω_{L_0} is the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{S}_{L_0}$. The Eisenstein series $E_{L_0}(\tau,s)$, which is precisely the *incoherent* Eisenstein series denoted $E_{L_0}(\tau,s;1)$ in [BY09, §2.2], transforms in the variable τ like a weight 1 modular form of representation $\omega^\vee_{L_0}$. It is initially defined for $\mathrm{Re}(s) \gg 0$, but has meromorphic continuation to all s, and vanishes at $s=0$. The usual Rankin-Selberg unfolding method shows that

$$L(F,\Theta_\Lambda,s) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \left\{\overline{F}, E_{L_0}(s)\otimes\Theta_\Lambda\right\} \frac{du\,dv}{v^{1-\frac{n}{2}}},$$

where the pairing on the right is (3.3), and we are using the isomorphism

$$\mathfrak{S}^\vee_{L_0} \otimes \mathfrak{S}^\vee_\Lambda \xrightarrow{\simeq} \mathfrak{S}^\vee_{L_0\oplus\Lambda}$$

to view $E_{L_0}(s)\otimes\Theta_\Lambda$ as a smooth modular form valued in $\mathfrak{S}^\vee_{L_0\oplus\Lambda}$.

The central derivative $E'_{L_0}(\tau,0)$ is a harmonic $\mathfrak{S}^\vee_{L_0}$-valued function transforming like a weight 1 modular form of representation $\omega^\vee_{L_0}$. Denote the holomorphic part of the central derivative by

$$\mathcal{E}_{L_0}(\tau) = \sum_{m\in\mathbb{Q}} a^+_{L_0}(m)\cdot q^m.$$

We will sometimes abbreviate $a^+_{L_0}(m,\mu) = a^+_{L_0}(m,\varphi_\mu)$, where $\varphi_\mu \in \mathfrak{S}_{L_0}$ is the characteristic function of $\mu \in L_0^\vee/L_0$. Note that our $a^+_{L_0}(m,\mu)$ is exactly the $\kappa(m,\mu)$ of [BY09, (2.25)].

Abbreviate $\boldsymbol{k} = C^+(V_0)$, and recall that $\boldsymbol{k}$ is a quadratic imaginary field. For any positive rational number m, define a finite set of primes

$$\mathrm{Diff}_{L_0}(m) = \{\text{primes } p<\infty : L_0\otimes_{\mathbb{Z}}\mathbb{Q}_p \text{ does not represent } m\}. \tag{3.4}$$

Note that all primes in this set are nonsplit in $\boldsymbol{k}$. Because L_0 is negative definite, the set $\mathrm{Diff}_{L_0}(m)$ has odd cardinality; in particular, $\mathrm{Diff}_{L_0}(m)\neq\emptyset$. For $m\in\mathbb{Z}_{>0}$, set

$$\rho(m) = \#\{\text{nonzero ideals } \mathfrak{m}\subset\mathcal{O}_{\boldsymbol{k}} : \mathrm{N}(\mathfrak{m}) = m\}.$$

We will extend ρ to a function on all rational numbers by setting $\rho(m)=0$ for all $m\in\mathbb{Q}\smallsetminus\mathbb{Z}_{>0}$.

The coefficients $a^+_{L_0}(m)$ were computed by Schofer [Sch09], but those formulas contain a minor misstatement. This misstatement was corrected when Schofer's formulas were reproduced in [BY09, Theorem 2.6], but the formula for the constant term $a^+_{L_0}(0)$ in [BY09] is misstated[3]. We record now the corrected formulas.

Proposition 3.3.1 (Schofer)**.** *Let $d_{\boldsymbol{k}}$, $h_{\boldsymbol{k}}$, and $w_{\boldsymbol{k}}$ be the discriminant of $\boldsymbol{k}$, the class number of $\boldsymbol{k}$, and number of roots of unity in $\boldsymbol{k}$, respectively, and assume that $d_{\boldsymbol{k}}$ is odd. The coefficients $a^+_{L_0}(m)$ are as follows:*

(i) The constant term is

$$a^+_{L_0}(0,\varphi) = \varphi(0)\cdot\left(\gamma + \log\left|\frac{4\pi}{d_{\boldsymbol{k}}}\right| - 2\frac{L'(\chi_{\boldsymbol{k}},0)}{L(\chi_{\boldsymbol{k}},0)}\right).$$

Here $\gamma=-\Gamma'(1)$ is the Euler-Mascheroni constant, and $\chi_{\boldsymbol{k}}$ is the quadratic Dirichlet character associated with $\boldsymbol{k}/\mathbb{Q}$.

(ii) If $m>0$ and $\mathrm{Diff}_{L_0}(m)=\{p\}$ for a single prime p, then

$$a^+_{L_0}(m,\varphi) = -\frac{w_{\boldsymbol{k}}}{2h_{\boldsymbol{k}}}\cdot\rho\left(\frac{m|d_{\boldsymbol{k}}|}{p^\epsilon}\right)\cdot\mathrm{ord}_p(pm)\cdot\log(p)\sum_{\substack{\mu\in L_0^\vee/L_0\\ Q(\mu)=m}} 2^{s(\mu)}\varphi(\mu).$$

On the right hand side, $s(\mu)$ is the number of primes $\ell\mid\mathrm{disc}(L_0)$ such that $\mu\in L_{0,\ell}$,

$$\epsilon = \begin{cases} 1 & \text{if } p \text{ is inert in } \boldsymbol{k} \\ 0 & \text{if } p \text{ is ramified in } \boldsymbol{k}, \end{cases}$$

[3]The formula for $a^+_{L_0}(0,0)=\kappa(0,0)$ in [BY09] appears to agree with the formula in [Sch09]. In fact it does not, as the two papers use different normalizations for the completed L-function $\Lambda(\chi,s)$. It is Schofer's formula that is correct.

and $Q(\mu) = m$ is understood as an equality in $\mathbb{Q}/\mathbb{Z}$.

(iii) If $m > 0$ and $|\mathrm{Diff}_{L_0}(m)| > 1$, or if $m < 0$, then $a^+_{L_0}(m) = 0$.

4. Complex multiplication cycles

Keep $L_0 \oplus \Lambda \subset L$ as in §3.3, so that L is a maximal lattice of signature $(n, 2)$, L_0 is a sublattice of signature $(0, 2)$, and

$$\Lambda = L_0^\perp = \{x \in L : x \perp L_0\}.$$

Abbreviate $V_0 = L_0 \otimes_\mathbb{Z} \mathbb{Q}$ and $V = L \otimes_\mathbb{Z} \mathbb{Q}$. The even part $C^+(V_0)$ of the Clifford algebra is isomorphic to a quadratic imaginary field, and we continue to assume that $C^+(L_0)$ is its maximal order.

In this section we construct an $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{Y}$ from the quadratic space L_0. The inclusion $L_0 \subset L$ then induces a morphism

$$\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]} \to \mathcal{M}_{\mathcal{O}_{\boldsymbol{k}}[1/2]},$$

where $\mathcal{M}$ is the stack over $\mathbb{Z}[1/2]$ determined by the quadratic space L. We define special divisors on $\mathcal{Y}$, compute their degrees explicitly, and compare them with the special divisors on $\mathcal{M}$.

4.1. Preliminaries.

Fix once and for all an isomorphism of $C^+(L_0)$ with the maximal order $\mathcal{O}_{\boldsymbol{k}}$ in a quadratic imaginary subfield $\boldsymbol{k} \subset \mathbb{C}$. Let $d_{\boldsymbol{k}}$, $h_{\boldsymbol{k}}$, and $w_{\boldsymbol{k}}$ be as in Proposition 3.3.1.

The $\mathbb{Z}/2\mathbb{Z}$-grading on $C(L_0)$ takes the form

$$C(L_0) = \mathcal{O}_{\boldsymbol{k}} \oplus L_0, \tag{4.1}$$

and L_0 is both a left and right $\mathcal{O}_{\boldsymbol{k}}$-module; the two action are related by $x\alpha = \overline{\alpha}x$ for every $x \in L_0$ and $\alpha \in \mathcal{O}_{\boldsymbol{k}}$.

There is a fractional $\mathcal{O}_{\boldsymbol{k}}$-ideal $\mathfrak{a}$ and an isomorphism $L_0 \xrightarrow{\simeq} \mathfrak{a}$ of left $\mathcal{O}_{\boldsymbol{k}}$-modules identifying the quadratic form on L_0 with the form $-\mathrm{N}(\cdot)/\mathrm{N}(\mathfrak{a})$ on $\mathfrak{a}$, and an easy exercise shows that

$$L_0^\vee = \mathfrak{d}_{\boldsymbol{k}}^{-1} L_0, \tag{4.2}$$

where $\mathfrak{d}_{\boldsymbol{k}}$ is the different of $\boldsymbol{k}/\mathbb{Q}$.

4.2. The 0-dimensional Shimura variety.

Let T be the group scheme over $\mathbb{Z}$ with functor of points

$$T(R) = (C^+(L_0) \otimes_\mathbb{Z} R)^\times$$

for any $\mathbb{Z}$-algebra R. Thus $T_\mathbb{Q} = \mathrm{GSpin}(V_0)$ is a torus of rank 2 and can be identified with the maximal subgroup of G that acts trivially on the subspace $\Lambda_\mathbb{Q} \subset V$. Set $K_0 = T(\widehat{\mathbb{Z}})$.

Using our fixed embedding $\mathcal{O}_{\boldsymbol{k}} \subset \mathbb{C}$, left multiplication makes $V_{0,\mathbb{R}}$ into a complex vector space. This determines an orientation on $V_{0,\mathbb{R}}$, and the oriented negative plane $V_{0,\mathbb{R}} \subset V_\mathbb{R}$ determines a point $\boldsymbol{h}_0 \in \mathcal{D}$. The isomorphism of (2.1) then associates to $\boldsymbol{h}_0$ an isotropic line in $V_{0,\mathbb{C}}$; explicitly, this is the line on which the *right* action of $\mathcal{O}_{\boldsymbol{k}}$ agrees with the inclusion $\mathcal{O}_{\boldsymbol{k}} \subset \mathbb{C} = \mathrm{End}_\mathbb{C}(V_{0,\mathbb{C}})$.

The pair $(T_{\mathbb{Q}}, \boldsymbol{h}_0)$ is a Shimura datum with reflex field $\boldsymbol{k} \subset \mathbb{C}$, and the 0-dimensional complex orbifold

$$Y(\mathbb{C}) = T(\mathbb{Q})\backslash\{\boldsymbol{h}_0\} \times T(\mathbb{A}_f)/K_0$$

is the complex fiber of a $\boldsymbol{k}$-stack Y. As in §2.2, the representation of $T_{\mathbb{Q}}$ on V_0 and the lattice

$$L_0 \otimes \widehat{\mathbb{Z}} \subset V_0 \otimes \mathbb{A}_f$$

determine a variation of polarized $\mathbb{Z}$-Hodge structures $(\boldsymbol{V}_{0,\mathrm{Be}}, \mathrm{Fil}^{\bullet}\boldsymbol{V}_{0,\mathrm{dR},Y(\mathbb{C})})$ of type $(-1,1)$, $(1,-1)$ over $Y(\mathbb{C})$.

Similarly, the representation of $T_{\mathbb{Q}}$ on $C(V_0)$ by left multiplication, together with the lattice

$$C(L_0) \otimes \widehat{\mathbb{Z}} \subset C(V_0) \otimes \mathbb{A}_f,$$

determine a variation of $\mathbb{Z}$-Hodge structures $(\boldsymbol{H}_{0,\mathrm{Be}}, \mathrm{Fil}^{\bullet}\boldsymbol{H}_{0,\mathrm{dR},\mathcal{Y}(\mathbb{C})})$ of type $(-1,0)$, $(0,-1)$. As in §2.3, this is the homology of an abelian scheme

$$\mathcal{A}_{0,Y} \to Y$$

of relative dimension 2, equipped with a right $C(L_0)$-action and a compatible $\mathbb{Z}/2\mathbb{Z}$-grading

$$\mathcal{A}_{0,Y} = \mathcal{A}_{0,Y}^{+} \times \mathcal{A}_{0,Y}^{-}.$$

As before, we will refer to this abelian scheme as the *Kuga-Satake abelian scheme* over Y.

4.3. **The integral model.** We now define integral models of Y and $\mathcal{A}_{0,Y}$ over $\mathcal{O}_{\boldsymbol{k}}$.

Definition 4.3.1. If S is an $\mathcal{O}_{\boldsymbol{k}}$-scheme, an $\mathcal{O}_{\boldsymbol{k}}$*-elliptic curve* over S is an elliptic curve $\mathcal{E} \to S$ endowed with an action of $\mathcal{O}_{\boldsymbol{k}}$ in such a way that the induced action on the $\mathcal{O}_S$-module $\mathrm{Lie}(\mathcal{E})$ is through the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$.

Remark 4.3.2. Our convention is that the $\mathcal{O}_{\boldsymbol{k}}$-action on any $\mathcal{O}_{\boldsymbol{k}}$-elliptic curve is written on the *right*.

Denote by $\mathcal{Y}$ the $\mathcal{O}_{\boldsymbol{k}}$-stack classifying $\mathcal{O}_{\boldsymbol{k}}$-elliptic curves over $\mathcal{O}_{\boldsymbol{k}}$-schemes. Every point $y \in \mathcal{Y}(\mathbb{C})$ has automorphism group $\mathcal{O}_{\boldsymbol{k}}^{\times}$, and

$$\sum_{y \in \mathcal{Y}(\mathbb{C})} \frac{1}{\#\mathrm{Aut}(y)} = \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}.$$

As in [KRY99, Proposition 5.1], the stack $\mathcal{Y}$ is finite étale over $\mathcal{O}_{\boldsymbol{k}}$. In particular $\mathcal{Y}$ is regular of dimension 1, and is flat over $\mathcal{O}_{\boldsymbol{k}}$.

The positive graded part $\mathcal{A}_{0,Y}^{+}$ of the Kuga-Satake abelian scheme, with its right action of $C^{+}(L_0) = \mathcal{O}_{\boldsymbol{k}}$, is an $\mathcal{O}_{\boldsymbol{k}}$-elliptic curve over Y (for the obvious extension of this notion over algebraic stacks.) Therefore, it defines a morphism $Y \to \mathcal{Y}$. By checking on complex points, we see that this map defines an isomorphism

$$Y \xrightarrow{\simeq} \mathcal{Y}_{\boldsymbol{k}}.$$

Again by checking on complex points, one can show that the natural map

$$\mathcal{A}^+_{0,Y} \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L_0) \to \mathcal{A}_{0,Y}$$

is an isomorphism, from which it follows that

$$\mathcal{A}^-_{0,Y} \xrightarrow{\simeq} \mathcal{A}^+_{0,Y} \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0.$$

From these observations, we can use the universal $\mathcal{O}_{\boldsymbol{k}}$-elliptic curve $\mathcal{E} \to \mathcal{Y}$ to extend the Kuga-Satake abelian scheme $\mathcal{A}_{0,Y} \to Y$ to an abelian scheme

$$\mathcal{A}_0 = \mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L_0) \tag{4.3}$$

over $\mathcal{Y}$ of relative dimension 2. By construction, $\mathcal{A}_0$ carries a right action of $C(L_0)$ and a $\mathbb{Z}/2\mathbb{Z}$-grading

$$\mathcal{A}_0 = \mathcal{A}_0^+ \times \mathcal{A}_0^-$$

induced by (4.1), where $\mathcal{A}_0^+ = \mathcal{E}$ and $\mathcal{A}_0^- = \mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$.

Remark 4.3.3. Note that the tensor product $\mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$ is with respect to the *left* action of $\mathcal{O}_{\boldsymbol{k}}$ on L_0, and $\mathcal{O}_{\boldsymbol{k}} \subset C(L_0)$ acts on $\mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$ through *right* multiplication on L_0. From the relation $x\alpha = \overline{\alpha}x$ for $x \in L_0$ and $\alpha \in \mathcal{O}_{\boldsymbol{k}}$, it follows that $\mathcal{O}_{\boldsymbol{k}}$ acts on $\mathrm{Lie}(\mathcal{A}_0^-)$ through the *conjugate* of the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_{\mathcal{Y}}$.

There are obvious analogues on $\mathcal{Y}$ of the sheaves $\boldsymbol{H}$ defined in §2.4. Denote by $\boldsymbol{H}_{0,\mathrm{dR}}$ the first de Rham homology of $\mathcal{A}_0$ relative to $\mathcal{Y}$. Similarly, denote by $\boldsymbol{H}_{0,p,\mathrm{et}}$ the first p-adic étale homology of $\mathcal{A}_0$ relative to $\mathcal{Y}[1/p]$, and by $\boldsymbol{H}_{0,\mathrm{crys}}$ the first crystalline homology of the special fiber of $\mathcal{A}_0$ at a prime p relative to $\mathcal{Y}_{\mathbb{F}_p}$. All of these sheaves come with right actions of $C(L_0)$, and $\mathbb{Z}/2\mathbb{Z}$-gradings $\boldsymbol{H}_0 = \boldsymbol{H}_0^+ \oplus \boldsymbol{H}_0^-$ in such a way that $\boldsymbol{H}_0^- = \boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$. As in the higher dimensional case, over $\mathcal{Y}(\mathbb{C})$ the de Rham and étale sheaves arise from the Betti realization $\boldsymbol{H}_{0,\mathrm{Be}}$.

Define

$$\boldsymbol{V}_0 \subset \underline{\mathrm{End}}_{C(L_0)}(\boldsymbol{H}_0)$$

to be the submodule of grade shifting endomorphisms. As can be checked from the complex uniformization, over $\mathcal{Y}(\mathbb{C})$ this definition recovers the local system $\boldsymbol{V}_{0,\mathrm{Be}}$ (for the Betti realization) and the vector bundle $\boldsymbol{V}_{0,\mathrm{dR},Y(\mathbb{C})}$ (for the de Rham realization.) Indeed, this amounts to the fact that the embedding of L_0 into $\mathrm{End}(C(L_0))$ via its left multiplication action identifies it with the space of grade shifting endomorphisms of $C(L_0)$ that commute with right multiplication by $C(L_0)$.

Restriction to $\boldsymbol{H}_0^+$ defines an isomorphism

$$\boldsymbol{V}_0 \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\boldsymbol{H}_0^+, \boldsymbol{H}_0^-). \tag{4.4}$$

As $\boldsymbol{H}_0^{\pm}$ is a locally free $\mathcal{O}_{\boldsymbol{k}} \otimes \mathbf{1}$-module of rank one, the same is true of $\boldsymbol{V}_0$. In particular, $\boldsymbol{V}_0$ is a locally free $\mathbf{1}$-module of rank two. In the de Rham case, the Hodge filtration on $\boldsymbol{H}_{0,\mathrm{dR}}$ determines a submodule $\boldsymbol{V}^1_{0,\mathrm{dR}} \subset \boldsymbol{V}_{0,\mathrm{dR}}$ by

$$\boldsymbol{V}^1_{0,\mathrm{dR}} = \boldsymbol{V}_{0,\mathrm{dR}} \cap \mathrm{Fil}^1 \underline{\mathrm{End}}(\boldsymbol{H}_{0,\mathrm{dR}}).$$

Definition 4.3.4. As in Definition 2.4.6, the line bundle $\boldsymbol{V}^1_{0,\mathrm{dR}} \subset \boldsymbol{V}_{0,\mathrm{dR}}$ is the *tautological bundle* on $\mathcal{Y}$, and its dual is the *cotautological bundle.*

The following proposition makes the cotautological bundle more explicit.

Proposition 4.3.5. *There are canonical isomorphisms of $\mathcal{O}_{\mathcal{Y}}$-modules*

$$(\boldsymbol{V}^1_{0,\mathrm{dR}})^\vee \xrightarrow{\simeq} \mathrm{Lie}(\mathcal{A}_0^+) \otimes \mathrm{Lie}(\mathcal{A}_0^-) \xrightarrow{\simeq} \mathrm{Lie}(\mathcal{E})^{\otimes 2} \otimes \boldsymbol{L}_0, \tag{4.5}$$

where $\boldsymbol{L}_0 = \mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$.

Proof. We have canonical isomorphisms of line bundles over $\mathcal{Y}$

$$\boldsymbol{H}^+_{0,\mathrm{dR}}/\mathrm{Fil}^0 \boldsymbol{H}^+_{0,\mathrm{dR}} \xrightarrow{\simeq} \mathrm{Lie}(\mathcal{A}_0^+)$$
$$\boldsymbol{H}^-_{0,\mathrm{dR}}/\mathrm{Fil}^0 \boldsymbol{H}^-_{0,\mathrm{dR}} \xrightarrow{\simeq} \mathrm{Lie}(\mathcal{A}_0^-),$$

and the (de Rham realization of the) unique principal polarization on $\mathcal{A}_0^-$ induces an isomorphism of $\mathcal{O}_{\mathcal{Y}}$-modules

$$\mathrm{Fil}^0 \boldsymbol{H}^-_{0,\mathrm{dR}} \xrightarrow{\simeq} (\boldsymbol{H}^-_{0,\mathrm{dR}}/\mathrm{Fil}^0 \boldsymbol{H}^-_{0,\mathrm{dR}})^\vee = \mathrm{Lie}(\mathcal{A}_0^-)^\vee.$$

The isomorphism (4.4) restricts to

$$\begin{aligned} \boldsymbol{V}^1_{0,\mathrm{dR}} &\xrightarrow{\simeq} \mathrm{Fil}^1 \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\boldsymbol{H}^+_{0,\mathrm{dR}}, \boldsymbol{H}^-_{0,\mathrm{dR}}) \\ &\xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\boldsymbol{H}^+_{0,\mathrm{dR}}/\mathrm{Fil}^0 \boldsymbol{H}^+_{0,\mathrm{dR}}, \mathrm{Fil}^0 \boldsymbol{H}^-_{0,\mathrm{dR}}) \\ &\xrightarrow{\simeq} \underline{\mathrm{Hom}}(\mathrm{Lie}(\mathcal{A}_0^+), \mathrm{Lie}(\mathcal{A}_0^-)^\vee), \end{aligned} \tag{4.6}$$

and dualizing yields the first isomorphism in (4.5). The second isomorphism is clear from the definition of $\mathcal{A}_0^\pm$. ☐

Remark 4.3.6. It is clear from (4.6) that the isotropic line $\boldsymbol{V}^1_{0,\mathrm{dR}} \subset \boldsymbol{V}_{0,\mathrm{dR}}$ is stable under the action of $\mathcal{O}_{\boldsymbol{k}}$ induced by (4.4), and that $\mathcal{O}_{\boldsymbol{k}}$ acts on this line through the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_{\mathcal{Y}}$.

4.4. **Special endomorphisms.** If $\mathcal{E}_1$ and $\mathcal{E}_2$ are any elliptic curves with right actions of $\mathcal{O}_{\boldsymbol{k}}$, we make $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ into a right $\mathcal{O}_{\boldsymbol{k}}$-module via

$$(f \cdot \alpha)(-) = f(-) \cdot \alpha$$

for all $f \in \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ and $\alpha \in \mathcal{O}_{\boldsymbol{k}}$.

With this convention, for any $\mathcal{Y}$-scheme S there is an isomorphism of right $\mathcal{O}_{\boldsymbol{k}}$-modules

$$\mathrm{Hom}(\mathcal{E}_S, \mathcal{E}_S \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0) \xrightarrow{\simeq} \mathrm{End}(\mathcal{E}_S) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0.$$

This restricts to an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathcal{E}_S, \mathcal{E}_S \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0) \xrightarrow{\simeq} \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}_S) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0,$$

which we rewrite as

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathcal{A}_{0,S}^{+}, \mathcal{A}_{0,S}^{-}) \xrightarrow{\simeq} \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}_S) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0. \tag{4.7}$$

Here the subscripts $\mathcal{O}_{\boldsymbol{k}}$ and $\overline{\mathcal{O}}_{\boldsymbol{k}}$ indicate $\mathcal{O}_{\boldsymbol{k}}$-linear and $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linear maps, respectively. The quadratic form deg on the left hand side of (4.7) corresponds to the quadratic form

$$x \otimes v \mapsto -\deg(x) \cdot Q(v) \tag{4.8}$$

on the right hand side (which is positive definite, as deg and $-Q$ are positive definite.)

Elements of $\mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}_S) \otimes_{\mathcal{O}_{\boldsymbol{k}}} V_0$ are *special quasi-endomorphisms* of $\mathcal{E}_S$. For every $\mu \in L_0^\vee$ define a collection of special quasi-endomorphisms by

$$V_\mu(\mathcal{E}_S) = \left\{ x \in \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}_S) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0^\vee : x - \mathrm{id} \otimes \mu \in \mathrm{End}(\mathcal{E}_S) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 \right\}.$$

This collection depends only on the coset $\mu \in L_0^\vee / L_0$. We will sometimes abbreviate

$$V(\mathcal{E}_S) = \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}_S) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$$

for the space defined by the coset $\mu = 0$.

It will be useful in §4.6 to have an alternate description of the spaces $V_\mu(\mathcal{E}_S)$ in terms of the Kuga-Satake abelian scheme (4.3), analogous to the definitions in §2.6. This requires a lemma (compare with Corollary 2.5.2).

Lemma 4.4.1. *For the each of the Betti, étale, and crystalline realizations* $\boldsymbol{V}_0$*, there is a canonical isomorphism*

$$\alpha_{L_0} : \boldsymbol{V}_0^\vee / \boldsymbol{V}_0 \xrightarrow{\simeq} (L_0^\vee / L_0) \otimes_{\mathbb{Z}} \mathbf{1}. \tag{4.9}$$

Proof. Recalling that $\boldsymbol{V}_0 \subset \underline{\mathrm{End}}(\boldsymbol{H}_0)$, the symmetric bilinear pairing $[x, y] = x \circ y + y \circ x$ on $\boldsymbol{V}_0$, valued in the subsheaf $\mathbf{1} \subset \underline{\mathrm{End}}(\boldsymbol{H}_0)$, defines a map $\boldsymbol{V}_0 \to \boldsymbol{V}_0^\vee$. Under the identification (4.4) this map becomes

$$\begin{aligned} \underline{\mathrm{End}}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 &\to \underline{\mathrm{End}}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0^\vee \\ f \otimes v &\mapsto -f^\vee \otimes [v, \cdot], \end{aligned}$$

where $f^\vee$ is the dual of f with respect to the canonical polarization pairing on $\boldsymbol{H}_0^+$. For the Betti, étale, or crystalline realizations, the cokernel of this map is identified with $\underline{\mathrm{End}}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} (L_0^\vee / L_0)$, and hence

$$\underline{\mathrm{End}}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} (L_0^\vee / L_0) \xrightarrow{\simeq} \boldsymbol{V}_0^\vee / \boldsymbol{V}_0. \tag{4.10}$$

On the other hand, the map

$$L_0^\vee \otimes \mathbf{1} \to \underline{\mathrm{End}}_{\mathcal{O}_{\boldsymbol{k}}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0^\vee$$

defined by $\mu \otimes 1 \mapsto \mathrm{id} \otimes \mu$ determines an isomorphism

$$(L_0^\vee / L_0) \otimes \mathbf{1} \xrightarrow{\simeq} \underline{\mathrm{End}}_{\mathcal{O}_{\boldsymbol{k}}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} (L_0^\vee / L_0). \tag{4.11}$$

Using $\mathfrak{d}_{\boldsymbol{k}} L_0^\vee = L_0$ and the relation $\alpha - \overline{\alpha} \in \mathfrak{d}_{\boldsymbol{k}}$ for every $\alpha \in \mathcal{O}_{\boldsymbol{k}}$, we have an equality

$$\underline{\mathrm{End}}_{\mathcal{O}_{\boldsymbol{k}}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} (L_0^\vee/L_0) = \underline{\mathrm{End}}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} (L_0^\vee/L_0) \tag{4.12}$$

of submodules of $\underline{\mathrm{End}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} (L_0^\vee/L_0)$. The desired isomorphism is the composition of (4.10), (4.11), and (4.12). □

For any $S \to \mathcal{Y}$ as above, let

$$V(\mathcal{A}_{0,S}) \subset \mathrm{End}(\mathcal{A}_{0,S})$$

be the submodule of $C(L_0)$-linear grade-shifting endomorphisms of $\mathcal{A}_{0,S}$. Equivalently, $x \in \mathrm{End}(\mathcal{A}_{0,S})$ lies in $V(\mathcal{A}_{0,S})$ if and only if all of its homological realizations lie in the subsheaf

$$\boldsymbol{V}_0 \subset \underline{\mathrm{End}}(\boldsymbol{H}_0).$$

As in §2.6, for each $\mu \in L_0^\vee/L_0$ define

$$V_\mu(\mathcal{A}_{0,S}) \subset V(\mathcal{A}_{0,S}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

as the subset of all x for which every homological realization of x lies in the subsheaf $\boldsymbol{V}_0^\vee \subset \boldsymbol{V}_0 \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\alpha_{L_0}(x) = \mu \otimes 1$. When $\mu = 0$ this is the space $V(\mathcal{A}_{0,S})$ defined above.

In particular, restriction to the positive graded part identifies

$$V(\mathcal{A}_{0,S}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathcal{A}_{0,S}^+, \mathcal{A}_{0,S}^-) \xrightarrow{\cong} \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}_S) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 \xrightarrow{\cong} V(\mathcal{E}_S),$$

where the middle isomorphism is (4.7). Tracing through the definitions, it is not hard to see that if one tensors both sides of this isomorphism with $\mathbb{Q}$, the resulting isomorphism identifies

$$V_\mu(\mathcal{A}_{0,S}) = V_\mu(\mathcal{E}_S). \tag{4.13}$$

4.5. **Degrees of special divisors.** In this subsection we define the special divisors

$$\mathcal{Z}_0(m,\mu) \to \mathcal{Y}$$

and compute their degrees.

For a non-negative rational number m and a class $\mu \in L_0^\vee/L_0$, define $\mathcal{Z}_0(m,\mu)$ to be the $\mathcal{O}_{\boldsymbol{k}}$-stack that assigns to an $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of pairs $(\mathcal{E}_S, x)$ in which $\mathcal{E}_S$ is an $\mathcal{O}_{\boldsymbol{k}}$-elliptic curve over S, and $x \in V_\mu(\mathcal{E}_S)$ satisfies $Q(x) = m$. Note that for $m = 0$ we have

$$\mathcal{Z}_0(0,\mu) = \begin{cases} \emptyset & \text{if } \mu \neq 0 \\ \mathcal{Y} & \text{if } \mu = 0. \end{cases}$$

The following theorem is due to Kudla-Rapoport-Yang [KRY99] in the special case where $\boldsymbol{k}$ has prime discriminant and $\mu = 0$. Subsequent generalizations and variants can be found in [KY13], [BHY15], and [How12].

Theorem 4.5.1. *Assume that $d_{\boldsymbol{k}}$ is odd, and recall the finite set of primes $\mathrm{Diff}_{L_0}(m)$ of (3.4). For $m > 0$ the stack $\mathcal{Z}_0(m,\mu)$ has the following properties:*

(i) If $|\mathrm{Diff}_{L_0}(m)| > 1$, or if $Q(\mu) \neq m$ in $\mathbb{Q}/\mathbb{Z}$, then $\mathcal{Z}_0(m,\mu) = \emptyset$.

(ii) If $\mathrm{Diff}_{L_0}(m) = \{p\}$ and $Q(\mu) = m$ in $\mathbb{Q}/\mathbb{Z}$, then $\mathcal{Z}_0(m,\mu)$ has dimension 0 and is supported in characteristic p. Furthermore,

$$\frac{\log(\mathrm{N}(\mathfrak{p}))}{\log(p)} \sum_{z \in \mathcal{Z}_0(m,\mu)(\overline{\mathbb{F}}_{\mathfrak{p}})} \frac{\mathrm{length}(\mathcal{O}_{\mathcal{Z}_0(m,\mu),z})}{\#\mathrm{Aut}(z)} = 2^{s(\mu)-1} \cdot \mathrm{ord}_p(pm) \cdot \rho\left(\frac{m|d_{\boldsymbol{k}}|}{p^{\epsilon}}\right),$$

where $\mathfrak{p}$ is the unique prime of $\boldsymbol{k}$ above p, and ϵ, ρ, and $s(\mu)$ are as in Proposition 3.3.1.

In either case

$$\sum_{\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}} \log(\mathrm{N}(\mathfrak{p})) \sum_{z \in \mathcal{Z}_0(m,\mu)(\overline{\mathbb{F}}_{\mathfrak{p}})} \frac{\mathrm{length}(\mathcal{O}_{\mathcal{Z}_0(m,\mu),z})}{\#\mathrm{Aut}(z)} = -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \cdot a_{L_0}(m,\mu).$$

Proof. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_{\boldsymbol{k}}$, and let p be the rational prime below it. Suppose we have a point

$$z = (\mathcal{E}, x) \in \mathcal{Z}_0(m,\mu)(\overline{\mathbb{F}}_{\mathfrak{p}}). \tag{4.14}$$

The existence of $x \in \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$ already implies that $\mathcal{E}$ is supersingular, p is nonsplit in $\boldsymbol{k}$, and $\mathrm{End}(\mathcal{E}) \otimes \mathbb{Q}$ is the quaternion division algebra ramified at ∞ and p.

Given any $\eta \in \mathbb{Q}^{\times}$ let $W_\eta = \boldsymbol{k}$ with the quadratic form $\eta\alpha\overline{\alpha}$. An exercise with quaternion algebras shows that the quadratic form deg on $\mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E})_{\mathbb{Q}}$ represents 1 everywhere locally, except at ∞ and p, and so

$$\mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E})_{\mathbb{Q}} \xrightarrow{\cong} W_\eta,$$

where η is chosen to be a norm from $\boldsymbol{k}$ everywhere locally except at ∞ and p. Recalling that $V_0 = L_0 \otimes_{\mathbb{Z}} \mathbb{Q}$, any choice of left $\boldsymbol{k}$-module generator $v \in V_0$ identifies $V_0 \xrightarrow{\cong} W_\gamma$, where $\gamma = Q(v)$.

The existence of x shows that the quadratic space

$$\mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}) \otimes_{\mathcal{O}_{\boldsymbol{k}}} V_0 \xrightarrow{\cong} W_\eta \otimes_{\boldsymbol{k}} W_\gamma \xrightarrow{\cong} W_{\eta\gamma} \tag{4.15}$$

represents m globally, and hence $\eta\gamma/m$ is a norm from $\boldsymbol{k}$. This implies that γ/m is a norm from $\boldsymbol{k}$ everywhere locally except at ∞ and p, from which we deduce $\mathrm{Diff}_{L_0}(m) = \{p\}$. The equality $m = Q(\mu)$ in $\mathbb{Q}/\mathbb{Z}$ can be checked directly, but it also follows from (2.19) and Proposition 4.6.3.

We have now proved that $\mathcal{Z}_0(m,\mu)(\overline{\mathbb{F}}_{\mathfrak{p}}) \neq \emptyset$ only when $\mathrm{Diff}_{L_0}(m) = \{p\}$ and $m = Q(\mu)$ in $\mathbb{Q}/\mathbb{Z}$, and hence have proved (i). The proof of (ii) requires a lemma.

Lemma 4.5.2. *The existence of a point (4.14) implies $\mu \in L_{0,p}$.*

Proof. By (4.2) we may assume that p is ramified in $\boldsymbol{k}$. Fix a generator $\delta_{\boldsymbol{k}} \in \mathfrak{d}_{\boldsymbol{k}}$, so that $L_0^\vee = \delta_{\boldsymbol{k}}^{-1} L_0$. Let E_p be the p-divisible group of $\mathcal{E}$ with its induced $\mathcal{O}_{\boldsymbol{k}}$-action.

If $\mu \notin L_{0,p}$ then $\mu\delta_{\boldsymbol{k}}$ generates $L_{0,p}$ as an $\mathcal{O}_{\boldsymbol{k},p}$-module. If we use this generator to identify

$$\mathrm{End}(E_p) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 \xrightarrow{\cong} \mathrm{End}(E_p)$$

and multiply both sides of the relation

$$x - \mathrm{id} \otimes \mu \in \mathrm{End}(E_p) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$$

on the right by $\delta_{\boldsymbol{k}}$, we obtain

$$y - \mathrm{id} \in \mathrm{End}(E_p) \cdot \delta_{\boldsymbol{k}}, \tag{4.16}$$

where we have set $y = x\delta_{\boldsymbol{k}} \in \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(E_p)$.

The ring $\mathrm{End}(E_p)$ is the maximal order in the ramified quaternion algebra over $\mathbb{Q}_p$, and this maximal order is a (noncommutative) discrete valuation ring with $\mathrm{End}(E_p) \cdot \delta_{\boldsymbol{k}}$ as its unique maximal ideal. Using $yv = \overline{v}y$ for all $v \in \mathcal{O}_{\boldsymbol{k},p}$, an exercise with quaternion algebras shows that y has reduced trace 0. Taking the reduced trace of (4.16) now implies that $-2 \cdot \mathrm{id}$ lies in the maximal ideal of $\mathrm{End}(E_p)$, which is absurd, as our assumption that $d_{\boldsymbol{k}}$ is odd implies $p > 2$. □

It follows from Lemma 4.5.2 and the Serre-Tate theorem that we may completely ignore μ in the deformation problem that computes the length of the étale local ring at z. Gross's results on canonical liftings [Gro86] therefore imply

$$\frac{\log(\mathrm{N}(\mathfrak{p}))}{\log(p)} \cdot \mathrm{length}(\mathcal{O}_{\mathcal{Z}_0(m,\mu),z}) = \mathrm{ord}_p(pm)$$

for every point (4.14).

If $\mathrm{ord}_p(m) < 0$ then the relation $m = Q(\mu)$ in $\mathbb{Q}/\mathbb{Z}$ implies $\mu \notin L_{0,p}$, and hence, using Lemma 4.5.2, both sides of the desired equality in (ii) are equal to 0. Thus it only remains to prove

$$\sum_{z \in \mathcal{Z}_0(m,\mu)(\overline{\mathbb{F}}_{\mathfrak{p}})} \frac{1}{\#\mathrm{Aut}(z)} = 2^{s(\mu)-1} \cdot \rho\left(\frac{m|d_{\boldsymbol{k}}|}{p^{\epsilon}}\right) \tag{4.17}$$

under the assumption $\mathrm{ord}_p(m) \geq 0$. The argument is very similar to that of [KY13, Proposition 3.1], [HY12, Proposition 2.18], [BHY15, Theorem 5.5], and [How12, Theorem 3.5.3], and hence we omit some details.

Fix one elliptic curve $\mathcal{E}$ over $\overline{\mathbb{F}}_{\mathfrak{p}}$ with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$. As the ideal class group of $\boldsymbol{k}$ acts simply transitively on the elliptic curves with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$, we find

$$\sum_{z \in \mathcal{Z}_0(m,\mu)(\overline{\mathbb{F}}_{\mathfrak{p}})} \frac{1}{\#\mathrm{Aut}(z)} = \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{\boldsymbol{k}})} \sum_{\substack{x \in V_\mu(\mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{a}) \\ \deg(x) = m}} \frac{1}{w_{\boldsymbol{k}}}.$$

As in the argument surrounding (4.15), the assumption $\mathrm{Diff}_{L_0}(m) = \{p\}$ implies that there is some vector

$$x \in \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}) \otimes_{\mathcal{O}_{\boldsymbol{k}}} V_0$$

with $Q(x) = m$, and the group $\boldsymbol{k}^1 = \{s \in \boldsymbol{k}^\times : s\bar{s} = 1\}$ acts transitively (by right multiplication on V_0) on the set of all such x. Using $t \mapsto t\bar{t}^{-1}$ to identify $\mathbb{Q}^\times \backslash \boldsymbol{k}^\times \xrightarrow{\simeq} \boldsymbol{k}^1$ and arguing as in the proof of [HY12, Proposition 2.18], we obtain a factorization into *orbital integrals*

$$\begin{aligned} \sum_{z \in \mathcal{Z}_0(m,\mu)(\overline{\mathbb{F}}_\mathfrak{p})} \frac{1}{\#\mathrm{Aut}(z)} &= \frac{1}{w_{\boldsymbol{k}}} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{\boldsymbol{k}})} \sum_{t \in \mathbb{Q}^\times \backslash \boldsymbol{k}^\times} \mathbf{1}_{V_\mu(\mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{a})}(xt\bar{t}^{-1}) \\ &= \frac{1}{2} \prod_\ell \sum_{t \in \mathbb{Q}_\ell^\times \backslash \boldsymbol{k}_\ell^\times / \mathcal{O}_{\boldsymbol{k},\ell}^\times} \mathbf{1}_{V_\mu(E_\ell)} \left(xt\bar{t}^{-1} \right). \end{aligned} \tag{4.18}$$

Here $\mathbf{1}$ indicates characteristic function, E_ℓ is the ℓ-divisible group of $\mathcal{E}$, and

$$V_\mu(E_\ell) = \{y \in \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(E_\ell) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0^\vee : y - \mathrm{id} \otimes \mu \in \mathrm{End}(E_\ell) \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0\}.$$

On the right hand side of the first equality of (4.18) we are viewing

$$xt\bar{t}^{-1} \in \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E}) \otimes_{\mathcal{O}_{\boldsymbol{k}}} V_0 \xrightarrow{\simeq} \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(\mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{a}) \otimes_{\mathcal{O}_{\boldsymbol{k}}} V_0$$

using the quasi-isogeny from $\mathcal{E}$ to $\mathcal{E} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{a}$ defined by $P \mapsto P \otimes 1$.

A choice of left $\mathcal{O}_{\boldsymbol{k},\ell}$-module isomorphism $L_{0,\ell} \xrightarrow{\simeq} \mathcal{O}_{\boldsymbol{k},\ell}$ identifies

$$V_\mu(E_\ell) = \{y \in \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(E_\ell) \cdot \mathfrak{d}_{\boldsymbol{k}}^{-1} : y - \mu \in \mathrm{End}(E_\ell)\},$$

where $\mu \in L_{0,\ell}^\vee = \mathfrak{d}_{\boldsymbol{k},\ell}^{-1}$ is viewed as a quasi-endomorphism of E_ℓ using the complex multiplication. From now on we use this identification to view $x \in \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(E_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $\mu \in \boldsymbol{k}_\ell$. Using (4.8), the condition $m = Q(\mu)$ in $\mathbb{Q}/\mathbb{Z}$ translates to $\mathrm{Nrd}(x) = -\mathrm{Nrd}(\mu)$ in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, where Nrd is the reduced norm on the quaternion algebra

$$\mathrm{End}(E_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = \boldsymbol{k}_\ell \oplus \left(\mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(E_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \right).$$

An explicit description (in terms of this decomposition) of the maximal order $\mathrm{End}(E_\ell)$ can be found in [KY13, Lemma 3.3].

Using the explicit description of $\mathrm{End}(E_\ell)$, the local factors on the right hand side of (4.18) can be computed directly. We demonstrate the method for a prime ℓ ramified in $\boldsymbol{k}$. For such an ℓ we have

$$\mathbb{Q}_\ell^\times \backslash \boldsymbol{k}_\ell^\times / \mathcal{O}_{\boldsymbol{k},\ell}^\times = \{1, t\}$$

where $t \in \mathcal{O}_{\boldsymbol{k},\ell}$ is any uniformizer. As $\ell > 2$, we may choose t so that $\bar{t} = -t$, and so

$$\sum_{t \in \mathbb{Q}_\ell^\times \backslash \boldsymbol{k}_\ell^\times / \mathcal{O}_{\boldsymbol{k},\ell}^\times} \mathbf{1}_{V_\mu(E_\ell)} \left(xt\bar{t}^{-1} \right) = \mathbf{1}_{V_\mu(E_\ell)}(x) + \mathbf{1}_{V_\mu(E_\ell)}(-x).$$

We now consider the two sub-cases $\ell = p$ and $\ell \neq p$.

If $\ell = p$, our assumption $\mathrm{ord}_p(m) \geq 0$ and the relation $m = Q(\mu)$ in $\mathbb{Q}/\mathbb{Z}$ imply that $\mu \in \mathcal{O}_{\boldsymbol{k},p}$, and that $\mathrm{Nrd}(x) \in \mathbb{Z}_p$. As $\mathrm{End}(E_p)$ is the maximal order in the ramified quaternion algebra over $\mathbb{Q}_p$, this implies that $x \in \mathrm{End}(E_p)$. From this it follows that both x and $-x$ lie in $V_\mu(E_p)$, and so

$$\sum_{t \in \mathbb{Q}_\ell^\times \backslash \boldsymbol{k}_\ell^\times / \mathcal{O}_{\boldsymbol{k},\ell}^\times} \mathbf{1}_{V_\mu(E_\ell)}\left(xt\bar{t}^{-1}\right) = 2.$$

Now suppose $\ell \neq p$. If $\mu \in \mathcal{O}_{\boldsymbol{k},\ell}$ then $\mathrm{Nrd}(\mu) \in \mathbb{Z}_\ell$, and hence $\mathrm{Nrd}(x) \in \mathbb{Z}_\ell$. The description of $\mathrm{End}(E_\ell)$ in [KY13, Lemma 3.3] now implies that $x \in \mathrm{End}(E_\ell)$, and again both x and $-x$ lie in $V_\mu(E_p)$. If $\mu \notin \mathcal{O}_{\boldsymbol{k},\ell}$ then $\ell \cdot \mathrm{Nrd}(\mu) \in \mathbb{Z}_\ell^\times$, and hence the relation $\mathrm{Nrd}(x) + \mathrm{Nrd}(\mu) \in \mathbb{Z}_\ell$ noted above implies $\ell \cdot \mathrm{Nrd}(x) \in \mathbb{Z}_\ell^\times$. Using the same relation, and the description of $\mathrm{End}(E_\ell)$ in [KY13, Lemma 3.3], one can show that exactly one of $x \pm \mu$ lies in $\mathrm{End}(E_\ell)$. Thus exactly one of x and $-x$ lies in $V_\mu(E_\ell)$, and we have proved

$$\sum_{t \in \mathbb{Q}_\ell^\times \backslash \boldsymbol{k}_\ell^\times / \mathcal{O}_{\boldsymbol{k},\ell}^\times} \mathbf{1}_{V_\mu(E_\ell)}\left(xt\bar{t}^{-1}\right) = \begin{cases} 1 & \text{if } \mu \notin L_{0,\ell} \\ 2 & \text{if } \mu \in L_{0,\ell}. \end{cases}$$

From similar calculations at the unramified primes, one can deduce the relation

$$\prod_\ell \sum_{t \in \mathbb{Q}_\ell^\times \backslash \boldsymbol{k}_\ell^\times / \mathcal{O}_{\boldsymbol{k},\ell}^\times} \mathbf{1}_{V_\mu(E_\ell)}(xt\bar{t}^{-1}) = 2^{s(\mu)} \cdot \rho\left(\frac{m|d_{\boldsymbol{k}}|}{p^\epsilon}\right)$$

(the right hand side has an obvious product factorization, and one verifies equality prime-by-prime.) The equality (4.17) follows immediately.

The final claim of the theorem follows by comparing the above formulas with Proposition 3.3.1. □

4.6. **Mapping to the orthogonal Shimura variety.** Recall from §2.3 the GSpin Shimura variety M over $\mathbb{Q}$ defined by the quadratic space L. In §2.4 we defined the integral model $\mathcal{M}$ over $\mathbb{Z}[1/2]$, and the Kuga-Satake abelian scheme $\mathcal{A} \to \mathcal{M}$. The integral model $\mathcal{M}$ carries over it the collection of sheaves

$$\boldsymbol{V} \subset \underline{\mathrm{End}}(\boldsymbol{H})$$

of (2.10).

The inclusion $L_0 \to L$ extends to a homomorphism $C(V_0) \to C(V)$ of Clifford algebras, which then induces a morphism $\mathrm{GSpin}(V_0) \to \mathrm{GSpin}(V)$. The theory of canonical models shows that the induced map of complex orbifolds $Y(\mathbb{C}) \to M(\mathbb{C})$ descends to a finite and unramified morphism of $\boldsymbol{k}$-stacks

$$Y \to M_{\boldsymbol{k}} \tag{4.19}$$

In this section we will extend this morphism to integral models, and explain the how the various structures (special cycles, sheaves, and Kuga-Satake abelian schemes) on the source and target are related.

As noted earlier, $\mathcal{Y}$ is finite étale over $\mathcal{O}_{\boldsymbol{k}}$. The proof of Proposition 4.6.1 below will require a slightly different perspective on this, couched in the language of local models.

Let $\mathrm{M}^{\mathrm{loc}}(L_0)$ be the quadric over $\mathbb{Z}$ that parameterizes isotropic lines in L_0. There is an isomorphism of $\mathbb{Z}$-schemes

$$\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}) \xrightarrow{\simeq} \mathrm{M}^{\mathrm{loc}}(L_0) \tag{4.20}$$

determined as follows. Consider the map

$$L_0 \otimes_{\mathbb{Z}} \mathcal{O}_{\boldsymbol{k}} \to L_0$$

defined by $v \otimes \alpha \mapsto v\overline{\alpha}$. The kernel of this map is an isotropic rank one $\mathcal{O}_{\boldsymbol{k}}$-module corresponding to a map $\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}) \to \mathrm{M}^{\mathrm{loc}}(L_0)$, which is the desired isomorphism (4.20).

Now, consider the functor $\mathcal{P}_0$ on $\mathcal{Y}$-schemes parameterizing $\mathcal{O}_{\boldsymbol{k}}$-equivariant isomorphisms of rank two vector bundles

$$\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{Y}} \xrightarrow{\simeq} \boldsymbol{H}^{+}_{0,\mathrm{dR}}.$$

This is represented over $\mathcal{Y}$ by a T-torsor $\mathcal{P}_0 \to \mathcal{Y}$. Indeed, we only need to observe that, étale locally on $\mathcal{Y}$, $\boldsymbol{H}^{+}_{0,\mathrm{dR}}$ is a free $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{Y}}$-module of rank one.

Suppose that we are given a scheme $U \to \mathcal{Y}$ over $\mathcal{Y}$ and a section $U \to \mathcal{P}_0$, corresponding to an $\mathcal{O}_{\boldsymbol{k}}$-equivariant isomorphism

$$\xi_0 : \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_U \xrightarrow{\simeq} \boldsymbol{H}^{+}_{0,\mathrm{dR}}|_U.$$

This determines an isomorphism

$$L_0 \otimes_{\mathbb{Z}} \mathcal{O}_U \xrightarrow{\simeq} \boldsymbol{H}^{+}_{0,\mathrm{dR}}|_U \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 = \boldsymbol{H}^{-}_{0,\mathrm{dR}}|_U.$$

Combining the above isomorphisms with the canonical isomorphism

$$L_0 \otimes_{\mathbb{Z}} \mathcal{O}_U \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_U, L_0 \otimes_{\mathbb{Z}} \mathcal{O}_U)$$

yields an $\mathcal{O}_{\boldsymbol{k}}$-linear isometry

$$\xi_0 : L_0 \otimes_{\mathbb{Z}} \mathcal{O}_U \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}\big(\boldsymbol{H}^{+}_{0,\mathrm{dR}}|_U, \boldsymbol{H}^{-}_{0,\mathrm{dR}}|_U\big) = \boldsymbol{V}_{0,\mathrm{dR}}.$$

The preimage $\xi_0^{-1}\big(\boldsymbol{V}^{1}_{0,\mathrm{dR}}\big) \subset L_0 \otimes \mathcal{O}_U$ is an isotropic line, and so corresponds to a point in $\mathrm{M}^{\mathrm{loc}}(L_0)(U)$. We therefore obtain a map $U \to \mathrm{M}^{\mathrm{loc}}(L_0)$. It is not hard to see that the composition of this map with the isomorphism (4.20) is just the structure map $U \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$. The main point is that, as in Remark 4.3.6, $\mathcal{O}_{\boldsymbol{k}}$ acts on $\boldsymbol{V}^{1}_{0,\mathrm{dR}}$ through the structure map $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_{\mathcal{Y}}$, and on the quotient $\boldsymbol{V}_{0,\mathrm{dR}}/\boldsymbol{V}^{1}_{0,\mathrm{dR}}$ through the conjugate of the structure map. Hence the quotient $\boldsymbol{V}_{0,\mathrm{dR}}/\boldsymbol{V}^{1}_{0,\mathrm{dR}}$ is obtained from $\boldsymbol{V}_{0,\mathrm{dR}}$ via base-change along the augmentation map

$$\mathcal{O}_{\boldsymbol{k}} \otimes \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}}$$

defined by $\alpha \otimes f \mapsto \overline{\alpha} f$.

In particular, since $\mathcal{Y}$ is étale over $\mathcal{O}_{\boldsymbol{k}}$, if U is étale over $\mathcal{Y}$, then the map $U \to \mathrm{M}^{\mathrm{loc}}(L_0)$ will also be étale. This can also be shown directly using Grothendieck-Messing theory.

The following result is analogous to Proposition 2.5.1.

Proposition 4.6.1. *The morphism (4.19) extends uniquely to a finite and unramified morphism*

$$\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]} \to \mathcal{M}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}, \tag{4.21}$$

and there is a canonical $C(L)$-equivariant isomorphism

$$\mathcal{A}_0|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \otimes_{C(L_0)} C(L) \xrightarrow{\simeq} \mathcal{A}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \tag{4.22}$$

of $\mathbb{Z}/2\mathbb{Z}$-graded abelian schemes over $\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$. In particular,

$$\boldsymbol{H}_0|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \otimes_{C(L_0)} C(L) \xrightarrow{\simeq} \boldsymbol{H}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}. \tag{4.23}$$

Furthermore, set $\boldsymbol{\Lambda} = \Lambda \otimes \mathbf{1}$, and recall the space of special endomorphisms

$$V(\mathcal{A}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}) \subset \mathrm{End}_{C(L)}(\mathcal{A}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}})$$

of §2.6. There is a canonical embedding $\Lambda \subset V(\mathcal{A}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}})$ with the following properties:

(i) Its homological realization exhibits $\boldsymbol{\Lambda} \subset \boldsymbol{V}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}$ as a local direct summand.

(ii) The injection

$$\underline{\mathrm{End}}_{C(L_0)}(\boldsymbol{H}_0|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}) \to \underline{\mathrm{End}}_{C(L)}(\boldsymbol{H}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}})$$

induced by (4.23) identifies $\boldsymbol{V}_0|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}$ with the submodule of all elements of $\boldsymbol{V}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}$ anticommuting with all elements of $\boldsymbol{\Lambda}$. Furthermore

$$\boldsymbol{V}_0|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \subset \boldsymbol{V}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}$$

is locally a direct summand.

(iii) In the de Rham case, the inclusion

$$\boldsymbol{V}_{0,\mathrm{dR}}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \subset \boldsymbol{V}_{\mathrm{dR}}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}$$

identifies

$$\boldsymbol{V}^1_{0,\mathrm{dR}}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} = \boldsymbol{V}^1_{\mathrm{dR}}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}.$$

Proof. Just as in (2.12), there is a $C(L)$-equivariant isomorphism

$$\mathcal{A}_0|_Y \otimes_{C(L_0)} C(L) \xrightarrow{\simeq} \mathcal{A}|_Y \tag{4.24}$$

of $\mathbb{Z}/2\mathbb{Z}$-graded abelian schemes over Y.

Extending the map of (4.19) of $\boldsymbol{k}$-stacks to a map (4.21) of $\mathcal{O}_{\boldsymbol{k}}[1/2]$-stacks is equivalent to extending the map of $\mathbb{Q}$-stacks $Y \to M$ to a map of $\mathbb{Z}[1/2]$-stacks $\mathcal{Y}_{\mathbb{Z}[1/2]} \to \mathcal{M}$.

To do this, fix a prime $p > 2$, and assume first that $L_{\mathbb{Z}_{(p)}}$ is self-dual. As observed in §2.4, an appropriate choice of $\delta \in C^+(L) \cap C(V)^\times$ determines a degree d polarization of the Kuga-Satake abelian scheme $\mathcal{A}_M \to M$, inducing a finite unramified map

$$i_\delta : M \to \mathcal{X}^{\mathrm{Siegel}}_{2^{n+1},d,\mathbb{Q}}$$

to a moduli stack of polarized abelian varieties. The integral model $\mathcal{M}_{\mathbb{Z}_{(p)}}$ is then simply the normalization of $\mathcal{X}^{\mathrm{Siegel}}_{2^{n+1},d,\mathbb{Z}_{(p)}}$ in M.

The composition

$$Y \to M \to \mathcal{X}^{\mathrm{Siegel}}_{2^{n+1},d,\mathbb{Q}}$$

determines a polarized abelian scheme over Y, which is precisely (4.24). The polarized abelian scheme $\mathcal{A}_0|_{\mathcal{Y}_{\mathbb{Z}_{(p)}}} \otimes_{C(L_0)} C(L)$ defines an extension of (4.24) to $\mathcal{Y}_{\mathbb{Z}_{(p)}}$, which shows that the above composition extends to a morphism

$$\mathcal{Y}_{\mathbb{Z}_{(p)}} \to \mathcal{X}^{\mathrm{Siegel}}_{2^{n+1},d,\mathbb{Z}_{(p)}}.$$

Since $\mathcal{Y}$ is regular, and hence normal, this extension must lift to a map

$$\mathcal{Y}_{\mathbb{Z}_{(p)}} \to \mathcal{M}_{\mathbb{Z}_{(p)}},$$

and by construction (4.24) extends to an isomorphism

$$(\mathcal{A}_0 \otimes_{C(L_0)} C(L))|_{\mathcal{Y}_{\mathbb{Z}_{(p)}}} \xrightarrow{\simeq} \mathcal{A}|_{\mathcal{Y}_{\mathbb{Z}_{(p)}}}.$$

The construction of the embedding $\Lambda \hookrightarrow \big(\mathcal{A}|_{\mathcal{Y}_{\mathbb{Z}_{(p)}}}\big)$, as well as the proofs of assertions (i), (ii) and (iii) over $\mathcal{Y}_{\mathbb{Z}_{(p)}}$, now proceed exactly as in Proposition 2.5.1.

It remains to show (under our self-duality assumption on $L_{\mathbb{Z}_{(p)}}$) that the map $\mathcal{Y}_{\mathbb{Z}_{(p)}} \to \mathcal{M}_{\mathbb{Z}_{(p)}}$ is finite and unramified. For this, using Proposition 2.4.2 and the above discussion of local models, we see that the map is étale locally on the source isomorphic to an étale neighborhood of $\mathrm{M}^{\mathrm{loc}}(L_0) \to \mathrm{M}^{\mathrm{loc}}(L)$. This shows that $\mathcal{Y}_{\mathbb{Z}_{(p)}}$ is unramified over $\mathcal{M}_{\mathbb{Z}_{(p)}}$. That it is also finite is immediate from the fact that $\mathcal{Y}$ is finite over $\mathcal{O}_{\boldsymbol{k}}$.

It remains to deal with the primes p where $L_{\mathbb{Z}_{(p)}}$ is not self-dual. For this, embed L as an isometric direct summand of a maximal lattice $\widetilde{L}$ of signature $(\tilde{n}, 2)$ for which $\widetilde{L}_{\mathbb{Z}_{(p)}}$ is self-dual. Let $\widetilde{\mathcal{M}}$ be the regular integral model over $\mathbb{Z}[1/2]$ for the Shimura variety $\widetilde{M}$ associated with $\widetilde{L}$. As in (2.4), let $\check{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ be the normalization of $\widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$ in M.

From what we have shown above, the composition

$$Y \to M \to \widetilde{M}$$

extends to a map $\mathcal{Y}_{\mathbb{Z}_{(p)}} \to \widetilde{\mathcal{M}}_{\mathbb{Z}_{(p)}}$, which, since $\mathcal{Y}$ is a normal stack over $\mathbb{Z}[1/2]$, must lift to a finite map $\mathcal{Y}_{\mathbb{Z}_{(p)}} \to \check{\mathcal{M}}_{\mathbb{Z}_{(p)}}$. As in the proof of Proposition 2.5.1, we find that that this lift must map $\mathcal{Y}_{\mathbb{Z}_{(p)}}$ to $\mathcal{M}_{\mathbb{Z}_{(p)}}$ and must also be unramified, being, étale locally on the source, isomorphic to an étale neighborhood of

$$\mathrm{M}^{\mathrm{loc}}(L_0)_{\mathbb{Z}[1/2]} \to \mathrm{M}^{\mathrm{loc}}(L). \tag{4.25}$$

The key point again is that the radical of $L_{0,\mathbb{F}_p}$, if non-zero, defines an $\mathbb{F}_p$-valued point of the quadric $\mathrm{M}^{\mathrm{loc}}(L_0)$ associated with L_0. Therefore (4.25) must map into the regular locus of the target.

Finally, the extension (4.22) of the isomorphism (4.24) once again follows from the property of Néron models, and the remaining assertions follow from Proposition 2.5.1 and from what we have already shown in the self-dual case. ☐

Remark 4.6.2. The proof of Proposition 4.6.1 actually proves the stronger claim that the morphism

$$\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]} = \mathcal{Y}_{\mathbb{Z}[1/2]} \to \mathcal{M}_{\mathbb{Z}[1/2]}$$

is finite and unramified.

Now suppose that S is a scheme over $\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$. Then Proposition 4.6.1 gives us an isometric embedding $\Lambda \hookrightarrow V(\mathcal{A}_S)$. We obtain the following analogue of Proposition 2.6.2.

Proposition 4.6.3.

(i) There is a canonical isometry

$$V(\mathcal{E}_S) \xrightarrow{\simeq} \Lambda^{\perp} \subset V(\mathcal{A}_S).$$

(ii) The induced map $V(\mathcal{E}_S) \oplus \Lambda \hookrightarrow V(\mathcal{A}_S)$, tensored with $\mathbb{Q}$, restricts to an injection

$$V_{\mu_1}(\mathcal{E}_S) \times (\mu_2 + \Lambda) \hookrightarrow V_{\mu}(\mathcal{A}_S)$$

for every $\mu \in L^{\vee}/L$ and every

$$(\mu_1, \mu_2) \in \big(\mu + L\big)/\big(L_0 \oplus \Lambda\big) \subset (L_0^{\vee}/L_0) \oplus (\Lambda^{\vee}/\Lambda).$$

(iii) The above injections determine a decomposition

$$V_{\mu}(\mathcal{A}_S) = \coprod_{(\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)} V_{\mu_1}(\mathcal{E}_S) \times \big(\mu_2 + \Lambda\big).$$

Proof. Exactly as in the proof Proposition 2.6.2, Proposition 4.6.1 implies that there is a canonical isometry

$$V(\mathcal{A}_{0,S}) \xrightarrow{\simeq} \Lambda^{\perp} \subset V(\mathcal{A}_S).$$

Thus assertion (i) is clear from (4.13).

Given the definitions of the spaces involved in (ii) and (iii), to prove these assertions we have to compare the isomorphism (4.9) with the isomorphism constructed in the style of Corollary 2.5.2. More precisely, suppose that L is a unimodular lattice (we can reduce to this case in the usual way by replacing L by a larger lattice, if necessary.) Then, as in the proof of *loc. cit.*, we obtain from Proposition 4.6.1 canonical isomorphisms

$$(\boldsymbol{V}_0^{\vee}/\boldsymbol{V}_0)|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \xleftarrow{\simeq} \boldsymbol{V}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}}/\big(\boldsymbol{V}_0|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} + (\Lambda \otimes \mathbf{1})\big) \xrightarrow{\simeq} (\Lambda^{\vee}/\Lambda) \otimes \mathbf{1}.$$

Combining this with the canonical isomorphism $L_0^\vee/L_0 \xrightarrow{\simeq} \Lambda^\vee/\Lambda$, we obtain an isometry

$$\tilde{\alpha}_{L_0} : (\boldsymbol{V}_0^\vee/\boldsymbol{V}_0)|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \xrightarrow{\simeq} (L_0^\vee/L_0) \otimes \mathbf{1}.$$

To finish the proof of the proposition, we have to check that $\tilde{\alpha}_{L_0}$ agrees with the isomorphism α_{L_0} of (4.9).

For this, we first unravel $\tilde{\alpha}_{L_0}^{-1}$. It can be described as follows: Given an element $\mu \in L_0^\vee$, there exists $\lambda \in \Lambda^\vee$ such that $\mu+\lambda$ belongs to $L \subset L_0^\vee \oplus \Lambda^\vee$. Now, there is a natural embedding

$$\boldsymbol{V}|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \subset \boldsymbol{V}_0^\vee|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}} \oplus (\Lambda^\vee \otimes \mathbf{1}).$$

This is such that, after localizing if necessary, we can find a section $\tilde{f}$ of $\boldsymbol{V}_0^\vee$ so that $\tilde{f} + \lambda \otimes 1$ is a section of $\boldsymbol{V}$. Now, $\tilde{\alpha}_{L_0}^{-1}$ maps $\mu \otimes 1$ to the image of $\tilde{f}$ in $\boldsymbol{V}_0^\vee/\boldsymbol{V}_0$.

Now, let f_μ be a section of $\boldsymbol{V}_0^\vee$ such that $f_\mu - \mathrm{id} \otimes \mu$ is a section of $\underline{\mathrm{End}}(\boldsymbol{H}_0^+) \otimes_{\mathcal{O}_{\boldsymbol{k}}} \boldsymbol{L}_0$. Let $C(L_0^\vee \oplus \Lambda^\vee) \subset C(L)_{\mathbb{Q}}$ be the $C(L)$-submodule generated by $L_0^\vee \oplus \Lambda^\vee$. Since $\boldsymbol{H}|_{\mathcal{Y}}$ can be $C(L)$-equivariantly identified with $\boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L)$, we can view both f_μ and $\lambda \otimes 1$ as sections of

$$\underline{\mathrm{Hom}}_{C(L)}\big(\boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L), \boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L_0^\vee \oplus \Lambda^\vee)\big).$$

Explicitly, given a section h of $\boldsymbol{H}_0^+$, we have

$$\begin{aligned} f_\mu(h \otimes 1) = f_\mu(h) &\in \boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0^\vee \subset \boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L_0^\vee \oplus \Lambda^\vee); \\ (\lambda \otimes 1)(h \otimes 1) = h \otimes \lambda &\in \boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L_0^\vee \oplus \Lambda^\vee). \end{aligned}$$

Now, we have:

$$\begin{aligned} f_\mu(h \otimes 1) + (\lambda \otimes 1)(h \otimes 1) &= f_\mu(h) + h \otimes \lambda \\ &= (f_\mu(h) - h \otimes \mu) + h \otimes (\mu + \lambda). \end{aligned}$$

By the choice of f_μ, the first summand is a section of $\boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0$. Moreover, the second summand belongs to $\boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L)$. Therefore, the sum belongs to $\boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L)$.

The preceding paragraph shows that $f_\mu + \lambda \otimes 1$ is actually a section of the subsheaf

$$\underline{\mathrm{End}}_{C(L)}\big(\boldsymbol{H}_0^+ \otimes_{\mathcal{O}_{\boldsymbol{k}}} C(L)\big) = \underline{\mathrm{End}}_{C(L)}(\boldsymbol{H})|_{\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}},$$

and is therefore a section of $\boldsymbol{V}$. By comparing definitions, we see that this is exactly equivalent to showing $\alpha_{L_0}^{-1}(\mu \otimes 1) = \tilde{\alpha}_{L_0}^{-1}(\mu \otimes 1)$. □

We immediately obtain from Proposition 4.6.3 the following analogue of Proposition 2.7.2.

Proposition 4.6.4. *There is an isomorphism of $\mathcal{Y}$-stacks*

(4.26)
$$\mathcal{Z}(m,\mu) \times_{\mathcal{M}} \mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]} \xrightarrow{\simeq} \coprod_{\substack{m_1+m_2=m \\ (\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)}} \mathcal{Z}_0(m_1,\mu_1)_{\mathcal{O}_{\boldsymbol{k}}[1/2]} \times \Lambda_{m_2,\mu_2},$$

where

$$\Lambda_{m_2,\mu_2} = \{x \in \mu_2 + \Lambda : Q(x) = m_2\},$$

and $\mathcal{Z}_0(m_1,\mu_1)_{\mathcal{O}_{\boldsymbol{k}}[1/2]} \times \Lambda_{m_2,\mu_2}$ *denotes the disjoint union of* $\#\Lambda_{m_2,\mu_2}$ *copies of* $\mathcal{Z}_0(m_1,\mu_1)_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$.

Remark 4.6.5. The terms $\mathcal{Z}_0(m_1,\mu_1)_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$ in the decomposition (4.26) define 0-cycles on $\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$ when $m_1 \neq 0$. The remaining terms

$$\coprod_{(0,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)} \mathcal{Z}_0(0,0)_{\mathcal{O}_{\boldsymbol{k}}[1/2]} \times \Lambda_{m,\mu_2} = \coprod_{(0,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)} \mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]} \times \Lambda_{m,\mu_2}$$

are those with $(m_1,\mu_1) = (0,0)$, and account for the improper intersection between the cycles $\mathcal{Z}(m,\mu)_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$ and $\mathcal{Y}_{\mathcal{O}_{\boldsymbol{k}}[1/2]}$.

5. Degrees of metrized line bundles

As in previous sections, fix a maximal quadratic space L over $\mathbb{Z}$ of signature $(n,2)$, and a $\mathbb{Z}$-module direct summand $L_0 \subset L$ of signature $(0,2)$. Assume that the even Clifford algebra $C^+(L_0)$ is isomorphic to the maximal order in a quadratic imaginary field $\boldsymbol{k} \subset \mathbb{C}$. Set

$$\Lambda = \{x \in L : x \perp L_0\}.$$

Recall from §2.4 that we have associated with L an algebraic stack

$$\mathcal{M} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}[1/2]). \tag{5.1}$$

Similarly, in §4.3 we constructed from L_0 an algebraic stack

$$\mathcal{Y} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}[1/2]).$$

The functoriality results of Proposition 4.6.1 provide us with a finite and unramified morphism

$$i : \mathcal{Y} \to \mathcal{M}.$$

Of course the stack $\mathcal{M}$ was originally constructed over $\mathbb{Z}[1/2]$, and the stack $\mathcal{Y}$ over $\mathcal{O}_{\boldsymbol{k}}$. However, the restrictions of these stacks to the common base $\mathcal{O}_{\boldsymbol{k}}[1/2]$ are all that we will use in this section.

5.1. **Metrized line bundles.**

We will need some basic notions of arithmetic intersection theory on the stack $\mathcal{M}$. For more details see [GS90], [KRY04], [KRY06], or [Vis89]

Although the integral model over $\mathbb{Z}[1/2]$ described in §2.4 is regular, its base change (5.1) to $\mathcal{O}_{\boldsymbol{k}}[1/2]$ need not be. This is not a serious problem: the first part of the following proposition implies that $\mathcal{M}$ has a perfectly good theory of Cartier divisors. Whenever we speak of "divisors" on $\mathcal{M}$ we always mean "Cartier divisors."

Proposition 5.1.1. *The stack* $\mathcal{M}$ *is locally integral. If* D *denotes the greatest common divisor of* $\mathrm{disc}(L_0)$ *and* $\mathrm{disc}(L)$*, then the restriction of* $\mathcal{M}$ *to* $\mathcal{O}_{\boldsymbol{k}}[1/2D]$ *is regular.*

Proof. Using Proposition 2.4.2 and the definition of $\mathcal{M}$, to show that $\mathcal{M}$ is locally integral, it suffices to show that the regular locus of the quadric $\mathrm{M}^{\mathrm{loc}}(L)$ associated with the quadratic space L is integral after base change to $\mathcal{O}_{\boldsymbol{k}}$. In explicit terms, the complete local rings of this locus at its $\overline{\mathbb{F}}_p$-points are either formally smooth over $\mathbb{Z}_p$, or are isomorphic to one of the two following $W(\overline{\mathbb{F}}_p)$-algebras [MPar, (2.16)]:

$$W(\overline{\mathbb{F}}_p)\frac{[[u_1,\dots,u_{n+1}]]}{(\sum_i u_i^2+p)} \quad \text{or} \quad W(\overline{\mathbb{F}}_p)\frac{[[u_1,\dots,u_n,w]]}{(\sum_i u_i^2+p(w+1))}.$$

Since $n \geq 1$, it is easily checked that, in all three cases, tensoring with $\mathcal{O}_{\boldsymbol{k}}$ over $\mathbb{Z}$ gives us something locally integral.

The claim about regularity over $\mathcal{O}_{\boldsymbol{k}}[1/2D]$ follows from the smoothness of the integral model of §2.4 over $\mathbb{Z}[1/2\mathrm{disc}(L)]$ and the smoothness of $\mathcal{O}_{\boldsymbol{k}}[1/\mathrm{disc}(L_0)]$ over $\mathbb{Z}$. □

An *arithmetic divisor* on $\mathcal{M}$ is a pair $\widehat{\mathcal{Z}} = (\mathcal{Z}, \Phi)$ consisting of a Cartier divisor $\mathcal{Z}$ on $\mathcal{M}$ and a Green function Φ for $\mathcal{Z}(\mathbb{C})$ on $\mathcal{M}(\mathbb{C})$. Thus if $\Psi = 0$ is a local equation for $\mathcal{Z}(\mathbb{C})$, the function $\Phi + \log|\Psi|^2$, initially defined on the complement of $\mathcal{Z}(\mathbb{C})$, is required to extend smoothly across the singularity $\mathcal{Z}(\mathbb{C})$. A *principal arithmetic divisor* is an arithmetic divisor of the form

$$\widehat{\mathrm{div}}(\Psi) = (\mathrm{div}(\Psi), -\log|\Psi|^2)$$

for a rational function Ψ on $\mathcal{M}$. The group of all arithmetic divisors is denoted $\widehat{\mathrm{Div}}(\mathcal{M})$, and its quotient by the subgroup of principal arithmetic divisors is the *arithmetic Chow group* $\widehat{\mathrm{CH}}^1(\mathcal{M})$ of Gillet-Soulé.

A *metrized line bundle* on $\mathcal{M}$ is a line bundle endowed with a smoothly varying Hermitian metric on its complex points. The isomorphism classes of metrized line bundles form a group $\widehat{\mathrm{Pic}}(\mathcal{M})$ under tensor product.

There is an isomorphism

$$\widehat{\mathrm{CH}}^1(\mathcal{M}) \xrightarrow{\cong} \widehat{\mathrm{Pic}}(\mathcal{M})$$

defined as follows. Given an arithmetic divisor $(\mathcal{Z}, \Phi)$, the line bundle $\mathcal{L} = \mathcal{O}(\mathcal{Z})$ is endowed with a canonical rational section s with divisor $\mathcal{Z}$. This section is nothing more than the constant function 1 on $\mathcal{M}$, viewed as a section of $\mathcal{L}$. We endow $\mathcal{L}$ with the metric defined by $-\log||s||^2 = \Phi$. For the inverse construction, start with a metrized line bundle $\widehat{\mathcal{L}}$ on $\mathcal{M}$ and let s be any nonzero rational section. The associated arithmetic divisor, well-defined modulo principal arithmetic divisors, is

$$\widehat{\mathrm{div}}(s) = (\mathrm{div}(s), -\log||s||^2).$$

Of course there is a similar discussion with $\mathcal{M}$ replaced by $\mathcal{Y}$. As $\mathcal{Y}$ is smooth of relative dimension 0 over $\mathcal{O}_{\boldsymbol{k}}[1/2]$, all divisors on $\mathcal{Y}$ are supported in nonzero characteristics; thus a Green function for a divisor on $\mathcal{Y}$ is *any* complex-valued function on the 0-dimensional orbifold $\mathcal{Y}(\mathbb{C})$. In particular any arithmetic divisor $(\mathcal{Z},\Phi)$ decomposes as $(\mathcal{Z},0)+(0,\Phi)$, and $\mathcal{Z}$ can be further decomposed as the difference of two effective Cartier divisors.

To define the *arithmetic degree* (as in [KRY04] or [KRY06]) of an arithmetic divisor $\widehat{\mathcal{Z}}$ on $\mathcal{Y}$ we first assume that $\widehat{\mathcal{Z}}=(\mathcal{Z},0)$ with $\mathcal{Z}$ effective. Then

$$\widehat{\deg}(\widehat{\mathcal{Z}})=\sum_{\mathfrak{p}\subset\mathcal{O}_{\boldsymbol{k}}[1/2]}\log \mathrm{N}(\mathfrak{p})\sum_{z\in\mathcal{Z}(\overline{\mathbb{F}}_{\mathfrak{p}})}\frac{\mathrm{length}(\mathcal{O}_{\mathcal{Z},z})}{\#\mathrm{Aut}(z)}$$

where $\mathcal{O}_{\mathcal{Z},z}$ is the étale local ring of $\mathcal{Z}$ at z. If $\widehat{\mathcal{Z}}=(0,\Phi)$ is purely archmedean, then

$$\widehat{\deg}(\widehat{\mathcal{Z}})=\sum_{y\in\mathcal{Y}(\mathbb{C})}\frac{\Phi(y)}{\#\mathrm{Aut}(y)}.$$

The arithmetic degree extends linearly to all arithmetic divisors, and defines a homomorphism

$$\widehat{\deg}:\widehat{\mathrm{Pic}}(\mathcal{Y})\to\mathbb{R}/\mathbb{Q}\log(2).$$

We now define a homomorphism

$$[\cdot:\mathcal{Y}]:\widehat{\mathrm{Pic}}(\mathcal{M})\to\mathbb{R}/\mathbb{Q}\log(2),$$

the *arithmetic degree along* $\mathcal{Y}$, as the composition

$$\widehat{\mathrm{Pic}}(\mathcal{M})\xrightarrow{i^*}\widehat{\mathrm{Pic}}(\mathcal{Y})\xrightarrow{\widehat{\deg}}\mathbb{R}/\mathbb{Q}\log(2).$$

5.2. **Specialization to the normal bundle.** Fix a positive $m\in\mathbb{Q}$ and a $\mu\in L^\vee/L$, and denote by

$$\mathcal{Z}(m,\mu)\to\mathcal{M} \tag{5.2}$$

the stack obtained from (2.21) by base change from $\mathbb{Z}[1/2]$ to $\mathcal{O}_{\boldsymbol{k}}[1/2]$. The fact that (5.2) and $i:\mathcal{Y}\to\mathcal{M}$ are not closed immersions is a minor nuisance. The following definition is introduced to address technical difficulties arising from this defect.

Definition 5.2.1. A *sufficiently small étale open chart* of $\mathcal{M}$ is a scheme U together with an étale morphism $U\to\mathcal{M}$ such that

- (i) the natural map $\mathcal{Z}(m,\mu)_U\to U$ restricts to a closed immersion on every connected component $Z\subset\mathcal{Z}(m,\mu)_U$,
- (ii) the natural map $\mathcal{Y}_U\to U$ restricts to a closed immersion on every connected component $Y\subset\mathcal{Y}_U$.

Remark 5.2.2. The stack $\mathcal{M}$ admits a covering by sufficiently small étale open charts. This is a consequence of [Vis89, Lemma 1.19] and the fact that $\mathcal{Z}(m,\mu)\to\mathcal{M}$ and $\mathcal{Y}\to\mathcal{M}$ are finite and unramified. Note that these two morphisms are relatively representable, and so for any $U\to\mathcal{M}$ as above the pull-backs $\mathcal{Z}(m,\mu)_U$ and $\mathcal{Y}_U$ are actually schemes.

Remark 5.2.3. When we think of $\mathcal{Z}(m,\mu)$ as a Cartier divisor on $\mathcal{M}$, its pull-back to U is simply the sum of the connected components of $\mathcal{Z}(m,\mu)_U$, each viewed as closed subscheme of U.

Remark 5.2.4. In §4 the notation Y was used for the generic fiber of $\mathcal{Y}$. Throughout §5 the notation Y is only used for a connected component of $\mathcal{Y}_U$, and so no confusion should arise.

Fix a sufficiently small $U \to \mathcal{M}$ and a connected component $Y \subset \mathcal{Y}_U$. The smoothness of $\mathcal{Y}$ over $\mathcal{O}_{\boldsymbol{k}}$ implies that Y is a regular integral scheme of dimension one. Let $I \subset \mathcal{O}_U$ be the ideal sheaf defining the closed immersion $Y \to U$, and recall that the *normal bundle* of $Y \subset U$ is the $\mathcal{O}_Y$-module

$$N_Y U = \underline{\mathrm{Hom}}(I/I^2, \mathcal{O}_Y).$$

As $Y \to U$ may not be a regular immersion, the $\mathcal{O}_Y$-module I/I^2 may not be locally free. However, we will see below in Proposition 5.4.1 that $N_Y U$ is locally free of rank n, and so defines a vector bundle $N_Y U \to Y$. By taking the disjoint union over all connected components $Y \subset \mathcal{Y}_U$ we obtain a vector bundle on $\mathcal{Y}_U$. Letting U vary over a cover of $\mathcal{M}$ by sufficiently small étale open charts and glueing defines the *normal bundle*

$$\pi : N_{\mathcal{Y}}\mathcal{M} \to \mathcal{Y}. \tag{5.3}$$

Recall the Hejhal-Poincaré series

$$F_{m,\mu}(\tau) \in H_{1-\frac{n}{2}}(\omega_L)$$

of §3.2. The discussion of §3.2 endows the divisor $\mathcal{Z}(m,\mu) = \mathcal{Z}(F_{m,\mu})$ with a Green function

$$\Phi_{m,\mu} = \Phi(F_{m,\mu}, \cdot),$$

which is defined at every point of $\mathcal{M}(\mathbb{C})$, but is discontinuous at points of the divisor $\mathcal{Z}(m,\mu)$. The corresponding arithmetic divisor is denoted

$$\widehat{\mathcal{Z}}(m,\mu) = (\mathcal{Z}(m,\mu), \Phi_{m,\mu}) \in \widehat{\mathrm{Div}}(\mathcal{M}). \tag{5.4}$$

From the arithmetic divisor (5.4) we will construct a new arithmetic divisor

$$\sigma(\widehat{\mathcal{Z}}(m,\mu)) \in \widehat{\mathrm{Div}}(N_{\mathcal{Y}}\mathcal{M}) \tag{5.5}$$

called the *specialization to the normal bundle*. Fix a cover of $\mathcal{M}$ by sufficiently small étale open charts. Given a chart $U \to \mathcal{M}$ in the cover, fix a connected component $Y \subset \mathcal{Y}_U$ and decompose $\mathcal{Z}(m,\mu)_U = \coprod Z$ as the union of its connected components. By refining our cover, we may assume that each closed subscheme $Z \to U$ is defined by a single equation $f_Z = 0$.

Lemma 5.2.5. *For every connected component $Z \subset \mathcal{Z}(m,\mu)_U$ the intersection*

$$Y \cap Z := Y \times_U Z$$

satisfies one of the following (mutually exclusive) properties:

(i) $Y \cap Z$ has dimension 0, and f_Z restricts to a nonzero section of $\mathcal{O}_U/I$,
(ii) $Y \cap Z = Y$, and f_Z defines a section of $I \subset \mathcal{O}_U$ with nonzero image under the natural map $I/I^2 \to \underline{\mathrm{Hom}}(N_U Y, \mathcal{O}_Y)$.

Proof. As Y is an integral scheme of dimension one, its closed subscheme $Y \cap Z$ is either all of Y or of dimension 0. Given this, the only nontrivial thing to check is that when $Y \cap Z = Y$, the image of f_Z under $I/I^2 \to \underline{\mathrm{Hom}}(N_U Y, \mathcal{O}_Y)$ is nonzero. This can be checked after restricting to the complex fiber, where it follows from the smoothness of the divisor $Z(\mathbb{C})$ on $U(\mathbb{C})$. This smoothness can be checked using the complex uniformization (2.4). □

If $Y \cap Z$ has dimension 0, then restricting f_Z to Y and pulling back via $N_Y U \to Y$ results in a function $\sigma(f_Z)$ on $N_Y U$, which is homogeneous of degree zero. On the other hand, if $Y \cap Z = Y$ then the image of f_Z in $\underline{\mathrm{Hom}}(N_U Y, \mathcal{O}_Y)$ defines a function $\sigma(f_Z)$ on $N_U Y$, which is homogeneous of degree one. In either case, define an effective Cartier divisor

$$\sigma(\mathcal{Z}(m,\mu)) = \sum_{Z \subset \mathcal{Z}(m,\mu)_U} \mathrm{div}\big(\sigma(f_Z)\big) \tag{5.6}$$

on $N_Y U$.

The function

$$\phi_{m,\mu} = \Phi_{m,\mu} + \sum_{Z \subset \mathcal{Z}(m,\mu)_U} \log |f_Z|^2,$$

initially defined on the complex fiber of $U \smallsetminus \mathcal{Z}(m,\mu)_U$, extends smoothly to all of $U(\mathbb{C})$. By restricting to $\mathcal{Y}(\mathbb{C})$ and then pulling back via (5.3) we obtain a smooth function $\pi^* i^* \phi_{m,\mu}$ on $(N_Y U)(\mathbb{C})$. Define $\sigma(\Phi_{m,\mu})$ by the relation

$$\pi^* i^* \phi_{m,\mu} = \sigma(\Phi_{m,\mu}) + \sum_{Z \subset \mathcal{Z}(m,\mu)_U} \log |\sigma(f_Z)|^2. \tag{5.7}$$

The resulting arithmetic divisor

$$\big(\sigma(\mathcal{Z}(m,\mu)), \sigma(\Phi_{m,\mu})\big) \in \widehat{\mathrm{Div}}(N_Y U)$$

does not depend on the choice of f_Z's used in its construction. Using étale descent these arithmetic divisors glue together to form the desired arithmetic divisor (5.5).

Proposition 5.2.6. *The composition*

$$\widehat{\mathrm{Pic}}(\mathcal{M}) \xrightarrow{i^*} \widehat{\mathrm{Pic}}(\mathcal{Y}) \xrightarrow{\pi^*} \widehat{\mathrm{Pic}}(N_{\mathcal{Y}} \mathcal{M})$$

sends the metrized line bundle defined by $\widehat{\mathcal{Z}}(m,\mu)$ to the metrized line bundle defined by $\sigma(\widehat{\mathcal{Z}}(m,\mu))$

Proof. Let $U \to \mathcal{M}$ be a sufficiently small étale open chart, and write

$$\mathcal{Z}(m,\mu)_U = \coprod Z$$

as the union of its connected components. Each connected component determines a line bundle $\mathcal{O}(Z) = f_Z^{-1} \mathcal{O}_U$, and

$$\mathcal{Z}(m,\mu)_U \xrightarrow{\simeq} \bigotimes \mathcal{O}(Z)$$

as line bundles on U.

Let $\pi^*i^*\mathcal{O}(Z)$ be the pullback of $\mathcal{O}(Z)$ via $N_Y U \xrightarrow{\pi} Y \xrightarrow{i} U$. If we denote by s_Z the constant function 1 on U viewed as a section of $\mathcal{O}(Z)$, then

$$\sigma(s_Z) = \sigma(f_Z)\pi^*i^*(f_Z^{-1}s_Z)$$

is a nonzero section of $\pi^*i^*\mathcal{O}(Z)$, and does not depend on the choice of f_Z. The tensor product $\sigma(s) = \otimes\sigma(s_Z)$ over all Z is a global section of the metrized line bundle $\pi^*i^*\widehat{\mathcal{Z}}(m,\mu)_U$, and these section glue together over an étale cover of $\mathcal{M}$ to define a global section $\sigma(s)$ of $\pi^*i^*\widehat{\mathcal{Z}}(m,\mu)$.

Tracing through the definitions shows that $\widehat{\mathrm{div}}(\sigma(s)) = \sigma(\widehat{\mathcal{Z}}(m,\mu))$, as arithmetic divisors, and so

$$\pi^*i^*\widehat{\mathcal{Z}}(m,\mu) \xrightarrow{\simeq} \sigma(\widehat{\mathcal{Z}}(m,\mu))$$

as metrized line bundles. □

Remark 5.2.7. The construction (5.5) and Proposition 5.2.6 will be our main tools for computing improper intersections. The technique is based on the thesis of J. Hu [Hu99], which reconstructs the arithmetic intersection theory of Gillet-Soulé [GS90] using Fulton's method of deformation to the normal cone.

Letting $C_{\mathcal{Y}}\mathcal{M}$ denote the normal cone of $\mathcal{Y}$ in $\mathcal{M}$, Hu[4] constructs a *specialization to the normal cone* map

$$\sigma : \widehat{\mathrm{Div}}(\mathcal{M}) \to \widehat{\mathrm{Div}}(C_{\mathcal{Y}}\mathcal{M}),$$

and shows that

$$[\widehat{\mathcal{Z}} : \mathcal{Y}] = [\sigma(\widehat{\mathcal{Z}}) : \mathcal{Y}]$$

for any arithmetic divisor $\widehat{\mathcal{Z}}$. Here the intersection pairing on the right is defined using the canonical closed immersion $\mathcal{Y} \to C_{\mathcal{Y}}\mathcal{M}$.

The normal bundle $N_{\mathcal{Y}}\mathcal{M}$ is essentially a first order approximation to $C_{\mathcal{Y}}\mathcal{M}$, and our construction (5.5) and Proposition 5.2.6 amount to truncating Hu's theory to first order. In order to use this to compute arithmetic intersections, it is essential to know that our divisors satisfy Lemma 5.2.5, which says, loosely speaking, that the functions f_Z vanish along Y to at most order 1. This guarantees that the specialization (5.5) to the normal bundle does not lose too much information about the divisor $\mathcal{Z}(m,\mu)$.

To be clear: we are exploiting here a special property of the divisors $\mathcal{Z}(m,\mu)$ and their relative positions with respect to $\mathcal{Y}$. It would not be very useful to define the specialization (5.5) for an arbitrary divisor on $\mathcal{M}$, as truncating Hu's specialization map to first order would lose essential higher order information.

[4]Hu works in greater generality, and allows for the cycles $\mathcal{Z}$ and $\mathcal{Y}$ to have arbitrary codimension.

5.3. **The cotautological bundle.** Recall from (2.10) the vector bundle $\boldsymbol{V}_{\mathrm{dR}}$ on $\mathcal{M}$ with its isotropic line $\boldsymbol{V}^1_{\mathrm{dR}}$, and from (4.4) the line bundle $\boldsymbol{V}_{0,\mathrm{dR}}$ on $\mathcal{Y}$. Proposition 4.6.1 implies that $\boldsymbol{V}_{0,\mathrm{dR}} \subset i^*\boldsymbol{V}_{\mathrm{dR}}$ in such a way that $\boldsymbol{V}^1_{0,\mathrm{dR}} = i^*\boldsymbol{V}^1_{\mathrm{dR}}$.

Denote by $\boldsymbol{T} = (\boldsymbol{V}^1_{\mathrm{dR}})^\vee$ and $\boldsymbol{T}_0 = (\boldsymbol{V}^1_{0,\mathrm{dR}})^\vee$ the cotautological bundles on $\mathcal{M}$ and $\mathcal{Y}$, respectively, so that $\boldsymbol{T}_0 = i^*\boldsymbol{T}$. Under the complex uniformization

$$\Gamma_g \backslash \mathcal{D} \to \mathcal{M}(\mathbb{C})$$

of (2.3) the fiber of $\boldsymbol{V}^1_{\mathrm{dR}}$ at $z \in \mathcal{D}$ is identified with the tautological line $\mathbb{C}z \subset V_{\mathbb{C}}$, which we endow with the metric

$$||z||^2 = \frac{-1}{4\pi e^\gamma}[z, \overline{z}]. \tag{5.8}$$

Here, as before, $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant. Dualizing, we obtain a metric on the cotautological bundle, and the resulting metrized line bundle is denoted

$$\widehat{\boldsymbol{T}} \in \widehat{\mathrm{Pic}}(\mathcal{M}).$$

We endow $\boldsymbol{T}_0$ with the unique metric for which

$$\widehat{\boldsymbol{T}}_0 \xrightarrow{\cong} i^*\widehat{\boldsymbol{T}}.$$

Proposition 5.3.1. *The metrized cotautological bundle satisfies*

$$\frac{w_{\boldsymbol{k}}}{h_{\boldsymbol{k}}} \cdot [\widehat{\boldsymbol{T}} : \mathcal{Y}] = 2\frac{L'(\chi_{\boldsymbol{k}}, 0)}{L(\chi_{\boldsymbol{k}}, 0)} + \log\left|\frac{d_{\boldsymbol{k}}}{4\pi}\right| - \gamma.$$

Proof. This is virtually identical to the proof of [BHY15, Theorem 6.4], and so we only give an outline. Let $\mathcal{E} \to \mathcal{Y}$ be the universal $\mathcal{O}_{\boldsymbol{k}}$-elliptic curve. If we endow the $\mathcal{O}_{\mathcal{Y}}$-module $\mathrm{Lie}(\mathcal{E})$ with the Faltings metric, the Chowla-Selberg formula implies

$$\frac{w_{\boldsymbol{k}}}{h_{\boldsymbol{k}}} \cdot \widehat{\deg}\, \widehat{\mathrm{Lie}}(\mathcal{E}) = \frac{L'(\chi_{\boldsymbol{k}}, 0)}{L(\chi_{\boldsymbol{k}}, 0)} + \log(2\pi) + \frac{1}{2}\log|d_{\boldsymbol{k}}|.$$

Now view L_0 as an $\mathcal{O}_{\boldsymbol{k}}$-module through left multiplication, and endow the corresponding line bundle $\boldsymbol{L}_0$ on $\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$ with the metric

$$||x||^2 = -16\pi^3 e^\gamma \cdot Q(x).$$

Pulling back $\boldsymbol{L}_0$ to $\mathcal{Y}$ via the structure morphism yields a metrized line bundle $\widehat{\boldsymbol{L}}_0$ on $\mathcal{Y}$ satisfying

$$\frac{w_{\boldsymbol{k}}}{h_{\boldsymbol{k}}} \cdot \widehat{\deg}\, \widehat{\boldsymbol{L}}_0 = -\log(16\pi^3 e^\gamma).$$

By keeping track of the metrics, the isomorphism (4.5) defines an isomorphism of metrized line bundles

$$\widehat{\boldsymbol{T}}_0 \xrightarrow{\cong} \widehat{\mathrm{Lie}}(\mathcal{E})^{\otimes 2} \otimes \widehat{\boldsymbol{L}}_0$$

on $\mathcal{Y}$, and the claim follows. ☐

Comparing with Proposition 3.3.1 proves:

Corollary 5.3.2. *The coefficient* $a^+_{L_0}(0) \in \mathfrak{S}^\vee_{L_0}$ *satisfies*

$$a^+_{L_0}(0,\mu) = -\frac{w_{\boldsymbol{k}}}{h_{\boldsymbol{k}}} \cdot \begin{cases} [\widehat{\boldsymbol{T}} : \mathcal{Y}] & \text{if } \mu = 0 \\ 0 & \text{otherwise} \end{cases}$$

for every $\mu \in L_0^\vee / L_0$.

5.4. **The normal bundle at CM points.** We can consider the orthogonal complement $(\boldsymbol{V}^1_{0,\mathrm{dR}})^\perp \subset i^*\boldsymbol{V}_{\mathrm{dR}}$: Since $\mathcal{Y}$ is regular of dimension 1, this complement is torsion-free and is therefore a vector sub-bundle of $i^*\boldsymbol{V}_{\mathrm{dR}}$ of co-rank 1 (Note that the corresponding assertion over $\mathcal{M}$ may not be true; the orthogonal complement of $\boldsymbol{V}^1_{\mathrm{dR}}$ in $\boldsymbol{V}_{\mathrm{dR}}$ will in general not be a vector bundle unless L is unimodular.)

Set $\mathrm{Fil}^0 i^*\boldsymbol{V}_{\mathrm{dR}} = (\boldsymbol{V}^1_{0,\mathrm{dR}})^\perp$ and $\mathrm{gr}^0_{\mathrm{Fil}} i^*\boldsymbol{V}_{\mathrm{dR}} = \mathrm{Fil}^0 i^*\boldsymbol{V}_{\mathrm{dR}}/\boldsymbol{V}^1_{0,\mathrm{dR}}$.

Proposition 5.4.1. *There is a canonical isomorphism of* $\mathcal{O}_{\mathcal{Y}}$*-modules*

$$N_{\mathcal{Y}}\mathcal{M} \xrightarrow{\simeq} \boldsymbol{T}_0 \otimes \mathrm{gr}^0_{\mathrm{Fil}} i^*\boldsymbol{V}_{\mathrm{dR}}.$$

In particular, the normal bundle is locally free of rank n*, and is therefore relatively representable over* $\mathcal{Y}$*.*

Proof. Fix a sufficiently small étale open chart $U \to \mathcal{M}$, and let $Y \subset \mathcal{Y}_U$ be a connected component. Let $Y[\varepsilon] = Y \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[\varepsilon])$ be the scheme of dual numbers relative to Y, and let Def_i be the Zariski sheaf of infinitesimal deformations of $i : Y \to U$. By this we mean the sheaf associating to an open subscheme $S \subset Y$ the set $\mathrm{Def}_i(S)$ of morphisms $j_S : S[\varepsilon] \to U$ such that $j_S|_S = i_S$. Here we are writing i_S for the inclusion $S \hookrightarrow Y \hookrightarrow U$.

Lemma 5.4.2. *There is a canonical isomorphism of functors*

$$\mathrm{Def}_i \xrightarrow{\simeq} N_Y U. \tag{5.9}$$

Proof. The morphism (5.9) is defined in the usual way: every $j_S \in \mathrm{Def}_i(S)$ defines a map $\mathcal{O}_U/I^2 \to \mathcal{O}_S[\varepsilon]$, whose restriction

$$I/I^2 \to \mathcal{O}_S \cdot \varepsilon \xrightarrow{\simeq} \mathcal{O}_S$$

defines an element of $(N_Y U)(S)$.

To construct an inverse to (5.9), first recall that $\mathcal{Y}$ is formally étale over $\mathcal{O}_{\boldsymbol{k}}[1/2]$. This implies that the surjection $\mathcal{O}_U/I^2 \to \mathcal{O}_Y$ admits a canonical section, and so

$$\mathcal{O}_U/I^2 \xrightarrow{\simeq} \mathcal{O}_Y \oplus I/I^2$$

as sheaves of rings on $\mathcal{O}_Y$. Any S-point $s \in (N_Y U)(S)$ therefore defines a homomorphism

$$\mathcal{O}_U/I^2 \xrightarrow{\simeq} \mathcal{O}_Y \oplus I/I^2 \xrightarrow{1 \oplus s\cdot\epsilon} \mathcal{O}_S[\epsilon],$$

which in turn determines a deformation $j_S : S[\epsilon] \to U$ of i_S. □

Lemma 5.4.3. *There is a canonical isomorphism of sheaves of sets*

$$\mathrm{Def}_i \xrightarrow{\simeq} (\boldsymbol{V}^1_{0,\mathrm{dR}})^\vee \otimes_{\mathcal{O}_Y} \mathrm{gr}^0_{\mathrm{Fil}}\,(i^*\boldsymbol{V}_{\mathrm{dR}})$$

on Y*.*

Proof. Let $S \subset Y$ be an open subscheme, and let $j_S \in \mathrm{Def}_i(S)$. The pullback $j_S^*\boldsymbol{V}_{\mathrm{dR}}$ is a locally free sheaf of $\mathcal{O}_{S[\varepsilon]}$-modules of rank $n+2$, endowed with an $\mathcal{O}_{S[\varepsilon]}$-submodule $j_S^*\boldsymbol{V}^1_{\mathrm{dR}}$ of rank one, locally a direct summand. As $\boldsymbol{V}_{\mathrm{dR}}$ is a vector bundle with integrable connection, the retraction $S[\varepsilon] \to S$ induces a canonical isomorphism

$$j_S^*\boldsymbol{V}_{\mathrm{dR}} \xrightarrow{\simeq} i_S^*\boldsymbol{V}_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S[\varepsilon]}.$$

We claim that, under this isomorphism, $j_S^*\boldsymbol{V}^1_{\mathrm{dR}}$ maps into $\mathrm{Fil}^0 i_S^*\boldsymbol{V}_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S[\varepsilon]}$; in other words, the image of $j_S^*\boldsymbol{V}^1_{\mathrm{dR}}$ is orthogonal to $i_S^*\boldsymbol{V}^1_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S[\varepsilon]}$. This is easily deduced from the fact that both this image and $i_S^*\boldsymbol{V}^1_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S[\varepsilon]}$ are isotropic lines lifting $i_S^*\boldsymbol{V}^1_{\mathrm{dR}}$.

Consider the composition

$$j_S^*\boldsymbol{V}^1_{\mathrm{dR}} \hookrightarrow \mathrm{Fil}^0 i_S^*\boldsymbol{V}_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S[\varepsilon]} \to \mathrm{gr}^0_{\mathrm{Fil}}\big(i_S^*\boldsymbol{V}_{\mathrm{dR}}\big) \otimes_{\mathcal{O}_S} \mathcal{O}_{S[\varepsilon]}.$$

The reduction of this map modulo (ε) is 0, and therefore it must factor through a map

$$i_S^*\boldsymbol{V}^1_{\mathrm{dR}} \to \varepsilon \cdot \left(\mathrm{gr}^0_{\mathrm{Fil}}\big(i_S^*\boldsymbol{V}_{\mathrm{dR}}\big) \otimes_{\mathcal{O}_S} \mathcal{O}_{S[\varepsilon]}\right) \xrightarrow{\simeq} \mathrm{gr}^0_{\mathrm{Fil}}\big(i_S^*\boldsymbol{V}_{\mathrm{dR}}\big).$$

Thus, we have produced a canonical morphism

$$\mathrm{Def}_i \to (\boldsymbol{V}^1_{0,\mathrm{dR}})^\vee \otimes_{\mathcal{O}_Y} \mathrm{gr}^0_{\mathrm{Fil}}\,(i^*\boldsymbol{V}_{\mathrm{dR}})\,.$$

To show that this is an isomorphism, we can, by (2.4.2) and the above discussion of local models, assume that the immersion $Y \subset U$ is isomorphic to an étale neighborhood of the closed immersion $\mathrm{M}^{\mathrm{loc}}(L_0) \hookrightarrow \mathrm{M}^{\mathrm{loc}}(L)$. Moreover, we can choose this isomorphism in such a way that it identifes $\boldsymbol{V}_{0,\mathrm{dR}}$ and $\boldsymbol{V}_{\mathrm{dR}}$ with the *trivial* vector bundles $\mathbf{1}\otimes L_0$ and $\mathbf{1}\otimes L$, respectively, and such that $\boldsymbol{V}^1_{0,\mathrm{dR}}$ and $\boldsymbol{V}^1_{\mathrm{dR}}$ get identified with the tautological isotropic lines in $\mathbf{1}\otimes L_0$ and $\mathbf{1}\otimes L$, respectively.

So, if $\boldsymbol{L}^1_0 \subset \mathbf{1}\otimes L_0$ is the tautological isotropic line over $\mathrm{M}^{\mathrm{loc}}(L_0)$, and $\boldsymbol{L}^0 \subset \mathbf{1}\otimes L$ is its orthogonal complement, we are reduced to showing the elementary fact that there is a canonical isomorphism

$$N_{\mathrm{M}^{\mathrm{loc}}(L_0)}(\mathrm{M}^{\mathrm{loc}}(L)) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\boldsymbol{L}^1_0, \boldsymbol{L}^0/\boldsymbol{L}^1_0)$$

of Zariski sheaves over $\mathrm{M}^{\mathrm{loc}}(L_0)$. We leave this as an exercise to the reader. □

The proof of Proposition 5.4.1 now follows by combining Lemmas (5.4.2) and (5.4.3) and glueing over an étale cover of $\mathcal{M}$.

□

We will have use for the following composition:

$$N_{\mathcal{Y}}\mathcal{M} \xrightarrow[(5.4.1)]{\simeq} \boldsymbol{T}_0 \otimes \mathrm{gr}^0_{\mathrm{Fil}} i^*\boldsymbol{V}_{\mathrm{dR}} \to \boldsymbol{T}_0 \otimes \big(i^*\boldsymbol{V}_{\mathrm{dR}}/\boldsymbol{V}_{0,\mathrm{dR}}\big). \tag{5.10}$$

Here, the map on the right is induced by the natural map

$$\mathrm{gr}^0_{\mathrm{Fil}} i^*\boldsymbol{V}_{\mathrm{dR}} \to i^*\boldsymbol{V}_{\mathrm{dR}}/\boldsymbol{V}^0_{\mathrm{dR}}.$$

Note that this last map is an isomorphism over the generic fiber of $\mathcal{Y}$. Indeed, it amounts to knowing that the inclusion $\boldsymbol{V}_{0,\mathrm{dR}} + \mathrm{Fil}^0 i^*\boldsymbol{V}_{\mathrm{dR}} \subset i^*\boldsymbol{V}_{\mathrm{dR}}$ is an isomorphism over the generic fiber. This follows from the self-duality of the quadratic form on $\boldsymbol{V}_{\mathrm{dR},M}$.

On the other hand, over the generic fiber, the map $\boldsymbol{\Lambda}_{\mathrm{dR}} \to i^*\boldsymbol{V}_{\mathrm{dR}}/\boldsymbol{V}^0_{\mathrm{dR}}$ is also an isomorphism. Therefore, for any point $y \in \mathcal{Y}(\mathbb{C})$, (5.10) induces a canonical isomorphism:

$$(N_{\mathcal{Y}}\mathcal{M})_y \xrightarrow{\simeq} \boldsymbol{T}_{0,y} \otimes \Lambda_{\mathbb{C}}. \tag{5.11}$$

We will now give an explicit description of this isomorphism. To begin, y determines an isotropic line $\mathbb{C}y \subset L_{\mathbb{C}}$, whose dual can be naturally identified with the fiber $\boldsymbol{T}_{0,y}$ of $\boldsymbol{T}_0$ at y. The construction of the isomorphism (5.11) proceeded by choosing an appropriate local trivialization of $\boldsymbol{V}_{\mathrm{dR}}$. In an analytic neighborhood of y, this simply means that we choose a local section $U \to \mathcal{D}$ of the complex analytic uniformization carrying y to $\mathbb{C}y$.

Note that $\mathcal{D}$ is contained in the complex quadric $\mathrm{M}^{\mathrm{loc}}(L_{\mathbb{C}})$. We therefore see that it is enough to describe the induced isomorphism:

$$T_{\mathbb{C}y}\mathcal{D} = T_{\mathbb{C}y}\mathrm{M}^{\mathrm{loc}}(L_{\mathbb{C}}) \xrightarrow{\simeq} (\mathbb{C}y)^{\vee} \otimes \Lambda_{\mathbb{C}}. \tag{5.12}$$

Here, $T_{\mathbb{C}y}\mathcal{D}$ is the tangent space of $\mathcal{D}$ at $\mathbb{C}y$.

We consider the local immersion

$$\mathrm{Hom}(\mathbb{C}y, \Lambda_{\mathbb{C}}) = (\mathbb{C}y)^{\vee} \otimes \Lambda_{\mathbb{C}} \to \mathrm{M}^{\mathrm{loc}}(L_{\mathbb{C}}),$$

which carries a section ξ to the isotropic line spanned by

$$y + \xi(y) - \frac{Q(\xi(y))}{[y,\overline{y}]}\overline{y}.$$

This carries 0 to $\mathbb{C}y$, and the derivative at 0 is exactly the inverse to (5.12).

5.5. **Specialization of the Green function.** Suppose we have a complex point $y \in \mathcal{Y}(\mathbb{C})$. Under the uniformization

$$\mathcal{Y}(\mathbb{C}) \xrightarrow{\simeq} T(\mathbb{Q})\backslash\{\boldsymbol{h}_0\} \times T(\mathbb{A}_f)/\widehat{\mathcal{O}}^{\times}_{\boldsymbol{k}}$$

of §4.2, the point y is represented by a pair $(\boldsymbol{h}_0, g)$ with

$$g \in T(\mathbb{A}_f) \subset G(\mathbb{A}_f),$$

and so its image in $\mathcal{M}(\mathbb{C})$ lies on the component $\Gamma_g\backslash\mathcal{D}$ of (2.3). Here $G = \mathrm{GSpin}(L)$. By mild abuse of notation we also let $\mathbb{C}y \in \mathcal{D}$ denote the isotropic line corresponding to the oriented negative plane $\boldsymbol{h}_0 = L_{0\mathbb{R}}$, so that $\mathbb{C}y$ is a lift of y under $\mathcal{D} \to \mathcal{M}(\mathbb{C})$, and $L_{\mathbb{C}} = \mathbb{C}y \oplus \mathbb{C}\overline{y} \oplus \Lambda_{\mathbb{C}}$.

From (2.4), we find that a neighborhood of y admits a complex analytic uniformization

$$\coprod_{\substack{x\in\mu_g+L_g\\ Q(x)=m}} \mathcal{D}(x) \to \mathcal{Z}(m,\mu)(\mathbb{C}).$$

The components $\mathcal{D}(x)$ passing through $\mathbb{C}y$ are precisely those for which x is orthogonal to y. But, since x is rational, this is equivalent to requiring that x be orthogonal to both y and $\overline{y}$; or, in other words, that x lies in $\Lambda^\vee$.

As the orthogonal transformation $g \in \mathrm{SO}(L\otimes \mathbb{A}_f)$ acts trivially on the direct summand $\Lambda\otimes\mathbb{A}_f$, we see that such x in fact lie in $\mu+L$. This shows that in a small enough analytic neighborhood U of $\mathbb{C}y$ there is an isomorphism of divisors

$$U\cap \sum_{\lambda\in\Lambda_{m,\mu}} \mathcal{D}(\lambda) \xrightarrow{\simeq} U\cap \mathcal{Z}(m,\mu)(\mathbb{C}), \tag{5.13}$$

where

$$\Lambda_{m,\mu} = \left\{ \lambda\in\Lambda^\vee : \begin{array}{c} Q(\lambda)=m \\ \lambda\in\mu+L \end{array} \right\}.$$

Recall that each isotropic line $\mathbb{C}z\in\mathcal{D}$ corresponds to a negative plane $\boldsymbol{h}_z\subset V_{\mathbb{R}}$. For each $\lambda\in\Lambda_{m,\mu}$ let $\lambda_z\in V_{\mathbb{R}}$ be the orthogonal projection of λ to $\boldsymbol{h}_z$. Simple linear algebra gives

$$Q(\lambda_z) = \frac{[\lambda,\bar{z}]\cdot[\lambda,z]}{[z,\bar{z}]}.$$

Define a function on $U\smallsetminus\mathcal{Z}(m,\mu)(\mathbb{C})$ by

$$\Phi^{\mathrm{reg}}_{m,\mu} = \Phi_{m,\mu} + \sum_{\lambda\in\Lambda_{m,\mu}} \log\left|4\pi e^\gamma\cdot Q(\lambda_z)\right|.$$

Recall that the function $\Phi_{m,\mu}(z)$ is defined, but discontinuous, at $z=y$. The following proposition gives us a method of computing the value of this function at the discontinuity $z=y$.

Proposition 5.5.1. *The function $\Phi^{\mathrm{reg}}_{m,\mu}$ extends smoothly to all of U, and this extension satisfies*

$$\Phi^{\mathrm{reg}}_{m,\mu}(y) = \Phi_{m,\mu}(y).$$

Proof. Let $\tau=u+iv\in\mathcal{H}$ be the variable on the upper half-plane. The function $\Phi_{m,\mu}(z)$ is defined, as in [BY09, (4.8)], as the constant term of the Laurent expansion at $s=0$ of

$$\lim_{T\to\infty}\int_{\mathcal{F}_T} \left\{F_{m,\mu}(\tau),\theta_L(\tau,z)\right\}\frac{du\,dv}{v^{s+2}}, \tag{5.14}$$

where

$$\mathcal{F}_T = \{\tau=u+iv\in\mathcal{H} : |u|\le 1/2,\ |\tau|\ge 1,\ v\le T\}.$$

Now substitute the definition [BY09, (2.4)] of $\theta_L(\tau,z)$ and the Fourier expansion of $F_{m,\mu}$ into (5.14), and argue as in [Bor98, Theorem 6.2]. Using the decomposition $F_{m,\mu} = F^+_{m,\mu} + F^-_{m,\mu}$ of [BY09, (3.4a) and (3.4b)] and the rapid decay as $v \to \infty$ of the Fourier coefficients of $F^-_{m,\mu}$, we find

$$\Phi_{m,\mu}(z) = \varphi_{m,\mu}(z) + \sum_{\lambda \in \Lambda_{m,\mu}} \mathrm{CT}_{s=0}\left[\int_1^\infty e^{4\pi v Q(\lambda_z)} \frac{dv}{v^{s+1}}\right] \tag{5.15}$$

for some smooth function $\varphi_{m,\mu}$ on U, where $\mathrm{CT}_{s=0}$ means take the constant term at $s = 0$.

Each $\lambda \in \Lambda_{m,\mu}$ is orthogonal to y, and so $Q(\lambda_y) = 0$. Thus setting $z = y$ in (5.15) and computing the constant term at $s = 0$ shows that

$$\Phi_{m,\mu}(y) = \varphi_{m,\mu}(y).$$

On the other hand, for any $z \in U \smallsetminus \mathcal{Z}(m,\mu)(\mathbb{C})$ we have $Q(\lambda_z) < 0$ for every $\lambda \in \Lambda_{m,\mu}$, and so

$$\begin{aligned}\sum_{\lambda \in \Lambda_{m,\mu}} \mathrm{CT}_{s=0}\left[\int_1^\infty e^{4\pi v Q(\lambda_z)} \frac{dv}{v^{s+1}}\right] &= \sum_{\lambda \in \Lambda_{m,\mu}} \int_1^\infty e^{4\pi v Q(\lambda_z)} \frac{dv}{v} \\ &= \sum_{\lambda \in \Lambda_{m,\mu}} \Gamma(0, -4\pi Q(\lambda_z)),\end{aligned} \tag{5.16}$$

where

$$\Gamma(0,x) = \int_x^\infty e^{-t}\,\frac{dt}{t} = -\gamma - \log(x) - \sum_{k=1}^\infty \frac{(-x)^k}{k \cdot k!}. \tag{5.17}$$

Comparing (5.16) with (5.17) shows that

$$\sum_{\lambda \in \Lambda_{m,\mu}} \log|4\pi e^\gamma \cdot Q(\lambda_z)| = -\sum_{\lambda \in \Lambda_{m,\mu}} \mathrm{CT}_{s=0}\left[\int_1^\infty e^{4\pi v Q(\lambda_z)} \frac{dv}{v^{s+1}}\right]$$

up to a smooth function on U, vanishing at $z = y$. Adding this equality to (5.15) proves that

$$\Phi^{\mathrm{reg}}_{m,\mu}(z) = \varphi_{m,\mu}(z)$$

up to a smooth function vanishing at $z = y$, and completes the proof. □

For any $y \in \mathcal{Y}(\mathbb{C})$, we have a canonical isomorphism (see 5.11)

$$(N_{\mathcal{Y}}\mathcal{M})_y \xrightarrow{\simeq} (\mathbb{C}y)^\vee \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}} = \mathrm{Hom}(\mathbb{C}y, \Lambda_{\mathbb{C}}). \tag{5.18}$$

For each $\lambda \in \Lambda_{m,\mu}$, define a function Φ_λ on $\mathrm{Hom}(\mathbb{C}y, \Lambda_{\mathbb{C}})$ by

$$\Phi_\lambda(\xi) = -\log\left|\frac{4\pi e^\gamma [\lambda, \xi(y)]^2}{[y, \bar{y}]}\right|.$$

This function has a logarithmic singularity along the hyperplane $(\mathbb{C}y)^\vee \otimes \lambda^\perp$. Letting y vary over $\mathcal{Y}(\mathbb{C})$ yields a function Φ_λ on the vector bundle $(N_{\mathcal{Y}}\mathcal{M})(\mathbb{C})$ having a logarithmic singularity along a sub-bundle of hyperplanes.

Proposition 5.5.2. *The Green function $\sigma(\Phi_{m,\mu})$ defined by (5.7) satisfies*

$$\sigma\big(\Phi_{m,\mu}\big) = \pi^* i^* \Phi_{m,\mu} + \sum_{\lambda \in \Lambda_{m,\mu}} \Phi_\lambda,$$

where $\pi^* i^* \Phi_{m,\mu}$ denotes the pullback of $\Phi_{m,\mu}$ via

$$(N_{\mathcal{Y}}\mathcal{M})(\mathbb{C}) \xrightarrow{\pi} \mathcal{Y}(\mathbb{C}) \xrightarrow{i} \mathcal{M}(\mathbb{C}).$$

Proof. As above, fix a point $y \in \mathcal{Y}(\mathbb{C})$ and let $U \subset \mathcal{D}$ be a neighborhood of the isotropic line $\mathbb{C}y \subset L_{\mathbb{C}}$. As in (5.4), we can define a holomorphic local immersion

$$\mathrm{Hom}(\mathbb{C}y, \Lambda_{\mathbb{C}}) = (\mathbb{C}y)^\vee \otimes \Lambda_{\mathbb{C}} \to \mathrm{M}^{\mathrm{loc}}(L_{\mathbb{C}})$$

by sending ξ to the $\mathbb{C}$-span of

$$y + \xi(y) - \frac{Q(\xi(y))\overline{y}}{[y, \overline{y}]}.$$

As we saw at the end of (5.4), the induced isomorphism on tangent spaces

$$(\mathbb{C}y)^\vee \otimes \Lambda_{\mathbb{C}} \to T_y\mathcal{D} \xrightarrow{\simeq} (N_{\mathcal{Y}}\mathcal{M})_y$$

agrees with the isomorphism of (5.18).

From now on we use the above map to identify U with an open neighborhood of the origin of $(\mathbb{C}y)^\vee \otimes \Lambda_{\mathbb{C}}$. Each divisor $\mathcal{D}(\lambda) \cap U$ appearing in (5.13) is identified with the zero locus of

$$f_\lambda(\xi) = [\lambda, \xi(y)] \cdot \left| \frac{4\pi e^\gamma}{[y, \overline{y}]} \right|^{1/2}.$$

This function is already linear, and so when we apply the construction $f_\lambda \mapsto \sigma(f_\lambda)$ of the discussion preceding (5.7) and identify $(\mathbb{C}y)^\vee \otimes \Lambda_{\mathbb{C}}$ with its own tangent space at the origin, we obtain

$$\sigma(f_\lambda)(\xi) = [\lambda, \xi(\boldsymbol{v})] \cdot \left| \frac{4\pi e^\gamma}{[\boldsymbol{v}, \overline{\boldsymbol{v}}]} \right|^{1/2}.$$

Thus $-\log|\sigma(f_\lambda)|^2 = \Phi_\lambda$.

If we define a smooth function on U by

$$\phi_{m,\mu} = \Phi_{m,\mu} + \sum_{\lambda \in \Lambda_{m,\mu}} \log|f_\lambda|^2,$$

then directly from the definition (5.7) and the paragraph above we see that

$$\sigma(\Phi_{m,\mu}) = \pi^* i^* \phi_{m,\mu} + \sum_{\lambda \in \Lambda_{m,\mu}} \Phi_\lambda.$$

Now use $\phi_{m,\mu}(y) = \Phi^{\mathrm{reg}}_{m,\mu}(y) = \Phi_{m,\mu}(y)$ to complete the proof. □

5.6. **The calculation of the pullback.** Recall that the Kuga-Satake abelian scheme $\mathcal{A} \to \mathcal{M}$ restricts to the abelian scheme

$$\mathcal{A}_{\mathcal{Y}} = \mathcal{E} \otimes_{C^+(L_0)} C(L)$$

over $\mathcal{Y}$, where $\mathcal{E} \to \mathcal{Y}$ is the universal elliptic curve.

By Proposition 4.6.3, every $\lambda \in \Lambda_{m,\mu}$ determines a canonical special endomorphism $\boldsymbol{\ell}_\lambda \in V_\mu(\mathcal{A}_{\mathcal{Y}})$, and the pair $(\mathcal{A}_{\mathcal{Y}}, \boldsymbol{\ell}_\lambda)$ is a $\mathcal{Y}$-valued point of $\mathcal{Z}(m,\mu)$. The corresponding map $\mathcal{Y} \to \mathcal{Z}(m,\mu)$ is finite and unramified, and so we may define an $\mathcal{O}_{\mathcal{Y}}$-submodule

$$\mathcal{Z}_\lambda = N_{\mathcal{Y}}\mathcal{Z}(m,\mu) \tag{5.19}$$

of $N_{\mathcal{Y}}\mathcal{M}$ in exactly the same way that $N_{\mathcal{Y}}\mathcal{M}$ was defined.

Proposition 4.6.1 provides us with a canonical morphism $\Lambda \to i^*\boldsymbol{V}_{\mathrm{dR}}$ such that $\Lambda^\perp = \boldsymbol{V}_{0,\mathrm{dR}} \subset i^*\boldsymbol{V}_{\mathrm{dR}}$. Thus, the bilinear pairing on $i^*\boldsymbol{V}_{\mathrm{dR}}$ gives us a map

$$i^*\boldsymbol{V}_{\mathrm{dR}}/\boldsymbol{V}_{0,\mathrm{dR}} \to \mathcal{O}_{\mathcal{Y}} \otimes \Lambda^\vee$$

of vector bundles over $\mathcal{Y}$. This map is injective, and using local trivializations, one sees that its image is identified precisely with the inclusion

$$\mathcal{O}_{\mathcal{Y}} \otimes (L/L_0) \hookrightarrow \mathcal{O}_{\mathcal{Y}} \otimes \Lambda^\vee.$$

This latter inclusion is the natural one induced by the bilinear pairing on L.

Now, by definition, $\Lambda_{m,\mu}$ is contained in $(L/L_0)^\vee \subset L^\vee$. Therefore, we obtain a morphism:

$$[\,\cdot\,,\lambda] : i^*\boldsymbol{V}_{\mathrm{dR}}/\boldsymbol{V}_{0,\mathrm{dR}} \to \mathcal{O}_{\mathcal{Y}} \otimes (L/L_0) \xrightarrow{1\otimes\lambda} \mathcal{O}_{\mathcal{Y}}.$$

of $\mathcal{O}_{\mathcal{Y}}$-modules.

Here is another way to describe this morphism. Recall from §4.6 that we have a canonical T-torsor over $\mathcal{Y}$ parameterizing $\mathcal{O}_{\boldsymbol{k}}$-equivariant trivializations of $\boldsymbol{H}^+_{0,\mathrm{dR}}$. Using this torsor of local trivializations, we get a canonical functor from representations of T to vector bundles over $\mathcal{Y}$ with integrable connection.

The map $[\,\cdot\,,\lambda]$ is now simply obtained by applying this functor to the T-invariant functional $L/L_0 \to \mathbb{Z}$ given by pairing with λ. In particular, if $t_{\mathrm{dR}}(\boldsymbol{\ell}_\lambda)$ is the de Rham realization of $\boldsymbol{\ell}_\lambda$ (it is a section of $i_S^*\boldsymbol{V}_{\mathrm{dR}}^\vee$), we find that functionals

$$i_S^*\boldsymbol{V}_{\mathrm{dR}} \to i_S^*\boldsymbol{V}_{\mathrm{dR}}/\boldsymbol{V}_{0,\mathrm{dR}} \xrightarrow{[\cdot,\lambda]} \mathcal{O}_S$$

and

$$[\,\cdot\,, t_{\mathrm{dR}}(\boldsymbol{\ell}_\lambda)] : i_S^*\boldsymbol{V}_{\mathrm{dR}} \to \mathcal{O}_S$$

are identical.

Proposition 5.6.1. *Fix any $\lambda \in \Lambda_{m,\mu}$.*

(i) The divisor $\mathcal{Z}_\lambda$ is canonically identified with the kernel of

$$N_{\mathcal{Y}}\mathcal{M} \xrightarrow{(5.10)} \boldsymbol{T}_0 \otimes (i^*\boldsymbol{V}_{\mathrm{dR}}/\boldsymbol{V}_{0,\mathrm{dR}}) \xrightarrow{[\cdot,\lambda]} \boldsymbol{T}_0.$$

(ii) The metrized line bundle $\pi^*\widehat{\boldsymbol{T}}_0$ *is represented by the arithmetic divisor*

$$(\mathcal{Z}_\lambda, \Phi_\lambda) \in \widehat{\mathrm{Div}}(N_{\mathcal{Y}}\mathcal{M}).$$

Proof. We work over a sufficiently small étale neighborhood $Y \subset U$ of $\mathcal{Y} \to \mathcal{M}$. Suppose that we are given an open subspace $S \subset Y$ and $j_S \in \mathrm{Def}_i(S)$. By the construction of the map (5.10), to prove (1), we have to show that $\boldsymbol{\ell}_\lambda$ lifts to an element in $V_\mu(j_S^*\mathcal{A})$ if and only if the composition

$$j_S^*\boldsymbol{V}^1_{\mathrm{dR}} \to i_S^*\boldsymbol{V}_{\mathrm{dR}}/\boldsymbol{V}_{0,\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S[\varepsilon]} \xrightarrow{[\,\cdot\,,\lambda]} \mathcal{O}_{S[\varepsilon]}$$

is identically 0. This follows from Lemma 2.7.4 and the identification we made above of $[\,\cdot\,,\lambda]$ with pairing against $t_{\mathrm{dR}}(\boldsymbol{\ell}_\lambda)$.

The second claim follows from the first, as the map $[\,\cdot\,,\lambda] : N_{\mathcal{Y}}\mathcal{M} \to \boldsymbol{T}_0$ defines a section s_λ of $\pi^*\boldsymbol{T}_0$ with $\widehat{\mathrm{div}}(s_\lambda) = (\mathcal{Z}_\lambda, \Phi_\lambda)$. Indeed, using the notation of (5.18), the norm of this section at a complex point $\xi \in (N_{\mathcal{Y}}\mathcal{M})_y$ in the fiber over $y \in \mathcal{Y}(\mathbb{C})$ is

$$-\log ||s_\lambda||^2 = -\log \left| \frac{[\lambda, \xi(y)]}{||y||} \right|^2 = \Phi_\lambda(\xi),$$

where the second equality is by definition (5.8) of the metric on $\widehat{\boldsymbol{T}}_0$. □

As in [BHY15, §7.3], we define a new metrized line bundle

$$\widehat{\mathcal{Z}}^\heartsuit(m,\mu) = \widehat{\mathcal{Z}}(m,\mu) \otimes \widehat{\boldsymbol{T}}^{\otimes - \#\Lambda_{m,\mu}} \in \widehat{\mathrm{Pic}}(\mathcal{M}).$$

Proposition 5.6.2. *The image of* $\widehat{\mathcal{Z}}^\heartsuit(m,\mu)$ *under the pullback*

$$\widehat{\mathrm{Pic}}(\mathcal{M}) \xrightarrow{i^*} \widehat{\mathrm{Pic}}(\mathcal{Y})$$

is represented by the arithmetic divisor

$$\Big(\sum_{\substack{m_1>0 \\ m_2 \geq 0 \\ m_1+m_2=m}} \sum_{\substack{\mu_1 \in L_0^\vee/L_0 \\ \mu_2 \in \Lambda^\vee/\Lambda \\ (\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)}} R_\Lambda(m_2,\mu_2)\cdot \mathcal{Z}_0(m_1,\mu_1) \ , \ i^*\Phi_{m,\mu} \Big) \in \widehat{\mathrm{Div}}(\mathcal{Y})$$

Proof. By Proposition 4.6.4 there is a decomposition of $\mathcal{Y}$-stacks

$$\mathcal{Z}(m,\mu) \times_{\mathcal{M}} \mathcal{Y} = \coprod_{\substack{m_1,m_2 \in \mathbb{Q}_{\geq 0} \\ m_1+m_2=m}} \coprod_{\substack{\mu_1 \in L_0^\vee/L_0 \\ \mu_2 \in \Lambda^\vee/\Lambda \\ (\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)}} \mathcal{Z}_0(m_1,\mu_1) \times \Lambda_{m_2,\mu_2}.$$

Using Remark 4.6.5 we may therefore decompose

$$\mathcal{Z}(m,\mu) \times_{\mathcal{M}} \mathcal{Y} = \mathcal{Z}_0^{\mathrm{prop}} \coprod (\mathcal{Y} \times \Lambda_{m,\mu}),$$

where

$$\mathcal{Z}_0^{\mathrm{prop}} = \coprod_{\substack{m_1>0 \\ m_2 \geq 0 \\ m_1+m_2=m}} \coprod_{\substack{\mu_1 \in L_0^\vee/L_0 \\ \mu_2 \in \Lambda^\vee/\Lambda \\ (\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)}} \mathcal{Z}_0(m_1,\mu_1) \times \Lambda_{m_2,\mu_2}$$

is a $\mathcal{Y}$-stack of dimension 0. Moreover, the induced map

$$\mathcal{Y} \times \Lambda_{m,\mu} \to \mathcal{Z}(m,\mu)$$

determines, for each $\lambda \in \Lambda_{m,\mu}$, a map $\mathcal{Y} \to \mathcal{Z}(m,\mu)$, which is none other than the map used in the definition (5.19) of $\mathcal{Z}_\lambda$.

Over a sufficiently small étale open chart $U \to \mathcal{M}$ we may fix a connected component $Y \subset \mathcal{Y}_U$, and decompose

$$\mathcal{Z}(m,\mu)_U = Z^{\mathrm{prop}} \coprod (Y \times \Lambda_{m,\mu})$$

where Z^{prop} is the union of all connected components $Z \subset \mathcal{Z}(m,\mu)_U$ for which $Z \cap Y$ has dimension 0. If we apply the specialization to the normal bundle construction of (5.6) separately to Z^{prop} and to each of the $\#\Lambda_{m,\mu}$ copies of Y, and then glue over an étale cover, we find the equality

$$\sigma\big(\mathcal{Z}(m,\mu)\big) = \pi^* \mathcal{Z}_0^{\mathrm{prop}} + \sum_{\lambda \in \Lambda_{m,\mu}} \mathcal{Z}_\lambda \tag{5.20}$$

of divisors on $N_{\mathcal{Y}}\mathcal{M}$.

By Propositions 5.2.6 and 5.6.1 the image of $\widehat{\mathcal{Z}}^{\heartsuit}(m,\mu)$ under

$$\widehat{\mathrm{Pic}}(\mathcal{M}) \xrightarrow{i^*} \widehat{\mathrm{Pic}}(\mathcal{Y}) \xrightarrow{\pi^*} \widehat{\mathrm{Pic}}(N_{\mathcal{Y}}\mathcal{M})$$

is

$$\pi^* i^* \widehat{\mathcal{Z}}^{\heartsuit}(m,\mu) = \sigma\big(\widehat{\mathcal{Z}}(m,\mu)\big) - \sum_{\lambda \in \Lambda_{m,\mu}} (\mathcal{Z}_\lambda, \Phi_\lambda),$$

and Proposition 5.5.2 and (5.20) allow us to rewrite this as an equality of metrized line bundles

$$\pi^* i^* \widehat{\mathcal{Z}}^{\heartsuit}(m,\mu) = \pi^*\big(\mathcal{Z}_0^{\mathrm{prop}}, i^*\Phi_{m,\mu}\big)$$

on $N_{\mathcal{Y}}\mathcal{M}$. Pulling back by the zero section $\mathcal{Y} \to N_{\mathcal{Y}}\mathcal{M}$ yields an isomoprhism

$$i^* \widehat{\mathcal{Z}}^{\heartsuit}(m,\mu) = (\mathcal{Z}_0^{\mathrm{prop}}, i^*\Phi_{m,\mu})$$

of metrized line bundles on $\mathcal{Y}$. □

5.7. **The main result.** We are now ready to prove our main result. For the reader's convenience, we review the *dramatis personae*.

The lattices $L_0 \oplus \Lambda \subset L$ determine a finite and unramified morphism of stacks $\mathcal{Y} \to \mathcal{M}$ over $\mathcal{O}_{\boldsymbol{k}}[1/2]$, and hence a linear functional

$$[\cdot : \mathcal{Y}] : \widehat{\mathrm{CH}}^1(\mathcal{M}) \to \mathbb{R}/\mathbb{Q}\log(2)$$

defined in §5.1.

Fix a weak harmonic Maass form $f(\tau) \in H_{1-n/2}(\omega_L)$. The holomorphic part

$$f^+(\tau) = \sum_{m \gg -\infty} c_f^+(m) \cdot q^m$$

is a formal q-expansion valued in the finite-dimensional vector space $\mathfrak{S}_L$ of complex functions on $L^\vee/L$. We assume that f^+ has integral principal part, so that the constructions of §3.2 provide us with an arithmetic divisor

$$\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(f)) \in \widehat{\mathrm{CH}}^1(\mathcal{M}).$$

We also have, from §5.3, the metrized cotautological bundle

$$\widehat{\boldsymbol{T}} \in \widehat{\mathrm{CH}}^1(\mathcal{M}).$$

Recalling the Bruinier-Funke differential operator

$$\xi : H_{1-\frac{n}{2}}(\omega_L) \to S_{1+\frac{n}{2}}(\overline{\omega}_L)$$

from (3.1), we may form the convolution L-function $L(\xi(f), \Theta_\Lambda, s)$ of $\xi(f)$ with the theta series $\Theta_\Lambda(\tau) \in M_{\frac{n}{2}}(\omega_\Lambda^\vee)$ of §3.3. The convolution L-function vanishes at $s = 0$, the center of the functional equation.

Recall the $\mathfrak{S}_{L_0 \oplus \Lambda}$-valued formal q-expansion $\mathcal{E}_{L_0} \otimes \Theta_\Lambda$ of §3.3. Using (3.2) and (3.3), we obtain a scalar-valued q-expansion

$$\{f^+, \mathcal{E}_{L_0} \otimes \Theta_\Lambda\} = \sum_{m_1, m_2, m_3 \in \mathbb{Q}} \big\{ c_f^+(m_1), a_{L_0}^+(m_2) \otimes R_\Lambda(m_3) \big\} \cdot q^{m_1+m_2+m_3}$$

with constant term

$$\mathrm{CT}\{f^+, \mathcal{E}_{L_0} \otimes \Theta_\Lambda\} = \sum_{m_1+m_2+m_3=0} \big\{ c_f^+(m_1), a_{L_0}^+(m_2) \otimes R_\Lambda(m_3) \big\}. \tag{5.21}$$

The following *CM value formula* was proved by Schofer [Sch09] for weakly holomorphic modular forms, and then generalized to harmonic weak Maass forms by Bruinier-Yang [BY09]. We repeat here the statement of [BY09, Theorem 4.7], with a sign error corrected. See also [BHY15, Theorem 5.3.6].

Theorem 5.7.1 (Schofer, Bruinier-Yang)**.** *The Green function $\Phi(f)$ satisfies*

$$\frac{w_{\boldsymbol{k}}}{h_{\boldsymbol{k}}} \sum_{y \in \mathcal{Y}(\mathbb{C})} \frac{\Phi(f, y)}{\#\mathrm{Aut}(y)} = -L'(\xi(f), \Theta_\Lambda, 0) + \mathrm{CT}\{f^+, \mathcal{E}_{L_0} \otimes \Theta_\Lambda\},$$

where

$$\xi : H_{1-\frac{n}{2}}(\omega_L) \to S_{1+\frac{n}{2}}(\overline{\omega}_L)$$

is the Bruinier-Funke differential operator of (3.1).

Remark 5.7.2. Recalling Remark 3.2.1, Theorem 5.7.1 holds even when some points of $\mathcal{Y}(\mathbb{C})$ lie on $\mathcal{Z}(f)(\mathbb{C})$, the divisor along which $\Phi(f)$ has its logarithmic singularities.

Inspired by Theorem 5.7.1, the following formula was conjectured by Bruinier-Yang [BY09].

Theorem 5.7.3. *Assume $d_{\boldsymbol{k}}$ is odd. Every weak harmonic Maass form $f \in H_{1-n/2}(\omega_L)$ with integral principal part satisfies*

$$[\widehat{\mathcal{Z}}(f) : \mathcal{Y}] + c_f^+(0,0) \cdot [\widehat{\boldsymbol{T}} : \mathcal{Y}] = -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \cdot L'(\xi(f), \Theta_\Lambda, 0),$$

where $c_f^+(0,0)$ is the value of $c_f^+(0) \in \mathfrak{S}_L$ at the trivial coset in $L^\vee/L$.

Proof. First assume $f(\tau) = F_{m,\mu}(\tau)$ is the Hejhal-Poincaré series of §3.2, so that

$$\widehat{\mathcal{Z}}(f) = \big(\mathcal{Z}(m,\mu), \Phi_{m,\mu}\big) = \widehat{\mathcal{Z}}^\heartsuit(m,\mu) \otimes \widehat{\boldsymbol{T}}^{\otimes R_\Lambda(m,\mu)}.$$

By Proposition 5.6.2 there is a decomposition

$$\begin{aligned}[\widehat{\mathcal{Z}}^\heartsuit(m,\mu) : \mathcal{Y}] &= \sum_{\substack{m_1+m_2=m \\ m_1>0 \\ (\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)}} R_\Lambda(m_2,\mu_2) \cdot \widehat{\deg}\, \mathcal{Z}_0(m_1,\mu_1) \\ &\quad + \sum_{y\in\mathcal{Y}(\mathbb{C})} \frac{\Phi_{m,\mu}(y)}{\#\mathrm{Aut}(y)},\end{aligned}$$

where we view $\mathcal{Z}_0(m_1,\mu_1) \in \widehat{\mathrm{Div}}(\mathcal{Y})$ as an arithmetic divisor with trivial Green function. Theorem 4.5.1 shows that the first term is

$$\begin{aligned}&\sum_{\substack{m_1+m_2=m \\ m_1>0 \\ (\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)}} R_\Lambda(m_2,\mu_2) \cdot \widehat{\deg}\, \mathcal{Z}_0(m_1,\mu_1) \\ &\quad = -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \sum_{\substack{m_1+m_2=m \\ m_1>0 \\ (\mu_1,\mu_2)\in(\mu+L)/(L_0\oplus\Lambda)}} a_{L_0}^+(m_1,\mu_1) R_\Lambda(m_2,\mu_2) \\ &\quad = -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \sum_{\substack{m_1+m_2=m \\ m_1>0}} \big\{c_f^+(-m), a_{L_0}^+(m_1) \otimes R_\Lambda(m_2)\big\}.\end{aligned}$$

For our special choice of $f(\tau)$ the constant term (5.21) simplifies to

$$\begin{aligned}\mathrm{CT}\big\{f^+, \mathcal{E}_{L_0} \otimes \Theta_\Lambda\big\} &= \big\{c_f^+(0), a_{L_0}^+(0) \otimes R_\Lambda(0)\big\} \\ &\quad + \sum_{m_1+m_2=m} \big\{c_f^+(-m), a_{L_0}^+(m_1) \otimes R_\Lambda(m_2)\big\}\end{aligned}$$

and so Theorem 5.7.1 becomes

$$\begin{aligned}\sum_{y\in\mathcal{Y}(\mathbb{C})} \frac{\Phi_{m,\mu}(y)}{\#\mathrm{Aut}(y)} &= -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \cdot L'(\xi(f), \Theta_\Lambda, 0) + \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \big\{c_f^+(0), a_{L_0}^+(0) \otimes R_\Lambda(0)\big\} \\ &\quad + \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \sum_{m_1+m_2=m} \big\{c_f^+(-m), a_{L_0}^+(m_1) \otimes R_\Lambda(m_2)\big\}.\end{aligned}$$

We have now proved

$$[\widehat{\mathcal{Z}}^{\heartsuit}(m,\mu):\mathcal{Y}] = -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot L'(\xi(f),\Theta_\Lambda,0) + \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot\big\{c_f^+(0), a_{L_0}^+(0)\otimes R_\Lambda(0)\big\} + \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\big\{c_f^+(-m), a_{L_0}^+(0)\otimes R_\Lambda(m)\big\}.$$

Corollary 5.3.2 implies both

$$c_f^+(0,0)\cdot[\widehat{\boldsymbol{T}}:\mathcal{Y}] = -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot\big\{c_f^+(0), a_{L_0}^+(0)\otimes R_\Lambda(0)\big\},$$

and

$$R_\Lambda(m,\mu)\cdot[\widehat{\boldsymbol{T}}:\mathcal{Y}] = -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot\big\{c_f^+(-m), a_{L_0}^+(0)\otimes R_\Lambda(m)\big\},$$

leaving

$$[\widehat{\mathcal{Z}}^{\heartsuit}(m,\mu):\mathcal{Y}] + [\widehat{\boldsymbol{T}}^{\otimes R_\Lambda(m,\mu)}:\mathcal{Y}] + c_f^+(0,0)[\widehat{\boldsymbol{T}}:\mathcal{Y}] = -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot L'(\xi(f),\Theta_\Lambda,0),$$

and proving the desired formula in the special case of $f = F_{m,\mu}$.

Now assume that f satisfies $f^+(\tau) = O(1)$. In particular, $\mathcal{Z}(f) = \emptyset$ and the Green function $\Phi(f)$ is a smooth function on $\mathcal{M}(\mathbb{C})$. The arithmetic intersection is purely archimedean, and Theorem 5.7.1 shows that

$$\begin{aligned}[\widehat{\mathcal{Z}}(f):\mathcal{Y}] &= \sum_{y\in\mathcal{Y}(\mathbb{C})}\frac{\Phi(f,y)}{\#\mathrm{Aut}(y)}\\ &= -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot L'(\xi(f),\Theta_\Lambda,0) + \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot \mathrm{CT}\big\{f^+,\mathcal{E}_{L_0}\otimes\Theta_\Lambda\big\}\\ &= -\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot L'(\xi(f),\Theta_\Lambda,0) + \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot\big\{c_f^+(0), a_{L_0}^+(0)\otimes R_\Lambda(0)\big\}.\end{aligned}$$

Corollary 5.3.2 implies that

$$\frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot\big\{c_f^+(0), a_{L_0}^+(0)\otimes R_\Lambda(0)\big\} = \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot c_f^+(0,0)\cdot a_{L_0}^+(0,0) = -c_f^+(0,0)\cdot[\widehat{\boldsymbol{T}}:\mathcal{Y}],$$

completing the proof in this case.

Finally, every weak harmonic Maass form can be written as a linear combination of the Hejhal-Poincaré series, and a form with holomorphic part $f^+(\tau) = O(1)$. Thus the claim follows from the linearity in f of both sides of the desired equality. □

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