The adic Hilbert eigenvariety

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July 24, 2015

1 Introduction

Robert Coleman constructed finite slope families of overconvergent modular forms parametrized by rigid analytic spaces over $\mathbb{Q}_p$. This was an extension of Hida's construction of ordinary families. One striking difference was that Hida's theory was completely integral and worked over formal schemes while Coleman's theory was rigid analytic. Nevertheless Coleman observed that the characteristic series of the U_p-operator had coefficients in the Iwasawa algebra and suggested to look for an integral or positive characteristic theory of overconvergent modular forms. Following Coleman's intuition, we obtained such a theory for modular forms in [3]. The present paper is an extension of [3] to the case of Hilbert modular forms.

Our main results are :

1. The construction of an integral family of sheaves of overconvergent modular forms, parametrized by the formal spectrum of the Iwasawa algebra. This overconvergent family extends the family of p-adic modular forms defined by Katz and used by Hida. See §6.4.

2. The construction of an eigenvariety sitting over the analytic space associated to the Iwasawa algebra. See §8.4.

Notations Let F be a totally real number field. Denote by g the degree $[F:\mathbb{Q}]$. Fix a prime p. Denote by $\mathfrak{P}_1, \ldots, \mathfrak{P}_f$ the prime ideals of $\mathcal{O}_F$ over p. For each i let f_i be the residual degree and e_i the ramification index. Write $\mathfrak{p} = \mathfrak{P}_1 \cdots \mathfrak{P}_f$ to for the product of all the primes of $\mathcal{O}_F$ above p. Set $q = p$ if $p > 2$ and $q = 4$ if $p = 2$.

2 The weight space

2.1 The Iwasawa algebra

Denote by $\mathbb{T} := \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_m$ and by Λ_F the completed group algebra $\mathbb{Z}_p[\![\mathbb{T}(\mathbb{Z}_p)]\!]$. We write

$$\kappa^{\mathrm{un}} \colon \mathbb{T}(\mathbb{Z}_p) \to \Lambda_F^*$$

for the universal character. Fix an isomorphism of topological groups

$$\rho\colon H \times \mathbb{Z}_p^g \to \mathbb{T}(\mathbb{Z}_p) = \big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*$$

where H is the torsion subgroup of $\mathbb{T}(\mathbb{Z}_p)$. Write Λ_F^0 for $\mathbb{Z}_p[\![\mathbb{Z}_p^g]\!] \cong \mathbb{Z}_p[\![T_1,\ldots,T_g]\!]$ where $1+T_i = e_i$, the i-th vector basis of $\mathbb{Z}_p^g$. It is a complete, regular, local ring with maximal ideal $\mathfrak{m}$. Furthermore $\Lambda_F \cong \Lambda_F^0[H]$ is a finite flat Λ_F^0-algebra. Actually, there is also a canonical projection map $\Lambda_F \to \Lambda_F^0$ obtained by sending all $h \in H$ to 1. We let $\kappa : \mathbb{T}(\mathbb{Z}_p) \to (\Lambda_F^0)^\star$ be the composition of κ^{un} and the above projection. We let $\chi : H \to \Lambda_F^*$ be the composition of the inclusion $H \hookrightarrow \mathbb{T}(\mathbb{Z}_p)$ and the universal character.

We denote by $\mathfrak{W}_F$, resp. $\mathfrak{W}_F^0$ the $\mathfrak{m}$-adic formal scheme defined by Λ_F, resp. Λ_F^0. Then we have a natural map $\mathfrak{W}_F \to \mathfrak{W}_F^0$ which is finite and flat.

Remark 2.1. In [2] the weight space has been defined over the ring of integers of a finite extension K of $\mathbb{Q}_p$ splitting F. The reason is that the classical weights are defined over K. Here we prefer to work over $\mathbb{Z}_p$. As a consequence it will turn out that the characteristic series of the U_p operator will have coefficients in the ring Λ_F defined above, with no need to extend scalars.

2.2 A blow up of the weight space

Consider the blow up $\widetilde{\operatorname{Spec}\Lambda_F}$ of Spec Λ_F with respect to the ideal $\mathfrak{m}$ and let $\mathfrak{t}\colon \widetilde{\mathfrak{W}}_F \to \mathfrak{W}_F$ be the associated $\mathfrak{m}$-adic formal scheme.

We describe in more detail the space $\widetilde{\mathfrak{W}}_F$. Notice that by the universal property of the blow up, the ideal sheaf $\mathfrak{I} := \mathfrak{t}^{-1}(\mathfrak{m}) \subset \mathcal{O}_{\widetilde{\mathfrak{W}}_F}$ is invertible. For every element $\alpha \in \mathfrak{m}$ denote by $\mathfrak{W}_\alpha = \mathfrak{D}_+(\alpha) = \operatorname{Spf}(B_\alpha) \subset \widetilde{\mathfrak{W}}_F$ the open affine formal subscheme where $\mathfrak{I}$ is generated by α ($\mathfrak{W}_\alpha$ is empty unless $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$). In particular the $\mathfrak{m}$-adic topology on B_α coincides with the α-adic topology.

One has variants $\widetilde{\mathfrak{W}}_F^0 \to \mathfrak{W}_F^0$ of the spaces introduced above and associated to the sub-algebra Λ_F^0 of Λ_F. We also have natural finite and flat morphisms $\widetilde{\mathfrak{W}}_F \to \widetilde{\mathfrak{W}}_F^0$. For every element $\alpha \in \mathfrak{m}$ we write $\mathfrak{W}_\alpha^0 = \operatorname{Spf}(B_\alpha^0) \subset \widetilde{\mathfrak{W}}_F^0$ for the open affine formal sub-scheme defined by α.

2.3 The adic weight space

Let $\widetilde{\mathcal{W}}_F$ be the analytic adic space associated to $\widetilde{\mathfrak{W}}_F$. For all open Spf A of $\widetilde{\mathfrak{W}}_F$, the associated open of $\widetilde{\mathcal{W}}_F$ is the open subset of analytic points $\operatorname{Spa}(A,A)^{an}$ of $\operatorname{Spa}(A,A)$. For every element $\alpha \in \mathfrak{m}$, let $\mathcal{W}_\alpha$ be the open subset of $\widetilde{\mathcal{W}}_F$ consisting of the analytic points of the adic space associated to $\mathfrak{W}_\alpha$. Then $\mathcal{W}_\alpha$ is affinoid equal to $\operatorname{Spa}\big(B_\alpha\big[\alpha^{-1}\big], B_\alpha\big) = \operatorname{Spa}(B_\alpha, B_\alpha)^{an}$.

Choosing generators $(p, T_1, \ldots, T_g)$ of $\mathfrak{m}$ then $\widetilde{\mathcal{W}}_F$ is covered by the affinoids

$$\mathcal{W}_p, \mathcal{W}_{T_1}, \cdots, \mathcal{W}_{T_g}.$$

We let $\mathcal{W}_F$ be the analytic adic space associated to $\mathfrak{W}_F$. Namely, $\mathcal{W}_F$ consists of the analytic points $\operatorname{Spa}(\Lambda_F,\Lambda_F)^{an} \subset \operatorname{Spa}(\Lambda_F,\Lambda_F)$. We denote by $t\colon \widetilde{\mathcal{W}}_F \to \mathcal{W}_F$ the morphism of analytic adic spaces associated to $\mathfrak{t}\colon \widetilde{\mathfrak{W}}_F \to \mathfrak{W}_F$.

Lemma 2.2. *The morphism $t\colon \widetilde{\mathcal{W}}_F \to \mathcal{W}_F$ is an isomorphism of adic spaces.*

Proof. For all $\alpha \in \mathfrak{m}$ the subset $\{x \in \mathcal{W}_F, 0 \neq |\alpha|_x \geq |\beta|_x, \forall \beta \in \mathfrak{m}\}$ of $\mathcal{W}_F$ equals $\mathcal{W}_\alpha$ by definition. Moreover, $\mathcal{W}_F$ is covered by the $\mathcal{W}_\alpha$. One concludes easily. □

Remark 2.3. Let us denote by $\mathcal{W}_F^{Berk}$ the subset of rank 1 points of $\mathcal{W}_F$. Then there is a map:

$$\begin{array}{rcl} \Theta : \mathcal{W}_F^{Berk} & \to & \mathbb{P}^g(\mathbb{R}) \\ x & \mapsto & (|p|_x, |T_1|_x, \cdots, |T_g|_x) \end{array}$$

with image included in $[0,1[^{g+1}$. This map may be helpful in order to understand $\mathcal{W}_F$. Let us denote by $(x_0, ..., x_g)$ the coordinates on $\mathbb{P}^g(\mathbb{R})$. Then $\Theta^{-1}(\{x_0 \neq 0\})$ is the set of rank one points on the usual (adic) weight space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ associated to Λ_F.

Let us denote by $\mathcal{W}_F^0$ the analytic adic space attached to $\mathfrak{W}_F^0$. For every element $\alpha \in \mathfrak{m}$ we denote by $\mathcal{W}_\alpha^0$ the associated analytic adic space to $\mathfrak{W}_\alpha^0$.

2.4 Properties of the universal character

2.4.1 Congruence properties

First of all we need to elaborate on the identification $\rho\colon H \times \mathbb{Z}_p^g \simeq \big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*$ of the previous section.

Lemma 2.4. *(1) The group H can be realized as a quotient of $\big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*/(1 + q\mathcal{O}_F \otimes \mathbb{Z}_p)$. Its prime to p part is isomorphic to $\big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*/(1 + \mathfrak{p}\mathcal{O}_F \otimes \mathbb{Z}_p)$.*

(2) Given $(a_1, \ldots, a_g) \in \mathbb{Z}_p^g$, we have $\kappa\big(\rho(a_1, \ldots, a_g)\big) = \prod_{i=1}^g \big(1 + T_i\big)^{a_i} \in (\Lambda_F^0)^*$.

Proof. (1) The group H is finite and its prime to p part maps isomorphically onto $\big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*/(1+\mathfrak{p}\mathcal{O}_F \otimes \mathbb{Z}_p)$ via ρ. Denote by L the quotient $\big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*/H$. The subgroup $1 + q\mathcal{O}_F \otimes \mathbb{Z}_p$ of $\big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*$ is isomorphic to $q\mathcal{O}_F$, and hence to $\mathbb{Z}_p^g$, via the logarithm. In particular it injects into L via the quotient map and the subgroup H of $\big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*$ injects into $\big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*/(1+q\mathcal{O}_F \otimes \mathbb{Z}_p)$. This proves the first claim.

(2) The standard basis elements $e_1, \ldots, e_g$ of $\mathbb{Z}_p^g$ map to $1 + T_1, \ldots, 1 + T_g$ in Λ_F^0. □

We define the following ideals in Λ_F^0 :

- $\mathfrak{m}_n = \big(\alpha^{p^{n-1}}, p\alpha^{p^{n-2}}, \ldots, p^{n-1}\alpha, \alpha \in (T_1, \ldots, T_g)\big)$ if $n \geq 1$.
- $\mathfrak{m}_0 = \mathfrak{m}_1 = (T_1, \ldots, T_g)$.

Lemma 2.5. *For every $n \in \mathbb{Z}_{\geq 1}$ we have that $\kappa\big(\rho(p^{n-1}\mathbb{Z}_p^g)\big) - 1 \subset \mathfrak{m}_n$. In particular we have for all $\mathbb{Z}_{\geq 1}$*

$$\kappa\big(1 + qp^{n-1}\mathcal{O}_F \otimes \mathbb{Z}_p\big) - 1 \subset \mathfrak{m}_n.$$

Moreover, $\kappa(\mathbb{T}(\mathbb{Z}_p)) - 1 \subset \mathfrak{m}_0$.

Proof. Note that $\kappa\big(\rho(p^{n-1}a_1, \ldots, p^{n-1}a_g)\big) = \prod_{i=1}^g \big(1+T_i\big)^{p^{n-1}a_i}$. One computes that $(1+T_i)^{p^{n-1}} - 1$ is contained in the ideal $\big(T_i^{p^{n-1}}, pT_i^{p^{n-2}}, \ldots, p^{n-1}T_i\big)$; see [3, Lemme 2.3].

Notice that κ is trivial on H so that it factors via $\rho(\mathbb{Z}_p^g) \cong \big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^*/H$. Furthermore $\big(1+qp^{n-1}\mathcal{O}_F \otimes \mathbb{Z}_p\big) = \big(1+q\mathcal{O}_F \otimes \mathbb{Z}_p\big)^{p^{n-1}}$ (using the logarithm). In particular $\big(1+qp^{n-1}\mathcal{O}_F \otimes \mathbb{Z}_p\big)$ is contained in $\rho\big(p^{n-1}\mathcal{O}_F \otimes \mathbb{Z}_p\big)$ via the identification above. The second claim follows. □

2.4.2 A key lemma

We introduce a formalism that will be used over and over in the paper. This formalism is inspired by Sen's theory. Let $n \in \mathbb{Z}_{\geq 1}$ and $A_0 \to A_1 \cdots \to A_n$ a tower of Λ_F^0-algebras which are domains. We assume that the group $(\mathcal{O}_F/p^n\mathcal{O}_F)^\star$ acts on A_n by automorphisms of Λ_F^0-algebras and that A_s is the sub-ring of A_n fixed by the kernel H_s of the map $(\mathcal{O}_F/p^n\mathcal{O}_F)^* \to (\mathcal{O}_F/p^s\mathcal{O}_F)^*$.

Let $h \in A_0$ and let $p_0 = 0 \leq p_1 \leq \cdots \leq p_n$ be a sequence of integers. Let $c_n \in h^{-p_n}A_n$ be an element. Set $c_s = \sum_{\sigma \in H_s} \sigma.c_n$. We assume that :

- $c_s \in h^{-p_s}A_s$ for all $s \geq 0$,
- $c_0 = 1$.

Set $b_s = \sum_{\sigma \in (\mathcal{O}_F/p^s\mathcal{O}_F)^*} \kappa(\tilde{\sigma})\sigma(c_s) \in h^{-p_s}A_s$ for $s \geq 1$ and $b_0 = 1$. Here $\tilde{\sigma} \in \mathbb{T}(\mathbb{Z}_p)$ is a lift of σ so that b_s depends on c_s and on the choices of lifts.

Lemma 2.6. *1. An other choices of lifts $\tilde{\sigma}$ would give an element b'_s and*

- $b'_s - b_s \in h^{-p_s}\mathfrak{m}_s A_s$ *if* $s \geq 1$, $p \geq 3$,
- $b'_s - b_s \in h^{-p_s}\mathfrak{m}_{s-1} A_s$ *if* $s \geq 2$, $p = 2$,
- $b'_1 - b_1 \in h^{-p_1}\mathfrak{m}_0 A_1$ *if* $p = 2$.

2. We have the following congruence relations :

- $b_s - b_{s-1} \in h^{-p_s}\mathfrak{m}_{s-1} A_s$ *if* $s \geq 1$, $p \geq 3$,
- $b_s - b_{s-1} \in h^{-p_s}\mathfrak{m}_{s-2} A_s$ *if* $s \geq 2$, $p \geq 3$,
- $b_1 - b_0 \in h^{-p_1}\mathfrak{m}_0 A_0$ *if* $p = 2$.

Proof. The first point follows from lemma 2.5. To prove the second point, assume that $s \geq 1$ and notice that

$$\begin{aligned} b_s &= \sum_{\tau \in (\mathcal{O}_F/p^{s-1}\mathcal{O}_F)^\star} \kappa(\tilde{\tau})\tau\Big(\sum_{\sigma \in 1+p^{s-1}\mathcal{O}_F/p^s\mathcal{O}_F} (\kappa(\tilde{\sigma})-1)\sigma(c_s) + c_{s-1}\Big) \\ &= \sum_{\tau \in (\mathcal{O}_F/p^{s-1}\mathcal{O}_F)^\star} \kappa(\tilde{\tau})\tilde{\tau}\Big(\sum_{\sigma \in 1+p^{s-1}\mathcal{O}_F/p^s\mathcal{O}_F} (\kappa(\tilde{\sigma})-1)\sigma(c_s)\Big) + b_{s-1} \end{aligned}$$

One concludes by applying the lemma 2.5 (we are also using the first point). □

2.4.3 Analyticity of the universal character

We now study the analytic properties of the universal character. The degree of analyticity depends on the p-adic valuation of $T_1, \cdots, T_g$. This motivates the following definition. For $\frac{r}{s} \in \mathbb{Q}_{\geq 1}$ we define the following rational open subsets of $\mathcal{W}_F^0$:

- $\mathcal{W}^0_{F, \leq \frac{r}{s}} = \{x \in \mathcal{W}^0_F, |\alpha^r|_x \leq |p^s|_x \neq 0, \ \forall \alpha \in \mathfrak{m}\}$,

- $\mathcal{W}^0_{F,\geq\frac{r}{s}} = \{x \in \mathcal{W}^0_F, \exists \alpha \in \mathfrak{m},\ |p^s|_x \leq |\alpha^r|_x \neq 0\}$.

Set $\mathcal{W}^0_{F,\leq\infty} := \mathcal{W}^0_F$. If $I = [a,b]$ is a closed interval with a, $b \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$, define $\mathcal{W}^0_{F,I} = \mathcal{W}^0_{F,\leq b} \cap \mathcal{W}^0_{F,\geq a}$. For all $\alpha \in \mathfrak{m}$ we let $\mathcal{W}^0_{\alpha,I} = \mathcal{W}^0_{F,I} \cap \mathcal{W}^0_\alpha$.

Remark 2.7. If $x \in \mathcal{W}^0_\alpha$ is a rank one point, then α is a pseudo-uniformizer on the residue field $k(x)$. Let us denote by $v_\alpha : k(x) \to \mathbb{R} \cup \{\infty\}$ the valuation on $k(x)$ normalized by $v_\alpha(\alpha) = 1$. Notice that the norm $p^{-v_\alpha(\cdot)}$ represents the equivalence class of $|.|_x$. Then $x \in \mathcal{W}^0_{\alpha,I}$ if and only if $v_\alpha(p) \in I$.

We now construct formal models. Take an element $\alpha \in \mathfrak{m}$. We define $B^0_{\alpha,I} = \mathrm{H}^0(\mathcal{W}_{\alpha,\mathrm{I}}, \mathcal{O}^+_{\mathcal{W}_{\alpha,\mathrm{I}}})$.

Set $\mathfrak{W}^0_{\alpha,I} = \mathrm{Spf}\ B^0_{\alpha,I}$. The analytic fiber of $\mathfrak{W}^0_{\alpha,I}$ is $\mathcal{W}^0_{\alpha,I}$. For various α's, the $\mathfrak{W}^0_{\alpha,I}$ glue to a formal scheme $\widetilde{\mathfrak{W}}^0_{F,I}$ with analytic fiber $\mathcal{W}^0_{F,I}$. Remark that $\widetilde{\mathfrak{W}}^0_{F,[1,\infty]} = \widetilde{\mathfrak{W}}^0_F$.

If $I \subset [0,\infty[$, then $\widetilde{\mathfrak{W}}^0_{F,I}$ is a p-adic formal scheme (the $\mathfrak{m}$-adic topology is the p-adic one). In the lemma below, $\mathbb{G}_m$, $\mathbb{G}_a$ are considered as functors on the category of p-adic formal schemes equipped with a structural morphism to $\mathfrak{W}^0_F$. Let $\epsilon = 1$ if $p \neq 2$ and $\epsilon = 3$ if $p = 2$. The group $\mathbb{T}(\mathbb{Z}_p) \cdot \big(1 + p^{n+\epsilon}\mathcal{O}_F \otimes \mathbb{G}_a\big)$ is a subgroup of $\mathbb{G}_m$.

Proposition 2.8. *Let $n \geq 0$ be an integer. Suppose that $I \subset [0, p^n]$. The character κ extends to a pairing*

$$\widetilde{\mathfrak{W}}^0_{F,I} \times \mathbb{T}(\mathbb{Z}_p) \cdot \big(1 + p^{n+\epsilon}\mathcal{O}_F \otimes \mathbb{G}_a\big) \longrightarrow \mathbb{G}_m.$$

It restricts to a pairing

$$\widetilde{\mathfrak{W}}^0_{F,I} \times \big(1 + p^{n+\epsilon+n'}\mathcal{O}_F \otimes \mathbb{G}^+_a\big) \longrightarrow 1 + qp^{n'}\mathbb{G}_a$$

for all $n' \in \mathbb{Z}_{\geq 0}$.

Proof. Easy and left to the reader. □

3 Hilbert modular varieties and the Igusa tower

3.1 Hilbert modular varieties

Fix an integer $N \geq 4$ and a prime p not dividing N. Let $\mathfrak{c}$ be a fractional ideal of F and let $\mathfrak{c}^+$ be the cone of totally positive elements. Denote by $\mathcal{D}_F$ the different ideal of $\mathcal{O}_F$. Let $M(\mu_N, \mathfrak{c})$ be the Hilbert modular scheme over $\mathbb{Z}_p$ classifying triples $(A, \iota, \Psi, \lambda)$ consisting of: (1) abelian schemes $A \to S$ of relative dimension g over S, (2) an embedding $\iota\colon \mathcal{O}_F \subset \mathrm{End}_S(A)$, (3) a closed immersion $\Psi\colon \mu_N \otimes \mathcal{D}_F^{-1} \to A$ compatible with $\mathcal{O}_F$-actions, (4) if $P \subset \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)$ is the sheaf for the étale topology on S of symmetric $\mathcal{O}_F$-linear homomorphisms from A to the dual abelian scheme $A^\vee$ and if $P^+ \subset P$ is the subset of polarizations, then λ is an isomorphism of étale sheaves $\lambda\colon (P, P^+) \cong (\mathfrak{c}, \mathfrak{c}^+)$, as invertible $\mathcal{O}_F$-modules with a notion of positivity. The triple is subject to the condition that the map $A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^\vee$ is an isomorphism abelian schemes (the so called Deligne-Pappas condition).

We write $\overline{M}(\mu_N, \mathfrak{c})$ and $\overline{M}^*(\mu_N, \mathfrak{c})$ for a projective toroidal compactification, respectively the minimal or Satake compactification of $M(\mu_N, \mathfrak{c})$. Let $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})$ (resp. $\overline{\mathfrak{M}}^*(\mu_N, \mathfrak{c})$) be the associated formal schemes. They are endowed with a semi-abelian scheme G with $\mathcal{O}_F$-action.

There exists greatest open (formal) subschemes $\overline{M}^{\mathrm{R}}(\mu_N, \mathfrak{c}) \subset \overline{M}(\mu_N, \mathfrak{c})$, resp. $\overline{\mathfrak{M}}^{\mathrm{R}}(\mu_N, \mathfrak{c}) \subset \overline{\mathfrak{M}}(\mu_N, \mathfrak{c})$ such that ω_G, the conormal sheaf to the identity of G, is an invertible $\mathcal{O}_{\overline{M}^{\mathrm{R}}(\mu_N,\mathfrak{c})} \otimes_{\mathbb{Z}} \mathcal{O}_F$-module, resp. $\mathcal{O}_{\overline{\mathfrak{M}}^{\mathrm{R}}(\mu_N,\mathfrak{c})} \otimes_{\mathbb{Z}} \mathcal{O}_F$-module (the so called Rapoport condition). The complement is empty if p does not divide the discriminant of F and, in general, it is of codimension 2 in the characteristic p special fiber of $\overline{M}(\mu_N, \mathfrak{c})$.

We denote by $\mathrm{Ha} \in \mathrm{H}^0(M^*(\mu_N, \mathfrak{c})_{\mathbb{F}_p}, \det \omega_G^{p-1})$ the Hasse invariant. We let $\mathrm{Hdg} \subset \mathcal{O}_{\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})}$ be the Hodge ideal defined by the Hasse invariant (see [3, §A.1] for a precise definition: locally on $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})$ it is the ideal generated by p and a (any) lift of a local generator of $\mathrm{Ha} \det \omega_G^{1-p}$).

3.2 Canonical subgroups

Let A_0 be a $\mathbb{Z}_p$-algebra and $\alpha \in A_0$ a non-zero element. We assume that A_0 satisfies the following:

$(*)$ A_0 is an integral domain, it is the α-adic completion of a $\mathbb{Z}_p$-algebra of finite type and $p \in \alpha A_0$.

Let $\overline{M}(\mu_N, \mathfrak{c}) \times_{\mathrm{Spec}\ \mathbb{Z}_p} \mathrm{Spec}\ A_0$ be the base change of the toroidal compactification via $\mathrm{Spec}\ A_0 \to \mathrm{Spec}\ \mathbb{Z}_p$ and let $\mathfrak{Y}$ be the associated formal scheme over $\mathrm{Spf}\ A_0$.

Definition 3.1. For every integer $r \in \mathbb{N}$ denote by $\mathfrak{Y}_r \to \mathfrak{Y}$ the formal scheme over $\mathfrak{Y}$ representing the functor which to any α-adically complete A_0-algebra R associates the equivalence classes of pairs $\big(h\colon \mathrm{Spf}\ R \to \mathfrak{X}, \eta \in \mathrm{H}^0(\mathrm{Spf}\ R, h^* \det \omega_G^{(1-p)p^{r+1}}\big)$ such that

$$\mathrm{Ha}^{p^{r+1}} \eta = \alpha \mod p^2.$$

Two pairs (h, η) et (h', η') are declared equivalent if $h = h'$ and $\eta = \eta'(1 + \frac{p^2}{\alpha} u)$ for some $u \in R$.

We also denote by $\mathfrak{Y}_r^{\mathrm{R}} \subset \mathfrak{Y}_r$ the open formal subscheme where the Rapoport condition holds (see §3).

Proposition 3.2. *Assume that $p \in \alpha^{p^k} A_0$. Then for every integer $1 \leq n \leq r + k$ one has a canonical sub-group scheme H_n of $G[p^n]$ over $\mathfrak{Y}_r$ and H_n modulo $p\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}$ lifts the kernel of the n-th power of Frobenius. Moreover H_n is finite flat and locally of rank p^{ng}, it is stable under the action of $\mathcal{O}_F$, and the Cartier dual H_n^D is étale locally over $A_0\big[\alpha^{-1}\big]$ isomorphic to $\mathcal{O}_F/p^n$ (as $\mathcal{O}_F$-module).*

Proof. All claims follow from [3, Appendix 1]. □

Proposition 3.3. *For every $r \in \mathbb{Z}_{\geq 2}$ the isogeny given by dividing by the canonical subgroup H_1 of level 1 defines a finite morphism $\phi\colon \mathfrak{Y}_r \to \mathfrak{Y}_{r-1}$. The restriction to the Rapoport locus $\phi\colon \mathfrak{Y}_r^{\mathrm{R}} \to \mathfrak{Y}_{r-1}^{\mathrm{R}}$ is finite and flat of degree p^g.*

Proof. This is the content of [3, Cor. A.2] which is written for general p-divisible groups. The last claim follows as relative Frobenius is finite and it is flat over the (smooth) Rapoport locus. □

3.3 The partial Igusa tower

3.3.1 Construction

We use the notations of the previous section. Let $A := A_0[\alpha^{-1}]$: it is a Tate ring in the sense of Huber [5] with ring of definition A_0. Let $A^+ \subset A$ be the normalization of A_0 in A. The fact that A_0 is noetherian implies that $\mathrm{Spa}(A, A^+)$ is an adic space; [6, Thm. 2.2]. We define

$$\mathcal{Y}_r := \mathfrak{Y}_r^{\mathrm{ad}} \times_{\mathrm{Spa}(A_0,A_0)} \mathrm{Spa}(A, A^+) :$$

here $\mathfrak{Y}_r^{\mathrm{ad}}$, resp. $\mathrm{Spa}(A_0, A_0)$ is the adic space associated to the formal scheme $\mathfrak{Y}_r$, resp. $\mathrm{Spf}\, A_0$, and the fibre product is taken in the category of adic spaces.

Assume that $p \in \alpha^{p^k} A_0$ and let $r \in \mathbb{N}$ and $n \in \mathbb{N}$ be an integer such that $1 \leq n \leq r + k$. It follows from Proposition 3.2 that H_n^D over $\mathcal{Y}_r$ is étale locally isomorphic to $\mathcal{O}_F/p^n\mathcal{O}_F$. We let $\mathcal{IG}_{n,r} \to \mathcal{Y}_r$ be the Galois cover for the group $(\mathcal{O}_F/p^n\mathcal{O}_F)^*$ classifying the isomorphisms $\mathcal{O}_F/p^n\mathcal{O}_F \to H_n^D$, as group schemes, equivariant for the $\mathcal{O}_F$-action.

We define $\mathfrak{IG}_{n,r} \to \mathfrak{Y}_r$ to be the formal scheme given by the normalization of $\mathfrak{Y}_r$ in $\mathcal{IG}_{n,r}$. See [3, §3.2] for details. Such morphism is finite and is endowed with an action of $(\mathcal{O}_F/p^n\mathcal{O}_F)^*$. One then gets a sequence of finite, $(\mathcal{O}_F/p^{r+k}\mathcal{O}_F)^*$-equivariant morphisms

$$\mathfrak{IG}_{r+k,r} \to \mathfrak{IG}_{r+k-1,r} \to \cdots \to \mathfrak{Y}_r.$$

The morphisms $h\colon \mathfrak{IG}_{n,r} \to \mathfrak{IG}_{n-1,r}$ are finite and étale over $\mathcal{Y}_r$. In particular there is a trace map $\mathrm{Tr}_{\mathfrak{IG}}\colon h_*\mathcal{O}_{\mathfrak{IG}_{n,r}} \to \mathcal{O}_{\mathfrak{IG}_{n-1,r}}$.

3.3.2 Ramification

Proposition 3.4. *We have*

$$\mathrm{Hdg}^{p^{n-1}} \mathcal{O}_{\mathfrak{IG}_{n-1,r}} \subset \mathrm{Tr}_{\mathfrak{IG}}\big(h_*\mathcal{O}_{\mathfrak{IG}_{n,r}}\big)$$

for every $1 \leq n \leq r + k$.

If p is unramified one further has $\mathrm{Tr}_{\mathfrak{IG}}(h_*\mathcal{O}_{\mathfrak{IG}_{1,r}}) = \mathcal{O}_{\mathfrak{Y}_r}$.

Proof. The claim for $n \geq 2$ follows arguing as in [3, Prop. 3.4]. We recall the argument. By normality the natural map $\mathcal{IG}_{n,r} \to H_n^D$ over $\mathcal{Y}_r$, associating to an isomorphism $\mathcal{O}_F/p^n\mathcal{O}_F \to H_n^D$ the image of $1 \in \mathcal{O}_F/p^n\mathcal{O}_F$, extends to a morphism of formal schemes $\mathfrak{IG}_{n,r} \to H_n^D$ over $\mathfrak{Y}_r$. In particular we get a commutative diagram of formal schemes over $\mathfrak{Y}_r$:

$$\begin{array}{ccc} \mathfrak{IG}_{n,r} & \longrightarrow & H_n^D \\ \downarrow & & \downarrow \\ \mathfrak{IG}_{n-1,r} & \longrightarrow & H_{n-1}^D, \end{array}$$

which is cartesian over the analytic fiber $\mathcal{Y}_r$. In particular $\mathfrak{IG}_{n,r} \to \mathfrak{IG}_{n-1,r}$ is the normalization of the fppf $(H_n/H_{n-1})^D$-torsor over $\mathfrak{IG}_{n-1,r}$ obtained by the fibre product of the diagram above. One reduces to prove the claimed result for the trace of the morphism $H_n^D \to H_{n-1}^D$ (over $\mathfrak{Y}_r$)

and this follows from the relation between different and the trace and a detailed analysis of the different of $(H_n/H_{n-1})^D$ given in [3, Cor. A.2].

We are left to discuss the case $n=1$. If p is unramified then the degree of $\mathcal{IG}_{1,r}\to\mathcal{Y}_r$ is prime to p and the second claim of the proposition follows immediately. If p is ramified we let $\mathfrak{p}$ be the product of all primes of $\mathcal{O}_F$ over p. We introduce a variant of $\mathcal{IG}_{1,r}$ by setting $\mathcal{IG}'_{1,r}$ to be the adic space over $\mathcal{Y}_r$ classifying isomorphisms $\mathcal{O}_F/\mathfrak{p}\mathcal{O}_F\to H_1\big[\mathfrak{p}\big]^D$, as group schemes, equivariant for the $\mathcal{O}_F$-action. Here $H_1\big[\mathfrak{p}\big]$ is the kernel of multiplication by $\mathfrak{p}$ on H_1.

We have a natural map of adic spaces $\mathcal{IG}_{1,r}\to\mathcal{IG}'_{1,r}\to\mathcal{Y}_r$. Taking normalizations we get morphisms of formal schemes $\mathfrak{IG}_{1,r}\to\mathfrak{IG}'_{1,r}\to\mathfrak{Y}_r$.

The degree of $\mathcal{IG}'_{1,r}\to\mathcal{Y}_r$ is the order of $\big(\mathcal{O}_F/\mathfrak{p}\mathcal{O}_F\big)^*$ which is prime to p so that $\mathrm{Tr}_{\mathfrak{IG}}(h_*\mathcal{O}_{\mathfrak{IG}'_{1,r}})=\mathcal{O}_{\mathfrak{Y}_r}$. We are left to estimate the image of the trace map associated to the morphism $\mathfrak{IG}_{1,r}\to\mathfrak{IG}'_{1,r}$. Arguing as at the beginning of the proof we get a commutative diagram of formal schemes over $\mathfrak{Y}_r$, which is cartesian over $\mathcal{Y}_r$:

$$\begin{array}{ccc} \mathfrak{IG}_{1,r} & \longrightarrow & H_1^D \\ \downarrow & & \downarrow \\ \mathfrak{IG}'_{1,r} & \longrightarrow & H_1[\mathfrak{p}]^D \end{array} .$$

Thus $\mathfrak{IG}_{1,r}$ is the normalization of a torsor under $(H_1/H_1[\mathfrak{p}])^D$. We have an exact sequence $0\to(H_1/H_1[\mathfrak{p}])^D\to H_1^D\to(H_1[\mathfrak{p}])^D\to 0$. It follows that Hdg is contained in the different of $(H_1/H_1[\mathfrak{p}])^D$ over $\mathfrak{Y}_r$ and we can conclude.

□

We immediately get the following

Corollary 3.5. *Let Spf R be an open of $\mathfrak{Y}_r$ such that the ideal sheaf* Hdg *is trivial and choose a generator* $\tilde{\mathrm{Ha}}$. *For every $0\le n\le r+k$ there exists elements $c_0=1$ and $c_n\in\tilde{\mathrm{Ha}}^{-\frac{p^n-1}{p-1}}\mathcal{O}_{\mathfrak{IG}_{n,r}}(Spf\ R)$ for $n\ge 1$ such that $\mathrm{Tr}_{\mathfrak{IG}}(c_n)=c_{n-1}$ for every $n\ge 1$.*

3.3.3 Frobenius

Recall from Proposition 3.3 that we have a Frobenius map $\phi\colon\mathfrak{Y}_r\to\mathfrak{Y}_{r-1}$.

Proposition 3.6. *There exists a $(\mathcal{O}_F/p^n\mathcal{O}_F)^*$-equivariant map $\phi\colon\mathfrak{IG}_{n,r}\to\mathfrak{IG}_{n,r-1}$ lifting the map $\phi\colon\mathfrak{Y}_r\to\mathfrak{Y}_{r-1}$.*

Proof. As $\mathfrak{IG}_{n,r}$ is constructed by normalizing $\mathfrak{Y}_r$ in $\mathcal{IG}_{n,r}$, it suffices to construct a lift $\phi\colon\mathcal{IG}_{n,r}\to\mathcal{IG}_{n,r'}$ at the level of adic spaces.

Notice that H_{n+1}/H_1 is the canonical subgroup H'_n of level n of $G'=G/H_1$ thanks to [3, cor. A.2]. As multiplication by p on H_{n+1} defines an isomorphism $H_{n+1}/H_1\cong H_n$ and hence an isomorphism $H_n\cong H'_n$ (over $\mathcal{Y}_r$!). Any $\mathcal{O}_F$-linear isomorphism map $\mathcal{O}_F/p^n\mathcal{O}_F\to H_n^D$ defines a $\mathcal{O}_F$-linear isomorphism $\Psi'\colon\mathcal{O}_F/p^n\mathcal{O}_F\to(H'_n)^D$. □

3.4 The basic constructions

Recall from §2.2 that we have introduced an $\mathfrak{m}$-adic formal scheme $\widetilde{\mathfrak{W}}_F^0$ providing a model of the weight space. It is characterized by the property that ideal generated by the maximal ideal of Λ_F^0 defines an invertible ideal sheaf of $\mathcal{O}_{\widetilde{\mathfrak{W}}_F^0}$.

For every element $\alpha \in \mathfrak{m}$ we have denoted by $\mathfrak{W}_\alpha^0 := \operatorname{Spf} B_\alpha^0$ the open formal affine subscheme of $\widetilde{\mathfrak{W}}_F^0$ defined by α. We have set $\mathcal{W}_\alpha^0 := \operatorname{Spa}\big(B_\alpha^0[\alpha^{-1}], B_\alpha^0\big)$ to be the analytic adic subspace of $\widetilde{\mathcal{W}}_F^0$ defined by $\mathfrak{W}_\alpha^0$.

Applying the construction of section 3.2 with $A_0 = B_\alpha^0$, one obtains a formal scheme $\mathfrak{X}_{r,\alpha}$ over $\mathfrak{W}_\alpha^0$. Set $\mathcal{X}_{r,\alpha}$ to be the associated analytic adic space over $\mathcal{W}_\alpha$.

For all choices of α, these formal schemes $\mathfrak{X}_{r,\alpha}$ glue into a formal scheme $\mathfrak{X}_r \to \widetilde{\mathfrak{W}}_F^0$. We let $\mathcal{X}_r \to \mathcal{W}_F^0$ be the analytic adic space associated to $\mathfrak{X}_r$.

Let $I = [p^k, p^{k'}] \subset [1, \infty]$ be an interval. We defined a formal scheme $\widetilde{\mathfrak{W}}_{F,I}^0 \to \widetilde{\mathfrak{W}}_F^0$. We let $\mathfrak{X}_{r,I} = \mathfrak{X}_r \times_{\widetilde{\mathfrak{W}}_F^0} \widetilde{\mathfrak{W}}_{F,I}^0$ and $\mathfrak{X}_{r,\alpha,I} = \mathfrak{X}_{r,\alpha} \times_{\mathfrak{W}_\alpha^0} \mathfrak{W}_{\alpha,I}^0$.

Let $n \in \mathbb{N}$ be an integer such that $1 \le n \le r + k$. Applying the considerations of §3.3 we obtain an étale cover of adic spaces

$$\mathcal{IG}_{n,r,\alpha,I} \to \mathcal{X}_{r,\alpha,I},$$

for the group $(\mathcal{O}_F/p^n\mathcal{O}_F)^*$, classifying the isomorphisms $\mathcal{O}_F/p^n\mathcal{O}_F \to H_n^D$, as group schemes, equivariant for the $\mathcal{O}_F$-action. This is a morphism of adic spaces associated to a morphism of formal schemes

$$\mathfrak{IG}_{n,r,\alpha,I} \to \mathfrak{X}_{r,\alpha,I}.$$

For various $\alpha \in \mathfrak{m}$ these adic spaces and formal schemes glue and we obtain $\mathcal{IG}_{n,r,I} \to \mathcal{X}_{r,I}$ and $\mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$.

3.4.1 Equations

In this subsection we give local equations for some of the spaces defined so far. We have :

$$B_\alpha^0 = \mathbb{Z}_p[[T_1, ..., T_g]]\langle \frac{p}{\alpha}, \frac{T_1}{\alpha}, ..., \frac{T_g}{\alpha} \rangle.$$

If $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$, this is a regular ring. Otherwise this ring is 0. Consider an interval $I = [p^k, p^h]$ with $k \ge 0$ an integer and $h \ge k$ an integer or $h = \infty$. Take $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$. If $1 \in I$ and $\alpha = p$, then $B_{\alpha,I}^0 = B_\alpha$. If $\alpha = p$ and $1 \notin I$, $B_{\alpha,I}^0 = 0$. Assume now that $\alpha \ne p$.

1. If $k' \ne \infty$ then $B_{\alpha,I}^0 = \mathbb{Z}_p[[T_1, ..., T_g]]\langle \frac{T_1}{\alpha}, ..., \frac{T_g}{\alpha}, u, v\rangle / \big(\alpha^{p^k} v - p, uv - \alpha^{p^{h-k}}\big)$,
2. If $k' = \infty$ then $B_{\alpha,I}^0 = \mathbb{Z}_p[[T_1, ..., T_g]]\langle \frac{T_1}{\alpha}, ..., \frac{T_g}{\alpha}, u\rangle / (\alpha^{p^k} v - p)$.

In the second case $B_{\alpha,I}^0$ is a regular ring and, in particular, it is normal. In the first case, one checks that $B_{\alpha,I}$ is normal by verifying that it is Cohen-Macaulay and regular in codimension 1 (Serre's criterion).

Let $U := \operatorname{Spf} A$ be a formal open affine subscheme of $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})$ over which ω_G is trivial. Let $\tilde{\mathrm{Ha}}$ be a lift of Ha. The inverse image of U in $\mathfrak{X}_{r,\alpha,[p^k,\infty]}$ is Spf R with $R := A \hat{\otimes}_{\mathbb{Z}_p} B_\alpha^0 \langle u, w\rangle / (w\tilde{\mathrm{Ha}}^{p^r+1} -$

$\alpha, \alpha^{p^k}u - p)$. Similarly for integers $0 \le k \le h$ the inverse image of U in $\mathfrak{X}_{r,\alpha,[p^k,p^h]}$ is $\mathrm{Spf}\ R' := A\hat{\otimes}_{\mathbb{Z}_p} B^0_\alpha\langle u, v, w\rangle/(w\tilde{\mathrm{Ha}}^{p^r+1} - \alpha, \alpha^{p^k}u_k - p, u_k v_h - \alpha^{p^{h-k}})$.

Lemma 3.7. *The rings R and R' are normal.*

Proof. By Deligne-Pappas, we know that the non-smooth locus of A has codimension at least 2. It follows easily that $A \otimes B^0_\alpha\langle u, w\rangle$ is Cohen-Macaulay. Since R is a complete intersection, it is Cohen-Macaulay. Let us check that R is regular in codimension 1. Let $\mathfrak{P}$ be a codimension 1 prime ideal of R. Then $R_{\mathfrak{P}}$ is easily seen to be regular if $\alpha \notin \mathfrak{P}$. Assume that α lies in $\mathfrak{P}$. Then $\mathfrak{P}$ is a generic point of

$$A/pA \otimes_{\mathbb{F}_p} B^0_\alpha/\alpha B^0_\alpha[u, w]/(w\mathrm{Ha}^{p^{r+1}}).$$

Either $\mathrm{Ha}^{p^{r+1}} \in \mathfrak{P}$ and in that case, $\mathfrak{P}$ maps to the generic point $\mathfrak{P}'$ of an irreducible component of $A/(pA, \mathrm{Ha})$. By [1] the ring $(A/pA)_{\mathfrak{P}'}$ is a DVR. Let t be a generator of the maximal ideal. Let $\hat{t}$ be a lift in R. Then $\hat{t}$ is a generator of the maximal ideal in $R_{\mathfrak{P}}$. If $w \in \mathfrak{P}$, then $\mathfrak{P}$ maps to the generic point of A/pA and one concludes as before. The case of the ring R' follows along similar lines. □

Corollary 3.8. *The formal schemes $\mathfrak{X}_{r,[p^k,p^h]}$ are normal.*

4 Overconvergent modular sheaves in characteristic 0

In this section we will construct sheaves of overconvergent forms over the adic space $\mathcal{W}_F \setminus \{|p| = 0\}$. This is already done in [2] but our goal is to provide canonical integral structures.

4.1 A modified integral structure on ω_G

Fix an interval $I = \left[p^k, p^{k'}\right]$ with k and k' integers such that $k' \ge k \ge 0$. Let $r \in \mathbb{Z}_{\ge 1}$ and fix a positive integer n with $n \le r + k$. Let G be the semi-abelian scheme over $\mathfrak{X}_{r,I}$. It follows from Proposition 3.2 that there exists a canonical subgroup $H_n \subset G[p^n]$.

Let $g_n \colon \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ be the partial Igusa tower defined in §3.4. Let ω_G be the sheaf of invariant differentials of G. It follows from [3, Cor. A.2] that the kernel of the map $\omega_G/p^n\omega_G \to \omega_{H_n}$ is annihilated by $\mathrm{Hdg}^{\frac{p^n-1}{p-1}}\omega_G$. We deduce that the projection map $\omega_G \to \omega_G/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\omega_G$ factors via ω_{H_n}. One then has a commutative diagram of $fppf$ sheaves of abelian groups over $\mathfrak{X}_{r,I}$:

$$\begin{array}{ccc} & & \omega_G \\ & & \downarrow \\ H_n^D & \xrightarrow{\mathrm{HT}} & \omega_{G_n} \\ & & \downarrow \\ & & \omega_G/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\omega_G \end{array}$$

where all vertical arrows are surjective and the horizontal arrow is the Hodge-Tate map.

Over $\mathfrak{IG}_{n,r,I}$ we have a universal section $P \in H_n^D$ which is the image of 1 via the universal morphism $\psi_n \colon \mathcal{O}_F/p^n\mathcal{O}_F \to H_n^D$.

Proposition 4.1. *Let $\mathcal{F}$ be the inverse image in ω_G of the $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-submodule of $\omega_G/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\omega_G$ spanned by $\mathrm{HT}(P)$. Then $\mathcal{F}$ is a locally free $\mathcal{O}_F \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-module of rank 1, the cokernel of $\mathcal{F} \subset \omega_G$ is annihilated by $\mathrm{Hdg}^{\frac{1}{p-1}}$ and the map $\mathrm{HT}\circ\psi_n$ defines an isomorphism of $\mathcal{O}_F\otimes\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-modules:*

$$\mathrm{HT}' \colon \mathcal{O}_F \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}/p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \cong \mathcal{F}/p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}\mathcal{F}.$$

Proof. This is a variant of [2], prop. 3.4. Let $U := \mathrm{Spf}\ R \subset \mathfrak{IG}_{n,r,I}$ be an open formal affine subscheme such that $\omega_G|_U$ is free of rank g as an R-module. Write $\overline{M} \in \mathrm{M}_{n\times n}\big(R/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}R\big)$ for the matrix of the linearization of the map HT over U. Thanks to [3, prop. A.3] it has determinant ideal equal to $\mathrm{Hdg}^{\frac{1}{p-1}}$. In particular $\mathrm{Hdg}^{\frac{1}{p-1}}\omega_G/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\omega_G$ lies in the span of $\mathrm{HT}(P)$.

Let $M \in \mathrm{M}_{n\times n}(R)$ be any lift of $\overline{M}$. Its determinant δ is $\mathrm{Hdg}^{\frac{1}{p-1}}$ (up to unit). Let $\mathcal{S} \subset \omega_G|_U$ be the submodule spanned by the columns of M. Then $\delta\omega_G \subset \mathcal{S}$. Since $p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\omega_G = p^n\mathrm{Hdg}^{-\frac{p^n-1}{p}}\cdot\delta\omega_G \subset \mathcal{S}$ one deduces that $\mathcal{F}|_U$ coincides with the $\mathcal{S}$. In particular it is a free R-module of rank g. By definition it is stable for the action of $\mathcal{O}_F$.

For every $x \in \mathcal{O}_F/p^n\mathcal{O}_F$ the image $y \in \mathrm{HT}(\psi_n(x))$ lies by construction in the image of $\mathcal{S}$ in $\omega_G/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\omega_G$. As $\mathrm{Hdg}^{-\frac{p^n}{p-1}} = \mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\cdot\mathrm{Hdg}^{\frac{1}{p-1}}$ and as $\omega_G/\mathcal{F}$ is annihilated by $\mathrm{Hdg}^{\frac{1}{p-1}}$ it follows that any two lifts y' and y'' in $\mathcal{S}$ differ by an element lying in $\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\cdot\mathrm{Hdg}^{\frac{1}{p-1}}\omega_G = \mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\cdot\mathcal{S}$. We then get a well defined map $\mathcal{O}_F/p^n\mathcal{O}_F \to \mathcal{F}/p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}\mathcal{F}$ inducing $\mathrm{HT}\circ\psi_n$ when composed with the projection to $\omega_G/p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}\mathcal{F}\omega_G$. This provides the HT'. By construction its restriction to U is a surjective map, of free $R/p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}R$-modules of rank g and, hence it is an isomorphism. It follows that $\mathcal{S} = \mathcal{F}|_U$ is a free $\mathcal{O}_F\otimes R$-module of rank 1 concluding the proof of the Proposition.

∎

We denote by $f_n \colon \mathfrak{F}_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ the torsor for the group $1+p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_a$ defined by

$$\mathfrak{F}_{n,r,I}(R) := \big\{\omega \in \mathcal{F},\ \omega = \mathrm{HT}'(1) \text{ in } \mathcal{F}/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\mathcal{F}\big\}.$$

One has an action of $(\mathcal{O}_F\otimes\mathbb{Z}_p)^*$ on $\mathfrak{F}_{n,r,I}$, lifting the action of $(\mathcal{O}_F/p^n\mathcal{O}_F)^*$ on $\mathfrak{IG}_{n,r,I}$, given by $\lambda\cdot(\omega,1) = (\lambda\omega,\lambda)$. We then get an action of the group $(\mathcal{O}_F\otimes\mathbb{Z}_p)^*\cdot\big(1+p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_a\big)$ on $\mathfrak{F}_{n,r,I}$.

4.2 The sheaves of overconvergent forms

Fix an interval $I = \big[p^k, p^{k'}\big]$ with k and k' integers such that $k' \geq k \geq 0$. Let $r, n \in \mathbb{Z}_{\geq 0}$. We assume that $r \geq 3$, $r+k \geq n \geq k'+2$ (resp. $r+k \geq n \geq k'+4$ if $p=2$). Set $n' = n-k'-2$ (resp. $n' = n-k'-4$ if $p=2$).

Lemma 4.2. *We have $p\mathrm{Hdg}^{-p^n} \subset \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$. In particular $p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}} \subset p^{k'+1}\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ (resp. $\subset p^{k'+3}\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ if $p=2$).*

Proof. The claim is local on $\mathfrak{IG}_{n,r,I}$. Let $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$. We prove the claim over an open $U = \operatorname{Spf} R \subset \mathfrak{IG}_{n,r,\alpha,I}$ over the open $\mathfrak{W}^0_{\alpha,I} = \operatorname{Spf} B^0_{\alpha,I}$ of $\widetilde{\mathfrak{W}}^0_F$. By construction we have $p/\alpha^{p^k} \in B^0_{\alpha,I}$ and $\alpha \mathrm{Hdg}^{-p^{r+1}} \subset R$. Hence $p\mathrm{Hdg}^{-p^{r+k+1}} \subset R$. In particular $p\mathrm{Hdg}^{-p^n} \subset R$. The second claim follows easily. □

Proposition 2.8 implies that the character κ extends to a character

$$\kappa\colon (\mathcal{O}_F \otimes \mathbb{Z}_p)^* \cdot \left(1 + p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}} \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_a\right) \to \mathbb{G}_m$$

over $\mathfrak{W}^0_{F,I}$.

Define $\mathfrak{w}^1_{n,r,I} = f_{n,*}\mathcal{O}_{\mathfrak{F}_{n,r,I}}[\kappa^{-1}]$ as the subsheaf of $f_{n,*}\mathcal{O}_{\mathfrak{F}_{n,r,I}}$ of sections transforming according to the character κ^{-1} under the action of $1 + p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}} \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_a$. It is an invertible sheaf over $\mathfrak{IG}_{n,r,I}$. Define $\mathfrak{w}_{n,r,I} \subset g_{n,*}\mathfrak{w}^1_{n,r,I}$ as the subsheaf of $(g_n \circ f_n)_*\mathcal{O}_{\mathfrak{F}_{n,r,I}}$ of κ^{-1}-equivariant sections for the action of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^* \cdot \left(1 + p^n \mathrm{Hdg}^{-\frac{p^n}{p-1}} \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_a\right)$.

Proposition 4.3. *The sheaf $\mathfrak{w}_{n,r,I}$ is an invertible $\mathcal{O}_{\mathfrak{X}_{r,I}}$-module of rank 1.*

The rest of this section is devoted to the proof of the Proposition. We follow [3, §5] closely. We start with the following:

Lemma 4.4. *Let $(\mathcal{O}_{\mathfrak{X}_{r,I}})^{00}$ be the ideal of topologically nilpotent elements of $\mathcal{O}_{\mathfrak{X}_{r,I}}$. Suppose that $r \geq 1$ (resp. $r \geq 2$ if $p = 2$). Then $\kappa((\mathcal{O}_F \otimes \mathbb{Z}_p)^*) - 1 \subset \mathrm{Hdg}\big(\mathcal{O}_{\mathfrak{X}_{r,I}}\big)^{00}$ and for every integer ℓ such that $2 \leq \ell \leq r + k$ we have*

$$\kappa(1 + p^{\ell-1}\mathcal{O}_F \otimes \mathbb{Z}_p) - 1 \subset \mathrm{Hdg}^{\frac{p^\ell - 1}{p-1}} \big(\mathcal{O}_{\mathfrak{X}_{r,I}}\big)^{00}.$$

Proof. We deal with the case $p \neq 2$ leaving to the reader the case $p = 2$. The claim is local on $\mathfrak{X}_{r,I}$. We restrict ourselves to an open formal affine subscheme $U = \operatorname{Spf} R$ mapping to the open $\mathfrak{W}^0_\alpha$ of $\widetilde{\mathfrak{W}}^0_F$ defined by an element $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$. By construction, $\kappa((\mathcal{O}_F \otimes \mathbb{Z}_p)^*) - 1 \subset \alpha B^0_\alpha$ and since $\alpha \mathrm{Hdg}^{-1} \subset \big(\mathcal{O}_{\mathfrak{X}_{r,I}}\big)^{00}$, we can conclude that the first point holds. Using lemma 2.5 we see that for $\ell \geq 2$, we have that $\kappa(1 + p^{\ell-1}\mathcal{O}_F \otimes \mathbb{Z}_p) - 1 \subset (\alpha^{p^{\ell-2}}, p) B^0_\alpha$. Arguing as in lemma 4.2 we deduce from the assumption that $\ell \leq r + k$ that $p\mathrm{Hdg}^{-p^\ell} \in \big(\mathcal{O}_{\mathfrak{X}_{r,I}}\big)^{00}$. On the other hand as $r \geq 1$, then $\alpha \in \mathrm{Hdg}^{p^2}\mathcal{O}_{\mathfrak{X}_{r,I}}$ so that $\alpha^{p^{\ell-2}} \in \mathrm{Hdg}^{p^\ell}\mathcal{O}_{\mathfrak{X}_{r,I}}$. As $\frac{p^\ell-1}{p-1} < p^\ell$, it follows that $\alpha^{p^{\ell-2}} \mathrm{Hdg}^{-\frac{p^\ell-1}{p-1}} \subset \big(\mathcal{O}_{\mathfrak{X}_{r,I}}\big)^{00}$. □

We also have the following:

Lemma 4.5. *The inclusion $\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \to f_{n,*}\mathcal{O}_{\mathfrak{F}_{n,r,I}}$ defines an isomorphism*

$$\mathcal{O}_{\mathfrak{IG}_{n,r,I}}/qp^{n'}\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \to \mathfrak{w}^1_{n,r,I}/qp^{n'}\mathfrak{w}^1_{n,r,I}.$$

Proof. Consider an open formal affine subscheme $U = \operatorname{Spf} R \subset \mathfrak{IG}_{n,r,I}$ mapping to the open formal subscheme $\mathfrak{W}^0_\alpha$ of $\widetilde{\mathfrak{W}}^0_F$ defined by some $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$. Assume that $\omega_G|_U$ is free. The choice of an element $\tilde{s} \in \mathcal{F}|_U$ lifting $s := \mathrm{HT}'(1)$ defines a section of the morphism $\mathfrak{F}_{n,r,I}|_U \cong \mathfrak{IG}_{n,r,I}|_U$ and hence an isomorphism $f_{\tilde{s}}\colon \mathfrak{w}^1_{n,r,I}|_U \to \mathcal{O}_{\mathfrak{IG}_{n,r,I}}|_U$ given by evaluating the functions at $\tilde{s}$.

Two different lifts $\tilde{s}$ and $\tilde{s'}$ differ by an element of $1+p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}\mathcal{O}_F\otimes R$ thanks to proposition 4.1. Proposition 4.2 implies that $1+p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}\mathcal{O}_F\otimes R \subset 1+p^{k'+1+n'}\mathcal{O}_F\otimes R$ (resp. $1+p^{k'+3+n'}\mathcal{O}_F\otimes R$ if $p=2$). As $I=[p^k,p^{k'}]$ we conclude from Proposition that $\kappa(1+p^{k'+1+n'}\mathcal{O}_F\otimes R)\subset 1+p^{n'+1}R$ (and similarly $\kappa(1+p^{k'+3+n'}\mathcal{O}_F\otimes R)\subset 1+qp^{n'}R$ for $p=2$). Thus $f_{\tilde{s}}\equiv f_{\tilde{s'}}$ modulo $qp^{n'}$. This provides the inverse to the isomorphism of the lemma. □

Let $U=\mathrm{Spf}\ R$ be an open affine formal subscheme of $\mathfrak{X}_{r,I}$. Suppose that ω_G is free over U. Thanks to Corollary 3.5 for every non-negative integer n such that $0\leq n\leq r+k$ there exists elements $c_0=1$ and $c_n\in\tilde{\mathrm{Ha}}^{-\frac{p^n-1}{p-1}}\mathcal{O}_{\mathfrak{IG}_{n,r}}(\mathrm{Spf}\ R)$ for $n\geq 1$ such that $\mathrm{Tr}_{\mathfrak{IG}}(c_n)=c_{n-1}$ for every $n\geq 1$. If n satisfies $r+k\geq n\geq k'+3$ (resp. $n\geq k'+4$ if $p=2$) we define a projector:

$$\begin{aligned} e_{c_n}: g_{n,*}\mathfrak{w}^1_{n,r,I}(R) &\to \tilde{\mathrm{Ha}}^{-\frac{p^n-1}{p-1}}\mathfrak{w}_{n,r,I}(R)\\ s &\mapsto \sum_{\sigma\in(\mathcal{O}_F/p^n\mathcal{O}_F)^*}\kappa(\sigma)\sigma(c_ns)\end{aligned}$$

The following lemma proves Proposition 4.3:

Lemma 4.6. *Let $s\in g_{n,*}\mathfrak{w}^1_{n,r,I}(R)$ be an element such that $s\equiv 1 \mod p$ (in the sense of Lemma 4.5). Then $e_{c_n}(s)\in\mathfrak{w}_{n,r,I}(R)$ and $\mathfrak{w}_{n,r,I}(R)$ is the free R-module generated by $e_{c_n}(s)$.*

Proof. The proof is entirely analogous to the proof of [3, Lemme 5.4]. Write $s=1+ph$ for a section $h\in\mathfrak{F}_{n,r,I}(R)$. We get

$$e_{c_n}(s)=\sum_{\sigma\in(\mathcal{O}_F/p^n\mathcal{O}_F)^*}\kappa(\tilde{\sigma})\tilde{\sigma}(c_n)+p\sum_{\sigma\in(\mathcal{O}_F/p^n\mathcal{O}_F)^*}\kappa(\tilde{\sigma})\tilde{\sigma}(c_nh).$$

In this formula, $\tilde{\sigma}$ is an arbitrary lift of σ dans $\mathbb{T}(\mathbb{Z}_p)$. Since $\tilde{\mathrm{Ha}}^{p^{r+k+1}}\mid p$ and $\frac{p^n-1}{p-1}<p^{r+k+1}$, it follows that $p\sum_{\sigma\in(\mathcal{O}_F/p^n\mathcal{O}_F)^*}\kappa(\tilde{\sigma})\tilde{\sigma}(c_nh)\in R^{00}\mathfrak{F}_{n,r,I}(R)$ where R^{00} is the ideal of topologically nilpotent elements in R.

We need to show that

$$\sum_{\sigma\in(\mathcal{O}_F/p^n\mathcal{O}_F)^*}\kappa(\tilde{\sigma})\tilde{\sigma}(c_n)\in 1+R^{00}\mathfrak{F}_{n,r,I}(R).$$

This follows from lemmas 2.6 and 4.4. As a consequence, $e_{c_n}(s)$ belongs to $\mathfrak{w}_{n,r,I}(R)$ and one checks easily that it is a generator using the normality of R as in [3, Lemme 5.4]. □

4.3 Properties of $\mathfrak{w}_{n,r,I}$

4.3.1 Functorialities

Fix intervals $I'\subset I$, r' and r such that $r'\geq r$ and integers $n'\geq n$ so that (I',r',n') and (I,r,n) satisfy the assumptions given at the beginning of §4.2 . We then have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{F}_{n',r',I'} & \longrightarrow & \mathfrak{F}_{n,r,I} \\ \downarrow & & \downarrow \\ \mathfrak{IG}_{n',r',I'} & \longrightarrow & \mathfrak{IG}_{n,r,I} \\ \downarrow & & \downarrow \\ \mathfrak{X}_{r',I'} & \xrightarrow{\iota} & \mathfrak{X}_{r,I} \end{array}$$

which induces a morphism of $\mathcal{O}_{\mathfrak{X}_{r',I'}}$-modules:

$$\iota^*\mathfrak{w}_{n,r,I} \to \mathfrak{w}_{n',r',I'}.$$

Proposition 4.7. *The morphism above is an isomorphism.*

Proof. Consider $\mathcal{O}_{\mathfrak{X}_{r',I'}} \to \iota^*\mathfrak{w}_{n,r,I}^{-1} \otimes \mathfrak{w}_{n',r',I'}$. This last sheaf is the subsheaf of $(g_{n'} \circ f_{n'})_*\mathcal{O}_{\mathfrak{F}_{n',r',I'}}$ consisting of section on which $(\mathcal{O}_F \otimes \mathbb{Z}_p)^* \cdot \left(1 + p^{n'}\mathrm{Hdg}^{-\frac{p^{n'}}{p-1}}\mathcal{O}_F \otimes \mathbb{Z}_p\right)$ acts trivially. This coincides with the sheaf $\mathcal{O}_{\mathfrak{X}_{r',I'}}$ by normality of $\mathfrak{X}_{r',I'}$. The composite map

$$\mathcal{O}_{\mathfrak{X}_{r',I'}} \to \iota^*\mathfrak{w}_{n,r,I}^{-1} \otimes \mathfrak{w}_{n',r',I'} \to \mathcal{O}_{\mathfrak{X}_{r',I'}}$$

is the identity. This proves the claim. □

We simplify the notations and write $\mathfrak{w}_I$ instead of $\mathfrak{w}_{n,r,I}$.

4.3.2 Frobenius

Propositions 3.3 and 3.6 provides compatible morphisms $\phi\colon \mathfrak{X}_{r,I} \to \mathfrak{X}_{r-1,I}$ and $\mathfrak{IG}_{n+1,r,I} \to \mathfrak{IG}_{n,r-1,I}$ obtained by composing the projection $\mathfrak{IG}_{n+1,r,I} \to \mathfrak{IG}_{n,r,I}$ and the Frobenius map $\phi : \mathfrak{IG}_{n,r,I} \to \mathfrak{IG}_{n,r-1,I}$. Let us recall the description of the morphism $\mathfrak{IG}_{n+1,r,I} \to \mathfrak{IG}_{n,r-1,I}$. Let $F\colon G \to G/H_1 = G'$ be the canonical isogeny on the semi-abelian schemes G over $\mathfrak{X}_{r,I}$ and G' over $\mathfrak{X}_{r-1,I}$. Such morphism induces a surjective morphism of canonical subgroups $H_{n+1} \to H_{n+1}/H_1 \cong H'_n$ of G and G' respectively. Dualizing we get an injective morphism $F^D\colon H_n'^D \to H_{n+1}^D$. The map $\phi\colon \mathfrak{IG}_{n+1,r,I} \to \mathfrak{IG}_{n,r',I}$ associates to a morphism $\psi\colon \mathcal{O}_F/p^{n+1}\mathcal{O}_F \to H_{n+1}^D$ the morphism $\psi'\colon \mathcal{O}_F/p^n\mathcal{O}_F \to H_n'^D$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{O}_F/p^{n+1}\mathcal{O}_F & \xrightarrow{\psi} & H_{n+1}^D \\ {\scriptstyle \times p}\uparrow & & \uparrow{\scriptstyle F^D} \\ \mathcal{O}_F/p^n\mathcal{O}_F & \xrightarrow{\psi'} & H_n'^{,D} \end{array}$$

We then get the commutative diagram:

$$\begin{array}{ccc} \mathfrak{F}_{n+1,r,I} & \longrightarrow & \mathfrak{F}_{n,r',I} \\ \downarrow & & \downarrow \\ \mathfrak{IG}_{n+1,r,I} & \longrightarrow & \mathfrak{IG}_{n,r',I} \\ \downarrow & & \downarrow \\ \mathfrak{X}_{r,I} & \xrightarrow{\phi} & \mathfrak{X}_{r',I} \end{array}$$

where the morphism $\mathfrak{F}_{n+1,r,I} \to \mathfrak{F}_{n,r-1,I}$ is given by mapping a differential $w \in \mathcal{F}$ to $pw \in \mathcal{F}' \subset \omega_{G'}$. We can check that this is well defined by using the following cummutative diagram:

$$\begin{array}{ccc} H_n'^{,D} & \longrightarrow & H_{n+1}^D \\ \downarrow{\scriptstyle HT} & & \downarrow{\scriptstyle HT} \\ \omega_{G'} & \xrightarrow{F^*} & \omega_G \end{array}$$

We then get a morphism $\phi^*\mathfrak{w}_I \to \mathfrak{w}_I$.

Proposition 4.8. *The morphism $\phi^*\mathfrak{w}_I \to \mathfrak{w}_I$ is an isomorphism.*

Proof. The proof is analogous to the proof of Proposition 4.7. □

5 Perfect overconvergent modular forms

In this section we define a sheaf of perfect overconvergent modular forms over the weight space $\mathfrak{W}_F^0$. In the next section we will show how we can undo the perfectisation.

5.1 The anti-canonical tower

Let $I = [p^k, p^{k'}] \subset [1, +\infty]$ and $r, n \in \mathbb{Z}_{\geq 0}$ and $n \leq r + k$. As explained in §4.3.2 we have compatible morphisms

$$\begin{array}{ccc} \mathfrak{IG}_{n,r+1,I} & \longrightarrow & \mathfrak{IG}_{n,r,I} \\ \downarrow & & \downarrow \\ \mathfrak{X}_{r+1,I} & \xrightarrow{\phi} & \mathfrak{X}_{r,I} \end{array}$$

Taking the limits we get formal schemes $\mathfrak{IG}_{n,\infty,I} \to \mathfrak{X}_{\infty,I}$ over $\widetilde{\mathfrak{W}}_F^0$. Varying n we also get a tower of formal schemes $\cdots \to \mathfrak{IG}_{n+2,\infty,I} \to \mathfrak{IG}_{n+1,\infty,I} \to \mathfrak{IG}_{n,\infty,I}$. Let $\mathfrak{IG}_{\infty,\infty,I}$ be the projective limit. As the index r varies now, we denote by $G_r \to \mathfrak{X}_{r,I}$ the semi-abelian scheme and by $\mathrm{Hdg}_r \subset \mathcal{O}_{\mathfrak{X}_{r,I}}$ the Hodge ideal defined by G_r.

Recall from §3.3 that associated to the finite morphism $\mathfrak{IG}_{n,r,I} \to \mathfrak{IG}_{n-1,r,I}$ we have a trace map $\mathrm{Tr}_{\mathfrak{IG}}\colon \mathcal{O}_{\mathfrak{IG}_{n,r,I}} \to \mathcal{O}_{\mathfrak{IG}_{n-1,r,I}}$. These are compatible for varying r and define a trace map $\mathrm{Tr}_{\mathfrak{IG}}\colon \mathcal{O}_{\mathfrak{IG}_{n,\infty,I}} \to \mathcal{O}_{\mathfrak{IG}_{n-1,\infty,I}}$.

Proposition 5.1. *We have* $\mathrm{Hdg}_s \mathcal{O}_{\mathfrak{IG}_{n-1,\infty,I}} \subset \mathrm{Tr}_{\mathfrak{IG}}\big(\mathcal{O}_{\mathfrak{IG}_{n,\infty,I}}\big)$ *for every* $s \geq 1$.

Proof. Thanks to Proposition 3.4 we have $\mathrm{Hdg}_s^{p^{n-1}} \mathcal{O}_{\mathfrak{IG}_{n-1,s,I}} \subset \mathrm{Tr}_{\mathfrak{IG}}\big(h_* \mathcal{O}_{\mathfrak{IG}_{n,s,I}}\big)$. It follows from [3, Cor. A.2] that $\mathrm{Hdg}_{s+1}^p = \mathrm{Hdg}_s$. Since $\mathrm{Hdg}_s^{p^{n-1}} \mathcal{O}_{\mathfrak{IG}_{n-1,\infty,I}} \subset \mathrm{Tr}_{\mathfrak{IG}}\big(\mathcal{O}_{\mathfrak{IG}_{n,\infty,I}}\big)$ and s is arbitrary, the claim follows.

□

5.2 Tate traces

Let $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$. Denote by $\mathfrak{IG}_{\infty,\infty,\alpha,I} \to \mathfrak{X}_{\infty,\alpha,I}$ the base change of the formal schemes above to $\mathfrak{W}^0_{\alpha,I} \to \widetilde{\mathfrak{W}}^0_F$.

Let $h_r \colon \mathfrak{X}_{\infty,\alpha,I} \to \mathfrak{X}_{r,\alpha,I}$ be the projection map onto the r-th factor.

Proposition 5.2. *One has Tate traces:*

$$\mathrm{Tr}_r \colon (h_r)_* \mathcal{O}_{\mathfrak{X}_{\infty,\alpha,I}}[1/\alpha] \to \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}[1/\alpha]$$

such that $f = \lim_{r\to\infty} \mathrm{Tr}_r(f)$. *Moreover*

$$\mathrm{Tr}_r\big((h_r)_* \mathcal{O}_{\mathfrak{X}_{\infty,\alpha,I}}\big) \subset \alpha^{-1} \cdot \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$$

as soon as $p^r(p-1) > 2g+1$.

Proof. The proof follows closely the proof of [3, Proposition 6.2]. We first provide the analogue of [3, §6.3.2 & §6.3.3] which reduce the proof to [3, Lemme 6.1].

For every non negative integer $k \geq r+1$ define $B^0_{\alpha,I,p^{-k}} := B^0_{\alpha,I}\big[\alpha^{p^{-k}}\big]$. One proves as in §3.4.1 that it is a normal ring. Let $\mathfrak{W}^0_{\alpha,I,p^{-k}}$ be the associated α-adic formal scheme and let $\mathcal{W}^0_{\alpha,I,p^{-k}} := \mathrm{Spa}\big(B^0_{\alpha,I,p^{-k}}\big[\alpha^{-1}\big], B^0_{\alpha,I,p^{-k}}\big)$ be the associated analytic adic space. Define $\mathcal{X}_{r,\alpha,I,p^{-k}}$ to be the product $\mathcal{X}_{r,\alpha,I} \times_{\mathcal{W}^0_{\alpha,I}} \mathcal{W}^0_{\alpha,I,p^{-k}}$. Define $\mathfrak{X}_{r,\alpha,I,p^{-(r+1)}}$ to be the normalization of $\mathfrak{X}_{r,\alpha,I}$ in $\mathcal{X}_{r,\alpha,I,p^{-(r+1)}}$ (see §3.3). For general $k \geq r+1$ let $\mathfrak{X}_{r,\alpha,I,p^{-k}}$ be the base change of $\mathfrak{X}_{r,\alpha,I,p^{-(r+1)}}$ via the map $\mathfrak{W}^0_{\alpha,I,p^{-k}} \to \mathfrak{W}^0_{\alpha,I,p^{-(r+1)}}$. The associated analytic adic space is $\mathcal{X}_{r,\alpha,I,p^{-k}}$. We have morphisms

$$\mathfrak{X}_{r,\alpha,I,p^{-k}} \to \mathfrak{X}_{r,\alpha,I,p^{-(r+1)}} \to \mathfrak{X}_{r,\alpha,I} \to \overline{\mathfrak{M}}(\mu_N,\mathfrak{c}) \times \mathfrak{W}^0_{\alpha,I} \to \overline{\mathfrak{M}}(\mu_N,\mathfrak{c}). \tag{1}$$

5.2.1 An explicit description

Let $U := \mathrm{Spf}\ A \subset \overline{\mathfrak{M}}(\mu_N,\mathfrak{c})$ be an open formal affine so that the sheaf ω_G is trivial. We will describe the fiber of the above chain of morphisms over U. Choose a lift of the Hasse invariant viewed as a scalar $\tilde{\mathrm{Ha}}$.

The fiber of U in $\mathfrak{X}_{r,\alpha,I,p^{-(r+1)}}$ is the formal spectrum of

$$R := A\widehat{\otimes} B^0_{\alpha,I,p^{-(r+1)}}\langle \frac{\alpha^{1/p^{r+1}}}{\tilde{\mathrm{Ha}}}\rangle = A\widehat{\otimes} B^0_{\alpha,p^{-(r+1)}}\langle u,v,w\rangle/\big(w\tilde{\mathrm{Ha}} - \alpha^{1/p^{r+1}}, \alpha^{p^k}v - p, uv - \alpha^{p^{k'-k}}\big).$$

Here the variable u and the equation $uv - \alpha^{p^{k'-k}}$ are missing in case $k' = \infty$. Arguing as in the proof of Lemma 3.7 it follows that R is a normal ring.

Set $R_k := R \otimes_{B^0_{\alpha,I,p^{-(r+1)}}} B^0_{\alpha,I,p^{-k}} = A\widehat{\otimes} B^0_{\alpha,I,p^{-k}}\langle \frac{\alpha^{1/p^{r+1}}}{\tilde{\mathrm{Ha}}}\rangle$. It is finite and free as R-module with basis α^{a/p^k} for $0 \le a \le p^{k+1-r} - 1$. The associated formal scheme Spf R_k is the open of $\mathfrak{X}_{r,\alpha,I,p^{-k}}$ over the open $U \subset \mathfrak{X}$.

Then the restriction of the diagram (1) to U is given by the ring homomorphisms:

$$A \to A\widehat{\otimes} B^0_{\alpha,I} \to A\widehat{\otimes} B^0_{\alpha,I}\langle \frac{\alpha}{\tilde{\mathrm{Ha}}^{p^{r+1}}}\rangle \to R \to R_k. \tag{2}$$

5.2.2 Frobenius

The Frobenius morphism of Proposition 3.3 defines a cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_{r,\alpha,I,p^{-k}} & \longrightarrow & \mathcal{X}_{r,\alpha,I} \\ \downarrow \phi & & \downarrow \phi \\ \mathcal{X}_{r-1,\alpha,I,p^{-k}} & \longrightarrow & \mathcal{X}_{r-1,\alpha,I}. \end{array} \tag{3}$$

Due to Proposition 3.3 the morphism $\mathcal{X}_{r,\alpha,I,p^{-k}} \to \mathcal{X}_{r-1,\alpha,I,p^{-k}}$ is finite. Hence we get a commutative diagram

$$\begin{array}{ccccc} \mathfrak{X}_{r,\alpha,I,p^{-k}} & \longrightarrow & \mathfrak{X}_{r,\alpha,I,p^{-(r+1)}} & \longrightarrow & \mathfrak{X}_{r,\alpha,I} \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ \mathfrak{X}_{r-1,\alpha,I,p^{-k}} & \longrightarrow & \mathfrak{X}_{r-1,\alpha,I,p^{-(r+1)}} & \longrightarrow & \mathfrak{X}_{r-1,\alpha,I} \end{array}$$

Over $U = \mathrm{Spf}\ A$ the morphism $\mathfrak{X}_{r,\alpha,I,p^{-k}} \to \mathfrak{X}_{r-1,\alpha,I,p^{-k}}$ is given by

$$S_k := A\widehat{\otimes} B^0_{\alpha,I,p^{-k}}\langle \frac{\alpha^{1/p^r}}{\tilde{\mathrm{Ha}}}\rangle \to A\widehat{\otimes} B^0_{\alpha,I,p^{-k}}\langle \frac{\alpha^{1/p^{r+1}}}{\tilde{\mathrm{Ha}}}\rangle =: R_k. \tag{4}$$

It is finite an modulo $p\alpha^{-1/p^r}$ is induced by the absolute Frobenius on A/pA. Indeed this holds true modulo $p\tilde{\mathrm{Ha}}^{-1}$ due to [3, Cor. A.2] and $p\tilde{\mathrm{Ha}}^{-1} = \big(p\alpha^{-1/p^r}\big) \cdot \big(\alpha^{-1/p^r}\tilde{\mathrm{Ha}}^{-1}\big)$.

5.2.3 The unramified case

We first assume that p is unramified in F. This implies that A is formally smooth over $\mathbb{Z}_p$. It follows from [3, Lemme 6.1] applied to the extension $S_k \subset R_k$ that $S_k[\alpha^{-1}] \subset R_k[\alpha^{-1}]$ is a finite and flat extension and that $\mathrm{Tr}(R_k) \subset p^g \alpha^{-\frac{(2g+1)}{p^r}} S_k$. This implies that for all $r' \ge r$ and $k \ge r' + 1$,

$$\mathrm{Tr}_{\phi^{r'-r}}\big(\mathcal{O}_{\mathfrak{X}_{r',\alpha,I,p^{-k}}}\big) \subset p^g \alpha^{-\frac{2g+1}{p^r(p-1)}} \mathcal{O}_{\mathfrak{X}_{r,\alpha,I,p^{-k}}}.$$

In particular, defining

$$\mathrm{Tr}_r := \frac{1}{p^{sg}} \mathrm{Tr}_{\phi^s} \colon h_{r,*}\mathcal{O}_{\mathfrak{X}_{r+s,\alpha,I}}\big[\alpha^{-1}\big] \to \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}\big[\alpha^{-1}\big],$$

we deduce that, if $p^r(p-1) > 2g+1$, the image of $h_{r,*}\mathcal{O}_{\mathfrak{X}_{r+s,\alpha,I}}$ is contained in $\alpha^{-1}\mathcal{O}_{\mathfrak{X}_{r,\alpha,I,p^{-k}}} \cap \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}[\alpha^{-1}]$ which is $\alpha^{-1}\cdot\mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$ since $\mathfrak{X}_{r,\alpha,I,p^{-k}} \to \mathfrak{X}_{r,\alpha,I}$ is a finite and dominant morphism and $\mathfrak{X}_{r,\alpha,I}$ is normal. The Proposition is easily deduced from this. □

5.2.4 The general case

We now don't assume anymore that p is unramified in F. In this situation $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})$ is not formally smooth. Nevertheless the Rapoport locus $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})^R \subset \overline{\mathfrak{M}}(\mu_N,\mathfrak{c})$ is the smooth locus and its complement is of codimension at least 2. We let $\mathfrak{X}^R_{r,\alpha,I,p^{-k}} \subset \mathfrak{X}_{r,\alpha,I,p^{-k}}$ the open formal subscheme where the Rapoport condition holds.

Arguing as in the unramified case, we obtain a map $\mathrm{Tr}_r := \frac{1}{p^{sg}}\mathrm{Tr}_{\phi^s}\colon h_{r,*}\mathcal{O}_{\mathfrak{X}^R_{r+s,\alpha,I}} \to \alpha^{-1}\mathcal{O}_{\mathfrak{X}^R_{r,\alpha,I}}$.

Lemma 5.3. *The formal scheme $\mathfrak{X}^R_{r,\alpha,I}$ is Zariski dense in $\mathfrak{X}_{r,\alpha,I}$.*

Proof. This follows easily on the equations. Note that we use here crucially that the complement of the Rapoport locus is of codimension 1 in the non-ordinary locus in the special fiber of $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})$. □

Consider the following commutative diagram :

$$\begin{array}{ccc} \mathfrak{X}^R_{r+s,\alpha,I} & \longrightarrow & \mathfrak{X}_{r+s,\alpha,I} \\ \downarrow{\scriptstyle \phi^s} & & \downarrow{\scriptstyle \phi^s} \\ \mathfrak{X}^R_{r,\alpha,I} & \longrightarrow & \mathfrak{X}_{r,\alpha,I} \end{array}$$

We claim that it induces a commutative diagram :

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{X}_{r+s,\alpha,I}} & \longrightarrow & \mathcal{O}_{\mathfrak{X}^R_{r+s,\alpha,I}} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_r} \\ \alpha^{-1}\mathcal{O}_{\mathfrak{X}_{r,\alpha,I}} & \longrightarrow & \alpha^{-1}\mathcal{O}_{\mathfrak{X}^R_{r,\alpha,I}} \end{array}$$

Indeed, let Spf R be an open formal subscheme of $\mathfrak{X}_{r,\alpha,I}$. Take $f \in \mathcal{O}_{\mathfrak{X}_{r+s,\alpha,I}}(R)$. Then $\alpha\mathrm{Tr}_r(f)$ is in $\mathcal{O}_{\mathfrak{X}^R_{r,\alpha,I}}(R)$. Moreover, as the morphisme $\phi^s : R[\frac{1}{\alpha}] \to \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}(R)[\frac{1}{\alpha}]$ is finite flat, we deduce that $\alpha\mathrm{Tr}_r(f) \in R[\frac{1}{p}]$. Since R is normal, $R = \cap_{\mathfrak{Q}} R_{\mathfrak{Q}}$ where $\mathfrak{Q}$ runs over all codimension 1 prime ideals in R. Thus we are left to check that $\alpha\mathrm{Tr}_r(f) \in R_{\mathfrak{Q}}$ whenever $p \in \mathfrak{Q}$.

If $\alpha \notin \mathfrak{Q}$ as $w\tilde{\mathrm{Ha}}^{p^{r+1}} - \alpha = 0$, we deduce that the image of $\mathfrak{Q}$ in $\overline{\mathfrak{M}}(\mu_N,\mathfrak{c})$ lies in the ordinary locus and in particular in the Rapoport locus. Thus $\alpha\mathrm{Tr}_r(f) \in R_{\mathfrak{Q}}$. If $\alpha \in \mathfrak{Q}$, then $\mathfrak{Q}$ is a generic point of the special fiber of R and since the Rapoport locus is Zariski dense, $\mathfrak{Q}$ lies in the Rapoport locus. Thus $\alpha\mathrm{Tr}_r(f) \in R_{\mathfrak{Q}}$ in this case also.

5.3 The sheaf of perfect overconvergent modular forms

Lemma 5.4. *Let $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$. The sheaf $\mathcal{O}_{\mathfrak{X}_{\infty,\alpha,I}}$ is integrally closed in $\mathcal{O}_{\mathfrak{X}_{\infty,\alpha,I}}[1/\alpha]$. Moreover*

$$\left(\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,\alpha,I}}\right)^{(\mathcal{O}_F\otimes\mathbb{Z}_p)^*} = \mathcal{O}_{\mathfrak{X}_{\infty,\alpha,I}}.$$

Proof. (1) Let $U := \mathrm{Spf}\, R$ be an open formal subscheme of $\mathfrak{X}_{\infty,\alpha,I}$. For s large enough it is the inverse image of an open formal subscheme $\mathrm{Spf}\, R_s$ of $\mathfrak{X}_{s,\alpha,I}$. Let $f \in R[1/\alpha]$. For s large enough we have $f = f_s + h$ with $f_s \in R_s[1/\alpha]$ and $h \in R$ so that we may assume that $f \in R_s[1/\alpha]$. Let $f^n + f^{n-1}a_{n-1} + \cdots + a_0 = 0$ with $a_0, \ldots, a_{n-1} \in\in R$ be an integral relation.

Applying Tr_s one gets $f^n + f^{n-1}\mathrm{Tr}_h(a_{n-1}) + \cdots + \mathrm{Tr}_h(a_0) = 0$ and as $a_i \in R$ for s large enough we have $\mathrm{Tr}_s(a_i) \in R \cap \alpha^{-1}R_s$. For $h \geq 0$ the morphism $R_s \to R_{s+h}$ is a finite dominant morphism of normal rings. Hence $R_s/\alpha \to R_{s+h}/\alpha$ is injective so that $R_s/\alpha \to R/\alpha$ is injective as well. We deduce that $\alpha\mathrm{Tr}_s(a_i) \in \alpha R_s$ and hence that $\mathrm{Tr}_s(a_i) \in R_s$. Thus f is integral over R_s and, hence, $f \in R_s$ proving the first claim of the corollary.

(2) The inverse image of $\mathrm{Spf}\, R$ in $\mathfrak{IG}_{n,\infty,\alpha,I}$ is equal to $\mathrm{Spf}\, R_n$ with R_n integral over R and $R\left[\alpha^{-1}\right] \subset R_n\left[\alpha^{-1}\right]$ finite and étale. In particular $\left(R_n\left[\alpha^{-1}\right]\right)^{(\mathcal{O}_F/p^n\mathcal{O}_F)^*} = R\left[\alpha^{-1}\right]$ so that $R_n^{(\mathcal{O}_F/p^n\mathcal{O}_F)^*}$ contains R and is integral over R and, hence by the first claim, it must be equal to R. Let R_∞ be the inverse image of $\mathrm{Spf}\, R$ in $\mathfrak{IG}_{\infty,\infty,\alpha,I}$. Consider an element $x \in R_\infty$ fixed by $(\mathcal{O}_F \otimes \mathbb{Z}_p)^*$. There exists n large enough and $x_n \in R_n$ such that $x - x_n = \alpha x'$ for some $x' \in R_\infty$. In particular x' is fixed by $1 + p^n\mathcal{O}_F \otimes \mathbb{Z}_p$. Thanks to Proposition 5.1 for every $n' \geq n$ there exists an element $c_{n'} \in R_n'$ such that $\mathrm{Tr}_{R_{n'}/R_n}(c_{n'}) = \alpha$. In particular the higher cohomology groups of $(1 + p^n\mathcal{O}_F/p^{n'}\mathcal{O}_F)$ acting on $R_{n'}$ are annihilated by α.

For every s there exists $n(s) \geq n$ such that $x' \in (R_{n(s)}/\alpha^s)^{1+p^n\mathcal{O}_F\otimes\mathbb{Z}_p}$ and hence there exists $y_s \in R_n$ such that $y_s \equiv \alpha x'$ modulo α^s. We deduce that y_s converges to an element y for $s \to \infty$ such that $\alpha x' = y$. Hence $x \in R_n^{(\mathcal{O}_F/p^n\mathcal{O}_F)^*}$ which is R by the first part of the argument. □

Define $\mathfrak{w}_I^{\mathrm{perf}}$ to be the subsheaf of $\mathcal{O}_{\mathfrak{X}_{\infty,I}}$-modules of $\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,I}}$ consisting of those sections transforming according to the character κ^{-1} for the action of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^*$. Then:

Proposition 5.5. *The sheaf $\mathfrak{w}_I^{\mathrm{perf}}$ is an invertible $\mathcal{O}_{\mathfrak{X}_{\infty,I}}$-module. Moreover for every subinterval $J \subset I$ the pull-back of $\mathfrak{w}_I^{\mathrm{perf}}$ via the natural morphism $\iota_{J,I} \colon \mathfrak{X}_{\infty,I} \to \mathfrak{X}_{\infty,J}$ coincides with $\mathfrak{w}_J^{\mathrm{perf}}$.*

Proof. We prove the first claim. Let $U := \mathrm{Spf}\, R$ be an open formal subscheme of $\mathfrak{X}_{\infty,\alpha,I}$. Suppose that Hdg_1 is a principal ideal over $\mathrm{Spf}\, R$ with generator $\tilde{\mathrm{Ha}}_1$. Let $\mathrm{Spf}\, R_n$ (resp. $\mathrm{Spf}\, R_\infty$) be the inverse image of U in $\mathfrak{IG}_{n,\infty,\alpha,I}$ (resp. in $\mathfrak{IG}_{\infty,\infty,\alpha,I}$). Due to Corollary 5.4 it suffices to exhibit a invertible element $x \in R_\infty$ such that $\sigma(x) = \kappa^{-1}(\sigma)x$ for every $\sigma \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^*$.

Proposition 5.1 implies that there exists elements $c_n \in \tilde{\mathrm{Ha}}_1^{-1} R_n$ such that $\mathrm{Tr}_{R_n/R_{n-1}}(c_n) = c_{n-1}$ and $c_0 = 1$. Define $b_n := \sum_{\sigma\in(\mathcal{O}_F/p^n\mathcal{O}_F)^*} \kappa(\tilde{\sigma})\sigma(c_n) \in \tilde{\mathrm{Ha}}_1^{-1} R_n$ for $n \geq 1$. Here $\tilde{\sigma} \in \mathbb{T}(\mathbb{Z}_p)$ is a lift of σ. It follows from lemma 2.6 that b_n converges to an element $b_\infty \in R_\infty$ such that $\sigma(b_\infty) = \kappa^{-1}(\sigma)b_\infty$ for every $\sigma \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^*$ and $b_\infty \equiv 1$ modulo $\frac{\alpha}{\tilde{\mathrm{Ha}}_1} R_\infty$ so that $b_\infty$is invertible in R_∞ as claimed.

The last claim is proven as in §4.3. □

6 Descent

In this section we prove that the sheaf $\mathfrak{w}_I^{\mathrm{perf}}$ defined in §5.3 can be descended to a finite level.

6.1 Comparison with the sheaf $\mathfrak{w}_I$

Consider an interval $I = [p^k, p^{k'}]$ with k and k' non negative integers. Thanks to proposition 4.3 we have an invertible sheaf $\mathfrak{w}_I$ over $\mathfrak{X}_{r,I}$. Recall that we have a projection map $h_r \colon \mathfrak{X}_{\infty,I} \to \mathfrak{X}_{r,I}$. Then:

Proposition 6.1. *There exists a canonical isomorphism* $\mathfrak{w}_I^{\mathrm{perf}} \simeq h_r^* \mathfrak{w}_I$.

Proof. Over $\mathfrak{X}_{\infty,\alpha,I}$ we have a chain of isogenies

$$\cdots G_{n+r} \xrightarrow{F} G_{n+r-1} \to \cdots G_r$$

where G_s is the versal semi-abelian scheme over $\mathfrak{X}_{s,I}$. Denote by $C_{n,r} \hookrightarrow G_r[p^n]$ the kernel of $(F^n)^D \colon G_r[p^n]^D \to G_{r+n}[p^n]^D)$. Clearly $C_{n,r} = H_n(G_{r+n})^D$. The isogeny $F \colon G_{r+1} \to G_r$ induces a morphism $C_{n,r+1} \to C_{n,r}$ which is generically an isomorphism. Over $\mathfrak{IG}_{n,\infty,I}$ we have a universal morphism $\mathcal{O}_F/p^n\mathcal{O}_F \to H_n(G_s)^D$ for every $s \geq n-k$. The map $C_{n,s} \to G_s[p^n]/H_n(G_s) \simeq H_n(G_s)^D$ is generically an isomorphism as both group schemes are generically étale. Composing we then get a $\mathcal{O}_F$-equivariant map $\mathcal{O}_F/p^n\mathcal{O}_F \to C_{n,s}$ for every $s \geq n-k$. Using the morphisms $C_{n,r+1} \to C_{n,r}$ we get a $\mathcal{O}_F$-equivariant morphism $\mathcal{O}_F/p^n\mathcal{O}_F \to C_{n,r}$ for every n which is generically an isomorphism. Passing to the projective limit we get a $\mathcal{O}_F$-equivariant map $\mathcal{O}_F \otimes \mathbb{Z}_p \to \lim_n C_{n,r}$. Let HT^{un} be the image of 1 in $\lim_n C_{n,r}$ via the Hodge-Tate map $\lim_n G_r[p^n] \to \omega_{G_r}$. Then HT^{un} defines a $(\mathcal{O}_F \otimes \mathbb{Z}_p)^*$-equivariant map:

$$\mathfrak{IG}_{\infty,\infty,I} \to \mathfrak{F}_{n,r,I}$$

fitting in the commutative diagram:

$$\begin{array}{ccc}
\mathfrak{IG}_{\infty,\infty,I} & \longrightarrow & \mathfrak{F}_{n,r,I} \\
\downarrow & & \downarrow \\
\mathfrak{IG}_{n,\infty,I} & \longrightarrow & \mathfrak{IG}_{n,r,I} \\
\downarrow & & \downarrow \\
\mathfrak{X}_{\infty,I} & \xrightarrow{h_r} & \mathfrak{X}_{r,I}
\end{array}$$

We then get an injective homomorphism $h_r^*\mathfrak{w}_I \to \mathfrak{w}_I^{\mathrm{perf}}$. Moreover

$$\mathfrak{w}_I^{\mathrm{perf}} \otimes h_r^* \mathfrak{w}_I^{-1} \subset \left(\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,I}}\right)^{(\mathcal{O}_F \otimes \mathbb{Z}_p)^*} = \mathcal{O}_{\mathfrak{X}_{\infty,I}}$$

thanks to Corollary 5.4. □

Corollary 6.2. *We have a Tate trace map* $\mathrm{Tr}_r \colon h_{r,*}\mathfrak{w}_I^{\mathrm{perf}} \to \alpha^{-1}\mathfrak{w}_I$*, which is functorial in* I.

6.2 Descent along the anti-canonical tower

Fix $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$. Let $I = [1, \infty]$.

Lemma 6.3. *The natural morphisms of sheaves*

$$\mathcal{O}_{\mathfrak{X}_{r,\alpha,I}} \longrightarrow \lim_{k+1 \geq k' \geq k \geq 0} \mathcal{O}_{\mathfrak{X}_{r,\alpha,[p^k,p^{k'}]}}, \qquad \mathcal{O}_{\mathfrak{X}_{\infty,\alpha,I}} \longrightarrow \lim_{k+1 \geq k' \geq k \geq 0} \mathcal{O}_{\mathfrak{X}_{\infty,\alpha,[p^k,p^{k'}]}}$$

are isomorphisms.

Proof. We prove the first statement. The second follows arguing as in [3, Lemme 6.6] using the Tate traces constructed in Proposition 5.2.

Let $U := \operatorname{Spf} A$ be a formal open affine subscheme of $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})$ over which ω_G is trivial. Let $\tilde{\mathrm{Ha}}$ be a lift of Ha. Arguing as in the proof of Lemma 3.7 one deduces that the inverse image of U in $\mathfrak{X}_{r,\alpha,I}$ is Spf R with $R := A \otimes_{\mathbb{Z}_p} B_\alpha\langle u, w\rangle / (w\tilde{\mathrm{Ha}}^r - \alpha, \alpha u - p)$ and that for integers $1 \leq h < k$ the inverse image of U in $\mathfrak{X}_{r,\alpha,[p^h,p^k]}$ is Spf $R_{h,k}$ with $R_{h,k} := R\langle u_h, v_k\rangle / (\alpha^{p^h} u_h - p, u_h v_k - \alpha^{p^k - p^h})$.

There are maps $R_{h,k} \to R_{h',k'}$ for $h \leq h' \leq k \leq k'$ given by $u_h \mapsto \alpha^{p^{h'} - p^h} u_{h'}$ and $v_k \mapsto \alpha^{p^{k'} - p^k} v_{k'}$. The argument in [3, Lemme 6.4] shows that map $R \to \lim_k R_{1,k}$ is an isomorphism. This proves the claim. □

Theorem 6.4. *The sheaf $\mathfrak{w}_I^{\mathrm{perf}}$ descends to an invertible sheaf $\mathfrak{w}_I$ over $\mathfrak{X}_{r,\alpha,I}$ for $r \geq \sup\{4, 1 + \log_p 2g + 1\}$ if $p \geq 3$ and $r \geq \sup\{6, 1 + \log_p 2g + 1\}$ if $p = 2$. More precisely $\mathfrak{w}_I$ is the subsheaf of $\mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$-modules of $\mathfrak{w}_I^{\mathrm{perf}}$ characterized by the fact that for every interval $J = [p^k, p^{k'}]$ with $k' \geq k \geq 0$ integers and denoting $\iota_{J,I} \colon \mathfrak{X}_{r,\alpha,J} \to \mathfrak{X}_{r,\alpha,I}$ the natural morphism, then $\iota_{J,I}^* \mathfrak{w}_I$ is the sheaf $\mathfrak{w}_J$ of Proposition 4.3 compatibly with the identification of Proposition 6.1.*

Moreover $\mathfrak{w}_I$ is free of rank 1 over every formal affine subscheme $U \subset \overline{\mathfrak{M}}(\mu_N, \mathfrak{c})$ such that $\omega_G|_U$ is trivial.

Proof. The proof is analogous to the proof of [3, Thm. 6.4]. We set

$$\mathfrak{w}_I := \lim_{k+1 \geq k' \geq k \geq 0} \mathfrak{w}_{[p^k,p^{k'}]},$$

where the limit is taken over integers k, k'. Let $U := \operatorname{Spf} A$ be a formal open affine subscheme of $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})$ where ω_G is trivial. Let $W := \operatorname{Spf} B$ be the inverse image of U in $\mathfrak{X}_{r,\alpha,I}$. We prove that $\mathfrak{w}_I|_W$ is a free $\mathcal{O}_W$-module of rank 1 and it descends $\mathfrak{w}_I^{\mathrm{perf}}|_W$.

We prove the claim for the minimal r possible, i.e., $r = 4$ if $p \geq 3$ and $r = 6$ for if $p = 2$. Let $\tilde{\mathrm{Ha}}_r$ be a lift of the Hasse invariant over U. Thanks to Corollary 3.5 and Proposition 5.1 we can find elements $c_n \in \tilde{\mathrm{Ha}}_r^{-\frac{p^n-1}{p-1}} \mathcal{O}_{\mathfrak{IG}_{n,r,\alpha,I}}(W)$ for integers $n \leq r$ and elements $c_n \in \tilde{\mathrm{Ha}}_r^{-p^r} \mathcal{O}_{\mathfrak{IG}_{n,\infty,\alpha,I}}(W)$ for general $r \leq n$ so that $c_0 = 1$ and $\mathrm{Tr}_{\mathfrak{IG}}(c_n) = c_{n-1}$. Define $b_n = \sum_{\sigma \in (\mathcal{O}_F/p^n\mathcal{O}_F)^*} \kappa(\tilde{\sigma})\sigma(c_n)$ where $\tilde{\sigma}$ is a lift of σ in $(\mathcal{O}_F \otimes \mathbb{Z}_p)^*$.

Using lemma 2.6, we deduce that :

- The sequence b_n converges to an element b_∞,
- $b_\infty = 1 \mod \tilde{\mathrm{Ha}}^{-p^r} \alpha$, and in particular b_∞ generates $\mathfrak{w}_I^{\mathrm{perf}}(W)$,

- $b_\infty = b_r \mod \mathfrak{m}_{r-1}\tilde{\mathrm{Ha}}^{-p^r}$ (resp. $\mathfrak{m}_{r-2}\tilde{\mathrm{Ha}}^{-p^r}$ if $p=2$).

Consider an interval $J=[p^k,p^{k+1}]$ and the commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{IG}_{\infty,\infty,\alpha,I} & \longleftarrow & \mathfrak{IG}_{\infty,\infty,\alpha,J} & \longrightarrow & \mathfrak{F}_{k+r,r,\alpha,J} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{IG}_{k+r,\infty,\alpha,I} & \longleftarrow & \mathfrak{IG}_{k+r,\infty,J} & \longrightarrow & \mathfrak{IG}_{k+r,r,\alpha,J} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{X}_{\infty,\alpha,I} & \longleftarrow & \mathfrak{X}_{\infty,\alpha,J} & \xrightarrow{h_r} & \mathfrak{X}_{r,\alpha,J}
\end{array}$$

Let $T := \mathrm{Spf}\ C$ be the inverse image of W in $\mathfrak{X}_{r,\alpha,J}$. Then $\mathfrak{w}_J|_T$ admits a generator f as C-module constructed as follows. For every $0\le n\le r+k$ take $c'_n \in \tilde{\mathrm{Ha}}_r^{-\frac{p^n-1}{p-1}}\mathcal{O}_{\mathfrak{IG}_{n,r,\alpha,J}}(T)$ such that $c'_0=1$, $\mathrm{Tr}_{\mathfrak{IG}}(c'_n)=c'_{n-1}$ and $c'_n=c_n$ if $n\le r$. Take a section $s\in\mathcal{O}_{\mathfrak{F}_{k+r,r,\alpha,J}}(T)$ which is 1 mod p^2 and which generates $\mathfrak{w}_{r+k,r,J}(T)$ (see lemma 4.5, and note that $1=k+r-(k+1)-2$ if $p\neq 2$ and $1=k+r-(k+1)-4$ if $p=2$). . Let $f:=\sum_{\sigma\in(\mathcal{O}_F/p^{k+r}\mathcal{O}_F)^*}\kappa(\tilde\sigma)\sigma\big(c'_{r+k}s\big)$.

Then

$$f=\sum_{\sigma\in(\mathcal{O}_F/p^{k+r}\mathcal{O}_F)^*}\kappa(\tilde\sigma)\sigma\big(c'_{r+k}\big) \mod \tilde{\mathrm{Ha}}^{-\frac{p^{r+k}-1}{p-1}}p^2$$

and it follows from 2.6 that

$$\sum_{\sigma\in(\mathcal{O}_F/p^{k+r}\mathcal{O}_F)^*}\kappa(\tilde\sigma)\sigma\big(c'_{r+k}\big)=\sum_{\sigma\in(\mathcal{O}_F/p^{r}\mathcal{O}_F)^*}\kappa(\tilde\sigma)\sigma\big(c_{r}\big) \mod (\mathfrak{m}_{r-1}\tilde{\mathrm{Ha}}^{-\frac{p^r-1}{p-1}},\cdots,\mathfrak{m}_{r+k-1}\tilde{\mathrm{Ha}}^{-\frac{p^{r+k}-1}{p-1}})$$

$$\big(\text{resp.} \mod (\mathfrak{m}_{r-2}\tilde{\mathrm{Ha}}^{-\frac{p^r-1}{p-1}},\cdots,\mathfrak{m}_{r+k-2}\tilde{\mathrm{Ha}}^{-\frac{p^{r+k}-1}{p-1}}) \text{ if } p=2\big).$$

Over the interval $[p^k,p^{k+1}]$, we have $p\in\alpha^{p^k}B^0_{\alpha,I}$, and it follows that

$$\mathfrak{m}_nB^0_{\alpha,I}\subset(\alpha^{p^{n-1}},\alpha^{p^k+p^{n-2}},\cdots,\alpha^{(n-1)p^k+1}).$$

Moreover, $\alpha\in\tilde{\mathrm{Ha}}^{-p^{r+1}}T$. Assume $p\neq 2$. We claim that

$$(\mathfrak{m}_{r-1}\tilde{\mathrm{Ha}}^{-\frac{p^r-1}{p-1}},\cdots,\mathfrak{m}_{r+k-1}\tilde{\mathrm{Ha}}^{-\frac{p^{r+k}-1}{p-1}})\subset(\alpha^2).$$

It is enough to check that :

$$(\mathfrak{m}_{r-1}\alpha^{-1},\mathfrak{m}_r\alpha^{-1},\cdots,\mathfrak{m}_{r+k-1}\alpha^{-p^{k-1}})\subset(\alpha^2)$$

where the term $\mathfrak{m}_{r+k-1}\alpha^{-p^{k-1}}$ is missing if $k=0$.
This boils down to the set of inequalities :

- $1+2\le p^{r-2};\ 1+2\le p^k+p^{r-3};\ 1+2\le 2p^k+p^{r-4}.$

- for $r \leq n \leq r+k-1$:

$$p^{n-r+1}+2 \leq p^{n-1};\ p^{n-r+1}+2 \leq p^{n-2}+p^k; \ldots;\ p^{n-r+1}+2 \leq (n-1)p^k+1.$$

When $p=2$, one proves similarly that

$$(\mathfrak{m}_{r-2}\tilde{\mathrm{Ha}}^{-\frac{p^r-1}{p-1}}, \cdots, \mathfrak{m}_{r+k-2}\tilde{\mathrm{Ha}}^{-\frac{p^{r+k}-1}{p-1}}) \subset (\alpha^2).$$

It follows that

$$f - b_\infty \in (\alpha^2, \tilde{\mathrm{Ha}}^{-\frac{p^{r+k}-1}{p-1}}p^2)\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,\alpha,J}}(T)$$

so that $b_\infty = (1+\alpha^2 u + \tilde{\mathrm{Ha}}^{-\frac{p^{r+k}-1}{p-1}}p^2 v)f$ for some elements $u, v \in \mathcal{O}_{\mathfrak{X}_{\infty,\alpha,J}}(T)$. Taking the Tate trace $\mathrm{Tr}_r \colon h_{r,*}\mathcal{O}_{\mathfrak{X}_{\infty,\alpha,J}} \to \alpha^{-1}\mathcal{O}_{\mathfrak{X}_{r,\alpha,J}}$ constructed in Proposition 5.2 we conclude that $\mathrm{Tr}_r(b_\infty) = f(1+\alpha^2\mathrm{Tr}_r(u) + \tilde{\mathrm{Ha}}^{-\frac{p^{r+k}-1}{p-1}}p^2\mathrm{Tr}_r(v))$. As $\mathrm{Tr}_r(u), \mathrm{Tr}_r(v) \in \alpha^{-1}\mathcal{O}_{\mathfrak{X}_{r,\alpha,J}}(T)$ by Proposition 5.2, we deduce that $\mathrm{Tr}_r(b_\infty)$ is a generator of $\mathfrak{w}_J|_T$. Since the construction of $\mathrm{Tr}_r(b_\infty)$ is functorial in J we get that $\mathfrak{w}_I(W) = \mathrm{Tr}_r(b_\infty) \cdot \lim_{k+1 \geq k' \geq k \geq 0} \mathcal{O}_{\mathfrak{X}_{r,\alpha,[p^k,p^{k'}]}}(W) = Tr_r(b_\infty)B$ thanks to Lemma 6.3. The theorem follows. □

Proposition 6.5. *The sheaves $\mathfrak{w}_I$ over each $\mathfrak{X}_{r,\alpha,I}$ glue to a sheaf still denoted $\mathfrak{w}_I$ over $\mathfrak{X}_{r,I}$. Let $\phi\colon \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ be the Frobenius. We have an isomorphism*

$$\mathfrak{w}_I \simeq \phi^*\mathfrak{w}_I.$$

Proof. Thanks to Theorem 6.4 the sheaf $\mathfrak{w}_I$ over $\mathfrak{X}_{r,\alpha,I}$ is canonically determined by the sheaves $\mathfrak{w}_I^{\mathrm{perf}}$ and the sheaves $\mathfrak{w}_J$ for $J = [p^k, p^{k'}]$. These glue for varying α and have compatible Frobenius morphisms by §4.3.2. The claim follows. □

6.3 Descent to the Iwasawa algebra

Let $\mathfrak{Z} = \overline{\mathfrak{M}}(\mu_N, \mathfrak{c}) \times \mathfrak{W}_F^0$. For all $r \geq 0$, let $\mathfrak{Z}_r$ be the $\mathfrak{m}$-adic formal scheme representing the functor which associates to any $\mathfrak{m}$-adically complete Λ_F^0-algebra R associates the equivalence classes of tuples $\big(h\colon \mathrm{Spf}\ R \to \overline{\mathfrak{M}}(\mu_N,\mathfrak{c}), \eta_p, \eta_1, \cdots, \eta_g \in \mathrm{H}^0(\mathrm{Spf}\ R, h^*\det\omega_G^{(1-p)p^{r+1}})\big)$ such that

$$\mathrm{Ha}^{p^{r+1}}\eta_p = p \mod p^2,\ \mathrm{Ha}^{p^{r+1}}\eta_1 = T_1 \mod p^2, \cdots,\ \mathrm{Ha}^{p^{r+1}}\eta_g = T_g \mod p^2.$$

Two tuples $(h, \eta_p, \eta_1, \cdots, \eta_g)$ and $(h', \eta_p, \eta_1, \cdots, \eta_g)$ are declared equivalent if $h = h'$ and

$$\eta_p = \eta_p'(1+\frac{p^2}{\alpha}u_p),\ \eta_1 = \eta_1'(1+\frac{p^2}{\alpha}u_1), \cdots,\ \eta_g = \eta_g'(1+\frac{p^2}{\alpha}u_g)$$

for some $u_p, u_1, \cdots, u_g \in R$.

There is a cartesian diagram of formal schemes :

$$\begin{array}{ccc} \mathfrak{X}_r & \xrightarrow{g} & \mathfrak{Z}_r \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{W}}_F^0 & \longrightarrow & \mathfrak{W}_F^0 \end{array}$$

Theorem 6.6. *The sheaf $g_\star \mathfrak{w}_I$ is an invertible sheaf over $\mathfrak{Z}_r$ and $g^\star g_\star \mathfrak{w}_I = \mathfrak{w}_I$.*

Proof. Let $U := \operatorname{Spf} A$ be a formal open affine subscheme of $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})$ where ω_G is trivial. Let $W := \operatorname{Spf} B$ be the inverse image of U in $\mathfrak{X}_r$. It suffices to prove that $\mathfrak{w}_I|_W$ is a free $\mathcal{O}_W$-module of rank 1. Let W^{perf} be the inverse image of W in $\mathfrak{X}_\infty$. We will actually prove that $\mathfrak{w}_I^{\mathrm{perf}}|_{W^{\mathrm{perf}}}$ is a free $\mathcal{O}_{W^{\mathrm{perf}}}$-module of rank 1 and find a generator b_∞ whose trace will be a generator of $\mathfrak{w}_I|_W$.

We apply the construction of §3.3 with $A_0 = \mathbb{Z}_p$. We thus obtain formal schemes $\mathfrak{Y}_s$ together with partial Igusa towers $\mathfrak{IG}\mathfrak{Y}_{n,s} \to \mathfrak{Y}_s$ for $n \leq s$.

Passing to the limit over Frobenius, we obtain $\mathfrak{IG}\mathfrak{Y}_{n,\infty} \to \mathfrak{Y}_\infty$. There is an obvious commutative diagram :

$$\begin{array}{ccc} \mathfrak{IG}_{n,r} & \longrightarrow & \mathfrak{IG}\mathfrak{Y}_{n,r} \\ \downarrow & & \downarrow \\ \mathfrak{X}_r & \longrightarrow & \mathfrak{Y}_r \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{W}}_F^0 & \longrightarrow & \operatorname{Spf} \mathbb{Z}_p \end{array}$$

We can find elements $c_n' \in \tilde{\mathrm{Ha}}_r^{-\frac{p^n-1}{p-1}} \mathcal{O}_{\mathfrak{IG}\mathfrak{Y}_{n,r}}(W)$ for integers $n \leq r$ and elements $c_n' \in \tilde{\mathrm{Ha}}_r^{-p^r} \mathcal{O}_{\mathfrak{IG}\mathfrak{Y}_{n,\infty}}(W)$ for general $r \leq n$ so that $c_0' = 1$ and $\mathrm{Tr}_{\mathfrak{IG}}(c_n') = c_{n-1}'$. We call c_n the pull back of c_n' in $\mathcal{O}_{\mathfrak{IG}_{n,\infty}}(W)$. We can now repeat the proof of thm 6.4 using these elements c_n to obtain a trivialisation b_∞ of $\mathfrak{w}_I^{\mathrm{perf}}|_{W^{\mathrm{perf}}}$ whose trace gives the trivialisation of $\mathfrak{w}_I|_W$.

□

We set $\mathfrak{w}^\kappa = g_\star \mathfrak{w}_I$.

6.4 The main theorem

Let H be the torsion subgroup of $\mathbb{T}(\mathbb{Z}_p)$. Let $\chi : H \to \Lambda_F^*$. For all $r \geq 0$, let

$$\mathfrak{M}_r = \mathfrak{Z}_r \times_{\mathfrak{W}_F^0} \mathfrak{W}_F.$$

Let $\mathcal{M}_r$ be the analytic adic space associated to $\mathfrak{M}_r$. In other words, $\mathcal{M}_r$ is the open subset of the $\overline{M}(\mu_N, \mathfrak{c}) \times_{\operatorname{Spec} \mathbb{Z}_p} \mathcal{W}_F$ defined by the conditions:

$$|\tilde{\mathrm{Ha}}^{p^{r+1}}| \geq \sup_{\alpha \in \mathfrak{m}} \{|\alpha|\}$$

where $\tilde{\mathrm{Ha}}$ is a local lift of the Hasse invariant. Over $\mathcal{M}_r$ we have a canonical subgroup of level 2, H_2. Let $\mathcal{IGM}_{2,r}$ be the torsor of trivializations of H_1^D. Let $\mathfrak{IG}\mathfrak{M}_{2,r} \xrightarrow{h} \mathfrak{M}_r$ be the normalization. It carries an action of $(\mathcal{O}_F/p^2\mathcal{O}_F)^*$.

Let $\mathfrak{w}^\chi$ be the subsheaf of $(h_2)_\star \mathcal{O}_{\mathfrak{IG}\mathfrak{M}_{2,r}}$ where $(\mathcal{O}_F/p^2\mathcal{O}_F)^*$ acts via $\chi^{-1} \colon H \to \Lambda_F^*$ composed with the projection $(\mathcal{O}_F/p^2\mathcal{O}_F)^* \to H$. This is a coherent sheaf, invertible over the ordinary locus and over the analytic fiber $\mathcal{M}_r$.

We define $\mathfrak{w}^{\kappa^{\mathrm{un}}} := (s^\star \mathfrak{w}^\kappa) \otimes \mathfrak{w}^\chi$ where $s : \mathfrak{M}_r \to \mathfrak{Z}_r$ is the projection. We let $\omega^{\kappa^{\mathrm{un}}}$ be the associated sheaf over $\mathcal{M}_r$.

Theorem 6.7. *The sheaf $\omega^{\kappa^{\mathrm{un}}}$ over $\mathcal{M}_r$ enjoys the following properties :*

1. *the restriction of $\omega^{\kappa^{\mathrm{un}}}$ to the classical analytic space $\mathcal{M}_r \times_{Spa(\mathbb{Z}_p,\mathbb{Z}_p)} Spa(\mathbb{Q}_p,\mathbb{Z}_p)$ is the sheaf defined in [2, Def. 3.6];*
2. *for all locally algebraic weight $k\chi : \mathbb{T}(\mathbb{Z}_p) \to \mathcal{O}_{\mathbb{C}_p}$ where k is an algebraic weight and χ is a finite character, then $\omega^{\kappa^{\mathrm{un}}}|_{k\chi} = \omega^k(\chi)$ is the sheaf of weight k modular forms and nebentypus χ.*
3. *If $i : \mathcal{M}_{r+1} \to \mathcal{M}_r$ is the inclusion and $\phi : \mathcal{M}_{r+1} \to \mathcal{M}_r$ is the Frobenius, then we have a canonical isomorphism $i^\star\omega^{\kappa^{\mathrm{un}}} \simeq \phi^\star\omega^{\kappa^{\mathrm{un}}}$.*

It is not straightforward to describe the sections of the sheaf $\omega^{\kappa^{\mathrm{un}}}$. Over some open subset $\mathcal{U} \hookrightarrow \mathcal{M}_r$ such that every component of $\mathcal{U}$ contains an ordinary point we have nevertheless the following description.

Proposition 6.8. *An overconvergent modular form f of weight κ^{un} over $\mathcal{U}$ is a functorial rule which associates to a tuple $\big((R,R^+), x : Spa(R,R^+) \to \mathcal{U}, \psi : \mathbb{T}(\mathbb{Z}_p) \to x^*\mathrm{T_p}(\mathrm{G^D})\big)$ an element in $f(x,\psi) \in R$ where :*

1. *(R,R^+) is an complete affinoide Tate algebra,*
2. *$x^*\mathrm{T_p}(\mathrm{G^D})$ is the pull back via x of the Tate module of the dual semi-abelian scheme G^D,*
3. *The morphisme ψ is $\mathbb{T}(\mathbb{Z}_p)$-equivariant, and induces an isomorphisme*

   $$\mathcal{O}_F \otimes \mathbb{F}_p \xrightarrow{\psi} x^\star G^D[p] \to x^\star H_1^D$$

   where H_1^D is the canonical quotient of $G^D[p]$,
4. *For all $\sigma \in \mathbb{T}(\mathbb{Z}_p)$, $f(x,\psi\circ\sigma) = (\kappa^{\mathrm{un}})^{-1}(\sigma) f(x,\psi)$,*
5. *If x factors through the ordinary locus, then $f(\psi,x) = f(\psi',x)$ whenever the maps*

   $$\mathbb{T}(\mathbb{Z}_p) \xrightarrow{\psi} \mathrm{T_p}(\mathrm{G^D}) \to \mathrm{T_p}(\mathrm{H^D_\infty}) \quad \text{and} \quad \mathbb{T}(\mathrm{\mathbb{Z}_p}) \xrightarrow{\psi'} \mathrm{T_p}(\mathrm{G^D}) \to \mathrm{T_p}(\mathrm{H^D_\infty})$$

   coincide where $\mathrm{T_p}(\mathrm{H^D_\infty})$ is the canonical quotient of $\mathrm{T_p}(\mathrm{G^D})$ given by the canonical subgroup of infinite order,
6. *There exists a rational cover $Spa(R,R^+) = \cup Spa(R_i,R_i^+)$ and a bounded and open subring $(R_i)_0 \subset R_i^+$ such that $x^\star G|_{Spa(R_i,R_i^+)}$ comes from a semi-abelian scheme G_0 over $Spf\ R_{i,0}$ and the morphisme $\psi|_{Spa(R_i,R_i^+)}$ comes from a group scheme morphisme $\psi_0 : \mathbb{T}(\mathbb{Z}_p) \to \mathrm{T_p}(\mathrm{G_0^D})$.*

Proof. Denote by α a topologically nilpotent unit in R. All conditions except 5. provide compatible sections f_i of $\mathrm{H}^0(\mathrm{Spf}\ R_{i,0}, \mathfrak{w}_I^{\mathrm{perf}})[1/\alpha]$. We claim that $\mathrm{Tr}_r(f_i) = f_i$ and that the f_i's glue to a section of $\omega^{\kappa^{\mathrm{un}}}$ over $\mathrm{Spa}(R,R^+)$. This holds true over the ordinary locus by 5. Using the assumption that every connected component of $\mathcal{U}$ contains an ordinary point we deduce the claim. □

7 Overconvergent forms in characteristic p

Specializing the sheaf $\omega^{\kappa^{\mathrm{un}}}$ of theorem 6.7 to characteristic p points of $\mathcal{W}_F$ we obtain sheaves of overconvergent modular forms in characteristic p. The goal of this section is to describe them directly.

7.1 The Igusa tower in characteristic p

Let $\overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ be the special fiber of $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})$ and denote by $\overline{M}_{\mathrm{ord}}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ the ordinary locus. It is an open dense subscheme of $\overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$, smooth over Spec $\mathbb{F}_p$. Fix a positive integer n. Over $\overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ we have a canonical subgroup of level n, denoted by H_n. It is the kernel of the n-th power frobenius map $F^n : G \to G^{(p^n)}$. The canonical subgroup over $\overline{M}_{\mathrm{ord}}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ is of multiplicative type and its dual H_n^D is étale locally isomorphic to $\mathcal{O}_F/p^n\mathcal{O}_F$. We denote by $\overline{\mathrm{IG}}_{n,\mathrm{ord}} \to \overline{M}_{\mathrm{ord}}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ the étale and Galois cover for the group $(\mathcal{O}_F/p^n\mathcal{O}_F)^*$ of trivializations of H_n^D. Passing to the projective limit over n we get a scheme $\overline{\mathrm{IG}}_{\infty,\mathrm{ord}}$ over $\overline{M}_{\mathrm{ord}}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$. For every n define $\overline{\mathrm{IG}}_n \to \overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ to be the normalization of $\overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ in $\overline{\mathrm{IG}}_{n,\mathrm{ord}}$. The scheme $\overline{\mathrm{IG}}_n$ is finite over $\overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ and carries an action of the group $(\mathcal{O}_F/p^n\mathcal{O}_F)^*$. It is characterized by the following universal property:

Lemma 7.1. *For every normal $\mathbb{F}_p$-algebra R and every R-valued point $x \in \overline{M}(\mu_N, \mathfrak{c})(R)$ such that the ordinary locus (Spec $R)_{\mathrm{ord}}$ is dense in Spec R, the R-valued points of $\overline{\mathrm{IG}}_n$ over x consist of the $\mathcal{O}_F$-equivariant morphisms $\mathcal{O}_F/p^n\mathcal{O}_F \to H_{n,x}^D$, which are isomorphisms over (Spec $R)_{\mathrm{ord}}$. Here $H_{n,x}$ is the pull back of the canonical subgroup to Spec R via x.*

Let $h_{n+1} : \overline{\mathrm{IG}}_{n+1} \to \overline{\mathrm{IG}}_n$ (with the convention $\overline{\mathrm{IG}}_0 = \overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$). Let us denote by $\mathrm{Tr}_{\mathrm{IG}} : (h_{n+1})_\star \mathcal{O}_{\overline{\mathrm{IG}}_{n+1}} \to \mathcal{O}_{\overline{\mathrm{IG}}_n}$ the trace of this morphism.

Lemma 7.2. *For all $n \geq 0$, we have* $\mathrm{Hdg}^{p^n} \subset \mathrm{Tr}_{\mathrm{IG}}\big((h_{n+1}) \star \mathcal{O}_{\overline{\mathrm{IG}}_n}\big)$.

Proof. Similar to the proof of proposition 3.4 □

Corollary 7.3. *Let Spec A be an open subset of $\overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ such that the sheaf* Hdg *is trivial. Let us identify* Ha *with a generator of* Hdg. *There is a sequence of elements $c_0 = 1$, $c_n \in \mathrm{Ha}^{-\frac{p^n-1}{p-1}}\mathcal{O}_{\overline{\mathrm{IG}}_n}$(Spec A) for $n \geq 1$ such that $\mathrm{Tr}_{\mathrm{IG}}(c_n) = c_{n-1}$.*

7.2 Formal schemes attached to the Hilbert modular variety in characteristic p

Let $\overline{\mathfrak{m}}$ be the maximal ideal $(T_1, \cdots, T_g)$ of $\Lambda_F^0/p\Lambda_F^0$. We set $\mathfrak{W}_{F,\{\infty\}}^0 = \mathrm{Spf}\ \Lambda_F^0/p\Lambda_F^0$. Recall that $\widetilde{\mathfrak{W}}_{F,\{\infty\}}^0$ is the blow-up of $\mathfrak{W}_{F,\{\infty\}}^0$ along $\overline{\mathfrak{m}}$.

In section 6.3 we defined an $\mathfrak{m}$-adic formal scheme $\mathfrak{Z}_r \to \mathfrak{W}_F^0$. We set $\mathfrak{Z}_{r,\{\infty\}} = \mathfrak{Z}_r \times_{\mathfrak{W}_F^0} \mathfrak{W}_{F,\{\infty\}}^0$. Its ordinary locus is $\mathfrak{Z}_{\mathrm{ord},\{\infty\}} = \overline{M}_{\mathrm{ord}}(\mu_N, \mathfrak{c})_{\mathbb{F}_p} \times_{\mathrm{Spec}\ \mathbb{F}_p} \mathrm{Spf}\ \mathfrak{W}_{F,\{\infty\}}^0$.

In §3.4 we defined a formal scheme $\mathfrak{X}_{r,\{\infty\}}$ over $\widetilde{\mathfrak{W}}_{F,\{\infty\}}^0$ and it follows from the definitions that :

$$\mathfrak{X}_{r,\{\infty\}} = \mathfrak{Z}_{r,\{\infty\}} \times_{\mathfrak{W}^0_{F,\{\infty\}}} \widetilde{\mathfrak{W}}^0_{F,\{\infty\}}.$$

Let $\mathfrak{IGZ}_{n,r,\{\infty\}} = \overline{\mathrm{IG}}_n \times_{\mathrm{Spec}\ \mathbb{F}_p} \mathfrak{Z}_{r,\{\infty\}}$ be the partial Igusa tower of level n over $\mathfrak{Z}_{r,\{\infty\}}$. Passing to the limit over n, we get an $\overline{\mathfrak{m}}$-adic formal scheme $h : \mathfrak{IGZ}_{\infty,r,\{\infty\}} \to \mathfrak{Z}_{r,\{\infty\}}$ which carries an action of $\mathbb{T}(\mathbb{Z}_p)$.

Lemma 7.4. *1. The formal scheme $\mathfrak{IGZ}_{n,r,\{\infty\}}$ is normal.*

2. We have :

$$\mathfrak{IG}_{n,r,\{\infty\}} = \mathfrak{IGZ}_{n,r,\{\infty\}} \times_{\mathfrak{Z}_{r,\{\infty\}}} \mathfrak{X}_{r,\{\infty\}}$$

where $\mathfrak{IG}_{n,r,\{\infty\}}$ is defined in §3.4.

3. $\left(h_\star \mathcal{O}_{\mathfrak{IGZ}_{\infty,r,\{\infty\}}}\right)^{\mathbb{T}(\mathbb{Z}_p)} = \mathcal{O}_{\mathfrak{Z}_{r,\{\infty\}}}$.

Proof. Easy and left to the reader. □

Proposition 7.5. *For any normal, $\overline{\mathfrak{m}}$-adically complete torsion free $\Lambda^0_F/p\Lambda^0_F$-algebra, the R-valued points $\mathfrak{IGZ}_{n,r,\infty}(R)$ classify tuples $\left(x, \eta_{T_1}, \cdots, \eta_{T_g}, \psi_n\right)$ where*

- *$x \in \overline{M}(\mu_N, \mathfrak{c})(R)$,*
- *$\eta_{T_1}, \cdots, \eta_{T_g} \in \mathrm{H}^0(\mathrm{R}, \det \omega_{\mathrm{G}}^{(1-\mathrm{p})\mathrm{p}^{\mathrm{r}+1}})$ satisfy $\mathrm{Ha}^{p^{r+1}} \eta_{T_i} = T_i$,*
- *$\psi_n \colon \mathcal{O}_F/p^n\mathcal{O}_F \to H_n^D$ is a $\mathcal{O}_F$-linear morphism of group schemes which is an isomorphism over $(\mathrm{Spec}\ R)_{\mathrm{ord}}$.*

7.3 p-adic modular forms in characteristic p

Let $\overline{\kappa} \colon (\mathcal{O}_F \otimes \mathbb{Z}_p)^* \to \left(\Lambda^0_F/p\Lambda^0_F\right)^*$ be the reduction modulo p of the character κ. Following Katz [7] we define

$$\mathfrak{w}_{\{\infty\}} := \mathcal{O}_{\mathfrak{IGZ}_{\infty,\mathrm{ord},\{\infty\}}}[\overline{\kappa}^{-1}].$$

It follows from loc. cit. that it is an invertible sheaf of $\mathcal{O}_{\mathfrak{Z}_{\mathrm{ord},\{\infty\}}}$-modules. Moreover the Frobenius on $\mathfrak{IGZ}_{\infty,\mathrm{ord},\{\infty\}}$ defines an isomorphism $\phi^*\mathfrak{w}_{\{\infty\}} \simeq \mathfrak{w}_{\{\infty\}}$.

7.4 Overconvergent forms

Let $h \colon \mathfrak{IGZ}_{\infty,r,\{\infty\}} \to \mathfrak{Z}_{r,\{\infty\}}$ be the structural morphism. As in the previous section we define the subsheaf $\mathfrak{w}_{\{\infty\}} := h_*\mathcal{O}_{\mathfrak{IGZ}_{\infty,r,\{\infty\}}}[\overline{\kappa}^{-1}]$ of $h_*\mathcal{O}_{\mathfrak{IG}_{\infty,r,\{\infty\}}}$. It is a sheaf of $\mathcal{O}_{\mathfrak{Z}_{r,\{\infty\}}}$-modules. Our main theorem is

Theorem 7.6. *Assume that $r \geq 2$ (resp. $r \geq 3$ if $p = 2$). Then the sheaf $\mathfrak{w}_{\{\infty\}}$ is an invertible sheaf of $\mathcal{O}_{\mathfrak{Z}_{r,\{\infty\}}}$-modules. Its restriction to $\mathfrak{Z}_{\mathrm{ord},\{\infty\}}$ is the sheaf defined in §7.3.*

Proof. The proof is local on $\overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$. Let Spec A be an open subset of $\overline{M}(\mu_N, \mathfrak{c})_{\mathbb{F}_p}$ where the Hodge ideal is trivial. We denote abusively Ha a generator. Let Spf R be the inverse image of Spec A in $\mathfrak{Z}_{r,\{\infty\}}$, Spf R_n the inverse image in $\mathfrak{IGU}_{n,r,\{\infty\}}$ and Spf R_∞ the inverse image in $\mathfrak{IGZ}_{\infty,r,\{\infty\}}$. By lemma 7.4, $R_\infty^{(\mathcal{O}_F\otimes\mathbb{Z}_p)^*} = R$. Thus to prove the theorem it suffices to show that there exists an invertible element $x \in R_\infty$ such that $\sigma(x) = \overline{\kappa}^{-1}(\sigma)x$ for every $\sigma \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^*$.

By Corollary 7.3 there exists elements $c_n \in \mathrm{Ha}^{-\frac{p^n-1}{p-1}} R_n$ such that $\mathrm{Tr}_{R_n/R_{n-1}}(c_n) = c_{n-1}$ and $c_0 = 1$. Define $b_n := \sum_{\sigma\in G_n} \overline{\kappa}(\tilde{\sigma})\sigma(c_n) \in \mathrm{Ha}^{-\frac{p^n-1}{p-1}-p} R_n$ for $n \geq 1$. Here $\tilde{\sigma} \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^*$ is a lift of $\sigma \in (\mathcal{O}_F/p^n\mathcal{O}_F)^*$. By Lemma 2.6, we deduce that

- $b_n - b_{n-1} \in (T_1^{p^{n-1}}, \cdots, T_g^{p^{n-1}})\mathrm{Ha}^{-\frac{p^n-1}{p-1}} R_n$ if $n \geq 1$ and $p \geq 3$,
- $b_n - b_{n-1} \in (T_1^{p^{n-2}}, \cdots, T_g^{p^{n-2}})\mathrm{Ha}^{-\frac{p^n-1}{p-1}} R_n$ if $n \geq 2$ and $p = 2$,
- $b_1 - 1 \in (T_1, \cdots, T_g)\mathrm{Ha}^{-1} R_1$ for all p.

One then concludes that $\{b_n\}_n$ is a Cauchy sequence of elements of R_∞ converging to a unit $b_\infty := \lim_n b_n$ of R_∞ having the property that $\sigma(b_\infty) = \overline{\kappa}^{-1}(\sigma)b_\infty$ for every $\sigma \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^*$. □

7.5 Comparison with the sheaf $\mathfrak{w}_{[1,\infty]}$

In this section we work over $\mathfrak{X}_{r,\{\infty\}}$ and prove that the specialization at $\{\infty\}$ of the sheaf $\mathfrak{w}_{[1,\infty]}$ of theorem 6.4 equals the pull back to $\mathfrak{X}_{r,\{\infty\}}$ of the sheaf $\mathfrak{w}_{\{\infty\}}$ defined in §7.4.

Proposition 7.7. *Let $\alpha \in \mathfrak{m}$. For all integer $k_0 \geq 1$, the obvious inclusion $\mathfrak{w}_{[p^{k_0},\infty]}$ in $\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,\alpha,[p^{k_0},\infty]}}$ factorizes modulo $\alpha^{p^{k_0}-p^{k_0-1}-1}$ into a morphisme*

$$\mathfrak{w}_{[p^{k_0},\infty]} \to \mathcal{O}_{\mathfrak{IG}_{r+k_0,r,\alpha,[p^{k_0},\infty]}}/\alpha^{p^{k_0}-p^{k_0-1}-1}.$$

The restriction of $\mathfrak{w}_I$ to $\mathfrak{X}_{r,\alpha,\{\infty\}}$ is a subsheaf of $\mathcal{O}_{\mathfrak{IG}_{\infty,r,\alpha,\{\infty\}}}$ which identifies canonically to $\mathfrak{w}_{\{\infty\}}$.

Proof. Fix an integer k_0. Let Spf B be an open affine of $\mathfrak{X}_{r,\alpha,[p^{k_0},\infty]}$. Assume that Hdg is trivial on Spf B, generated by $\tilde{\mathrm{Ha}}$. Fix elements $c_0 = 1$ and, for $1 \leq n \leq k_0 + r$, $c_n \in \tilde{\mathrm{Ha}}^{-\frac{p^n-1}{p-1}} \mathcal{O}_{\mathfrak{IG}_{n,r,\alpha,[p^{k_0},\infty]}}(\mathrm{Spf}\ B)$ such that $\mathrm{Tr}_{\mathfrak{IG}}(c_n) = c_{n-1}$. Complete the sequence by choosing, for $n \geq r + k_0 + 1$, elements $c_n \in \tilde{\mathrm{Ha}}^{-p^{r+k_0}} \mathcal{O}_{\mathfrak{IG}_{n,\infty,\alpha,[p^{k_0},\infty]}}(\mathrm{Spf}\ B)$ satisfying $\mathrm{Tr}_{\mathfrak{IG}}(c_n) = c_{n-1}$. Set $b_n = \sum_{\sigma\in(\mathcal{O}_F/p^n\mathcal{O}_F)^*} \kappa(\tilde{\sigma})\sigma.c_n$, where $\tilde{\sigma}$ is a lift of σ in $\mathbb{T}(\mathbb{Z}_p)$. The sequence b_n converges in $\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,[p^{k_0},\infty]}}$ to a generator b_∞ of the sheaf $\mathfrak{w}_I^{\mathrm{perf}}$. By lemma 2.6, for all $n \geq r + k_0$, $b_n = b_{r+k_0}$ mod $\mathfrak{m}_{r+k_0-1}\tilde{\mathrm{Ha}}^{-p^{r+k_0}}$ (resp. mod $\mathfrak{m}_{r+k_0-2}\tilde{\mathrm{Ha}}^{-p^{r+k_0}}$ if $p = 2$). It follows that $b_\infty = b_{r+k_0}$ mod $\alpha^{p^{k_0}-p^{k_0-1}}$.

Fix now an intervalle $[p^k, p^{k+1}]$ with $k \geq k_0$. Let Spf C be the inverse image of Spf B in $\mathfrak{X}_{r,\alpha,[p^k,p^{k+1}]}$. Fix elements $c'_n \in \tilde{\mathrm{Ha}}^{-\frac{p^n-1}{p-1}} \mathcal{O}_{\mathfrak{IG}_{n,r,\alpha,[p^k,p^{k+1}]}}(\mathrm{Spf}\ C)$ for $r + k_0 + 1 \leq n \leq r + k$ satisfying $\mathrm{Tr}_{\mathfrak{IG}}(c'_n) = c'_{n-1}$ for $n \geq r + k_0 + 2$ and $\mathrm{Tr}_{\mathfrak{IG}}(c'_{r+k_0+1}) = c_{r+k_0}$. There is a generator

f of the sheaf $\mathfrak{w}_I$ over Spf C such that $f = \sum_{\sigma\in(\mathcal{O}/p^{r+k}\mathcal{O})^*} \kappa(\tilde\sigma)\sigma.c'_{r+k} \mod p\tilde{\mathrm{Ha}}^{-p^{r+k}}$. In C, $\alpha^{p^k} \mid p$ and $\tilde{\mathrm{Ha}}^{-p^{r+k}} \mid \alpha^{p^{k-1}}$. It follows that $f = \sum_{\sigma\in(\mathcal{O}/p^{r+k}\mathcal{O})^*} \kappa(\tilde\sigma)\sigma.c'_{r+k} \mod \alpha^{p^{k_0}-p^{k_0-1}}$. Using lemma 2.6 one more time, we deduce that $f = b_{r+k_0} \mod \alpha^{p^{k_0}-p^{k_0-1}}$. As a consequence, in $\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,\alpha,[p^k,p^{k+1}]}}(\mathrm{Spf}\ C)$ we have $f = b_\infty = b_{r+k_0} \mod \alpha^{p^{k_0}-p^{k_0-1}}$. Using Tate's traces, we get $f = \mathrm{Tr}_r(b_\infty) = b_\infty \mod \alpha^{p^{k_0}-p^{k_0-1}-1}\mathfrak{w}^{perf}_{[p^k,p^{k+1}]}(\mathrm{Spf}\ C)$. As this relation holds on all intervals $[p^k,p^{k+1}]$, it follows that $\mathrm{Tr}_r(b_\infty) = b_\infty \mod \alpha^{p^{k_0}-p^{k_0-1}-1}\mathfrak{w}^{perf}_{[p^{k_0},\infty]}(\mathrm{Spf}\ B)$. As a result, $\mathrm{Tr}_r(b_\infty) = b_{r+k_0} \mod \alpha^{p^{k_0}-p^{k_0-1}-1}\mathcal{O}_{\mathfrak{IG}_{\infty,\infty,\alpha,[p^{k_0},\infty]}}(\mathrm{Spf}\ B)$.

It follows that we can define a morphisme $\mathfrak{w}_{[p^{k_0},\infty]} \to \mathcal{O}_{\mathfrak{IG}_{r+k_0,r,\alpha,[p^{k_0},\infty]}}/\alpha^{p^{k_0}-p^{k_0-1}-1}$ which on Spf B sends $\mathrm{Tr}_r(b_\infty)$ on b_{r+k_0}. It factorizes the map $\mathfrak{w}_{[p^{k_0},\infty]} \to \mathcal{O}_{\mathfrak{IG}_{\infty,\infty,\alpha,[p^{k_0},\infty]}}/\alpha^{p^{k_0}-p^{k_0-1}-1}$. Comparing the definition of b_{r+k_0} and the construction of the sheaf $\mathfrak{w}_{\{\infty\}}$ we obtain that the restriction of $\mathfrak{w}_{[1,\infty]}$ to $\mathfrak{X}_{r,\{\infty\}}$ is $\mathfrak{w}_{\{\infty\}}$.

□

7.6 Analytic overconvergent modular forms

We now let $\mathcal{M}_{r,\{\infty\}} = \mathcal{M}_r \times_{\mathcal{W}_F} \mathcal{W}_{F,\{\infty\}}$. Concretely, $\mathcal{M}_{r,\{\infty\}}$ is the open subset of $\overline{M}(\mu_N,\mathfrak{c})\times_{\mathrm{Spec}\ \mathbb{F}_p} \mathcal{W}_{F,\{\infty\}}$ where

$$|\mathrm{Ha}^{p^{r+1}}| \geq \sup_{\alpha\in\overline{\mathfrak{m}}} |\alpha|.$$

Denote by $\overline{\kappa}^{\mathrm{un}} : \mathbb{T}(\mathbb{Z}_p) \to \Lambda_F/\Lambda_F$ the universal character.

Let $\omega^{\overline{\kappa}^{\mathrm{un}}}$ be the pull back to $\mathcal{M}_{r,\{\infty\}}$ of $\omega^{\kappa^{\mathrm{un}}}$. An r-overconvergent modular forms of weight $\overline{\kappa}^{\mathrm{un}}$ is a global section $\omega^{\overline{\kappa}^{\mathrm{un}}}$. Here is the promised description.

Proposition 7.8. *An r-overconvergent modular form f of weight $\overline{\kappa}^{\mathrm{un}}$ is a functorial rule which associates to a tuple $\big((R,R^+), x : Spa(R,R^+) \to \mathcal{M}_{r,\{\infty\}}, \psi : \mathbb{T}(\mathbb{Z}_p) \simeq \lim_n x^*H_n^D\big)$ an element in $f(x,\psi) \in R$ where :*

- *(R,R^+) is a complete affinoide Tate algebra,*
- *$x^*H_n^D$ is the pullback of the dual canonical subgroup of level n to $Spa(R,R^+)$,*
- *The isomorphisme ψ is $\mathbb{T}(\mathbb{Z}_p)$ equivariant,*
- *For all $\sigma \in \mathbb{T}(\mathbb{Z}_p)$, $f(x,\psi\circ\sigma) = (\overline{\kappa}^{\mathrm{un}})^{-1}(\sigma)f(x,\psi)$.*
- *There exists a rational cover $Spa(R,R^+) = \cup Spa(R_i,R_i^+)$ and a bounded and open subring $(R_i)_0 \subset R_i^+$ such that $x^\star G|_{Spa(R_i,R_i^+)}$ comes from a semi-abelian scheme G_0 over Spf $R_{i,0}$ and the isomorphisme $\psi|_{Spa(R_i,R_i^+)}$ comes from a group scheme morphisme $\psi_0 : \mathbb{T}(\mathbb{Z}_p) \to \lim_n(G_0[F^n])^D$ defined over Spf$R_{i,0}$ (where $[F^n]$ means the kernel of the n-th power Frobenius isogeny).*

Proof. Easy and left to the reader. □

8 Overconvergent arithmetic Hilbert modular forms

8.1 The Shimura variety $\overline{M}(\mu_N, \mathfrak{c})_G$

Consider the group $G := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ and $G^* := G \times_{\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m} \mathbb{G}_m$, where the morphism $G \longrightarrow \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ is the determinant. So far we have worked on the Shimura variety associated to the group G^*. From the point of view of automorphic forms it is usefull to work with the Shimura variety defined by the group G.

Let $\mathcal{O}_F^{+,*}$ be the group of totally real units of $\mathcal{O}_F$ and let $U_N \subset \mathcal{O}_F^*$ be the group of units congruent to 1 modulo N. Consider the finite group $\Delta = \mathcal{O}_F^{+,*}/U_N^2$. For $\epsilon \in \mathcal{O}_F^{+,*}$ we have an action $[\epsilon]\colon \overline{M}(\mu_N, \mathfrak{c}) \to \overline{M}(\mu_N, \mathfrak{c})$ given by multiplying the polarization λ by ϵ. The action factorizes through Δ (see [2], intro., p. 6).

Lemma 8.1. *The group Δ acts freely on $\overline{M}(\mu_N, \mathfrak{c})$. One can form the quotient $\overline{M}(\mu_N, \mathfrak{c})_G := \overline{M}(\mu_N, \mathfrak{c})/\Delta$. The quotient map $\overline{M}(\mu_N, \mathfrak{c}) \to \overline{M}(\mu_N, \mathfrak{c})_G$ is finite étale with group Δ.*

Proof. Since $M(\mu_N, \mathfrak{c})$ is a projective scheme, it can be covered by open affine subschemes fixed by the action of Δ. Thus we can form the quotient $\overline{M}(\mu_N, \mathfrak{c})_G$. We now show that Δ acts freely on $M(\mu_N, \mathfrak{c})$. This can be proven over algebraically closed field k. The freeness of the action amounts to prove the following. Consider an abelian variety with real multiplication $(A, \iota, \Psi, \lambda)$ over k as in §3 and a totally positive unit $\epsilon \in \mathcal{O}_F^{+,*}$. Let $\alpha\colon A \to A$ be an automorphism commuting with the $\mathcal{O}_F$-action, the level N structure Ψ and such that $\lambda \circ \alpha = \epsilon\alpha^\vee \circ \lambda$. Then $\alpha \in U_N$ is a totally positive unit, congruent to 1 modulo N. As α respects the level N structure, it suffices to prove that α is an endomorphism lying in $\mathcal{O}_F$. Suppose not. Then $E := F[\alpha] \subset \mathrm{End}^0(A)$ is a commutative algebra of dimension at least $2g$. It must be a field, else A would decompose as a product of at least two abelian varieties of dimension $< g$, with real multiplication by F which is impossible. As a maximal commutative subalgebra of $\mathrm{End}^0(A)$ has dimension $\leq 2g$, it follows that E is a CM field of degree 2 over F. Moreover the Rosati involution associated to any $\mathcal{O}_F$-invariant polarization induces complex conjugation on E. As the rank of the group of units in $\mathcal{O}_E$ is equal to the rank of the group of units of $\mathcal{O}_F$ by Dirichlet's unit theorem, it follows that there exists an integer $n \geq 2$ such that $\alpha^n \in \mathcal{O}_F^*$ and $\alpha^{n-1} \notin \mathcal{O}_F^*$. Hence $\zeta = (\alpha/\bar{\alpha})$ is a primitive n-root of unity in $\mathcal{O}_E$. It preserves every $\mathcal{O}_F$-equivariant polarization λ as $\lambda^{-1} \circ \zeta^\vee \circ \lambda \circ \zeta = \overline{\zeta}\zeta = 1$. As $N \geq 4$ it follows from Serre's lemma that $\zeta = 1$ leading to a contradiction. We are left to show that Δ acts freely on the boundary $D := \overline{M}(\mu_N, \mathfrak{c})_G \setminus M(\mu_N, \mathfrak{c})_G$. Recall that the boundary is the union of its connected components parametrized by the cusps of the minimal compactification. Each connected component of the boundary is stratified. More precisely, for each connected component, there is a polyedral decomposition Σ of the cone of totally positive elements M^+ inside a fractional ideal $M \subset F$ determined by the cusp. To all cone $\sigma \in \Sigma$ corresponds a stratum $S_\sigma \subset D$. By construction of the toroidal compactification, if $\epsilon \in U_N$, then $S_{\epsilon^2\sigma} = S_\sigma$. We now claim that the action of Δ on the set of all strata in D is free. This follows from the fact that the stabilizor of $\sigma \in \Sigma$ is a finite sub-group of $\mathcal{O}_F^{+,*}$, thus trivial. This concludes the proof.

□

8.2 Descending the sheaf $\mathfrak{w}^{\kappa^{\mathrm{un}}}$

We follow closely [2]; see especially the Introduction and §4. First of all the weight space associated to G is the formal scheme $\mathfrak{W}_F^G$ defined by the Iwasawa algebra $\Lambda_F^G := \mathbb{Z}_p[\![(\mathcal{O}_F \otimes \mathbb{Z}_p)^* \times \mathbb{Z}_p^*]\!]$. There is a natural map of formal schemes $\mathfrak{W}_F^G \to \mathfrak{W}_F$ defined by the group homomorphism $(\mathcal{O}_F \otimes \mathbb{Z}_p)^* \to (\mathcal{O}_F \otimes \mathbb{Z}_p)^* \times \mathbb{Z}_p^*$ given by $t \mapsto \big(t^2, \mathrm{Nm}_{F/\mathbb{Q}}(t)\big)$. This induces a map of analytic adic spaces $\iota \colon \mathcal{W}_F^G \to \mathcal{W}_F$. We let $\kappa_G^{\mathrm{un}} := (\nu, w) \colon (\mathcal{O}_F \otimes \mathbb{Z}_p)^* \times \mathbb{Z}_p^* \to \Lambda_F^{G,*}$ be the universal character.

Consider the formal scheme $\mathfrak{M}_r \times_{\mathfrak{W}_F} \mathfrak{W}_F^G$. Let $\mathfrak{w}^{\kappa_G^{\mathrm{un}}}$ be the pull back of the universal sheaf to $\mathfrak{M}_r \times_{\mathfrak{W}_F} \mathfrak{W}_F^G$.

As a consequence of lemma 8.1 the group Δ acts freely on the formal schemes $\mathfrak{M}_r \times_{\mathfrak{W}_F} \mathfrak{W}_F^G$. We denote by $\mathfrak{M}_{r,G}$ the quotients by the group Δ. We now claim that the action of Δ on $\mathfrak{M}_r \times_{\mathfrak{W}_F} \mathfrak{W}_F^G$ can be lifted to an action on the sheaf $\mathfrak{w}^{\kappa_G^{\mathrm{un}}}$.

Since $\mathfrak{w}^{\kappa_G^{\mathrm{un}}}$ is defined without any reference to polarization we have an isomorphism $\mu_\epsilon \colon [\epsilon]^* \mathfrak{w}^{\kappa_G^{\mathrm{un}}} \to \mathfrak{w}^{\kappa_G^{\mathrm{un}}}$ for all $\epsilon \in \mathcal{O}_F^{+,*}$. We modify the action by multiplying μ_ϵ by $\nu(\epsilon)$. One then verifies, see [2, §4.1], that this action factorizes through the group Δ. By finite étale descent we obtain a sheaf that we continue to denote $\mathfrak{w}^{\kappa_G^{\mathrm{un}}}$ over $\mathfrak{M}_{r,G}$. We let $\mathcal{M}_{r,G}$ be the analytic fiber of $\mathfrak{M}_{r,G}$ and denote by $\omega^{\kappa_G^{\mathrm{un}}}$ the invertible sheaf on $\mathcal{M}_{r,G}$ associated to $\mathfrak{w}^{\kappa_G^{\mathrm{un}}}$.

8.3 The cohomology of the sheaf $\omega^{\kappa_G^{\mathrm{un}}}(-D)$

There is an obvious map $\mathfrak{M}_{r,G} \to \overline{\mathfrak{M}}(\mu_N, \mathfrak{c})_G$. Let D be the boundary divisor in $\overline{\mathfrak{M}}(\mu_N, \mathfrak{c})_G$. We also denote by D its inverse image in $\mathfrak{M}_{r,G}$.

Recall that $M^*(\mu_N, \mathfrak{c})$ is the minimal compactification of $M(\mu_N, \mathfrak{c})$. Certainly, the construction of $\mathfrak{M}_r$ admits a variant where one uses $M^*(\mu_N, \mathfrak{c})$ as a starting point instead of $\overline{M}(\mu_N, \mathfrak{c})$. Let us denote by $\mathfrak{M}_r^\star$ the resulting formal schemes.

We also define

$$\mathfrak{M}_{r,G}^\star := \big(\mathfrak{M}_r^\star \times_{\mathfrak{W}_F} \mathfrak{W}_F^G\big)/\Delta.$$

Let $h : \mathfrak{M}_{r,G} \to \mathfrak{M}_{r,G}^*$ be the canonical projection. The main result of the section is the following cohomology vanishing :

Theorem 8.2. *We have $R^i h_\star \mathfrak{w}^{\kappa_G^{\mathrm{un}}}(-D) = 0$ for all $i > 0$.*

Proof. This is a variant of [2], th. 3.17. Recall that $\overline{M}_{\mathrm{ord}}(\mu_N, \mathfrak{c})_G$ be the quotient by Δ of $\overline{M}(\mu_N, \mathfrak{c})$. Let $M^*(\mu_N, \mathfrak{c})$ be the minimal compactification and let $M^*(\mu_N, \mathfrak{c})_G$ be its quotient by Δ. We have an map $h' : \overline{M}(\mu_N, \mathfrak{c})_G \to M^*(\mu_N, \mathfrak{c})_G$. Let $\mathcal{L}$ be a torsion invertible sheaf on $\overline{M}(\mu_N, \mathfrak{c})_G$. Then we claim that $\mathrm{R^i h'_\star}\mathcal{L}(-\mathrm{D}) = 0$ for $i > 0$. This follows from [2], prop. 6.4. Note that in that reference the proposition is stated for the trivial sheaf, but the proof works without any change for a torsion sheaf.

The map $h : \mathfrak{M}_{r,G} \to \mathfrak{M}_{r,G}^*$ is an isomorphism away from the cusp and in particular away from the ordinary locus, so we are left to prove the statement for the map $h_{\mathrm{ord}} : \mathfrak{M}_{\mathrm{ord,G}} \to \mathfrak{M}_{\mathrm{ord,G}}^*$ over the ordinary locus. In this case, the sheaf $\mathfrak{w}^{\kappa_G^{\mathrm{un}}}(-D)$ is invertible. Recall that the ring Λ_F^G is semi-local and complete. Let $\mathfrak{n}$ be a maximal ideal of Λ_F^G corresponding to a character $\eta : (\mathcal{O}_F/\mathfrak{p}\mathcal{O}_F)^* \times (\mathbb{F}_p)^\times \to \mathbb{F}_q^\times$ where $\mathbb{F}_q$ is a finite extension of $\mathbb{F}_p$. We are left to prove the vanishing for the sheaf $\mathcal{F} := \mathfrak{w}^{\kappa_G^{\mathrm{un}}}(-D)/\mathfrak{n}$ over $\mathfrak{M}_{\mathrm{ord,G}}$. This is an invertible sheaf on its support $\overline{M}_{\mathrm{ord}}(\mu_N, \mathfrak{c})_{G,\mathbb{F}_q} \hookrightarrow \mathfrak{M}_{\mathrm{ord,G}}$. Moreover $\mathcal{F}^{\otimes q-1} = \mathcal{O}_{\overline{M}_{\mathrm{ord}}(\mu_N,\mathfrak{c})_{G,\mathbb{F}_q}}$ because the order of the character η divides $q-1$, and we can conclude. □

Corollary 8.3. *Let $g : \mathcal{M}_{r,G} \to \mathcal{W}_F^G$ be the projection to the weight space.*

1. *We have the vanishing $\mathrm{R}^i g_\star \omega^{\kappa_G^{\mathrm{un}}}(-D) = 0$ for all $i > 0$.*
2. *For all point $\kappa \in \mathcal{W}_G$,*

$$\kappa^* g_\star \omega^{\kappa_G^{\mathrm{un}}}(-D) = \mathrm{H}^0\big(\mathcal{M}_{r,G}, \kappa^\star \omega^{\kappa_G^{\mathrm{un}}}(-D)\big)$$

   is the space of r-overconvergent cuspidal arithmetic modular forms of weight κ.

3. *There exists an finite covering of the weight space $\mathcal{W}_F^G = \cup_i Spa(R_i, R_i^+)$ such that*

$$g_\star \omega^{\kappa_G^{\mathrm{un}}}(-D)(Spa(R_i, R_i^+))$$

   is a projective Banach R_i-module.

Proof. We have a sequence of maps

$$g : \mathcal{M}_{r,G} \overset{g_1}{\to} \mathcal{M}_{r,G}^* \overset{g_2}{\to} \mathcal{W}_F^G.$$

The map g_2 is affine and has no higher cohomology. Moreover, thm 8.2 implies that $\mathrm{R^i(g_2)_\star\omega^{\kappa^{un}}(-D)} = 0$. The second point follows easily.

For the last point, fix some open $\mathrm{Spa}(R_i, R_i^+) \subset \mathcal{W}_F^G$. Since the sheaf $\omega^{\kappa^{-\mathrm{un}}}(-D)$ is invertible, there is a finite covering $\cup \mathrm{Spa}(S_i, S_j^+)$ of $\mathcal{M}_{r,G}|_{\mathrm{Spa}(R_i,R_i^+)}$ such that the sheaf $\omega^{\kappa_G^{\mathrm{un}}}(-D)$ is trivial on this covering. Moreover, it is easy to see that each S_i is a projective Banach R_i-module. We can form the chech complex associated to the covering $\cup \mathrm{Spa}(S_i, S_j^+)$. By 1., this complex is a resolution of $g_\star \omega^{\kappa_G^{\mathrm{un}}}(-D)(\mathrm{Spa}(R_i, R_i^+))$. Moreover, each term appearing in the complex is a projective Banach module. Thus $g_\star \omega^{\kappa_G^{\mathrm{un}}}(-D)(\mathrm{Spa}(R_i, R_i^+))$ is a projective Banach module.

□

8.4 The adic eigenvariety

In [2], thm 5.1, we constructed an eigenvariety over $\mathcal{W}_F^G \setminus \{|p| = 0\}$. We can now extend it to $\mathcal{W}_F^G$.

Let $\mathrm{Frac}(F)^{(p)}$ be the group of fractional ideals prime to p. Let $\mathrm{Princ}(F)^{+,(p)}$ be the group of positive elements which are p-adic units. The quotient $\mathrm{Frac}(F)^{(p)}/\mathrm{Princ}(F)^{+,(p)} = \mathrm{Cl}^+(F)$ is the strict class group of F.

For all $\mathfrak{c} \in \mathrm{Frac}(F)^{(p)}$ we have defined an adic space $\mathcal{M}_r$ that we denote now by $\mathcal{M}_r(\mathfrak{c})$ to recall the dependance on $\mathfrak{c}$ and a sheaf $\omega^{\kappa^{\mathrm{un}}}$ over $\mathcal{M}_r(\mathfrak{c})$. Let $g_\mathfrak{c} : \mathcal{M}_r(\mathfrak{c}) \to \mathcal{W}_F^G$ be the projection. For all $x \in \mathrm{Princ}(F)^{+,(p)}$ we can canonically identify $(g_\mathfrak{c})_\star \omega^{\kappa^{\mathrm{un}}}(-D)$ and $(g_{x\mathfrak{c}})_\star \omega^{\kappa^{\mathrm{un}}}(-D)$ as in [2], def. 4.6.

We thus obtain a sheaf of projective Banach modules $\bigoplus_{\mathfrak{c} \in \mathrm{Cl}^+(F)} (g_\mathfrak{c})_\star \omega^{\kappa^{\mathrm{un}}}(-D)$. This sheaf carries an action of the Hecke algebra $\mathcal{H}^p$ of level prime to p as well as an action of the U_p operator and of the operators $U_{\mathfrak{P}_i}$ (see [2], §4.3). Moreover, the U_p-operator is compact. Applying [3], Appendice B, we obtain the following theorem.

Theorem 8.4. *1. The characteristic series*

$$P = \det\left(1 - XU_p | \bigoplus_{\mathfrak{c}\in \mathrm{Cl}^+(F)} (g_{\mathfrak{c}})_\star \omega^{\kappa^{\mathrm{un}}}(-D)\right)$$

takes values in $\Lambda_F[[X]]$.

2. *The spectral variety* $\mathcal{Z} = V(P) \to \mathcal{W}_F$ *is locally finite flat and partially proper over the weight space.*

3. *There is an eigenvariety* $\mathcal{E} \to \mathcal{Z}$ *finite and torsion free over* $\mathcal{Z}$ *which parametrizes finite slope eigensystems of overconvergent modular forms.*

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