# Katz type $p$-adic $L$-functions for primes $p$ non-split in the CM field.

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# 1 Introduction.

The celebrated article of N. Katz "p-Adic Interpolation of Real Analytic Eisenstein Series", 1976, (see [K2]) attaches a $p$-adic $L$-function to a pair $(K, p)$ consisting of a quadratic imaginary field $K$ and a prime integer $p$, satisfying a number of assumptions of which the most important is that the prime $p$ is split in $K$. Katz' $p$-adic $L$-function associated to a pair $(K, p)$ as above can be seen to interpolate $p$-adically the central critical values of the complex Rankin $L$-functions of a fixed Eisenstein series twisted by a family of certain algebraic Hecke characters. This article was very influential and produced, during the last 40 years, a large number of papers extending these ideas to other similar situations and/or trying to prove properties of these new "Katz type" $p$-adic $L$-functions.

An exemple is the work of Bertolini-Darmon-Prasana (see [BDP]) who studied Katz type $p$-adic $L$-functions in which the Eisenstein series was replaced by a cuspform. More precisely they defined an anticyclotomic $p$-adic $L$-function attached to an elliptic cuspidal eigenform $f$ and an imaginary quadratic field which interpolates the central critical values of the Rankin $L$-functions of $f$ twisted by anticyclotomic characters of higher infinity type and proved $p$-adic Gross-Zagier formulae for these $p$-adic $L$-functions, in the case $p$ is split in the quadratic imaginary field.

We will now be more precise.

*Classical L-values.*

In what follows we fix a classical, elliptic eigenform $F$ of weght $k \geq 2$, level $\Gamma_1(N)$ for $N \geq 5$ and character $\epsilon$ and a quadratic imaginary field $K$. We assume that there is an ideal $\mathfrak{N}$ of $\mathcal{O}_K$ whose norm is $N$ and such that we have a ring isomorphism $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$ (the so called

Heegner hypothesis); we choose and fix such an ideal. We consider the $L$-functions $L(F, \chi^{-1}, s)$ of the complex variable $s$, where $\chi$ is an algebraic Hecke character of $K$ such that $s = 0$ is central critical value.

This happens as follows:

a) If $F = E_{k,\epsilon}$ is an Eisensetin series then the set of Hecke characters we consider is: $\Sigma_{cc}(\mathfrak{N})$ associated to the triple $\big(\epsilon_{\mathfrak{N}}, (k_1, k_2)\big)$, where $k_1, k_2 \in \mathbb{Z}$, $k_1 + k_2 = k$ and $1 \leq k_1 \leq k-1$ or $k_1 \leq 0$.

This set naturally decomposes into a disjoint union: $\Sigma_{cc}(\mathfrak{N}) = \Sigma_{cc}^{(1)}(\mathfrak{N}) \cup \Sigma_{cc}^{(2)}(\mathfrak{N})$, where $\Sigma_{cc}^{(1)}(\mathfrak{N})$ is the set of the algebraic Hecke characters in $\Sigma_{cc}(\mathfrak{N})$ with infinity type $(k-j-1, 1+j)$ with $0 \leq j \leq k-2$ and $\Sigma_{cc}^{(2)}(\mathfrak{N})$ contains the algebraic Hecke characters with infinity type $(k+j, -j)$, $j \geq 0$.

b) If $F = f$ is a cuspform we denote by $\Sigma_{cc}(\mathfrak{N})$ the set of central critical algebraic Hecke characters of conductor dividing $c\mathfrak{N}$ (where $c$ is an auxiliary odd rational integer prime to $Np$ and to the discriminant of $K$) and satisfying the extra assumption that for every prime integer $q$, the epsilon factor $\epsilon_q(f, \chi^{-1}) = 1$.

Looking at the infinity types of the characters in $\Sigma_{cc}(\mathfrak{N})$ we see that we also have a natural decomposition $\Sigma_{cc}(\mathfrak{N}) = \Sigma_{cc}^{(1)}(\mathfrak{N}) \cup \Sigma_{cc}^{(2)}(\mathfrak{N})$ where $\Sigma_{cc}^{(1)}(\mathfrak{N})$ is the subset of characters of $\Sigma_{cc}(\mathfrak{N})$ having infinity type $(k-1-j, 1+j)$, with $0 \leq j \leq k-2$ and $\Sigma_{cc}^{(2)}(\mathfrak{N})$ is the subset of characters of $\Sigma_{cc}(\mathfrak{N})$ having infinity type $(k+j, -j)$ for $j \geq 0$.

For $F$ as above and $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ we have an explicit description of the algebraic part of the special value of $L(F, \chi^{-1}, 0)$ as follows provided by Damerell's theorem in the Eisenstein case and by a result of Waldspurger's in the cuspidal case:

$$(*) \quad L_{\mathrm{alg}}(F, \chi^{-1}) := \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \delta_k^j(F) \Big( \mathfrak{a} * (A_0, t_0, \omega_0) \Big) \in \overline{\mathbb{Q}}$$

and

$$\big(L_{\mathrm{alg}}(F, \chi^{-1})\big)^{\iota} = C(F, \chi) \frac{L(F, \chi^{-1}, 0)}{\Omega^{\iota(k+2j)}},$$

where $\iota = 1$ if $F$ is an Eisenstein series and $\iota = 2$ if $F$ is a cuspform, $C(F, \chi)$ is an explicit constant which varies if $F$ is an Eisentein series or a cuspform, $\Omega$ is a complex period, $c = 1$ if $F$ is an Eisenstein series and $c$ is an auxiliary odd integer if $F$ is a cuspform, $\mathcal{O}_c$ is the order in $\mathcal{O}_K$ of conductor $c$, $\chi_j = \chi \cdot \mathbf{N}^j$, where $\mathbf{N}$ is the norm Hecke character of $K$, and $(A_0, t_0, \omega_0)$ is a triple consisting of an elliptic curve with CM by $\mathcal{O}_c$, $t_0$ is a generator of $A_0[\mathfrak{N}]$ and $\omega_0$ is a generator of the invarint differentials on $A_0$. Finally, one of the most important players in the above formula is the weight $k$ Shimura-Maass differential operator $\delta_k$.

*$p$-Adic interpolation of the values $L_{\mathrm{alg}}(F, \chi^{-1})$.*

As we mentioned at the beginning of this introduction if $F$ is an Eisentein series, respectively a cusp form and $p$ is split in $K$, Katz in [K2] and respectively Bertolini-Darmon-Prasana in [BDP] constructed $p$-adic $L$-functions interpolating $p$-adically the values $L_{\mathrm{alg}}(F, \chi^{-1})$ for $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$.

A legitimate question is then: what about if $p$ is non-split in $K$? When $F = E_{k,\epsilon}$ is an Eisensetin series and $p$ is a prime integer inert in $K$, the $L$-values $L_{\mathrm{alg}}(E_{k,\epsilon}, \chi^{-1})$ have been

extensively studied by Fujiwara in [Fu], Bannai-Kobayashi in [BK] and by Bannai-Kobayashi-Yasuda in [BKY]. One knows, thanks to work of Katz [K4], that the values $L_{\mathrm{alg}}(E_{k,\epsilon}, \chi^{-1})$ are $p$-adic integers for $\chi$ algebraic Hecke characters of infinity type $(k+j,-j)$, where $k \geq 2$ and $j \geq 0$ are integers. It was observed by Fujiwara that their $p$-adic valuations tend to infinity as $k$ or $j$ tend to infinity; this was done by carefully computing explicit lower bounds for these valuations. This makes it clear that the special $L$-values cannot be "naively" $p$-adically interpolated and one would need to divide those $L$-values by naturally appearing $p$-adic periods in order to compensate for the growth of the $p$-adic valuations.

The main goal of the present article is to define the $p$-adic $L$-functions of Katz and of Bertolini-Darmon-Prasana in the cases when the prime $p \geq 5$ is either inert or ramified in the quadratic imaginary field $K$. Our construction is purely geometric and is based on the idea of an analytic continuation of certain overconvergent de Rham classes, which we will try to explain further. We also prove formulae describing special values of our new $p$-adic $L$-functions at characters outside the interpolation family, i.e. we prove $p$-adic Gross-Zagier formulae for our $p$-adic $L$-functions à la Bertolini-Darmon-Parsana and a Kronecker limit formula for our $p$-adic $L$-functions à la Katz, in the cases in which $p$ is either inert or ramified in $K$. But most importantly, our geometric constructions naturally produce $p$-adic periods which divide the special classical $L$-values when they are compared with the special values of the $p$-adic $L$-functions. We have computed upper and lower bounds for these $p$-adic periods and they seem to agree with the lower bounds of the $p$-adic valuations of the special $L$-values of Fujuwara and Bannai-Kobayashi-Yasuda.

We will first explain the main new ideas for the construction of the $p$-adic $L$-functions and we'll then come back to these $p$-adic periods.

*The p-Adic L-function.*

It was already mentioned that our goal is to interpolate $p$-adically the algebraic numbers $L_{\mathrm{alg}}(F, \chi^{-1})$ for $\chi \in \Sigma^{(2)}_{cc}(\mathfrak{N})$ described by formula $(*)$ above. In this regard it would be helpful to find an algebraic meaning of the Shimura-Maass differential operator $\delta_k$. This operator is analytically defined on (real analytic) modular forms of weight $k$ for $\Gamma_1(N)$ on the upper half plane by the formula: $\delta_k(f) := \frac{1}{2\pi i}\Big(\frac{\partial}{\partial z} + \frac{k}{z-\overline{z}}\Big)(f)$ and acts on the $q$-expansion of $f$ as the differential operator $q\frac{d}{dq}$. In the case when $p$ is split in $K$, for every $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)$ the elliptic curve $\mathfrak{a} * A_0 := A_0/A_0[\mathfrak{a}]$ is ordinary at $p$ so that the isomorphism class of the triple $\mathfrak{a} * (A_0, t_0)$ represents a point of $X_1(N)$, seen now as a rigid analytic curve over $\mathbb{Q}_p$, which lies in the ordinary locus of this curve. As the differential operator $q\frac{d}{dq}$ can be defined for $p$-adic modular forms over the ordinary locus, see for example [K2] or [K3], in this case the definition of the $p$-adic $L$-function can be performed using $p$-adic modular forms.

Let us remark that if $p$ is not split in $K$ then all the elliptic curves $\mathfrak{a} * A_0$, for $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)$, have supersingular reduction and so one cannot work with $q$-expansions or $p$-adic modular forms anymore. This is one of the reasons why for the last forty years the case $p$ non-split in $K$ was not addressed.

In trying to define these $p$-adic $L$-functions the first question one has is: what is the right $p$-adic analogue of the Shimura-Maass operator $\delta_k$, or even better of $\delta_k^n$, for $n \geq 1$? Classically, the Shimura-Maass operator can be seen more geometrically as follows: let $\mathrm{H}_{\mathbb{E}}$ denote the relative

de Rham cohomology sheaf of the universal generalised elliptic curve over the modular curve $X_1(N)$ (over $\mathbb{C}$) and $\nabla_k\colon \mathrm{Sym}^k(\mathrm{H}_{\mathbb{E}}) \longrightarrow \mathrm{Sym}^k(\mathrm{H}_{\mathbb{E}}) \otimes \omega_{\mathbb{E}}^2$ be the Gauss-Manin connection on the $k$-th symmetric power of $\mathrm{H}_{\mathbb{E}}$. Here $\omega_{\mathbb{E}}$ is the sheaf of the relative invariant differentials of the universal generalized elliptic curve over $X_1(N)$.

Next, Hodge theory provides a canonical splitting of the Hodge filtration of $\mathrm{H}_{\mathbb{E}}$, therefore we have a canonical projection $\Psi_{\mathrm{Hodge}}\colon \mathrm{Sym}^k(\mathrm{H}_{\mathbb{E}}) \to \omega_{\mathbb{E}}^k \otimes_{\mathcal{O}_X} C_X^\infty$, where $C_X^\infty$ is the sheaf of $C^\infty$-functions on $X_1(N)$. It is proven in [K3] that

• the Shimura-Maass operator $\delta_k\colon \mathrm{H}^0(X_1(N), \omega_{\mathbb{E}}^k) \to \mathrm{H}^0(X_1(N), \omega_{\mathbb{E}}^{k+2})$ coincides with $\theta_{\mathrm{Hodge}} := \Psi_{\mathrm{Hodge}} \circ \nabla_k$ (see [K3, Formula (2.3.26)]);

• the $n$-th iterate $\delta_k^n = \theta_{\mathrm{Hodge}}^n\colon \mathrm{H}^0(X_1(N), \omega_{\mathbb{E}}^k) \to \mathrm{H}^0(X_1(N), \omega_{\mathbb{E}}^{k+2n})$ coincides with $\Psi_{\mathrm{Hodge}} \circ (\nabla_k)^n$ (see [K3, Thm. 2.3.8]);

• if we specialize at a CM point of $X_1(N)$ corresponding to an elliptic curve $E$ with CM by an order in an imaginary quadratic field $K$, with CM type $\iota\colon K \to \mathbb{C}$, the splitting $\Psi_{\mathrm{Hodge}}$ is algebraic and coincides with the projection $\mathrm{Sym}^k(\mathrm{H}_E) \to \omega_E^k$ onto the eigenspace on which $K$ acts via $\iota^k$ (see [K3, Key Lemma 5.1.27], specially Formula (5.1.38)).

These three results imply that the value $L_{\mathrm{alg}}(F, \chi^{-1})$ defined in formula $(*)$ is an algebraic number (Damerell's theorem), and even an algebraic integer; see Katz [K4]. They also suggest that in order to $p$-adically interpolate these values we need to $p$-adically interpolate the $p$-adic overconvergent analogue of $\delta_k^n$, namely

"(a canonical splitting of the Hodge filtration) $\circ\, (\nabla_k)^n$".

The goal of the present paper is to explain how this can be achieved. As in the $p$-split in $K$ situation (see [BDP]) our $p$-adic $L$-functions will be functions on the space $\widehat{\Sigma}^{(2)}$, the $p$-adic completion of $\Sigma_{cc}^{(2)}(\mathfrak{N})$. We remark (see section 2.1) that the natural map $w\colon \Sigma_{cc}^{(2)}(\mathfrak{N}) \to \mathbb{Z}$ sending $\chi$ to $j$, where the infinity type of $\chi$ is $(k+j, -j)$, is continuous and therefore it extends to a natural map $w\colon \widehat{\Sigma}^{(2)} \to W(\mathbb{Q}_p)$, where $W(\mathbb{Q}_p)$ is the space of $\mathbb{Q}_p$-valued $p$-adic weights i.e. $W(\mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*, \mathbb{Z}_p^*)$.

To start with, for every $\chi \in \widehat{\Sigma}^{(2)}$ the following expression makes sense and it is a continuous function of $\chi$:

$$\hat{\Pi}_F(\chi) := (\chi \cdot \mathbf{N}^{w(\chi)})^{-1} \otimes (\nabla_k)^{w(\chi)}(F^{[p]}),$$

where $F^{[p]}$ is the overconvergent modular form whose $q$-expansion is $F^{[p]}(q) = \sum_{(n,p)=1} a_n q^n$ if the $q$-expansion of $F$ is $F(q) = \sum_{n=0}^\infty a_n q^n$. The right factor of the tensor product is an overconvergent section of a sheaf of de Rham classes of weight $k + 2w(\chi)$, denoted $\mathbb{W}_{k+2w(\chi)}$ (see section 4.4 for more details.) The construction of $\mathbb{W}_{k+2w(\chi)}$ and the definition of $(\nabla_k)^{w(\chi)}$ are recalled in §4.1 which reviews the main results of [AI].

Now we need to worry about the images of these sections at the points corresponding to isomorphism classes of pairs $\mathfrak{a} * (A_0, t_0)$ and finding the appropriate splittings of the Hodge filtration. The two cases we are looking at, i.e. the cases $p$ inert in $K$ and $p$ ramified in $K$, are different but relatively similar so we will content ourselves in the introduction to explain the inert case. For the case $p$ ramified in $K$ we leave it to the reader to make herself/himself the suitable changes or look the details in chapter §6 of this article.

To start with, we denote by $K_p$ the $p$-adic completion of $K$ and let us first see what happens when we look at the point corresponding to $(A_0, t_0)$. As we already mentioned, when $p$ is inert in $K$, $A_0$ seen as an elliptic curve over an extension of $K_p$ has supersingular reduction and moreover it has no canonical subgroup. Therefore the point corresponding to $(A_0, t_0)$ does not belong to the analytic space (or to the formal scheme) where $\mathbb{W}_{k+2w(\chi)}$ is defined and so we can't evaluate directly $\hat{\Pi}_F(\chi)$ at such a point if $\chi$ is not algebraic.

The key observations are the following: we choose a subgroup $D \subset A_0[p^2]$ of order $p^2$ which is generically cyclic and denote by $C := D[p]$, by $A' := A_0/C$ and by $C' := D/C \subset A'[p]$. We choose a subgroup $D' \subset A'[p^2]$ of order $p^2$, generically cyclic and such that $D'[p] = C'$. Then the elliptic curve $A'$ has a canonical subgroup $\mathrm{H}' := A_0[p]/C \subset A'[p]$ such that $A'/\mathrm{H}' \cong A_0$ and the image of $D'$ is $D$. Now we denote by $C'_1, C'_2, \ldots, C'_{p-1}$ the subgroups of $A'$ of order $p$ distinct from $H'$ and $C'$, for each $1 \leq i \leq p-1$ and let $D_i$ be the image of $D'$ in $A_i := A'/C_i$ for each $1 \leq i \leq p-1$. Similarly, let $t'$ be the image of $t_0$ in $A'$ and $t_i$ the images of $t'$ in $A_i$ for $1 \leq i \leq p-1$. Finally we choose, to the expense of maybe enlarging the field over which $A_0$ is defined, a trivialization $P'$ of the group scheme $\big(D'[p]\big)^\vee$, the Cartier dual of $D'[p]$, which determines a trivialization $P$ of $\big(D[p]\big)^\vee$ and for each $1 \leq i \leq p-1$ trivializations $P_i$ of $\big(D_i[p]\big)^\vee$.

Then $x' := (A', t', D', P')$, $x := (A_0, t, D, P)$, $x_i := (A_i, t_i, D_i, P_i)$, $1 \leq i \leq p-1$, correspond naturally to points of the modular curve $X(pN, p^2)$ (see section 4.2) and the $U = U_p$-correspondence on that curve has the property: $U(x') = \{x, x_1, x_2, ..., x_{p-1}\}$. Moreover the points $x_1, x_2, ..., x_{p-1}$ are situated in a region of $X(pN, p^2)$ where we can define $\nabla_k^\nu(F^{[p]})$, therefore keeping in mind that $U\big(\nabla_k^\nu(F^{[p]})\big) = 0$, we put:

$$\nabla_k^\nu(F^{[p]})(A_0, t_0, D, P) := -\sum_{i=1}^{p-1} \lambda_i^* \Big( \big(\nabla_k^\nu(F^{[p]})\big)_{x_i} \Big).$$

Here we denoted by $\lambda_i \colon A' \longrightarrow A_i$ the natural isogenies given by division by $C_i$. We remark that the finite group $\mathrm{Pic}(\mathcal{O}_{cp^2})$ acts on the quadruple $(A_0, t_0, D, P)$ in the natural way and if $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{cp^2})$ denoting by $\mathfrak{a} * \lambda_i$ the natural isogeny $\mathfrak{a} * A' \longrightarrow \mathfrak{a} * A_i$ for $1 \leq i \leq p-1$ we set:

$$\big(\nabla_k^\nu(F^{[p]})\big)\big(\mathfrak{a} * (A_0, t_0, D, P)\big) := -\sum_{i=1}^{p-1} (\mathfrak{a} * \lambda_i^*)\big((\nabla_k^\nu(F^{[p]}))_{\mathfrak{a} * x_i}\big).$$

Now we need to discuss the splittings of the Hodge filtrations and we start by looking at $\nabla_k^\nu(F^{[p]})(A_0, t_0, D, P) \in \mathrm{H}^0(A', \mathbb{W}_{A', k+2\nu})$, where the notations $A', A_i$ etc are as above and so far they depend on the choice of the subgroup $D$ and the trivialization $P'$. As $A'$ is the image by an isogeny of degree $p$ of $A_0$ whose $p$-divisible group has multiplication by $\mathcal{O}_{K_p}$, the $p$-divisible group of $A'$ has multiplication by the order of conductor $p$ in $\mathcal{O}_{K_p}$. Therefore the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A')$ is not split but it contains a submodule of finite index, let's denote it $\mathrm{H}^\flat$ which is a direct sum of $\mathrm{H}^0(A', \omega_{A'})$ with another submodule. The VBMS formalism, recalled in §4, produces a submodule $\mathbb{W}^\flat_{A', k+2\nu}$ of $\mathbb{W}_{A', k+2\nu}$ together with a canonical splitting $\Psi_{A'} \colon \mathbb{W}^\flat_{A', k+2\nu} \longrightarrow \mathrm{H}^0(A', \mathfrak{w}_{A'}^{k+2\nu})$ and one proves that the images of the sections above of $\mathbb{W}_{A', k+2\nu}$ land in $\mathbb{W}^\flat_{A', k+2\nu}$. Now let $\omega_0$ be an invariant differential of $A_0$ such that its inverse image $\omega' \in \mathrm{H}^0(A', \omega_{A'})$ with respect to the isogeny $A' \longrightarrow A_0$ satisfies $\omega' \equiv \mathrm{dlog}(P')$ (modulo a suitable power of $p$). By the theory of [AI] it defines a generator $(\omega')^{k+2\nu}$ of $\mathrm{H}^0(A', \mathfrak{w}_{A'}^{k+2\nu})$, then we define $\big(\nabla_k^\nu(F^{[p]})\big)\big(A_0, t_0, D, P, \omega_0\big)$ to be the unique element of $R$ such that

$$\Big(\big(\nabla_k^\nu(F^{[p]})\big)\big(A_0,t_0,D,P,\omega_0\big)\Big)(\omega')^{k+2\nu} = -\Psi_{A'}\Big(\sum_{i=1}^{p-1}\lambda_i^*\big((\nabla_k^\nu(F^{[p]}))_{x_i}\big)\Big).$$

We define similarly $\nabla_k^\nu(F^{[p]})\big(\mathfrak{a}*(A_0,t_0,D,P,\omega_0)\big)$ for $\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{cp^2})$. Therefore we can now define the $p$-adic $L$-function as follows. Let $\chi\in\widehat{\Sigma}^{(2)}$ with $\nu:=w(\chi)\in W(\mathbb{Q}_p)$, then we have:

$$L_p(F,\chi^{-1}) := M\cdot\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{cp^2})}\chi_\nu^{-1}(\mathfrak{a})(\nabla_k)^\nu\big(F^{[p]}\big)\big(\mathfrak{a}*(A_0/R,t_0,D,P,\omega_0)\big),$$

where $M$ is a certain constant, see section §5.3.

*Interpolation properties.*

We will only illustrate the interpolation properties in the case $p$ is inert in $K$, the ramified case can be seen in section §6.3. Let $\chi\in\Sigma_{cc}^{(2)}(\mathfrak{N})$, i.e. $\chi$ is an algebraic Hecke character with $w(\chi)=j\geq 0$. We denote by $\Omega_p(k,j)$ the appropriate $p$-adic period (remark that it depends on $k,j$) and define the factor

$$\mathcal{E}_p(a_p,\chi) := 1-\frac{(p-1)a_p^2}{\chi(p)(p+1)}-\frac{1}{p^2}.$$

Here $a_p$ is the $T_p$ eigenvalue of $F$. Then we have the following relationship between the values of the $p$-adic $L$ function and of the complex $L$-function at $\chi$ (see Proposition 5.11):

$$L_p(F,\chi^{-1}) = \frac{p^k\mathcal{E}_p(a_p,\chi)L_{\mathrm{alg}}(F,\chi^{-1})}{\Omega_p(k,j)}.$$

Although the factor $\mathcal{E}_p(a_p,\chi)$ looks like an Euler factor it is not one such. In the inert and ramified cases these factors are different from the automorphic Euler factors at $p$, respectively at $\mathfrak{p}$ as they appear in [Ja] and we believe that the fact that the corresponding factor agrees with the automorphic Euler factor at $p$ in the split case is an accident.

*Relation with the results of Katz, Fujiwara and Bannai, Kobayashi, Yasuda.*

As mentioned earlier in this Introduction, the algebraic parts of the special values of the complex $L$-function, $L_{\mathrm{alg}}(E_{k,\epsilon},\chi^{-1})$, for $E_{k,\epsilon}$ an Eisenstein series of weight $k\geq 2$, character $\epsilon$ and $\chi\in\Sigma_{cc}^{(2)}(\mathfrak{N})$ of weight $j\geq 0$, have been extensively studied by Fujiwara in [Fu], Bannai-Kobaiashi in [BK] and by Bannai-Kobayashi-Yasuda in [BKY]. It follows from [Fu, Thm. 2], see also [BKY, Thm. 6.8], that the $p$-adic valuations of these $L$-values tend to infinity as $k$ or $j$ tend to infinity; more precisely Fujiwara obtains lower bounds growing approximately as $\dfrac{2j}{p^2-1}$ for $j\gg k$ and as $\dfrac{pk}{p^2-1}$ for $k\gg j$.

On the other hand our $p$-adic periods $\Omega_p(k,j)$ are $p$-adic integers whose $p$-adic valuations also tend to infinity as $k$ and $j$ tend to infitiny (see Proposition 5.11); more precisely we have

$$\frac{pk}{p^2-1}+\frac{2j}{p(p^2-1)} \leq v_p\big(\Omega_p(k,j)\big) \leq \frac{pk+2j}{p^2-1}.$$

Both the $p$-adic valuations of $L_{\mathrm{alg}}(F,\chi^{-1})$ and of $\Omega_p(k,j)$ are related to the $p$-adic periods of the CM elliptic curves. For the first this was observed already by Bannai-Kobaiashi in [BK]. For $\Omega_p(k,j)$ this is related to the theory developed in [AI], and recalled in §4.1, that essentially uses the $p$-adic Hodge theory of the elliptic curve. We found this coincidence quite inspiring.

We remark that Katz defined two variable $L$-function in the Eisenstein case letting $k$ vary as well. The method in our paper should provide such a construction, also for more general families of overconvergent forms. The starting point is having families of overconvergent forms with large enough radius of overconvergence. For classical forms, as we consider in the present paper, this is not an issue. More generally, even for the first non-trivial example of a genuine family, Serre's Eisenstein family, this is not known and it is the object of an ongoing project of the first author and Johannes Sprang.

*Special values $L_p(F,\chi^{-1})$, for $\chi\in\Sigma_{cc}^{(1)}(\mathfrak{N})$.*

We will illustrate this result in the case $p$-inert in $K$ and $F=f$ is a cuspform of even weight $k\geq 2$ and leave the reader to look at the last section of the article for the case $p$-ramified in $K$ and the cases in which $F=E_{2,\epsilon}$, i.e. $F$ is the Eisenstein series of weight 2 and character $\epsilon$.

We first remark that for $\chi\in\Sigma_{cc}^{(1)}(\mathfrak{N})$, as the sign of the functional equation of $L(f,\chi^{-1},s)$ is $-1$, we have $L_{\mathrm{alg}}(f,\chi^{-1})=0$. Therefore the formula for $L_p(f,\chi^{-1})$ obtained in terms of the Abel-Jacobi image of a certain generalized Heegner cycle defined in [BDP] will be called, as in [BDP], a $p$-adic Gross-Zagier formula.

We let $(A,t_A)$ be an elliptic curve with CM by $\mathcal{O}_K$ and $\Gamma_1(\mathfrak{N})$-level structure, let $\omega_A$ denote a generator of the invariant differentials of $A$ and $\eta_A$ an element of $\mathrm{H}^1_{\mathrm{dR}}(A)$ such that $\langle\omega_A,\eta_A\rangle=1$ via the Poincaré pairing. Morever, $\varphi_0\colon A\to A_0$ is a cyclic isogeny of degree $c$ so that $A_0$ has CM by $\mathcal{O}_c$. For every $\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)$ we have an isogeny $\varphi_{\mathfrak{a}}\colon A_0\to\mathfrak{a}*A_0$ and $\mathrm{AJ}_p(\Delta_{\varphi_{\mathfrak{a}}\varphi_0})$ denotes the $p$-adic Abel-Jacobi map of the generalised Heegner cycle denoted $\Delta_{\varphi_{\mathfrak{a}}\varphi_0}$ constructed in [BDP, §2] supported in the fiber of the modular point $\varphi_{\mathfrak{a}}\circ\varphi_0\colon A\to\mathfrak{a}*A_0$.

Let $\chi\in\Sigma_{cc}^{(1)}(\mathfrak{N})$ be a character of infinity type $(k-1-j,1+j)$ with $0\leq j\leq r:=k-2$. Then $\chi$ can be seen as a $p$-adic character of $\widehat{\Sigma}^{(2)}$, i.e we can evaluate the $p$-adic $L$-functions on it. We have

$$L_p(f,\chi^{-1})=\frac{\mathcal{E}_p(a_p,\chi)}{\Omega_p(k,-j)}\left(\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}\frac{c^{-j}}{j!}\chi^{-1}(\mathfrak{a})\mathbf{N}(\mathfrak{a})\mathrm{AJ}_F(\Delta_{\varphi_{\mathfrak{a}}\varphi_0})\big(\omega_f\wedge\omega_A^j\wedge\eta_A^{r-j}\big)\right).$$

Let us remark that, as in the split case, the formula does not involve the derivative of the $p$-adic $L$-function but its value.

We remark that Daniel Kriz has a different construction of $p$-adic $L$-functions in the cases in which $p$ is not split in $K$ by defining a $p$-adic analogue of the Shimura-Maass operator $\delta_k$ on the functions of the perfectoid tower of modular curves of level $Np^\infty$, more precisely on the functions on the supersingular locus of that perfectoid tower. See [Kr]. Our work has been inspired by Kriz's report on this construction. Unfortunately in loc. cit. no interpolation formula is proven so that, for the time being, we cannot compare Kriz's $p$-adic $L$-functions with ours.

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# 2 Classical $L$-values.

## 2.1 Algebraic Hecke characters and their $p$-adic avatars.

We start by choosing embeddings $\iota_\infty\colon K \hookrightarrow \mathbb{C}$ and $\iota_p\colon K \hookrightarrow \mathbb{C}_p$ and an isomorphism $\eta\colon \mathbb{C}_p \cong \mathbb{C}$ such that $\eta \circ \iota_p = \iota_\infty$.

Recall that a Hecke character of $K$ is a continuous homomorphism $\chi\colon \mathbb{A}_K^*/K^* \longrightarrow \mathbb{C}^*$, where $\mathbb{A}_K^*$ is the group of ideles of $K$ and $K^*$ is embedded diagonally in $\mathbb{A}_K^*$. We say that $\chi$ is *algebraic* if the restriction of $\chi$ to the infinity component $\chi_\infty\colon (K \otimes_{\mathbb{Z}} \mathbb{R})^* \to \mathbb{C}^*$ of $\mathbb{A}_K^*$ has the form $\chi_\infty(z \otimes 1) = \iota_\infty(z)^{-n}\iota_\infty(\overline{z})^{-m}$ for a pair of integers $(n, m)$, which is called the *infinity type* of $\chi$.

We denote by $\alpha_\infty\colon \mathbb{A}_K^*/K^* \to \mathbb{C}^*$ the continuous characater whose infinity component is $\chi_\infty$ and is trivial on all the other components. Let us also denote by $\alpha_p$ the character $\alpha_p\colon \mathbb{A}_K^*/K^* \to \mathbb{C}_p^*$ which is trivial on all comonents different from $p$ and $\alpha_p\colon (K \otimes_{\mathbb{Z}} \mathbb{Q}_p)^* \to \mathbb{C}_p^*$ is the continuous character $\alpha(z \otimes 1) = \iota_p(z)^{-n}\iota_p(\overline{z})^{-m}$.

Every algebraic Hecke character $\chi$ as above has a *$p$-adic avatar*, namely the character also denoted $\chi\colon \mathbb{A}_K^*/K^* \to \mathbb{C}_p^*$ defined by: $\chi \cdot \alpha_\infty^{-1} \cdot \alpha_p$. It is continuous for the natural toplogy on $\mathbb{A}_K^*$ and the $p$-adic topology on $\mathbb{C}_p^*$. Let us remark that $\eta$ and $\eta^{-1}$ allow us to pass from the $p$-adic to the infinity incarnations of $\chi$ and back.

## 2.2 Classical values of $L(F, \chi^{-1}, s)$.

### 2.2.1 The case $F = E_{k,\epsilon}$ is an Eisenstein series.

We start by reviewing some of the results of Katz [K2]. Consider a fractional ideal $M$ of $\mathcal{O}_K$ and an isomorphism $\alpha\colon (\mathbb{Z}/N\mathbb{Z})^2 \cong \frac{1}{N}M/M$. Let $E_M$ be the elliptic curve $\mathbb{C}/M$ with full $\Gamma(N)$-level structure $\beta_\alpha\colon \mu_N \times \mathbb{Z}/N\mathbb{Z} \cong E[N]$ defined by $\alpha$ (we identify $\mathbb{Z}/N\mathbb{Z} \cong \mu_N$ via the choice of primitive root of unity $\zeta_N$ given by the Weil pairing of $\alpha(1,0)$ and $\alpha(0,1)$). Let $\omega$ be a generator of the differentials of the Nèron model of $E_M$ over a number field $L$. We assume that $L$ contains the $N$-roots of unity and we let $W$ be the completion of $\mathcal{O}_L$ at a prime above $p$. Fix an $\mathcal{O}_L$-valued function $\gamma\colon (\mathbb{Z}/N\mathbb{Z})^2 \to \mathcal{O}_L$.

For any positive integer $k \geq 1$, assuming that $\sum_j \gamma(0,j) = \sum_j \gamma(j,0) = 0$ if $k = 2$, Katz defines in [K2, Def. 3.6.5 & Thm. 3.6.9] Eisenstein series $G_{k,0,\gamma}$ of weight $k$ and full level $\Gamma(N)$ over $L$. For every $\ell \geq 0$ set

$$L_{\mathrm{alg}}(\gamma, k+\ell, -\ell; M, \alpha) := \delta_k^\ell\big(2G_{k,0,\gamma}\big)(E_M, \beta_\alpha, \omega).$$

It is proven in [K2, Cor. 4.1.3] that this is an algebraic integer and in fact lies in $L$ (Damerell's theorem). In loc. cit., one finds the Weil operator instead of the Shimura-Maass operator $\delta_k$ in the previous formula but it follows from [K2, (2.3.38)] that the two operators coincide. In particular, $N^\ell\delta_k^\ell\big(2G_{k,0,\gamma}\big) = 2G_{k+2\ell,-\ell,\gamma}$ by [K2, (3.6.8)] and one finds in [K2, (8.6.8)] the explicit formula

$$L_{\mathrm{alg}}(\gamma, k+\ell, -\ell; M, \alpha) := \frac{(-1)^k(k+\ell-1)!N^k\pi^\ell}{a(M)^\ell\Omega^{k+2\ell}}\left(\sum_{0\neq m\in M}\frac{g(m)\overline{m}^\ell}{m^{k+\ell}\mathbf{N}(m)^s}\right)|_{s=0}, \tag{1}$$

where $a(M)$ is the covolume of $M \subset \mathbb{C}$, $\mathbf{N}$ is the norm map on $K$, $\Omega$ is a complex period defined by $\omega$ by the equality $\omega = \Omega dz$ (for $z$ the coordinate on the Poincaré upper half space) and $g$ is a function on $M/NM$ described explicitly in [K2, §8.7] associated to $\gamma$. We will compute later the function $g$ in the case that $\gamma$ is associated to a character showing that, up to a Gauss sum, the associated $g$ is itself a character so that formula (1) will provide the link to special values of Hecke $L$-series.

**Remark 2.1.** (a) Even if in this paper we applied iterations of the Gauss-Manin connection to classical modular forms, it is explained in [AI] how to apply those to $p$-adic families. Taking $G_{k,f}$ to be a family in $k$ we will get in principle a 2-variable construction analogous to the one of Katz. The only remaining issue is to prove that such families have a radius of overconvergence large enough so that the evaluation of $\delta_k^\nu\big(2G_{k,f}^{[p]}\big)$ at $(E_M, \beta_\alpha, \omega_p)$ still makes sense.

(b) Katz has a more general definition allowing non-trivial levels at $p$ that we cannot deal with using the techniques introduced so far.

### $L$-functions of Hecke Grossencharacters.

Next we explain, following [K2, §9.4], how to express the value at 0 of the $L$-function associated to an algebraic Hecke character (equivalently a grossencharacter of $K$ of type $A_0$ in the terminology used in loc. cit.) and of conductor $\mathfrak{N}$ as a combination of special values of real analytic Eisenstein series (of full level $\Gamma(N)$). This allows Katz, working with special values of $p$-adic Eisenstein series, to construct two variable $p$-adic $L$-functions in the case that $p$ splits in $K$. We extend his construction, for a one variable family, to the case that $p \geq 5$ is not split.

From now on we assume that $N$ satisfies the Heegner hypothesis and we choose an ideal $\mathfrak{N} \subset \mathcal{O}_K$ such that $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{N}) = N\mathbb{Z}$ and $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$. We also fix a character $\epsilon\colon (\mathbb{Z}/N\mathbb{Z})^* \to L^*$ of parity $k$, i.e., such that $\epsilon(-x) = (-1)^k\epsilon(x)$. We assume that $\epsilon$ is non-trivial if $k = 2$. Fix a generator $t \in \mathfrak{N}^{-1}/\mathcal{O}_K$. Write $\epsilon_{\mathfrak{N}}$ for the character induced by $\epsilon$ via the identification $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$. Let $\chi$ be an algebraic Hecke character, of infinity type $(k_1, k_2)$, of conductor dividing $\mathfrak{N}$ and of finite type $\epsilon_{\mathfrak{N}}$, i.e., we assume that $\chi$ restricted to the finite ideles prime to $\mathfrak{N}$ factors via $(\mathcal{O}_K/\mathfrak{N})^*$ and coincides with $\epsilon_{\mathfrak{N}}$. We denote by $\Sigma_{cc}^{(2)}(\mathfrak{N})$ the set of such characters such that $(k_1, k_2) = (k + j, -j)$ with $k$ and $j$ non-negative integers and $k \geq 1$.

Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ be integral invertible idelas of $\mathcal{O}_K$, representatives of the class group of $\mathcal{O}_K$, coprime to $p\mathfrak{N}$. Given $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ define $L(K, \chi)$ to be the value at 0 of the $L$-function attached to $K$ and the groessencharacter $\chi$ as in [K2, 9.4.31]:

$$L(K,\chi) = \frac{1}{|\mathcal{O}_K^*|}\left(\sum_{i=1}^{h}\frac{\chi(\mathfrak{a}_i)^{-1}}{\mathbf{N}(\mathfrak{a}_i)^{-s}}\sum_{0\neq\alpha\in\mathfrak{a}_i}\frac{\chi\big((\alpha)\big)}{\mathbf{N}(\alpha)^s}\right)|_{s=0}.$$

Here we view $\chi$ as a classical Hecke character, namely a character on fractional ideals prime to $\mathfrak{N}$; in particular, if $\alpha \in K$ is a non-zero element, prime to $\mathfrak{N}$ then

$$\chi\big((\alpha)\big) = \epsilon_{\mathfrak{N}}(\alpha)\iota(\alpha)^{k_1}\iota(\overline{\alpha})^{k_2}.$$

Define $\gamma_\epsilon\colon (\mathbb{Z}/N\mathbb{Z})^2 \to \mathcal{O}_L$ by $\gamma_\epsilon(m,n) = \epsilon(m)$. It follows from [K2, Thm. 3.6.9] that for $k \geq 2$

$$2G_{k,0,\gamma_\epsilon} = E_{k,\epsilon} = L(1-k,\epsilon) + 2\sum_{n\geq 1}\sigma_{k-1,\epsilon}(n)q^n, \qquad \sigma_{k-1,\epsilon}(n) = \sum_{d|n}\epsilon(d)d^{k-1}$$

is an Eisenstein series of weight $k$, level $\Gamma_1(N)$ and character $\epsilon$. Moreover, the function $g_\epsilon\colon (\mathbb{Z}/N\mathbb{Z})^2 \to \mathcal{O}_L$ associated to $\gamma_\epsilon$ is $g_\epsilon(n,m) := \frac{1}{N}\sum_{a\in\mathbb{Z}/N\mathbb{Z}} \epsilon(a)\zeta_N^{-na}$ (see [K2, (3.2.2)]. This coincides with

$$g_\epsilon(n,m) = \epsilon^{-1}(n)\frac{s(\epsilon)}{N}, \qquad s(\epsilon) = \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^*} \epsilon(a)\zeta_N^{-a},$$

where $s(\epsilon)$ is simply the Gauss sum associated to the character $\epsilon$. Set $A_0$ be the elliptic curve $E_{\mathcal{O}_K} = \mathbb{C}/\iota(\mathcal{O}_K)$, $t_0$ the generator of $A_0[\mathfrak{N}]$ defined by $t$ and $\omega_0$ a Néron differential. Let $a(\mathfrak{a}_i) = \mathbf{N}(\mathfrak{a}_i)a(\mathcal{O}_K)$ be the covolume of $\mathfrak{a}_i \subset \mathbb{C}$ and let $\Omega$ be such that $\omega_0 = \Omega dz$. Assume that $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$. Set

$$L_{\mathrm{alg}}\big(E_{k,\epsilon}, \chi^{-1}\big) := \sum_{\mathfrak{a}\in \mathrm{Pic}(\mathcal{O}_K)} \chi_j^{-1}(\mathfrak{a})\delta_k^j(E_{k,\epsilon})\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big),$$

where $\chi_j = \chi\cdot\mathbf{N}^j$. Here $\mathfrak{a}*(A_0,t_0,\omega_0) = (A, t_\mathfrak{a}, \omega)$ where $A = A_0/A_0[\mathfrak{a}]$, $t_\mathfrak{a}$ is the image of $t$ and $\omega_0$ is the pull-back of $\omega$ via the isogeny $\phi_\mathfrak{a}\colon A_0 \to A$.

**Proposition 2.2.** *For every $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ of infinity type $(k+j,-j)$ we have*

$$L\big(K, \chi^{-1}\big) = \frac{1}{|\mathcal{O}_K^*|}\frac{(-1)^k(k+j-1)!N^{k+1}\pi^j}{s(\epsilon)a(\mathcal{O}_K)^j\Omega^{k+2j}} L_{\mathrm{alg}}\big(E_{k,\epsilon}, \chi^{-1}\big).$$

*Proof.* This follows from (1) using that $\mathfrak{a}*A_0$ is the elliptic curve associated to $E_M = \mathbb{C}/\iota(\mathfrak{a}^{-1})$ with $M = \mathfrak{a}^{-1}$ and taking $\mathfrak{a} = \mathfrak{a}_1^{-1}, \ldots, \mathfrak{a}_h^{-1}$.

□

### 2.2.2 The case $F = f$ is a cuspform.

As before we assume that there is an ideal $\mathfrak{N}$ of $\mathcal{O}_K$ whose norm is $N$ and such that we have an isomorphism $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$; we choose and fix such an ideal. We also choose an odd integer $c \geq 1$ such that $(c, Nd_K) = 1$ and denote $\mathcal{O}_c$ the order of conductor $c$ in $\mathcal{O}_K$ and let $\mathfrak{N}_c := \mathfrak{N}\cap\mathcal{O}_c$. Then we have $\mathfrak{N}_c^{-1}/\mathcal{O}_c \cong \mathfrak{N}^{-1}/\mathcal{O}_K \cong \mathbb{Z}/N\mathbb{Z}$ and we choose and fix a generator $t \in \mathfrak{N}_c^{-1}/\mathcal{O}_c$.

Let $f$ be a normalized cuspidal newform of level $\Gamma_1(N)$, weight $k \geq 2$ and nebentype $\epsilon$. Following Definition 4.4 in [BDP] we denote by $\Sigma_{cc}^{(2)}(\mathfrak{N})$ the set of algebraic Hecke characters $\chi$ satisfying the following:

1. the infinity type is $(k+j,-j)$, with $j \geq 0$;
2. they are are central critical for $f$, i.e., we have $\epsilon_\chi = \epsilon$ or equivalently $\chi|_{\mathbb{A}_\mathbb{Q}^*} = \epsilon\mathbf{N}^k$ where $\mathbf{N}$ is the norm character;
3. the epsilon factor $\epsilon_q(f, \chi^{-1}) = +1$ at every finite place;
4. they are of finite type $(c, \mathfrak{N}, \epsilon)$, i.e., $c$ divides the conductor of $\chi$ and the restriction of $\chi$ to $\widehat{\mathcal{O}}_c^*$ coincides with the projection onto $(\mathcal{O}_c/\mathfrak{N}_c)^* \cong (\mathbb{Z}/N\mathbb{Z})^*$ composed with $\epsilon$.

If the $q$-expansion of $f$ is $f(q) = \sum_{n=1}^{\infty} a_n q^n$, the $L$-function of $f$ can be written

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_{\ell} (1 - \alpha_\ell \ell^{-s})^{-1} (1 - \beta_\ell \ell^{-s})^{-1},$$

where the product is over all positive prime integers $\ell$. The above equality defines the pairs $(\alpha_\ell, \beta_\ell)$, for all $\ell$. Every algebraic Hecke character $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ is central critical for $f$, i.e. $s = 0$ is a critical point for the $L$-function $L(f, \chi^{-1}, s)$.

Let now $A_0$ be an elliptic curve with $CM$ by $\mathcal{O}_c$, i.e. over $\mathbb{C}$ we have $A_0 = \mathbb{C}/\mathcal{O}_c$, let $t_0$ be the $\mathfrak{N}$-torsion point of $A_0$ corresponding the a generator $t \in \mathfrak{N}_c^{-1}/\mathcal{O}_c$ and let $\omega_0 := 2\pi i dw$, where $w$ is the coordinate in $\mathbb{C}$ seen as an invariant differential on $A_0$. Then the isomorphism class of the pair $(A_0, t_0)$ determines a point in $X_1(N)(\mathbb{C})$. Moreover for every ideal $\mathfrak{a}$ representing a class in $\mathrm{Pic}(\mathcal{O}_c)$ we have $\mathfrak{a} * (A_0, t_0, \omega_0) = (A, t_\mathfrak{a}, \omega)$ where $A = A_0/A_0[\mathfrak{a}]$, $t_\mathfrak{a}$ is the image of $t$ and $\omega_0$ is the pull-back of $\omega$ via the quotient map $\phi_\mathfrak{a} \colon A_0 \to A$. With these notations we have the following explicit form of Waldspurger's formula:

**Theorem 2.3.** [BDP, Thm. 5.4 & Thm. 5.5] *Assume that $c$ and $d_K$ are odd. Let $\chi \in \Sigma_{cc}(\mathfrak{N})^{(2)}$ with infinity type $(k+j, -j)$, $j \geq 0$. Let*

$$L_{\mathrm{alg}}(f, \chi^{-1}) := \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \theta_{\mathrm{Hodge}}^j(f)\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big),$$

*where $\chi_j = \chi \cdot \mathbf{N}^j$. Here $\mathbf{N}$ is the norm Hecke character and the infinite type of $\chi$ is $(k+j, -j)$ with $j \geq 0$. Then*

*a) we have*

$$\big(L_{\mathrm{alg}}(f, \chi^{-1})\big)^2 = \frac{w(f,\chi)^{-1} C(f, \chi, c) L(f, \chi^{-1}, 0)}{\Omega^{2(k+2j)}},$$

*where $w(f,\chi) \in \{\pm 1\}$, $C(f, \chi, c)$ is a precisely defined constant and $\Omega$ is a complex period.*

*b) $L_{\mathrm{alg}}(f, \chi^{-1})$ is an algebraic number.*

.

## 2.3 Conclusions.

Recall that we have fixed a quadratic imaginary field $K$ and an integer $N \geq 5$, with $N$ satisfying the Heegner assumptions, i.e., there is an ideal $\mathfrak{N}$ of $\mathcal{O}_K$ with the properties: $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{N}) = N\mathbb{Z}$ and $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$. We consider a prime $p \geq 5$ not dividing $N$ and fix a character $\epsilon \colon (\mathbb{Z}/N\mathbb{Z})^* \to \overline{\mathbb{Q}}^*$. We also choose an odd integer $c \geq 1$ such that $(c, pNd_K) = 1$ and we denote by $\mathcal{O}_c := \mathbb{Z} + c\mathcal{O}_K$, the order in $\mathcal{O}_K$ of conductor $c$. Let $F$ be an eigenform of *even* weight $k \geq 2$ for $\Gamma_1(N)$ with character $\epsilon$. We assume one of the following:

i. $F = E_{k,\epsilon}$ is an Eisenstein series; in this case we assume further that $\epsilon$ is non trivial if $k = 2$ and we take $c = 1$;

ii. $F$ is a normalized cuspform; in this case we assume that $d_K$ is odd.

We let $\Sigma_{cc}^{(2)}(\mathfrak{N})$ denote the space of Hecke characters defined in section 2.2.1 if $F$ is an Eisenstein series and in section 2.2.2 if $F$ is a cuspform.

Our goal in this article is to $p$-adically interpolate the family of special values $L_{\mathrm{alg}}(F, \chi^{-1})$ indexed by the algebraic Hecke characters $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$, as described in the previous sections, by a $p$-adic $L$-function. This $p$-adic $L$-function will be a function on the space $\widehat{\Sigma}^{(2)}$ obtained as follows. Let $S$ be a field extension of $K$ in $\overline{\mathbb{Q}}$ which contains the values $\chi(\mathfrak{a})$ for $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)$ and $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$. We think of $\Sigma_{cc}^{(2)}(\mathfrak{N})$ as the space of $p$-adic avatars of the algebraic Hecke characters in $\Sigma_{cc}^{(2)}(\mathfrak{N})$. Let $\mathfrak{p}$ be a prime of $S$ dividing $p\mathcal{O}_S$ and we denote by $\mathcal{O}_{S,\mathfrak{p}}$ the $\mathfrak{p}$-adic completion of $\mathcal{O}_S$.

Let $\mathbb{A}'_{K,f} \subset \mathbb{A}_K^*$ be the finite ideles of $K$ prime to $p$. Then if we denote by $\mathcal{F}(\mathbb{A}'_{K,f}, \mathcal{O}_{S,\mathfrak{p}})$ the $\mathcal{O}_{S,\mathfrak{p}}$-algebra of functions on $\mathbb{A}'_{K,f}$ with the topology of uniform convergence. We also denote by $W(\mathbb{Q}_p) := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*, \mathbb{Z}_p^*)$ the $\mathbb{Q}_p$-points of the weight space, let us recall that we have a natural inclusion $\mathbb{Z} \hookrightarrow W(\mathbb{Q}_p)$ sending $n \in \mathbb{Z}$ to the continuous character $t \to t^n$, for all $t \in \mathbb{Z}_p^*$.

Every Hecke character $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ with infinity type $(k+j, -j)$, can be uniquely written as $\chi = \chi' \cdot \alpha_p$, where $\chi' \colon \mathbb{A}'_{K,f} \to \mathcal{O}_{S,\mathfrak{p}}$ is the restriction of the $p$-adic avatar of $\chi$ and it is a finite order character while $\alpha_p \colon \mathcal{O}_{K_{\mathfrak{P}}}^* \to \mathcal{O}_{S,p}$ is the map $\alpha_p(z) = z^{-k-j}\overline{z}^j$ where we have denoted by $K_{\mathfrak{P}}$ the $\mathfrak{P}$-adic completion of $K$, $\mathfrak{P}$ being the unique prime of $K$ over $p\mathcal{O}_K$ and $\overline{z}$ denotes the conjugate of $z$ in $K_{\mathfrak{P}}$.

Then the above decomposition gives us an injective map

$$\Sigma_{cc}^{(2)}(\mathfrak{N}) \subset \mathcal{F}(\mathbb{A}'_{K,f}, \mathcal{O}_{S,\mathfrak{p}}) \times \mathbb{Z} \hookrightarrow \mathcal{F}\big(\mathbb{A}_{K,f}^*, \mathcal{O}_{S,\mathfrak{p}}\big) \times W(\mathbb{Q}_p)$$

where the first map is $\chi \to (\chi', j)$. This embedding endows $\Sigma_{cc}^{(2)}(\mathfrak{N})$ with the induced topology and we denote $\widehat{\Sigma}^{(2)}$ the completion of $\Sigma_{cc}^{(2)}(\mathfrak{N})$ with respect to this topology. Let us remark that the natural map, defined above $w \colon \Sigma_{cc}^{(2)}(\mathfrak{N}) \to \mathbb{Z}$ by $w(\chi) = j$ if the infinity type of $\chi$ is $(k+j, -j)$, is continuous, in the sense that if $\chi_1 = \chi_2 (\mathrm{mod}\ p^r)$ then $w(\chi_1) = w(\chi_2) (\mathrm{mod}\ (p-1)p^{r-1})$. In particular, the natural map $w \colon \Sigma_{cc}^{(2)}(\mathfrak{N}) \to \mathbb{Z} \hookrightarrow W(\mathbb{Q}_p)$ extends by continuity to a map

$$w \colon \widehat{\Sigma}^{(2)} \to W(\mathbb{Q}_p).$$

If $\chi \in \widehat{\Sigma}^{(2)}$ then $\chi_{w(\chi)} := \chi \cdot \mathbf{N}^{w(\chi)} \in \widehat{\Sigma}^{(2)}$ makes sense.

# 3 Preliminaries on CM-elliptic curves.

In this section we consider a quadratic imaginary field $K$ and a positive integer $c$. Fix an elliptic curve $(E, \iota)$ with CM by $\mathcal{O}_c$, the order of conductor $c$ in $\mathcal{O}_K$. The pair $(E, \iota)$ is then defined over the ring of integeres $\mathcal{O}_L$ of a suitable ring class field $L$ of $K$. Let $p$ be a prime not dividing $c$ and let $R$ be the completion of $\mathcal{O}_L$ at a prime above $p$. We review the theory of the canonical subgroup of order a power of $p$ theory for $E$ over $R$. If $p$ splits in $K$ then $E$ is ordinary over $R$ so that $E$ admits canonical subgroup of every level $n$, coinciding with the connected subgroup of $E[p^n]$. The cases when $p$ is either inert or ramified in $K$ are more subtle. If $E'$ is an elliptic curve over $R$ we write $\mathrm{Hdg}_{E'} \in R$ for any element lifting the Hasse invariant of the modulo $p$ reduction of $E'$.

## 3.1 Canonical subgroups for CM elliptic curves. The inert case.

Assume first that $p$ is inert in $K$. Then

**Lemma 3.1.** *Let $E$ be an elliptic curve over the ring of integers $R$ of a local field with* CM *by $\mathcal{O}_c$. Then*

*a) $E$ does not admit a canonical subgroup and $\mathrm{val}_p\big(\mathrm{Hdg}(E)\big) \geq \frac{p}{p+1}$;*

*b) if $C \subset E[p]$ is a subgroup of order $p$ and we define $E' := E/C$, then $E'$ admits a canonical subgroup $\mathrm{H}'$ of order $p$ and $\mathrm{val}_p\big(\mathrm{Hdg}(E)\big) \geq \frac{1}{p+1}$;*

*c) for any subgroup $C' \subset E'[p]$ of order $p$ with $C' \cap \mathrm{H}' = \{0\}$, if we set $E'' := E'/C'$ then we have $\mathrm{val}_p\big(\mathrm{Hdg}(E'')\big) = \frac{1}{p(p+1)}$ and $E''$ has a canonical subgroup of oirder $p^2$.*

*Proof.* We remark that $E$ cannot have a canonical subgroup $D$ of order $p$ as if it did then $D$ would be stabilized by the action of $\mathcal{O}_c$ by functoriality. Since $p$ is inert, coprime to $c$, we have $\mathcal{O}_c/p\mathcal{O}_c \cong \mathbb{F}_{p^2}$ and $D$ would be an $\mathbb{F}_{p^2}$-vector space, which it cannot be. It then follows from the Katz–Lubin theory of canonical subgroups [K1, Thm. 3.10.7] that $\mathrm{val}_p\big(\mathrm{Hdg}(E)\big) \geq \frac{p}{p+1}$ and for any subgroup $C \subset E[p]$ of order $p$ we have that $\mathrm{val}_p\big(\mathrm{Hdg}(E')\big) = \frac{1}{p+1}$. Moreover $E'$ admits a canonical subgroup $\mathrm{H}' = E[p]/C$ and for $C' \subset E'$ as in the statement of the lemma we have that $\mathrm{val}_p\big(\mathrm{Hdg}(E'')\big) = \frac{1}{p(p+1)}$ as claimed. □

## 3.2 Canonical subgroups for CM elliptic curves. The ramified case.

Assume that $p$ is ramified in $K$ and let $\mathfrak{P}$ be the prime of $\mathcal{O}_K$ over $p$. Let $\mathrm{H} := E[\mathfrak{P}]$; it is a subgroup scheme of $E[p]$ of order $p$. Then we have:

**Lemma 3.2.** *a) $\mathrm{val}_p\big(\mathrm{Hdg}(E)\big) = 1/2$ and $\mathrm{H}$ is the canonical subgroup of order $p$ of $E$.*

*b) For any subgroup $C \subset E[p]$ of order $p$ with $C \cap \mathrm{H} = \{0\}$, if we denote $E' := E/C$ then $\mathrm{val}_p\big(\mathrm{Hdg}(E')\big) = \dfrac{1}{2p}$ and $\mathrm{H}' := E[p]/C \cong \mathrm{H}$ is the canonical subgroup of order $p$ of $E'$.*

*c) For any subgroup $C' \subset E'[p]$ of order $p$ with $C' \cap \mathrm{H}' = \{0\}$, if we denote $E'' := E'/C'$ then $\mathrm{val}_p\big(\mathrm{Hdg}(E'')\big) = \dfrac{1}{2p^2}$.*

*Proof.* Note that $\mathrm{val}_p\big(\mathrm{Hdg}(E)\big)$ depends only on the underlying $p$-divisible group $E[p^\infty]$ of $E$. If $\pi$ denotes a uniformizer of $K_{\mathfrak{P}}$, then $[\pi]\colon E[p^\infty] \to E[p^\infty]$ identifies its kernel with $E[\mathfrak{P}]$ and the quotient $E[p^\infty]/E[\mathfrak{P}]$ with $E[p^\infty]$. Hence $\mathrm{val}_p\big(\mathrm{Hdg}(E)\big) = \mathrm{val}_p\big(\mathrm{Hdg}(E/E[\mathfrak{P}])\big)$ and by the Ktaz-Lubin theory [K1, Thm. 3.10.7] the only possibility is that this value is 1/2. Claims (b) and (c) follow again from the Katz-Lubin theory.

□

## 3.3 A technical lemma

Consider the elliptic curve $E'$ defined in Lemma 3.1 (in the inert case) or in Lemma 3.2 (in the ramified case). It is a quotient of $E$ and, in particular it has CM by $\mathcal{O}_a$ with $a = pc$. Moreover $E'$ admits a canonical subgroup $\mathrm{H}'$ and $E'/\mathrm{H}' \cong E$. We let $\lambda\colon E' \to E$ be the quotient map.

Let $C_i \subset E'[p]$ for$i = 0, \ldots, p-1$ denote the subgroups of order $p$ of $E'[p]$ distinct from $\mathrm{H}'$. Define $E_i := E'/C_i$, $\lambda_i \colon E' \to E_i$ be the quotient morphisms and $\lambda_i^\vee \colon E_i \to E'$ the dual isogeny. Summarizing we have the degree $p$ isogenies:

$$E_i \xleftarrow{\lambda_i} E' \xrightarrow{\lambda} E.$$

Let $\mathrm{H}_{E'}$ be the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(E'/R)$. It fits in the exact sequence, defined by the Hodge filtration,

$$0 \to \omega_{E'} \to \mathrm{H}_{E'} \to \omega_{E'}^\vee \to 0.$$

We denote by $\mathrm{H}_{E',\tau}$, $\mathrm{H}_{E',\overline{\tau}}$ the $R$-submodules of $\mathrm{H}_{E'}$ on which $\mathcal{O}_a$ acts via the CM type $\tau \colon K \to L$ and its complex conjugate $\overline{\tau}$, respectively.

The last technical result of this section, to be used in §4.4, concernes the relative positions of the $R$-submodules $\mathrm{H}_{E',\overline{\tau}}$ and $\left((\lambda_i^\vee)^*\right)^\vee(\omega_{E_i}^\vee)$ in $\omega_{E'}^\vee$. Here $\left((\lambda_i^\vee)^*\right)^\vee$ is the $R$-linear dual of the map $(\lambda_i^\vee)^* \colon \omega_{E'} \to \omega_{E_i}$ induced by pull-back by the dual isogeny $\lambda_i^\vee$.

**Lemma 3.3.** *We have*

*i.* $\mathrm{H}_{E',\tau} = \omega_{E'}$*, the invariant differentials of* $E'$*;*

*ii. the inclusion* $\mathrm{H}_{E',\overline{\tau}} \subset \omega_{E'}^\vee$ *has cokernel annihilated by* $\mathrm{Hdg}(E') \cdot \mathrm{diff}_{K/\mathbb{Q}}$*, where* $\mathrm{diff}_{K/\mathbb{Q}}$ *denotes the different ideal of* $K/\mathbb{Q}$*;*

*iii. the image of* $\left((\lambda_i^\vee)^*\right)^\vee\left(\omega_{E_i}^\vee\right) \subset \omega_{E'}^\vee$ *is contained in* $p\mathrm{Hdg}(E_i)^{-1}\omega_{E'}^\vee$*.*

*Proof.* (i) The first claim follows from the fact that $\mathcal{O}_a$ acts on $\omega_{E'}$ via $\tau$ by the definition of CM type and the Hodge filtration provides an exact sequence $0 \to \omega_{E'} \to \mathrm{H}_{E'} \to \omega_{E'}^\vee \to 0$ of $\mathcal{O}_c$-modules where $\mathcal{O}_c$ acts on $\omega_{E'}^\vee$ via $\overline{\tau}$.

(ii) Write $\widetilde{\mathrm{H}}_E := \mathrm{H}_{E,\tau} \oplus \mathrm{H}_{E,\overline{\tau}} \subset \mathrm{H}_E$ similarly to what has been done for $E'$. Then $\lambda$ induces morphisms

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_E & \longrightarrow & \mathrm{H}_E \\ \downarrow & & \downarrow \lambda^* \\ \widetilde{\mathrm{H}}_{E'} & \longrightarrow & \mathrm{H}_{E'} \end{array}$$

The map $\lambda^* \colon \mathrm{H}_E \to \mathrm{H}_{E'}$ respects the Hodge filtration. It coincides with pull-back of the differentials $\lambda^* \colon \omega_E \to \omega_{E'}$. On the quotient of the Hodge filtration, it induces the map

$$\left((\lambda^\vee)^*\right)^\vee \colon \omega_E^\vee \to \omega_{E'}^\vee,$$

where $\lambda^\vee \colon E' \to E$ is the dual isogeny.

To prove claim (ii) it suffices to show that $\omega_{E'}^\vee/\mathrm{H}_{E',\overline{\tau}}$ is annihilated by $\mathrm{Hdg}_{E'} \cdot \mathrm{diff}_{K/\mathbb{Q}}$. This follows if we prove that

(I) The cokernel of $\omega_E^\vee/\mathrm{H}_{E,\overline{\tau}}$ is annihilated by $\mathrm{diff}_{K/\mathbb{Q}}$;

(II) The map $\left((\lambda^\vee)^*\right)^\vee \colon \omega_E^\vee \to \omega_{E'}^\vee$ has cokernel annihilated by $\mathrm{Hdg}(E')$.

*Proof of claim (I).* Notice that $E$ is defined over the $p$-adic completion $R$ of $\mathcal{O}_K$. Let $R_0 = \mathbb{W}(k) \subset R$ be the ring of Witt vectors of the residue field $k$ of $R$. Since its ramification index is

$\leq 2 \leq p-1$, then $\mathrm{H}_E \otimes_{\mathcal{O}_K} R$ coincides with the base change via $R_0 \to R$ of crystalline cohomology $\mathrm{H}_{\mathrm{cris}}(E_0/R_0)$ of the special fiber $E_0$ of $E$ (see [BO]). It suffices to prove that $\mathrm{H}_{\mathrm{cris}}(E_0/R_0)$ is a projective $\mathcal{O}_K \otimes R_0$-module of rank 1. But this is true after inverting $p$ and $\mathcal{O}_K \otimes R_0$ is a Dedekind domain for which projective modules are torsion free modules. The claim then follows.

Let us observe the following elementary fact: if $e$ and $\overline{e}$ denote the idempotents $(1,0)$ and $(0,1)$ in $K \times K$ identified with $\mathcal{O}_K \otimes_{\mathbb{Z}} K$ via the isomorphism of $K$-algebras $\tau \times \overline{\tau} \colon \mathcal{O}_K \otimes_{\mathbb{Z}} K \cong K \times K$, then $\mathrm{diff}_{K/\mathbb{Q}} e$ and $\mathrm{diff}_{K/\mathbb{Q}} \overline{e}$ lie in $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathcal{O}_K$.

To conclude the proof of the lemma let $z \in \mathrm{H}_E$. We have $\mathrm{diff}_{K/\mathbb{Q}} ez \in \mathrm{H}_{E',\tau}$ and $\mathrm{diff}_{K/\mathbb{Q}} \overline{e} z \in \mathrm{H}_{E',\overline{\tau}}$ and their sum is $\mathrm{diff}_{K/\mathbb{Q}} z$ by the above observation. Thus the cokernel of $\widetilde{\mathrm{H}}_E \subset \mathrm{H}_E$, which is the cokernel of $\mathrm{H}_{E,\overline{\tau}} \subset \omega_E^\vee$, is is annihilated by $\mathrm{diff}_{K/\mathbb{Q}}$ as claimed.

*Proof of claim (II).* Since $\omega_E$ and $\omega_{E'}$ are free $R$-modules, it suffices to show that $(\lambda^\vee)^* \colon \omega_{E'} \to \omega_E$ has image equal to $\mathrm{Hdg}(E')\omega_E$. Since $\lambda \colon E' \to E$ is the quotient of $E'$ by its canonical subgroup, it coincides with Frobenius modulo $p\mathrm{Hdg}(E')^{-1}$. Hence $\lambda^\vee$ is Vershiebung modulo $p\mathrm{Hdg}(E')^{-1}$ so that the image of $(\lambda^\vee)^* \colon \omega_{E'} \to \omega_E$ coincides with $\mathrm{Hdg}(E')\omega_E$, modulo $p\mathrm{Hdg}(E')^{-1}$ (recall that the Hasse invariant is defined as the image of the Vershiebung map on differentials modulo $p$). Since $\mathrm{val}_p(\mathrm{Hdg}(E')) < \frac{1}{2}$ by Lemma 3.1 (in the inert case) or in Lemma 3.2 (in the ramified case), we deduce that $\mathrm{val}_p(\mathrm{Hdg}(E')) < \mathrm{val}_p(p\mathrm{Hdg}(E')^{-1})$. The conclusion follows.

(iii) By construction $\lambda_i \colon E' \to E_i$ is defined by modding out by a subgroup different from the canonical subgroup. In particular the dual isogeny $\lambda_i^\vee \colon E_i \to E'$ is the quotient by the canonical subgroup of $E_i$. Arguing as in the proof of (ii), we deduce that $\lambda_i^\vee$ is Frobenius modulo $p\mathrm{Hdg}(E_i)^{-1}$ so that the induced map $(\lambda_i^\vee)^* \colon \omega_{E'} \to \omega_{E_i}$ on differentials is 0 modulo $p\mathrm{Hdg}(E_i)^{-1}$. The conclusion follows.

□

## 3.4 Adelic description of CM elliptic curves.

Let $a$ and $N$ are coprime and $N \geq 5$. We assume that there exists an ideal $\mathfrak{N}$ of $\mathcal{O}_K$ whose norm is $N$. We denote by $\mathbb{A}_K^{(\mathfrak{N}a)}$ the group of finite adeles whose components at places dividing $\mathfrak{N}a$ lie in $\prod_{\mathcal{P}|\mathfrak{N}a} \widehat{\mathcal{O}}_{K,\mathcal{P}}^*$. If $c$ is a divisor of $a$ we denote by $\mathcal{O}_a$ the order of $\mathcal{O}_K$ of conductor $c$. Let $H^{c,\mathfrak{N}} \subset \mathbb{A}_K^{(\mathfrak{N}a)}$ be the compact open subgroup defined by the elements lying in $\prod_p (\mathcal{O}_K \otimes \mathbb{Z}_p)^*$, whose components in $\prod_{\mathcal{P}|\mathfrak{N}} \widehat{\mathcal{O}}_{K,\mathcal{P}}^*$ are congruent to 1 modulo $\mathfrak{N}$ and whose components in $\prod_{\mathcal{P}|c} \widehat{\mathcal{O}}_{K,\mathcal{P}}^*$ are in fact in $\prod_{p|c} (\mathcal{O}_a \otimes \mathbb{Z}_p)^*$. Set $K^{(\mathfrak{N}a)} := K^* \cap \mathbb{A}_K^{(\mathfrak{N}a)}$.

For $N = 1$ and $a = c$ the quotient $K^{(a)} \backslash \mathbb{A}_K^{(a)} / H^{c,1}$ is isomorphic to the group $\mathrm{Pic}(\mathcal{O}_c)$ of fractionals $\mathcal{O}_c$-ideals for the order $\mathcal{O}_c \subset \mathcal{O}_K$: the isomorphism is defined by $(r_\mathcal{P})_\mathcal{P} \mapsto \prod_\mathcal{P} \mathcal{P}^{\mathrm{val}_\mathcal{P}(r_\mathcal{P})}$.

In general, the set of elliptic curves with CM by $\mathcal{O}_c$ and $\Gamma_1(\mathfrak{N})$-level structure is a principal homogenoeus space under the action of $K^{(\mathfrak{N}a)} \backslash \mathbb{A}_K^{(\mathfrak{N}a)} / H^{c,\mathfrak{N}}$: given $(E, \iota, \psi_\mathfrak{N})$ such an elliptic curve and given $\mathfrak{a} = (a_\mathcal{P})_\mathcal{P} \in \mathbb{A}_K^{(\mathfrak{N}c)}$ we set $\mathfrak{a} * (E, \iota, \psi_\mathfrak{N}) := (E', \iota', \psi'_\mathfrak{N})$ to be the elliptic curve $E'$ whose Tate module $\mathbf{T}(E')$ is isomorphic to $\mathfrak{a}^{-1}(\mathbf{T}(E)) \subset \mathbf{T}(E) \otimes \mathbb{Q}$ (recall that $\mathbf{T}(E) = \prod_\ell \mathbf{T}_\ell(E)$ is a principal $\mathcal{O}_c \otimes \widehat{\mathbb{Z}}$-module so that this makes sense). By construction also $\mathbf{T}(E')$ is a principal $\mathcal{O}_c \otimes \widehat{\mathbb{Z}}$-module so that $E'$ has CM by $\mathcal{O}_c$ and the $\mathfrak{N}$-torsion of $E$ and $E'$ are canonically isomorphic so that $E'$ acquires a natural $\Gamma_1(\mathfrak{N})$-level structure $\psi'_\mathfrak{N}$.

# 4 Vector bundles with marked sections. The sheaves $\mathbb{W}_k$.

We only present the theory in the cases we need for the treatement of the $p$-adic $L$-functions attached to twists by Hecke characters of elliptic modular forms. The main references for this section are [AI] and [ICM18], where more general cases are presented and all the details are carefully spelled out.

Let $\mathfrak{S}$ be a formal scheme with ideal of definition $\mathcal{I}$ which is invertible (i.e. locally principal generated locally by non-zero divisors) and let $\mathcal{E}$ be a locally free $\mathcal{O}_{\mathfrak{S}}$-module of rank 1 or 2. Let also $s \in \mathrm{H}^0(\mathfrak{S}, \mathcal{E}/\mathcal{I}\mathcal{E})$ be a section such that $s\mathcal{O}_{\mathfrak{S}}/\mathcal{I}$ is a direct summand of $\mathcal{E}/\mathcal{I}\mathcal{E}$. We call the pair $(\mathcal{E}, s)$ a locally free sheaf with a marked section. We have

**Theorem 4.1** ([AI] Thm.). *a) The functor attaching to a morphism of formal schemes $\rho : \mathfrak{T} \to \mathfrak{S}$ (the ideal of definition of $\mathfrak{T}$ is $\rho^*(\mathcal{I})$) the set*

$$\mathbb{V}(\mathcal{E})\big(\rho : \mathfrak{T} \to \mathfrak{S}\big) := \mathrm{Hom}_{\mathcal{O}_{\mathfrak{T}}}\big(\rho^*(\mathcal{E}), \mathcal{O}_{\mathfrak{T}}\big)$$

*is represented by the formal vector bundle $\mathbb{V}(\mathcal{E}) := \mathrm{Spf}\big(\mathrm{Sym}(\mathcal{E})\big)$.*

*b) The subfunctor of $\mathbb{V}(\mathcal{E})$, denoted $\mathbb{V}_0(\mathcal{E}, s)$, which associates to every morphism of formal schemes $\rho : \mathfrak{T} \longrightarrow \mathfrak{S}$ as above, the set*

$$\mathbb{V}_0(\mathcal{E}, s)\big(\rho : \mathfrak{T} \to \mathfrak{S}\big) := \Big\{h \in \mathbb{V}(\mathcal{E})(\rho : \mathfrak{T} \to \mathfrak{S}) \mid \big(h(\mathrm{mod}\ \rho^*(\mathcal{I}))\big)(s) = 1\Big\},$$

*is represented by the open in the admissible formal blow-up of $\mathbb{V}(\mathcal{E})$ at the ideal $\mathcal{J} := (\tilde{s} - 1, \mathcal{I}) \subset \mathcal{O}_{\mathbb{V}(\mathcal{E})}$, where the inverse image of this ideal is generated by $\mathcal{I}$. Here $\tilde{s}$ is a lift to $\mathrm{H}^0(\mathcal{S}, \mathcal{E})$ of $s$.*

**Remark 4.2.** If $\mathcal{E}$ has extra structure like a connection or filtration compatible mod $\mathcal{I}$ with the section $s$, then if we denote by $\pi\colon \mathbb{V}_0(\mathcal{E}, s) \to \mathfrak{S}$, the $\mathcal{O}_{\mathfrak{S}}$-module $\pi_*\big(\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}\big)$ has the same type of extra structures.

**Remark 4.3.** Let us suppose that we have a morphism of formal schemes $\varphi\colon \mathfrak{S} \to \mathfrak{S}'$ such that the ideal of definition of $\mathfrak{S}'$, $\mathcal{I}'$ is invertible and the ideal of definition of $\mathfrak{S}$ is $\varphi^*(\mathcal{I}')$. Suppose we also have a pair $(\mathcal{E}', s')$ consisting of a locally free sheaf with a marked section on $\mathfrak{S}'$ and let $\mathcal{E} := \varphi^*(\mathcal{E}')$ and $s := \varphi^*(s')$. Then $(\mathcal{E}, s)$ is a locally free sheaf with a marked section on $\mathfrak{S}$. Moreover the functoriality of VBMS implies that the following natural diagram is cartesian:

$$\begin{array}{ccc} \mathbb{V}_0(\mathcal{E}, s) & \longrightarrow & \mathbb{V}_0(\mathcal{E}', s') \\ u \downarrow & & v \downarrow \\ \mathfrak{S} & \xrightarrow{\varphi} & \mathfrak{S}' \end{array}$$

In particular we have: $u_*\big(\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}\big) \cong \varphi^*\big(v_*(\mathcal{O}_{\mathbb{V}_0(\mathcal{E}',s')}\big)$.

## 4.1 Applications to modular curves. The sheaf $\mathbb{W}_k$.

Let $N \geq 5$ be an integer and $p \geq 5$ a prime integer such that $(N, p) = 1$ and consider the tower of modular curves

$$(*) \quad X(N, p^2) \to X(N, p) \to X_1(N)$$

over the ring of integers of a finite extension of $\mathbb{Q}_p$ to be made more precise later. Here the modular curves classify, in order from left to right: generalized elliptic curves with $\Gamma_1(N) \cap \Gamma_0(p^2)$, respectively $\Gamma_1(N) \cap \Gamma_0(p)$, respectively $\Gamma_1(N)$-level structures. We denote $\hat{X}(Np^2) \to \hat{X}(N,p) \to \hat{X}_1(N)$ the sequence of formal completions of these curves along their respective special fibers and denote by $\mathcal{X}(N,p^2) \to \mathcal{X}_1(N,p) \to \mathcal{X}_1(N)$ the rigid analytic generic fibers associated to the previous sequence of formal scheme.

Consider the ideal Hdg, called the Hodge ideal, defined as the ideal of $\mathcal{O}_{\hat{X}_1(N)}$ locally (on open affines $\mathrm{Spf}(R) \subset \hat{X}_1(N)$ which trivialize the sheaf $\omega_{\mathbb{E}}$) generated by $p$ and $\mathrm{Ha}^o(E/R,\omega)$, where $\mathrm{Ha}^o$ is a local lift of the Hasse invariant Ha and $\omega$ is a basis of $\omega_E$. For every integer $r \geq 2$ we denote by $\mathfrak{X}_r$ the formal open subscheme in the formal admissible blow-up of $\hat{X}(N,p)$ with respect to the sheaf of ideals $(p, \mathrm{Hdg}^r)$, where this ideal is generated by $\mathrm{Hdg}^r$. Let $\mathcal{X}_r$ denote the adic generic fiber of $\mathfrak{X}_r$; it is a disjoint union of two open affinoids of $\mathcal{X}(N,p)$ each one an open neighbourhood of the respective component of the ordinary locus in $\mathcal{X}(N,p)$. By construction Hdg is an invertible ideal in $\mathfrak{X}_r$. Recall from [AIPHS, App. A] that the natural generalized elliptic curve $\mathbb{E} \to \mathfrak{X}_r$ has a canonical subgroup $\mathrm{H_m} \subset \mathbb{E}[p]$ of order $p^m$, where $m$ depends on $r$. In this article we only need: if $r \geq 1$ then $m = 1$ and if $r \geq p+2$ then $m = 2$. We denote by $\mathfrak{X}_r^{(\infty)}$ (respectively $\mathfrak{X}_r^{(0)}$) the component of $\mathfrak{X}_r$ of points corresponding to triples $(E, \psi_N, C)$, $E$ a generalized elliptic curve, $\psi_N$ a level $\Gamma_1(N)$ structure and $C$ a subgroup scheme of order $p$ of $E[p]$, such that $C = \mathrm{H}_1$ (respectively such that $C \cap \mathrm{H}_1 = \{0\}$). For $r = 1$ we drop the subscript, i.e., we write $\mathfrak{X} := \mathfrak{X}_1$, $\mathcal{X} := \mathcal{X}_1$ etc.

Let us denote by $\pi \colon \mathcal{IG}_{m,r} := \mathrm{Isom}\big(\underline{Z/p^m\mathbb{Z}}, {\mathrm{H_m}}^D\big) \to \mathcal{X}_r$ the $m$-th layer of the adic analytic Igusa tower over $\mathcal{X}_r$, where ${\mathrm{H_m}}^D$ denotes the Cartier dual of $\mathrm{H_m}$. Then $\mathcal{IG}_{m,r}$ is finite, ètale, Galois cover of $\mathcal{X}_r$, with Galois group $(\mathbb{Z}/p^m\mathbb{Z})^*$ and we denote by $\mathfrak{IG}_{m,r}$ the normalization of $\mathfrak{X}_r$ in $\mathcal{IG}_{m,r}$. We denote by $\mathfrak{IG}_{m,r}^{(\infty)}$ respectively $\mathfrak{IG}_{m,r}^{(0)}$ the components of $\mathfrak{IG}_{m,r}$ over $\mathfrak{X}_r^{(\infty)}$, respectively $\mathfrak{X}_r^{(0)}$. We briefly recall some key results proved, for example, in [AIPHS, Appendix A]. Let $r \geq 1$.

i. the canonical subgroup $\mathrm{H}_1$ of the universal elliptic curve $\mathbb{E}$ over $\mathfrak{IG}_{1,r}$ is a lifting of the kernel of Frobenius modulo $p/\mathrm{Hdg}$;

ii. we have an isomorphism between $C := \mathbb{E}[p]/\mathrm{H}_1$ and $\mathrm{H}_1^D$ defined by the Weil pairing on $\mathbb{E}[p]$ and a canonical section $P^{\mathrm{univ}}$ of $C$;

iii. the map of invariant differentials associated to the inclusion $\mathrm{H}_1 \subset \mathbb{E}$ induces a map $\omega_{\mathrm{H}_1} \to \omega_{\mathbb{E}}/p\mathrm{Hdg}^{-1}\omega_{\mathbb{E}}$ so that via $\mathrm{dlog}_{\mathrm{H}_1} \colon L = \mathrm{H}_1^D \to \omega_{\mathrm{H}_1}$ we get a section $s := \mathrm{dlog}_{\mathrm{H}_1}(P^{\mathrm{univ}}) \in {\mathrm{H}_1}^0\big(\mathfrak{IG}_{1,r}, \omega_{\mathbb{E}}/p\mathrm{Hdg}^{-1}\omega_{\mathbb{E}}\big)$;

iv. a lift of the section $s$ at iii. above, $\tilde{s}$ spans the $\mathcal{O}_{\mathfrak{IG}_{1,r}}$-submodule

$$\big(\mathrm{Hdg}^{\frac{1}{p-1}}\omega_{\mathbb{E}}/p\mathrm{Hdg}^{-1}\omega_{\mathbb{E}}\big) \subset \big(\omega_{\mathbb{E}}/p\mathrm{Hdg}^{-1}\omega_{\mathbb{E}}\big).$$

Here Claim (iv) strengthens [AIPHS] but follows in this case by an explicit computation of $\mathrm{dlog}_{\mathrm{H}} \colon L = \mathrm{H}^D \to \omega_{\mathrm{H}}$ using Oort-Tate group schemes as in [Fa, Lemme 9].

We then get an invertible $\mathcal{O}_{\mathfrak{IG}_{1,r}}$-submodule $\Omega_{\mathbb{E}}$ of $\omega_{\mathbb{E}}$ as the span of any lift $\tilde{s}$ of $s$ such that, if we set $\underline{\delta} := \Omega_{\mathbb{E}}\omega_{\mathbb{E}}^{-1}$, then $\underline{\delta}$ is an invertible $\mathcal{O}_{\mathfrak{IG}_{1,r}}$-ideal with $\underline{\delta}^{p-1} = \pi^*(\mathrm{Hdg})$ (recall that $\pi \colon \mathfrak{IG}_{1,r} \to \mathfrak{X}_r$ is the natural projection).

From $\tilde{s}$ we also get a canonical section $s'$ of $\mathrm{H}^0\big(\mathfrak{IG}, \Omega_{\mathbb{E}}/p\mathrm{Hdg}^{-\frac{p}{p-1}}\Omega_{\mathbb{E}}\big)$ such that $s'$ defines a basis of $\Omega_{\mathbb{E}}/p\mathrm{Hdg}^{-\frac{p}{p-1}}\Omega_{\mathbb{E}}$ as $\mathcal{O}_{\mathfrak{IG}_1}/p\mathrm{Hdg}^{-\frac{p}{p-1}}\mathcal{O}_{\mathfrak{IG}_1}$-module.

We denote by $\mathrm{H}^{\#}_{\mathbb{E}}$ the push-out in the category of coherent sheaves on $\mathfrak{X}_r$ of the diagram

$$\begin{array}{ccc} \underline{\delta}^p\omega_E & \longrightarrow & \underline{\delta}^p\mathrm{H}_{\mathbb{E}} \\ \cap & & \\ \Omega_{\mathbb{E}}. & & \end{array}$$

We then have the following commutative diagram of sheaves with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\delta}^p\omega_{\mathbb{E}} & \longrightarrow & \underline{\delta}^p\mathrm{H}_{\mathbb{E}} & \longrightarrow & \underline{\delta}^p\omega_{\mathbb{E}}^{-1} & \longrightarrow & 0 \\ & & \cap & & \downarrow & & || & & \\ 0 & \longrightarrow & \Omega_{\mathbb{E}} & \longrightarrow & \mathrm{H}^{\#}_{\mathbb{E}} & \longrightarrow & \underline{\delta}^p\omega_{\mathbb{E}}^{-1} & \longrightarrow & 0 \\ & & \cap & & \cap & & \cap & & \\ 0 & \longrightarrow & \omega_{\mathbb{E}} & \longrightarrow & \mathrm{H}_{\mathbb{E}} & \longrightarrow & \omega_{\mathbb{E}}^{-1} & \longrightarrow & 0 \end{array}$$

It follows that $\mathrm{H}^{\#}_{\mathbb{E}}$ is a locally free $\mathcal{O}_{\mathfrak{X}_r}$-module of rank two and $(\Omega_{\mathbb{E}}, s) = (\underline{\delta}\omega_{\mathbb{E}}, s) \subset (\mathrm{H}^{\#}_{\mathbb{E}}, s)$ is a compatible inclusion of locally free sheaves with marked sections.

We recall the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{\mathbb{E}} & \longrightarrow & \mathrm{H}^{\#}_{\mathbb{E}} & \longrightarrow & \underline{\delta}^p\omega_{\mathbb{E}}^{-1} & \longrightarrow & 0 \\ & & || & & \cap & & \cap & & \\ 0 & \longrightarrow & \Omega_{\mathbb{E}} & \longrightarrow & \underline{\delta}\mathrm{H}_{\mathbb{E}} & \longrightarrow & \underline{\delta}\omega_{\mathbb{E}}^{-1} & \longrightarrow & 0 \end{array}$$

in which the right square is cartesian and so which defines $\mathrm{H}^{\#}_{\mathbb{E}}$ as pull-back.

We have natural actions of $\mathfrak{T}^{\mathrm{ext}} := \big(\mathbb{Z}_p^*\big(1 + \pi_*\big(p\mathrm{Hdg}^{-\frac{p}{p-1}}\mathcal{O}_{\mathfrak{IG}_1}\big)\big)$ on $u\colon \mathbb{V}_0(\mathrm{H}^{\#}_{\mathbb{E}}, s) \longrightarrow \mathfrak{X}$ and on $v\colon \mathbb{V}_0(\Omega_{\mathbb{E}}, s) \longrightarrow \mathfrak{X}$.

Let $\nu\colon \mathbb{Z}_p^* \longrightarrow R^*$ be an analytic weight, i.e., such that there exists $u \in R$ with the property that $\nu(t) = \exp u \log t$, for every $t \in 1 + p\mathbb{Z}_p$.

**Definition 4.4.** Denote by $\mathfrak{w}^{\nu} := v_*(\mathcal{O}_{\mathbb{V}_0(\Omega_{\mathbb{E}}, s)})[\nu]$ and $\mathbb{W}_{\nu} := u_*(\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^{\#}_{\mathbb{E}}, s)})[\nu]$ as the functions on which $\mathfrak{T}^{\mathrm{ext}}$ acts via $\nu$ (see [AI]). We denote by $\mathbb{W}_{\nu}^{(\infty)}$, respectively $\mathbb{W}_{\nu}^{(0)}$ the components of $\mathbb{W}_{\nu}$ over $\mathfrak{X}_r^{(\infty)}$, respectively over $\mathfrak{X}_r^{(0)}$.

The definition makes sense thanks to the following:

**Lemma 4.5.** *Assume that $p \geq 5$. Consider a $p$-adically complete and separated ring $A$ with a map $Spf(A) \to \mathfrak{IG}_{1,r}$. Write $h$ for the image of* Hdg *in $A$. Then for $x \in A$ we have that $\nu(1+x) := \exp(u\log(1+ph^{-\frac{p}{p-1}}x)$ is a well defined element of $A$, congruent to $0$ modulo $p^{\frac{1}{2(p-1)}}$.*

*Proof.* We have $\mathrm{val}_p\big(ph^{-\frac{p}{p-1}}\big) = 1 - \frac{p}{p-1}\mathrm{val}_p\big(h\big) \geq 1 - \frac{p}{2(p-1)} > \frac{3}{2(p-1)}$ as $p \geq 5$. Thus $y := \log\big(1 + ph^{-\frac{p}{p-1}}x\big)$ converges to an element of $A$ divisible by $p^{\frac{3}{2(p-1)}}$ for any $x \in A$.

Recall that $\mathrm{val}_p(n!) < \frac{n}{p-1}$. Thus $\mathrm{val}_p\big(y^n/n!\big) = n\mathrm{val}_p(y) - \mathrm{val}_p(n!) > n\big(\mathrm{val}_p(y) - \frac{1}{p-1}\big) \geq n\frac{1}{2(p-1)}$. Hence $\exp(y)$ converges to an element of $A$ congruent to $0$ modulo $p^{\frac{1}{2(p-1)}}$. □

As in [AI] one has the following:

**Proposition 4.6.** *The sheaf $\mathfrak{w}^\nu$ is an invertible $\mathcal{O}_{\mathfrak{X}_r}$-module and $\mathbb{W}_\nu^{(\infty)}$ has a natural, increasing filtration $\big(\mathrm{Fil}\big)_{n\geq 0}$ such that $\mathfrak{w}^\nu$ is identified with $\mathrm{Fil}_0$.*

*The Gauss-Manin connection $\nabla\colon \mathrm{H}_{\mathbb{E}} \to \mathrm{H}_{\mathbb{E}} \otimes \Omega^1_{\mathfrak{X}_r/R}\big(\log(\mathrm{cusps})\big)$ induces a connection $\nabla_\nu$ (with poles) on $\mathbb{W}_\nu^{(\infty)}$.*

*Proof.* Consider the map $v_0\colon \mathbb{V}_0(\Omega_{\mathbb{E}}, s) \longrightarrow \mathfrak{IG}_{1,r}$ and define $\mathfrak{w}^{\nu,0} := v_{0,*}(\mathcal{O}_{\mathbb{V}_0(\Omega_{\mathbb{E}},s)})[\nu]$. The map $\pi\colon \mathfrak{IG}_{1,r} \to \mathfrak{X}_r$ is of degree $p-1$ and is endowed with an action of $\mathbb{F}_p^*$. Then $\pi_*(\mathcal{O}_{\mathfrak{IG}_{1,r}})$ decomposes as a sum of $(p-1)$-invertible sheaves according to the residual action of $\mathbb{F}_p^*$. It suffices to prove all statements over $\mathfrak{IG}_{1,r}$. To define the filtration and the connection one uses the formalism of VBMS (see [AI, §3.3&§3.4])

Concerning the first statement, since $\mathcal{O}_{\mathfrak{IG}_{1,r}} = v_{0,*}\big(\mathcal{O}_{\mathbb{V}_0(\Omega_{\mathbb{E}},s)}\big)[0]$, it suffices to show that, affine locally on $\mathfrak{IG}_{1,r}$, there exists a section $\gamma \in \mathfrak{w}^{\nu,0}$ with $\gamma \equiv 1$ modulo $p^{\frac{1}{2(p-1)}}$. This follows from §4.1.1.

□

**Remark 4.7.** Let us remark that if we denote by $\mathfrak{X}'$ the open in the formal blow-up of $\hat{X}_1(N)$ at the ideal $(p, \mathrm{Hdg}^2)$, where the inverse image of this ideal is generated by $\mathrm{Hdg}^2$, then the natural projection $\mathfrak{X}_r^{(\infty)} \to \mathfrak{X}'$ induced from the projection $X(N,p) \to X_1(N)$ is an isomorphism. This implies that all results in [AI] for $\mathbb{W}_\nu, \nabla_\nu$ and the interations of powers of $\nabla_\nu$ hold as such for $\mathbb{W}_\nu^{(\infty)}, \nabla_\nu$ and the iterations of the powers of $\nabla_\nu$. See also the next subsection.

### 4.1.1 Local descriptions of the sheaves $\mathbb{W}_\nu^{(\infty)}$ and $\mathbb{W}_\nu^{(0)}$.

We have nice local descriptions of these sheaves on $\mathfrak{IG}_{1,r}^{(\infty)}$ and $\mathfrak{IG}_{1,r}^{(0)}$ respectively (see section §3.2.2 of [AI]). Let $\mathrm{Spf}(A) \subset \mathfrak{IG}_{1,r}^{(\flat)}$, where $\flat \in \{\infty, 0\}$, be an affine open such that $\omega_{\mathbb{E}}|_{\mathrm{Spf}(A)}$ is free and $p\mathrm{Hdg}^{-\frac{p}{p-1}}|_{\mathrm{Spf}(A)}$, $\underline{\delta}|_{\mathrm{Spf}(A)}$ are principal ideals generated respectively by $\beta$, $\delta$. It follows that the $A$-module $\mathrm{H}^{\#}_{\mathbb{E}}(\mathrm{Spf}(A))$ is free of rank two and let us choose a basis of it $\{f, e\}$ such that $f(\mathrm{mod}\ \beta A) = s^{(\flat)} = \mathrm{dlog}(P^{\mathrm{univ}})$, where let us recall $P^{\mathrm{univ}}$ extends the universal generator of $\mathrm{H}^D$ on $\mathcal{IG}_{1,r}^{(\flat)}$ to $\mathfrak{IG}_{1,r}^{(\flat)}$.

Then an $A$-point of $\mathbb{V}_0(\mathrm{H}^{\#}_{\mathbb{E}}, s)\big(\mathrm{Spf}(A)\big)$ can be seen as $x := af^\vee + be^\vee$, where $\{f^\vee, e^\vee\}$ is the dual basis of $\{f, e\}$ and $a$, $b \in A$ satisfy $(a-1) \in \beta A$, so if we denote $\pi : \mathbb{V}_0(\mathrm{H}^{\#}_{E}, s) \to \mathfrak{IG}^{(\flat)}$, we have $\pi_*\big(\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^{\#}_{E},s)}\big)\big(\mathrm{Spf}(A)\big) = A\langle Y, Z\rangle$ and the point $x = af^\vee + be^\vee \in \mathbb{V}_0(\mathrm{H}^{\#}_{E}, s)\big(\mathrm{Spf}(A)\big)$ corresponds to the algebra homomorphism $x\colon A\langle Y, Z\rangle \longrightarrow A$ sending $Y \to b, Z \to \dfrac{a-1}{\beta}$. We then have $\mathfrak{w}^\nu\big(\mathrm{Spf}(A)\big) = (1+\beta Z)^\nu A$ and

$$\mathbb{W}_\nu^{(\flat)}\big(\mathrm{Spf}(A)\big) = (1+\beta Z)^\nu A\langle \frac{Y}{1+\beta Z)}\rangle \subset A\langle Y, Z\rangle.$$

### 4.1.2 Powers of the Gauss-Manin connection.

In this section we fix an even integer $k \in \mathbb{Z}$, $k \geq 2$, $r \geq 1$ and an analytic weight $\nu\colon \mathbb{Z}_p^* \to R^*$. This means that there exist $s \in pR$ and a character $\epsilon\colon \mu_{p-1} \longrightarrow \mathbb{Z}_p^*$ such that for all $t \in \mathbb{Z}_p^*$ we have

$$\nu(t) = \epsilon([t])\exp\big(s \cdot \log\langle t\rangle\big)$$

where we denote by $([\ ],\langle\ \rangle)\colon \mathbb{Z}_p^* \cong \mu_{p-1}\times(1+p\mathbb{Z}_p)$ the canonical isomorphism. In particular, the pair of weights $\nu$, $k$ satisfies the Assumption 4.1 of [AI].

We work entirely on $\mathfrak{IG}_{1,r}^{(\infty)}$, with the sheaf $\mathbb{W}_k^{(\infty)}$ and its connection $\nabla_k$. Let $F\in \mathrm{H}^0\big(\mathfrak{IG}_{1,r}^{(\infty)},\mathfrak{w}^k\big)$.

**Lemma 4.8.** *For every $N\in\mathbb{N}$ we have $\mathrm{Hdg}^N(\nabla_k)^N(F)\in \mathrm{H}^0\big(\mathfrak{IG}_{1,r}^{(\infty)},\mathbb{W}_{k+2N}\big)$.*

*If $U(F)=0$, then for every $N\in\mathbb{N}$ we have*

$$\mathrm{Hdg}^{(p-1)pN+rN}\big(\nabla^{p-1}-\mathrm{Id}\big)^{pN}(F)\in p^N\mathrm{H}^0\big(\mathfrak{IG}_{1,r}^{(\infty)},\sum_{i=0}^{(p-1)pN}\mathbb{W}_{k+2i},\big).$$

*Proof.* To prove the first claim we recall that Lemma 3.20 and the proof of Theorem 3.18 in Section 3.4.1 of [AI] implies that there is a basis of $\mathrm{H}^{\#}_{\mathbb{E}}$ such that the image of $\mathrm{Hdg}\nabla_k$ is contained in $\mathrm{H}^0(\mathfrak{IG}_{1,r}^{(\infty)},\mathbb{W}_{k+2})$. In order to finish the proof we only need to show that the image of $\Omega_{\mathfrak{IG}_{1,r}^{(\infty)}/\Lambda_I}$ in $\Omega_{\mathfrak{IG}_{1,r}^{(\infty)}/\Lambda_I}[1/p]$ is contained in the pull-back of $\frac{1}{\mathrm{Hdg}}\Omega_{\mathfrak{X}/\Lambda_I}$ to $\mathfrak{IG}_{1,r}^{(\infty)}$. Let $\mathfrak{U}=\mathrm{Spf}(A)\subset\mathfrak{X}$ be an affine open such that $\omega_{\mathbb{E}}|_{\mathfrak{U}}$ is a free $A$-module of rank one and let $h$ be a generator of the ideal $\mathrm{Hdg}(\mathfrak{U})$. Let us denote $\mathrm{Spf}(B)\subset\mathfrak{X}_r$ and $\mathrm{Spf}(C)\subset\mathfrak{IG}_{1,r}^{(\infty)}$ the inverse images of $\mathfrak{U}$ in $\mathfrak{X}_r$ and $\mathfrak{IG}_{1,r}^{(\infty)}$. The relative description of these algebras is done in [AIPHS, Lemme 3.4] as follows: $B=A\langle Y\rangle(h^rY-p)$ and $C:=B\langle Z\rangle/(Z^{p-1}-h)=A\langle Z,Y\rangle/(Z^{p-1}-h,Z^{(p-1)r}Y-p)$. Therefore $\Omega^1_{C/\Lambda_I}$ is the $C$ module generated by $dZ,dY$ and $\Omega_{A/\Lambda_I}$, with the relations $h^rdY=rh^{r-1}dh$ and $(p-1)Z^{p-2}dZ=dh$. Therefore the image of $\Omega_{C/\Lambda_I}$ in $\Omega_{C/\Lambda_I}[1/p]$ is contained in $\frac{1}{h}C\otimes_A\Omega^1_{A/\Lambda_I}$.

It follows from Proposition 4.3 of [AI] that if $g:=\mathrm{Hdg}^{(p-1)pN}\big(\nabla^{p-1}-\mathrm{Id}\big)^{pN}(F)$, when restricted to the ordinary locus, lies in $p^N\mathrm{H}^0\big(\mathfrak{IG}_{1,r}^{(\infty),\mathrm{ord}},\sum_{i=0}^{(p-1)pN}\mathbb{W}_{k+2i},\big)$. Using Lemma 3.4 of [AI] we deduce that $\mathrm{Hdg}^{rN}g\in p^N\mathrm{H}^0\big(\mathfrak{IG}_{1,r}^{(\infty)},\sum_{i=0}^{(p-1)pN}\mathbb{W}_{k+2i},\big)$. The claim follows. □

We next recall one of the main results of [AI], namely Theorem 4.3, about $p$-adic iterate of the Gauss-Manin connection. As we will need precise bounds on the radius of convergence of such iterate we will briefly go through the proof:

**Theorem 4.9.** *Let $F\in\mathrm{H}^0(\mathfrak{IG}_{1,p+2}^{(\infty)},\mathfrak{w}^k)$ such that $U(F)=0$. Assume that $s\in p^2R$ and let $u\colon\mathbb{Z}_p^*\to R^*$ be a weight obtained adding to the weight $\nu$, as at the beginning of this section, an integral weight and a finite character. Then there exists a section $(\nabla_k)^u(F)$ of $\mathbb{W}_{k+2u}$ over $\mathcal{IG}_{1,p(p+1)}^{(\infty)}$ whose q-expansion coincides with $(\nabla_k)^u(F(q))$ (as computed in Theorem 4.3 of [AI]).*

*Proof.* Recall that $\mathcal{IG}_{1,p+2}^{(\infty)}$ is the rigid analytic fiber of $\mathfrak{IG}_{1,p+2}^{(\infty)}$. Then

$$\mathrm{H}^0\big(\mathfrak{IG}_{1,p+2}^{(\infty)},\mathfrak{w}^k\big)[p^{-1}]=\mathrm{H}^0\big(\mathcal{IG}_{1,p+2}^{(\infty)},\mathfrak{w}^k\big)$$

as $\mathfrak{w}^k$ is an invertible sheaf. In particular, it suffices to prove the Theorem assuming that $F\in\mathrm{H}^0\big(\mathcal{IG}_{1,p+2}^{(\infty)},\mathfrak{w}^k\big)^{U=0}$.

First of all we recall how $(\nabla_k)^u(F)$ is constructed as a section of $\mathbb{W}_{k+2u}$ over $\mathcal{IG}_{1,p(p+1)}^{(\infty)}$. As in Theorem 4.3 of [AI] one reduces to prove the Theorem for $u=\nu$. In this case the argument is as in the proof of Proposition 4.13 of [AI]. One writes $(\nabla_k)^u(F)$ as the limit of a sequence

$\{B(F,s)_n\}_n$ of sections of $\oplus_{i=0}^{n(p-1)}\mathbb{W}_{k+2i}$ which needs to be proven to be Cauchy. The key input in loc. cit. is Lemma 4.12, that in our case is replaced by the following. Consider a positive integer $h$. Let $N = j_1 + \cdots + j_h$ be the sum of positive integers $j_1, \ldots, j_h$. Then

$$v_p(s)h + \frac{N}{p} - \sum_i v_p(j_i) - \frac{h}{p-1} > w\frac{N}{p}$$

with $w := 1 - \frac{1}{2p}$. As $N = j_1 + \cdots + j_h$ it suffices to prove the claim for $h = 1$, i.e., that for every positive integer $j$ we have $2 + \frac{1}{2p^2}j > v_p(j) + \frac{1}{p-1}$. If $v_p(j) = 0$ this is clear. Else write $j = \gamma p^r$ with $p$ not dividing $\gamma$ and $r \geq 1$ and the inequality becomes

$$2 + \frac{1}{2}\gamma p^{r-2} > r + \frac{1}{p-1}.$$

It suffices to prove it for $\gamma = 1$. For $r = 1$ or 2 this is clear. For $r \geq 3$ this follows as $p^{r-2} > 2(r-1)$ since $p \geq 5$. This concludes the proof of the Claim.

One then deduces from Lemma 4.8, arguing as in the proof of Corollary 4.11 of [AI], that there exists a positive integer $\gamma$ (independent of $N$) such that

$$\mathrm{Hdg}^{\gamma}\mathrm{Hdg}^{\frac{(p-1)p+(p+2)}{p}N}\big(B(F,s)_N - B(F,S)_{N-1}\big)$$

is a section of $\oplus_{i=0}^{N(p-1)}\mathbb{W}_{k+2i}$ which is zero modulo $p^{\frac{wN}{p}}$ for every $N \geq 2$. We conclude that the sequence $\{B(F,s)_n\}_n$ is Cauchy sequence in $u_*(\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^{\#}_{\mathbb{E}},s)})[p^{-1}]$ (see definition 4.4 for the notation) whenever $p^{\frac{w}{p}}/\mathrm{Hdg}^{\frac{(p-1)p+(p+2)}{p}}$ is a nilpotent element. In particular, this holds true after restricting to $\mathcal{IG}^{(\infty)}_{1,p(p+1)}$ if we prove the following:

$$wp(p+1) = p^2 + p - \frac{p+1}{2} \geq (p-1)p + (p+2) = p^2 + 2,$$

i.e., that $2p - 4 \geq p + 1$ which is true as $p \geq 5$ by hypothesis. We deduce that the sequence converges to a section $\lim_{N\to\infty} B(F,s)_N$ of $u_*(\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^{\#}_{\mathbb{E}},s)})[p^{-1}]$ that one checks to be a section $(\nabla_k)^u(F)$ of $u_*(\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^{\#}_{\mathbb{E}},s)})[k+2\nu] = \mathbb{W}_{k+2\nu}$. By construction it has the required $q$-expansion.

□

**Remark 4.10.** Notice that, in particular, any analytic character $u\colon \mathbb{Z}_p^* \to \mathbb{Z}_p^*$ satisfies the assumptions of the Corollary so that $(\nabla_k)^u(F)$ is defined for any $F$ as in the statement of the Theorem.

The statement of Theorem 4.9 also holds for $k\colon \mathbb{Z}_p^* \to R^*$ an analytic character satisfying the Assumptions 4.1 of [AI], namely $(\nabla_k)^u(F)$ exists as a section of $\mathbb{W}_{k+2u}$ on $\mathcal{IG}^{(\infty)}_{1,r}$ for some $r > 0$ but in this degree of generality, for the moment we can't precisely estimate $r$.

As consequence of the theorem we also have the following interpolation property. Take a homomorphism $\gamma\colon R \to \mathbb{Z}_p$ such that the induced character $\gamma(u)\colon \mathbb{Z}_p^* \to \mathbb{Z}_p^*$ is a classical positive weight $\ell$, i.e., it is given by raising elements of $\mathbb{Z}_p^*$ to the $\ell$-th power.

**Corollary 4.11.** *The specialization of $(\nabla_k)^u(F)$ at $\gamma$ is $\nabla_k^{\ell}(F)$ (the usual $\ell$-th iterate of the Gauss-Manin connection).*

*Proof.* See Corollary 4.6 of [AI]. □

## 4.2 $p$-Adic iterations of $\nabla_k$ on $\mathcal{X}(pN, p^2)$.

In order to define our $p$-adic $L$-functions à la Katz we need, for a reason which will be explained in section §5, to consider the inverse image of the pair $(\nabla_k, \mathbb{W}_k)$ from $\mathcal{X}(N,p)_1^{(\infty)}$ to a certain rigid analytic subspace of the modular curve $\mathcal{X}(pN, p^2)$ and iterate $p$-adically the powers of $\nabla_k$ there. We will first define the modular curve $X(pN, p^2)$ and study some of its $p$-adic properties.

### 4.2.1 The modular curve $X(pN, p^2)$.

Let us consider the morphisms of schemes over $\mathbb{Q}_p$:

$$\begin{array}{ccc} X(N,p^2) & & \\ f\downarrow & & \\ X(N,p) & \xleftarrow{g} & X_1(pN) \end{array}$$

defined as follows. Let $(E, \psi_N, D)$ be a triple representing a point of $X(N,p^2)_{\mathbb{Q}_p}$, i.e. $E$ is an elliptic curve over a $\mathbb{Q}_p$-algebra $S$, $\psi_N$ is a $\Gamma_1(N)$-level structure on $E$ and $D \subset E[p^2]$ is a cyclic subgroup scheme of order $p^2$. Then $f\big((E,\psi_N,D)\big) = (E,\psi_N, D[p]) \in X(N,p)\big(\mathrm{Spec}(S)\big)$.

Let now $(E,\psi_N,\psi_p)$ be a $\mathrm{Spec}(S)$-point of $X_1(pN)$, where as above $S$ is a $\mathbb{Q}_p$-algebra, $E$ an elliptic curve over $S$, $\psi_N$ a level $\Gamma_1(N)$-structure on $E$ and $\psi_p$ a $\Gamma_1(p)$-level structure on $E$ i.e. a embedding of group schemes over $S$, $\psi_p : \mu_{p,S} \hookrightarrow E[p]$. Then $g(E,\psi_N,\psi_p) := \big(E,\psi_N, \mathrm{Im}(\psi_p)\big) \in X(N,p)\big(\mathrm{Spec}(S)\big)$.

We denote by $X(pN,p^2) := X(N,p^2) \times_{X(N,p)} X_1(pN)$ and by $p_1, p_2$ the canonical morphisms from $X(pN,p^2)$ to $X(N,p^2)$ and $X_1(pN)$ respectively. Let us remark that $X(pN,p^2)$ is the modular curve over $\mathbb{Q}_p$ classifying (away from the cusps) quadruples $(E,\psi_N,D,P)$ consisting of an elliptic curve $E$ over a $\mathbb{Q}_p$-algebra $S$, a level $\Gamma_1(N)$-structure $\psi_N$ of $E$, a cyclic subgroup scheme of order $p^2$, $D \subset E[p^2]$ and a section $P$ of exact order $p$ (or a trivialization) of $\big(D[p]\big)^D\big(\mathrm{Spec}(S)\big)$ obtained as follows: the level $\Gamma_1(p)$ structure $\psi_p$ identifies $\psi_p : \mu_{p,S} \cong D[p]$ and so by dualizing we obtain an isomorphism $(\psi_p^D)^{-1} : \big(\mathbb{Z}/p\mathbb{Z}\big)_S \cong \big(D[p]\big)^D$. With these notations $P$ is the image of the section 1, i.e. $P := (\psi_p^D)^{-1}(1)$.

Let us remark that under these identifications the morphisms $p_1, p_2$ are defined as follows:

$$p_1 \colon X(pN,p^2) \longrightarrow X(N,p^2) \text{ is defined by } p_1(E,\psi_N,D,P) = (E,\psi_N,D)$$

and

$$p_2 \colon X(pN,p^2) \longrightarrow X_1(Np) \text{ is defined by } p_2(E,\psi_N,D,P) = (E,\psi_N,\psi_p),$$

where $\psi_p \colon \mu_p \cong D[p] \subset E[p]$ is the inverse dual of $\mathbb{Z}/p\mathbb{Z} \cong \big(D[p]\big)^D$ defined by $1 \mapsto P$.

### 4.2.2 The modular curve $X(pN, p^2)$ over strict neighbourhoods of the ordinary locus

Let us denote by $\mathcal{X}(pN,p^2)$ the rigid analytic modular curve over $\mathbb{Q}_p$ associated to the modular curve $X(pN,p^2)$ described in the previous section. Fix an integer $\epsilon$ such that $\epsilon \geq p+2$ and recall that we have denoted by

$$\mathcal{X}_\epsilon^{(\infty)} := \{x \in \mathcal{X}(N,p)_1^{(\infty)} | \quad \mathrm{val}_p(\mathrm{Hdg}(x)) \leq \frac{1}{\epsilon}\}.$$

Then $\mathcal{X}_\epsilon^{(\infty)}$ is an affinoid, strict neighbourhood of the ordinary locus in $\mathcal{X}(N,p)$ containing the cusp $\infty$ such that the natural generalized elliptic curve $\mathbb{E} \longrightarrow \mathcal{X}_\epsilon^{(\infty)}$ has a canonical subgroup $\mathrm{H}_2$ of order $p^2$.

We have a morphism of rigid spaces (the rigid analytic version of $f$ from section 4.2.1) $f\colon \mathcal{X}(N,p^2) \longrightarrow \mathcal{X}(N,p)$. In particular, $f$ is a finite étale morphism and we denote by $\mathcal{X}(N,p^2)_\epsilon := f^{-1}\big(\mathcal{X}_\epsilon^{(\infty)}\big)$; notice that also $\mathcal{X}(N,p^2)_\epsilon$ is an affinoid, admissible open of $\mathcal{X}(N,p^2)$.

We also have a morphism of rigid spaces (the rigid analytic version of the morphism $g$ from section 4.2.1) $g\colon \mathcal{X}_1(pN) \longrightarrow \mathcal{X}(N,p)$; this is also a finite étale morphism and we remark that $g^{-1}\big(\mathcal{X}_\epsilon^{(\infty)}\big) = \mathcal{IG}_{1,\epsilon}$, which denoted in section §4.1 the first level Igusa tower over $\mathcal{X}_\epsilon^{(\infty)}$.

This whole discussion implies that if we denote by $\pi\colon \mathcal{X}(pN,p^2) \longrightarrow \mathcal{X}(N,p)$ the natural morphism (i.e. $\pi = f\circ p_1 = g\circ p_2$) then

$$\mathcal{X}(pN,p^2)_\epsilon := \pi^{-1}(\mathcal{X}_\epsilon^{(\infty)}) \cong \mathcal{X}(N,p^2)_\epsilon \times_{\mathcal{X}_\epsilon^{(\infty)}} \mathcal{IG}_{1,\epsilon}.$$

We observe that the morphism $f\colon \mathcal{X}(N,p^2)_\epsilon \longrightarrow \mathcal{X}_\epsilon^{(\infty)}$ has a natural section $\sigma\colon \mathcal{X}_\epsilon^{(\infty)} \longrightarrow \mathcal{X}(N,p^2)_\epsilon$ defined by $\sigma(E,\psi_N,\mathrm{H}_1) := (E,\psi_N,\mathrm{H}_2)$ where the notations are as before, given that a point $x = (E,\psi_N,\mathrm{H}_1) \in X_\epsilon^{(\infty)}$ has the property that $E$ has a canonical subgroup $\mathrm{H}_2$ of order $p^2$ and that $\mathrm{H}_2[p] = \mathrm{H}_1$.

We denote by $\mathcal{X}(N,p^2)_{\epsilon,0} := \sigma\big(\mathcal{X}_\epsilon^{(\infty)}\big)$ and $\mathcal{X}(N,p^2)_{\epsilon,\neq 0} := \mathcal{X}(N,p^2)_\epsilon - \mathcal{X}(N,p^2)_{\epsilon,0}$. Then $\mathcal{X}(N,p^2)_{\epsilon,0} \amalg \mathcal{X}(N,p^2)_{\epsilon,\neq 0}$ is an admissible open affinoid cover of $\mathcal{X}(N,p^2)_\epsilon$. Set $\mathcal{X}(pN,p^2)_{\epsilon,0} := \mathcal{X}(N,p^2)_{\epsilon,0} \times_{\mathcal{X}_\epsilon^{(\infty)}} \mathcal{IG}_{\epsilon,1}$ and $\mathcal{X}(pN,p^2)_{\epsilon,\neq 0} := \mathcal{X}(N,p^2)_{\epsilon,\neq 0} \times_{\mathcal{X}_\epsilon^{(\infty)}} \mathcal{IG}_{\epsilon,1} \subset \mathcal{X}(pN,p^2)_\epsilon$; in particular $\mathcal{X}(pN,p^2)_{\epsilon,0} \amalg \mathcal{X}(pN,p^2)_{\epsilon,\neq 0}$ is an admissible affinoid open cover of $\mathcal{X}(pN,p^2)_\epsilon$. Let us fix a finite extension $L$ of $\mathbb{Q}_p$ which contains a primitive $p$-th root $\zeta_p$ of 1 (which we fix) and let us base change all our rigid spaces ($\mathcal{X}(pN,p^2)_\epsilon, \mathcal{X}(pN,p^2)_{p+2,\neq 0}$ etc.) to $L$. Our notations will not mark this base change. For every integer $j$ with $0 \leq j \leq p-1$ we define a morphism of rigid spaces $\rho_j\colon \mathcal{IG}_{\epsilon,1} \longrightarrow \mathcal{X}(pN,p^2)_\epsilon$ as follows:

a) if $j = 0$, the morphism $\rho_0$ is the natural morphism $\mathcal{IG}_{1,\epsilon} \longrightarrow \mathcal{X}(pN,p^2)_\epsilon$ induced by $\sigma$. It defines an isomorphism $\mathcal{IG}_{1,\epsilon} \cong \mathcal{X}(pN,p^2)_{\epsilon,0}$;

b) Let now $1 \leq j \leq p-1$.

Let $(E,\psi_N,\mathrm{H}_1,s) \in \mathcal{IG}_{1,\epsilon}$ be a point, i.e. $E$ is an elliptic curve, $\psi_N$ is a level $\Gamma_1(N)$ structure on $E$, $\mathrm{H}_1$ is the canonical subgroup of $E$ and $s\colon \mathbb{Z}/p\mathbb{Z} \cong \mathrm{H}_1^D$ is a trivialization of $\mathrm{H}_1^D$. Let $\varphi\colon E \longrightarrow \tilde{E} = E/\mathrm{H}_1$ be the natural isogeny. Then $\tilde{E}[p]$ has two natural subgroup schemes of order $p$: $\tilde{\mathrm{H}}_1 := \mathrm{H}_2/\mathrm{H}_1$ and $\tilde{\mathrm{H}}' = E[p]/\mathrm{H}_1 = \mathrm{Ker}\big(\varphi^\vee : \tilde{E} \longrightarrow E\big)$. As $\mathrm{H}_1$ is isotropic for the Weil pairing on $E$ we have $\tilde{\mathrm{H}}' = E[p]/\mathrm{H}_1 \cong \mathrm{H}_1^D$. Therefore the trivialization $s$ induces an isomorphism $\mathbb{Z}/p\mathbb{Z} \cong \mathrm{H}_1^D \cong \tilde{\mathrm{H}}'$ and we let $s$ be the image of $1 \in \mathbb{Z}/p\mathbb{Z}$. We also remark that the isogeny $\varphi^\vee : \tilde{E} \longrightarrow E$ induces an isomorphism between the canonical subgroups: $\tilde{\mathrm{H}}_1 \cong \mathrm{H}_1$. Therefore we obtain an isomorphism $s^D\colon \tilde{\mathrm{H}}_1 \cong \mathrm{H}_1 \cong \mu_p$. We have $\tilde{\mathrm{H}}_1 \cap \tilde{\mathrm{H}}' = \{0\}$ and we consider the natural morphism of group schemes $\alpha\colon \mathrm{H}' \times \tilde{\mathrm{H}}_1 \longrightarrow \tilde{E}[p]$ induced on points by $\alpha(P,Q) = P + Q$. We have the following commutative diagram of group schems with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathrm{H}}' & \longrightarrow & \tilde{\mathrm{H}}' \times \tilde{\mathrm{H}}_1 & \longrightarrow & \tilde{\mathrm{H}}_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow \alpha & & \downarrow \cong & & \\
0 & \longrightarrow & \tilde{\mathrm{H}}' & \longrightarrow & \tilde{E}[p] & \longrightarrow & \tilde{E}[p]/\tilde{\mathrm{H}}' & \longrightarrow & 0
\end{array}$$

It follows that $\alpha$ is an isomorphism of group schemes. Therefore the isomorphisms $s$, $s^D$ and the choice of a primitive $p$-th root $\zeta_p \in L$ of 1, mentioned above, define isomorphisms (see proposition 3.29 of [AI]):

$$\tilde{E}[p] \cong \tilde{\mathrm{H}}' \times \tilde{\mathrm{H}}_1 \cong \mathbb{Z}/p\mathbb{Z} \times \mu_p \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$$

For $1 \leq j \leq p-1$ above we define $\mathrm{H}_j \subset \tilde{E}[p]$ by the following cartesian diagram

$$\begin{array}{ccc} \mathrm{H}_j & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \\ \cap & & \downarrow \\ \tilde{E}[p] & \cong & \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \end{array}$$

where the right vertical arrow is defined by sending $x \in \mathbb{Z}/p\mathbb{Z}$ to $(jx, x)$. Now define $D_j := \varphi^{-1}(\mathrm{H}_j)$, where let us recall $\varphi\colon E \longrightarrow \tilde{E}$ is the isogeny whose kernel is $\mathrm{H}_1$. Therefore $D_j \subset E[p^2]$ is a cyclic subgroup scheme of order $p^2$ such that $D_j[p] = \mathrm{H}_1$ and $D_j \neq \mathrm{H}_2$. Moreover every cyclic subgroup scheme of order $p^2$ of $E$ with these properies is one of the $D_j$'s. Therefore for every $1 \leq j \leq p-1$ the association: $(E, \psi_N, \mathrm{H}_1, s) \to (E, \psi_N, D_j, s)$ defines a morphism of rigid spaces $\rho_j\colon \mathcal{IG}_{1,\epsilon} \longrightarrow \mathcal{X}(pN, p^2)_\epsilon$ such that $f \circ \rho_j$ is the natural projection $\mathcal{IG}_{1,\epsilon} \longrightarrow \mathcal{X}_\epsilon^{(\infty)}$.

We denote by $\mathcal{Z}_\epsilon := \amalg_{j=0}^{p-1} \mathcal{IG}_{1,\epsilon}$. Then the family of maps $\rho_j$, for $j = 0, 1, \ldots, p-1$, define a morphism of rigid spaces $\gamma\colon \mathcal{Z}_\epsilon \longrightarrow \mathcal{X}(pN, p^2)_\epsilon$ such that $f \circ \gamma$ is the morphism $\mathcal{Z}_\epsilon \longrightarrow \mathcal{X}_\epsilon^{(\infty)}$ defined by the natural projection $\mathcal{IG}_{1,\epsilon} \to \mathcal{X}_\epsilon^{(\infty)}$ on each component.

**Lemma 4.12.** *The morphism $\gamma\colon \mathcal{Z}_\epsilon \longrightarrow \mathcal{X}(pN, p^2)_\epsilon$ is an isomorphism.*

*Moreover the $U$ correspondence on $\mathcal{X}(pN, p^2)$ induces a correspondence from $\mathcal{X}(pN, p^2)_\epsilon$ to $\mathcal{X}(pN, p^2)_{\epsilon,0}$ compatible with the $U$ correspondence on $\mathcal{X}_\epsilon^{(\infty)}$ via the map $f$ and such that the composite with $\gamma$ and the identification $\mathcal{X}(pN, p^2)_{\epsilon,0} \cong \mathcal{IG}_{1,\epsilon}$ via $\gamma^{-1}$ define the $U$ correspondence on each component $\mathcal{IG}_{1,\epsilon}$ of $\mathcal{Z}_\epsilon$.*

*Proof.* We remark that $f\colon \mathcal{X}(N, p^2)_\epsilon \longrightarrow \mathcal{X}_\epsilon^{(\infty)}$ is an étale morphism of degree $p$ so by base-change the morphism $\theta\colon \mathcal{X}(pN, p^2)_\epsilon \longrightarrow \mathcal{IG}_{1,\epsilon}$ is étale of degree $p$. Since the morphism $\gamma\colon \mathcal{Z}_\epsilon \longrightarrow \mathcal{X}(pN, p^2)_\epsilon$ is an isomorphism on points, it is étale of degree 1 and hence an isomorphism as claimed.

The $U$ correspondence on $\mathcal{X}(pN, p^2)_\epsilon$ is defined as follows. Given a point $(E, \psi_N, D, P)$ of $\mathcal{X}(pN, p^2)_\epsilon$ then $U\big((E, \psi_N, D, P)\big)$ is obtained by considering all the isogenies $\tau\colon E \to E'$ of degree $p$ whose kernel do not intersect the canonical subgroup $\mathrm{H}_1$ of $E$. In particular $\tau$ identifies $D$ with a cyclic subgroup $D'$ of $E'$ of order $p^2$, the section $P$ of $(D[p])^D$ induces a section $P'$ of $(D'[p])^D$ and the composite of $\psi_N$ with $\tau$ defines a $\Gamma_1(N)$-level structure $\psi'_N$ on $E'$. As $U(E, \psi_N, D, P)$ consists of the collection of the $p$ points $(E', \psi'_N, D', P')$ obtained in this way, it is compatible with the $U$ correspondence on $\mathcal{IG}_{1,\epsilon}$ via the morphism $\theta$. This implies that it is compatible with the $U$ correspondence on $\mathcal{X}_\epsilon^{(\infty)}$ via the map $f$. In order to prove the last claim of the Lemma it then suffices to show that $U\big(\mathcal{X}(pN, p^2)_\epsilon\big) \subset \mathcal{X}(pN, p^2)_{\epsilon,0}$, i.e., that given $(E, \psi_N, D, P)$ and $\tau\colon E \to E'$ as above, the subgroup $D'$ coincides with the canonical subgroup $\mathrm{H}'_2$ of $E'$. As $\mathrm{H}'_2 = \tau(\mathrm{H}_2)$ this follows remarking that any two cyclic subgroups of order $p^2$ of $E$, having the same $p$-torsion not intersecting the kernel of $\tau$, have the same image via $\tau$ so that $D' = \tau(D) = \tau(\mathrm{H}_2) = \mathrm{H}'_2$. □

### 4.2.3 $p$-Adic iterations of powers of $\nabla_k$ on $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$

We define the pair $\big(\mathbb{W}_k,\nabla_k\big)$ on $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$ simply by pull-back via $\pi$ of the pair $\big(\mathbb{W}_k,\nabla_k\big)$ on $\mathcal{X}^{(\infty)}_{p+2}$. As in Theorem 4.9 we let $s\in p^2R$ and define the weight $\nu\colon \mathbb{Z}_p^*\longrightarrow R^*$ to be $\nu(t):=\exp\big(s(\log(t))\big)$. Take a weight $u\colon \mathbb{Z}_p^*\to R^*$ obtained by adding to $\nu$ an integral weight and a finite character.

We wish to define $\nabla_k^u(G)\in \mathrm{H}^0\big(\mathcal{X}(pN,p^2)_{p(p+1),\neq 0},\mathbb{W}_{k+2u}\big)$ but let us observe that the formal (integral) model of $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$ we would usually use, i.e. the normalization of the integral, formal model $\mathfrak{X}^{(\infty)}_{p+2}$ of $\mathcal{X}^{(\infty)}_{p+2}$ in $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$ is very complicated and its differentials are difficult to understand. Therefore we will not use the methods of Theorem 4.9 over this formal model and instead we will use the geometry of $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$ provided via the isomorphism $\gamma\colon \mathcal{IG}_{\epsilon,\neq 0}:=\amalg_{j=1}^{p-1}\mathcal{IG}_{1,\epsilon}\longrightarrow \mathcal{X}(N,p^2)_{\epsilon,\neq 0}$ of Lemma 4.12.

**Theorem 4.13.** *a) Let $G\in \mathrm{H}^0(\mathcal{X}(pN,p^2)_{p+2,\neq 0},\mathfrak{w}^k)$ be such that $U(G)=0$, Then there exists a section $(\nabla_k)^u(G)$ of $\mathbb{W}_{k+2u}$ over $\mathcal{X}(pN,p^2)_{p(p+1),\neq 0}$ whose pullback via $\gamma$ to $\amalg_{j=1}^{p-1}\mathcal{IG}_{1,\epsilon}$ coincides with the operator $(\nabla_k)^u$ of Theorem 4.9 applied on each component $\mathcal{IG}_{1,\epsilon}$ to the pull-back of $G$.*

*b) If the section $G$ is the inverse image via $p_1$ of a section $G'\in \mathrm{H}^0(\mathcal{X}(N,p^2)_{p+2,\neq 0},\mathfrak{w}^k)$, then under the same assumptions as in a) above we have that $(\nabla_k)^u(G)$ descends to a section which we'll denote $(\nabla_k)^u(G')$ of $\mathbb{W}_{k+2u}$ over $\mathcal{X}(N,p^2)_{p(p+1),\neq 0}$.*

*c) If the weight $u$ is a classical integral weight then $(\nabla_k)^u(G)$ coincides with the $u$-th iterate of the Gauss-Manin connection applied to $G$.*

*Proof.* Write $\gamma^*(G)=(G_j)_{j=1}^{p-1}$ with $G_j\in \mathrm{H}^0\big(\mathcal{IG}_{p+2,1},\mathfrak{w}^k\big)$ for all $1\leq j\leq p-1$. As $U(G)=0$ then $U(G_j)=0$ for every $1\leq j\leq p-1$ tahnks to Lemma 4.12; therefore as we know how to iterate $p$-adically powers on $\nabla_k$ on $\mathcal{IG}_{1,p+2}$ by Theorem 4.9 and the definition $(\nabla_k)^u(G):=(\gamma^{-1})^*\big((\nabla_k)^u(G_j)\big)_{j=1}^{p-1}\in \mathrm{H}^0\big(\mathcal{X}(pN,p^2)_{p(p+1),\neq 0},\mathbb{W}_{k+2u}\big)$ makes sense. Claim c) now follows from Corollary 4.11.

To prove b) we remark that the isomorphism $\gamma:\amalg_{j=1}^{p-1}\mathcal{IG}_{1,p+2}\longrightarrow \mathcal{X}(N,p^2)_{p+2,\neq 0}$ is compatible with the action of $(\mathbb{Z}/p\mathbb{Z})^*$ defined on the LHS as follows. If $x_j\in \big(\mathcal{IG}_{1,p+2}\big)_j$, for $1\leq j\leq p-1$ and $a\in(\mathbb{Z}/p\mathbb{Z})^*$, then $a*x_j:=(a\cdot x_j)_{(a^2j)\mathrm{mod}(p-1)}\in \big(\mathcal{IG}_{1,p+2}\big)_{(a^2j)\mathrm{mod}(p-1)}$, where $ax_j$ denotes the natural action of $a$ on $\mathcal{IG}_{1,p+2}$ as a Galois automorphism. Assume $G=p_1^*(G')$ with $G'\in \mathrm{H}^0\big(\mathcal{X}(N,p^2)_{p+2,\neq 0},\mathfrak{w}^k\big)$ and write as above $G=(G_j)_{j=1}^{p-1}$. Then as $G=p_1^*(G')$ is the inverse image of a section of a sheaf on $\mathcal{X}(N,p^2)_{p+2,\neq 0}$ the group $(\mathbb{Z}/p\mathbb{Z})^*$ acts trivially on it, i.e. if $a\in(\mathbb{Z}/p\mathbb{Z})^*$ we have:

$$a*\big(p_1^*(G')\big)=a*(G_j)_{j=1}^{p-1}=\Big((aG_j)_{a^2j(\mathrm{mod}\ p)}\Big)_{a^2j(\mathrm{mod}\ p),j=1,p-1}=p_1^*(G')$$

so that $aG_j=G_{a^2j(\mathrm{mod}\ p)}$ for every $1\leq j\leq p-1$. As $U(G_j)=0$ for every $1\leq j\leq p-1$ and as on $\mathcal{IG}_{1,p+2}$ we know how to iterate powers of $\nabla_k$ (see Theorem 4.9) we have: $\nabla_k^u(G_j)\in \mathrm{H}^0\big(\mathcal{IG}_{1,p(p+1)},\mathbb{W}_{k+2u}\big)$, for every $1\leq j\leq p-1$ so we define $\nabla_k(p_1^*(G')):=\big(\nabla_k(G_j)\big)_{j=1}^{p-1}$. Moreover as $\nabla_k$ is the inverse image of a connection also denoted $\nabla_k$ on $\mathcal{X}(N,p^2)_{p+2,\neq 0}$, for every $a\in(\mathbb{Z}/p\mathbb{Z})^*$ we have:

$$a*\Big(\nabla_k^u\big(p_1^*(G')\big)\Big)=a*\big(\nabla_k^u((G_j)\big)_{j=1}^{p-1}=\Big(\nabla_k^u\big((aG_j)_{a^2j(\mathrm{mod}\ p)}\Big)_{a^2j(\mathrm{mod}\ p)}=$$

$$= \Big(\nabla_k^u\big(G_{a^2j(\mathrm{mod}\ p)}\big)_{a^2j(\mathrm{mod}\ p)}\Big) = \big(\nabla_k^u(G_i)\big)_{i=1}^{p-1} = \nabla_k^u\big(p_1^*(G')\big).$$

Therefore $\nabla_k^u\big(p_1^*(G')\big)$ descends to a section denoted

$$\nabla_k^u(G') \in \mathrm{H}^0\big(\mathcal{X}(N,p^2)_{p(p+1),\neq 0}, \mathbb{W}_{k+2u}\big).$$

□

**Remark 4.14.** If in the hypothesis of Theorem 4.13 we assume $G \in \mathrm{H}^0\big(\mathfrak{X}(pN,p^2)_{p+2,\neq 0}, \mathfrak{w}^k\big)$ such that $U(G) = 0$, then we obtain

$$\big(\nabla_k\big)^u(G) \in p^{-a}\cdot \mathrm{H}^0\big(\mathfrak{X}(pN,p^2)_{p(p+1),\neq 0}, \mathbb{W}_{k+2u}\big) \subset \mathrm{H}^0\big(\mathcal{X}(pN,p^2)_{p(p+1),\neq 0}, \mathbb{W}_{k+2u}\big),$$

where $a \in \mathbb{N}$ is a constant independent of $u$.

## 4.3 The $p$-depletion operator on classical and overconvergent modular forms.

We have seen that for weights $k$ and $u$ as in Theorem 4.9, if $F$ is an overconvergent form of weight $k$ and level $\Gamma_1(N)$ defined over the rigid analytic fiber $\mathcal{X}_{p+1}$ of $\mathfrak{X}_{p+1}$ and such that $U(F) = 0$ (i.e. $F$ has infinite slope) then $(\nabla_k)^u(F)$ makes sense as a section of $\mathbb{W}_{k+2u}[p^{-1}]$ over $\mathfrak{X}^{(\infty)}_{p(p+1)}$, the strict neighborhood of $\mathfrak{X}^{(\infty)}_{p(p+1)}$ the ordinary locus in $\mathfrak{X}^{(\infty)}$ defined by $\mathfrak{X}^{(\infty)}\langle\frac{p}{\mathrm{Hdg}^{p(p+1)}}\rangle$. We will now recall how to obtain a large supply of such overconvergent modular forms of infinite slope.

If $F$ is an overconvergent modular form one has operators $U$, $V$ that can be defined geometrically on the overconvergent modular forms; see [AI, §3.6& §3.7]. On $q$-expansions if $F(q) = \sum_{n=0}^{\infty} a_n q^n$ then

$$U\big(f(q)\big) := \sum_{n=0}^{\infty} a_{np} q^n \text{ and } V\big(F(q)\big) := \sum_{n=0}^{\infty} a_n q^{pn} = F(q^p).$$

It is then easy to see that $U\circ V = \mathrm{Id}$ and we define $f^{[p]} := f - (V\circ U)(f) = \big((U\circ V) - (V\circ U)\big)(f)$. It is immediate that with this definition we have:

$$F^{[p]}(q) = \sum_{n=1,(n,p)=1}^{\infty} a_n q^n \text{ and } U(F^{[p]})(q) = U\big(F^{p]}(q)\big) = 0.$$

Recall from Katz-Lubin theory of the canonical subgroup that the universal generalized elliptic curve over $\mathcal{X}^{(\infty)}_{p+2}$ admits a canonical subgroup of order $p^2$ so that $\mathcal{X}^{(\infty)}_{p+2}$ can be identified with an open rigid analytic space of $\mathcal{X}(N,p^2)$ the rigid space associated to the modular curve $X(N,p^2)$ over $\mathbb{Q}_p$ of level $\Gamma_1(N)\cap\Gamma_0(p^2)$.

**Lemma 4.15.** *If $F$ is a classical eigenform form of weight $k \in \mathbb{Z}_{>0}$ of level $\Gamma_1(N)$, character $\epsilon$, then $F^{[p]}$ is a classical modular form of level $\Gamma_1(N)\cap\Gamma_0(p^2)$. In particular, it defines a section of $\mathrm{H}^0\big(\mathcal{X}^{(\infty)}_{p+2}, \mathfrak{w}^k[p^{-1}]\big)^{U=0}$ by restriction to $\mathcal{X}^{(\infty)}_{p+2}$.*

*Proof.* First of all we recall the following simple calculation (see section §5.2 [BDP]). Suppose that $T_p(F) = a_p(F)$. Then $a_pF = T_p(F) = U(F) + \epsilon(p)p^{k-1}V(F)$ and therfore

$$F^{[p]} = (1 - VU)(F) = \big(1 - a_pV + \epsilon(p)p^{k-1}V^2\big)(F).$$

One can define the operators $V^2$, respectively $V$ restricted to classical modular forms as the pull back via the morphisms $X(N, p^2) \to X_1(N)$ and respectively $X(N, p) \to X_1(N)$, over $\mathbb{Q}_p$, defined by $(E/S, \psi_N, C) \mapsto (E/C, \psi_N)$; here $(E/S, \psi_N, C)$ is an elliptic curve over $S$ with $\Gamma_1(N)$-level strcuture $\psi_N$ and a cyclic subgroup $C$ of order $p^2$ (respectively $p$). Indeed if one restricts these operators to a strict neighborhood of the ordinary locus where the elliptic curves admit canonical subgroups of order $p^2$ (respectively $p$), one obtains the operators $V^2$ and respectively $V$ defined in [AI, §3.7].

□

We conclude this section by studying the $p$-depletion operator on $\mathcal{X}(pN, p^2)_\epsilon$ (in the notation of §4.2.2). Let $\epsilon \geq p + 2$ be an integer, let $f\colon \mathcal{X}(pN, p^2)_\epsilon \to \mathcal{X}_\epsilon^{(\infty)}$ be the morphism $(E, \psi_N, D, s) \mapsto (E, \Psi_N, D[p])$ and consider the decomposition $\gamma\colon \mathcal{Z}_\epsilon = \amalg_{j=0}^{p-1}\mathcal{IG}_{1,\epsilon} \xrightarrow{\sim} \mathcal{X}(pN, p^2)_\epsilon$ of Lemma 4.12.

**Proposition 4.16.** *Let $F$ be a modular form of weight $k$ on $X_1(N)$. We denote $F^{[p]}$ the $p$-depleted modular form on $X(pN, p^2)$ defined in Lemma 4.15. Then, the restriction of $F^{[p]}$ to $\mathcal{X}(pN, p^2)_\epsilon$ (resp. its pull-back via $\gamma$) is the pull-back of the modular form $F^{[p]} = (1 - VU)(F)$ over $\mathcal{X}_\epsilon^{(\infty)}$ via $f$ (resp. via the forgetful map $\mathcal{IG}_{1,\epsilon} \to \mathcal{X}_\epsilon^{(\infty)}$).*

*Proof.* First of all we notice that $F^{[p]} = \big(1 - a_pV + \epsilon(p)p^{k-1}V^2\big)(F)$ coincides with $(1 - VU)(F)$, where $U$ is the $U$-operator on $X(N, p)$ and $V$ is induced by the morphism $v\colon X(pN, p^2) \to X(N, p)$ sending $(E, \psi_N, D, P) \mapsto (E', \Psi'_N, D')$ with $E' = E/D[p]$, $\psi'_N$ the level strcuture obtained from $\psi_N$ and $D' := D/D[p]$; infact both $F^{[p]}$ and $(1 - VU)(F)$ are modular forms on $X(pN, p^2)$ and they coincide on $\mathcal{X}(pN, p^2)_{\epsilon,0}$ ($\cong \mathcal{IG}_{1,\epsilon}$ via $\gamma$).

We have already proven in Lemma 4.12 that the $U$ correspondence on $\mathcal{X}(pN, p^2)_\epsilon$ is compatible with the $U$ correspondence on $\mathcal{X}_\epsilon^{(\infty)}$ via $f$. The morphism $\mathcal{X}(pN, p^2)_\epsilon \to \mathcal{X}(N, p)$ induced by $V$ is compatible with the excellent lift of Frobenius $\mathcal{X}_\epsilon \to \mathcal{X}_{\epsilon/p}$ defined by taking the quotient by the canonical subgroup. Via the isomorphism $\gamma$ it is compatible with the operator $\mathcal{IG}_{1,\epsilon} \to \mathcal{X}_{\epsilon/p}$ (on each component), sending $(E, \psi_N, P) \mapsto (E', \psi'_N)$ with $E' = E/\mathrm{H}_1$ the quotient by the canonical subgroup and $\psi'_N$ obtained from $\psi_N$. Hence $f^*\big(F^{[p]}\big) = f^*\big((1 - VU)(F)\big) = (1 - VU)(F)$ (and similalry on $\amalg_{j=0}^{p-1}\mathcal{IG}_{1,\epsilon}$) as claimed.

□

## 4.4 The $p$-adic $L$-function: the first approximation.

Let us recall the notations of section §2.2: $K$ is a quadratic imaginary field, $\Sigma_{cc}^{(2)}(\mathfrak{N})$ the set of algebraic Hecke characters $\chi$ of given conductor $c$ such that the infinity type of $\chi$ is $(k + j, -j)$ with fixed integer $k \geq 2$ and integer $j \geq 0$. We denoted by $\widehat{\Sigma}^{(2)}$ the $p$-adic completion of $\Sigma_{cc}^{(2)}(\mathfrak{N})$ and by $w\colon \widehat{\Sigma}^{(2)} \to W(\mathbb{Q}_p)$ the "weight map" defined in section 2.2. We summarize here that the weight map $w$ is determined by the following two properties: it is continuous and if $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N}) \subset \widehat{\Sigma}^{(2)}$ has infinity type $(k + j, -j)$ then $w(\chi) = j$.

Let $p \geq 5$ a prime integer and let us recall that we denoted by $F$ either the Eisenstein series $E_{k,\chi}$ or a normalized classical cuspform for $\Gamma_1(N)$, of weight $k \geq 2$ and nebentypus $\epsilon$. We assume that $N \geq 5$ is an integer prime to $p$ and denote by $\mathcal{O}_c$ the order of conductor $c$ of $\mathcal{O}_K$. We recall that $c = 1$ if $F$ is an Eisenstein series and $c$ and $d_K$ are odd integers such that $(c, Npd_K) = 1$ if $F$ is a cuspform.

We consider the function:

$$\Pi_F \colon \Sigma_{cc}(\mathfrak{N})^{(2)} \longrightarrow \Sigma_{cc}(\mathfrak{N})^{(2)} \otimes_{\mathbb{Z}_p} \mathbb{W}'_k \subset \widehat{\Sigma}^{(2)} \otimes_{\mathbb{Z}_p} \mathbb{W}'_k,$$

where $\mathbb{W}'_k = \sum_{i \in \mathbb{Z}} \mathrm{H}^0(\mathcal{X}(pN, p^2)_{p+2,\neq 0}, \mathbb{W}_{k+2i}) \subset \mathrm{H}^0(\mathcal{X}(pN, p^2)_{p+2,\neq 0}, \mathbb{W})$, defined by:

$$\Pi_F(\chi) = \chi_j^{-1} \otimes (\nabla_k)^j(F^{[p]}), \text{ for all } \chi \in \Sigma_{cc}^{(2)}(\mathfrak{N}) \text{ with infinite type } (k+j, -j), j \in \mathbb{N}.$$

We recall that $F^{[p]}$ is a classical modular form of weight $k$ on $\mathcal{X}(N, p^2)$ by the results of section §4.3 and here we denoted also by $F^{[p]}$ the inverse image of its restriction to $\mathcal{X}(N, p^2)_{p+2,\neq 0}$ by the morphism $p_1 : \mathcal{X}(pN, p^2)_{p+2,\neq 0} \longrightarrow \mathcal{X}(N, p^2)_{p+2,\neq 0}$. We recall that $\mathcal{X}(pN, p^2)_{p+2,\neq 0}$ is an admissible affinoid open of $\mathcal{X}(pN, p^2)$ defined in section §4.2.

**Lemma 4.17.** *The function $\Pi_F$ is continuous for its natural topology on its domanin and the $p$-adic topology on its target and so it extends to a continuous function $\hat{\Pi}_F \colon \widehat{\Sigma}^{(2)} \longrightarrow \widehat{\Sigma}^{(2)} \otimes_{\mathbb{Z}_p} \mathbb{W}'_k$.*

*Proof.* We directly define $\hat{\Pi}_F$ by the formula:

$$\hat{\Pi}_F(\chi) := \chi_{w(\chi)}^{-1} \otimes (\nabla_k)^{w(\chi)}(F^{[p]}) \in \widehat{\Sigma}^{(2)} \otimes_{Z_p} \mathrm{H}^0\big(\mathcal{X}(pN, p^2)_{p(p+1),\neq 0}, \mathbb{W}_{k+2w(\chi)}\big)$$

for all $\chi \in \widehat{\Sigma}^{(2)}$. We recall that $(\nabla_k)^{w(\chi)}(F^{[p]})$ was defined in section §4.2. It is clear that $\hat{\Pi}_F|_{\Sigma_{cc}^{(2)}(\mathfrak{N})} = \Pi_F$ and that $\hat{\Pi}_F$ is continuous.

□

**Remark 4.18.** Let us observe that denoting also $F$ the restriction of $F$ to $\mathfrak{X}_{p+2}^{(\infty)}$, we have that $F^{[p]} \in \mathrm{H}^0\big(\mathfrak{X}_{p+2}^{(\infty)}, \mathfrak{w}^k\big)^{U=0}$ and therefore we can see both $\Pi_F(\chi)$ and $\hat{\Pi}_F(\chi)$ as elements of

$$\widehat{\Sigma}^{(2)} \times p^{-a} \cdot \mathrm{H}^0\big(\mathfrak{X}(pN, p^2)_{p(p+1),\neq 0}, \mathbb{W}_{k+2w(\chi)}\big),$$

where $a$ is the constant of Remark 4.14.

Let us recall that we want to interpolate $p$-adically

$$L_{\mathrm{alg}}(F, \chi^{-1}) := \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \theta_{\mathrm{Hodge}}^j(F)\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big),$$

where $(A_0, t_0, \omega_0)$ is a triple consisting of: an elliptic curve $A_0$ with CM by $\mathcal{O}_c$, a $\Gamma_1(\mathfrak{N})$-level structure $t_0$ and a basis $\omega_0$ for the module of invariant differentials of $A_0$. The evaluation of $\theta_{\mathrm{Hodge}}^j(F)$ at the modular point $x_{\mathfrak{a}} := \big(\mathfrak{a} * (A_0, t_0, \omega_0)\big)$ is done by splitting the Hodge filtration using the CM action (see section 3, a)). In order to define the sought for $p$-adic $L$-function we will use $\hat{\Pi}_F$ but we need to address the following three probelms:

(I) *Evaluation at* CM *points.* Recall from theorem 4.9 that, in general, we can evaluate $(\nabla_k)^{w(\chi)}(F^{[p]})$ only at $\overline{\mathbb{Q}}_p$-valued points $x$ of $\mathcal{X}(pN,p^2)$ such that $\mathrm{val}_p\big(\mathrm{Hdg}^{p(p+1)}(\pi(x))\big) \leq 1$, where we recall $\pi : \mathcal{X}(pN,p^2) \longrightarrow \mathcal{X}(N,p)$ is defined on points by: $\pi(E,\psi_N,D,P) = (E,\psi_N,D[p])$. We recall the notations: $E$ is an elliptic curve, $\psi_N$ is a level $\Gamma_1(N)$-structure on $E$ and $D$ is a cyclic subgroup scheme of $E[p^2]$ of order $p^2$, $P$ is a trivialization of $\big(D[p]\big)^\vee$. Typically the points $\mathfrak{a} * A_0$ lie outside this range and we need to analytically continue $\hat{\Pi}_F(\chi)$ in order to be able to evaluate it at these points.

(II) *Splitting of the Hodge filtration.* We need to show that the evaluation of point (I) belongs to a subspace of sections of $\mathbb{W}_{k+2w(\chi)}$ where we can split the inclusion $\mathfrak{w}^{k+2w(\chi)}) \subset \mathbb{W}_{k+2w(\chi)}$. This is done using the CM action. We will call $\delta_k^{w(\chi)}(F^{[p]})$ the composite of $(\nabla_k)^{w(\chi)}(F^{[p]})$ with this projection.

(III) *Interpolation property.* For algebraic Hecke characters $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ we need to compare the evaluation of our $p$-adic $L$-function at $\chi$ with $L_{\mathrm{alg}}(F,\chi^{-1})$ as we wish our $p$-adic $L$-function to interpolate classical $L$-values. This will be achieved by computing certain correction factors which account on the one hand for the fact that in the definition of the $p$-adic $L$-function we use $F^{[p]}$ and in the classical values it is $F$ itself which appears. On the other hand the $p$-adic $L$-function is computed by evaluationg sections of sheaves on $\mathcal{X}(pN,p^2)$ while to compute $L_{\mathrm{alg}}(F,\chi^{-1})$ we evaluate sections of sheaves on $\mathcal{X}(N,p)$ and so the ramification of $\mathcal{X}(pN,p^2)$ over $\mathcal{X}(N,p)$ shows up in the correction factors.

The cases in which $p$ is inert and respectively ramified in $K$ are treated in the next sections.

# 5 The case: $p$ is inert in $K$.

We work under the assumptions of §2.3. In particular we have fixed a classical eigenform $F$ for $\Gamma_1(N)$ of weight $k \geq 2$ and nebentypus $\epsilon$. We suppose that $p$ is inert in $K$. With these assumptions we will address the three problems (I), (II), and (III) stated at the end of the previous section.

## 5.1 Evaluation at CM-points.

Fix an elliptic curve $E$ over the ring of integers $R$ of a finite extension of $\mathbb{Q}_p$. Assume it has CM by $\mathcal{O}_c$, a $\Gamma_1(\mathfrak{N})$-level structure $\psi_N$ and a subgroup $D \subset E[p^2]$ of order $p^2$ which is generically cyclic. As explained in section §3.1 $E$ cannot have a canonical subgroup.

We denote $C := D[p]$ and consider the elliptic curve $E' = E/C$ and denote by $C' := D/C$ seen as a subgroup scheme of $E'$. Let $\lambda\colon E' \to E$ be the isogeny dual to the projection $E \to E'$. As explained in Lemma 3.1 its kernel is the canonical subgroup $\mathrm{H}'$ of $E'$. Choose and fix a subgroup $D' \subset E'[p^2]$ of order $p^2$ such that $D'[p] = C'$ and $D'$ is cyclic over the fraction field of $R$. We denote the other subgroup schemes of order $p$ of $E'[p]$ (i.e. the subgroup schemes distinct from $C'$ and from $\mathrm{H}'$) by $C_1$, $C_2$, …, $C_{p-1}$. For $i = 1, 2, \ldots, p-1$ denote $E_i := E'/C_i$ and $D_i$ the subgroup schemes of order $p^2$ defined by the image of $D'$. We remark that the subgroup scheme defined by the image of $D'$ via $\lambda$ is $D$, the subgroup scheme we have started with. We recall that $D_i$ are cyclic group schemes of order $p^2$ such that $D_i[p] = \mathrm{H}_i$ the canonical subgroup of $E_i$, for all $1 \leq i \leq p-1$. Finally let us choose a trivialisation $P'$ of $\big(D'[p]\big)^\vee$ on $E'$ over the

ring of integers $R'$ of some finite extesion of the fraction field of $R$, call it $L$. We notice that the choice of $P'$ induces natural, compatible trivializations $P$ of $\big(D[p]\big)^\vee$ and for every $1 \leq i \leq p-1$, $P_i$ of $\big(D_i[p]\big)^\vee$.

These elliptic curves, level sugbroups and the chosen trivializations determine $L$-valued points of $\mathcal{X}(pN, p^2)$ as follows: $x'_L := [(E'/L, (\psi_{N,L}, D'_L, P'_L)]$, for every $i = 1, 2, \ldots, p-1$, we have $x_{i,L} := [(E_i/L, (\psi_{N,L}, D_{i,L}, P_{i,L})] \in \mathcal{X}(pN, p^2)_{p+2,\neq 0}$ and $x_L = [(E/L, \psi_{N,L}, D_L, P_L)]$. Here $\psi_N$ is the $\Gamma_1(N)$-level structure induced on $E'$ via $\lambda$ and then induced on the elliptic curves $E_i$ via the isogenies $\lambda_i$. All the points $x_{i,L}$, $1 \leq i \leq p-1$ naturally extend to sections of the formal model $\mathfrak{X}(pN, p^2)_{p+2,\neq 0} := \amalg_{i=1}^{p-1} \mathfrak{IG}_{1,p+2}$ over $\mathrm{Spf}(R')$, which will be denoted $x_i$.

If we let $U = U_p$ be the $U_p$-correspondence on $\mathcal{X}(pN, p^2)$ then $U(x'_L) = \{x_L, x_{1,L}, \ldots, x_{p-1,L}\}$. As mentioned above for every $1 \leq i \leq p-1$ we have sections $x_i := (E_i/R', \psi_N, D_i, P_i)$ of $\mathfrak{X}(pN, p^2)_{p+2,\neq 0}$ over $R'$, seen as morphisms of formal schemes $\varphi_{x_i} : \mathrm{Spf}(R') \longrightarrow \mathfrak{X}(pN, p^2)_{p+2,\neq 0}$. By Remark 4.18 we also have $\big(\nabla_k\big)^\nu(F^{[p]}) \in p^{-a}\mathrm{H}^0\big(\mathfrak{X}(pN, p^2)_{p(p+1),\neq 0}, \mathbb{W}_{k+2\nu}\big)$. In particular, we can define for $i = 1, \ldots, p-1$:

$$\varphi_{x_i}^*\big((\nabla_k)^\nu(F^{[p]})\big) \in \varphi_{x_i}^*\big(p^{-a}\mathbb{W}_{k+2\nu}\big) =: p^{-a}\mathbb{W}_{E_i,k+2\nu}, \text{ for all } 1 \leq i \leq p-1.$$

## 5.2 Splitting the Hodge filtration.

We keep the notations of the previous section: we started with an ellitpic curve $E/R'$ with CM by $\mathcal{O}_c$, we have a cyclic, degree $p$ isogeny $\lambda\colon E' \to E$ and $p-1$ quotients $\lambda_i\colon E' \to E_i$ for $i = 1, \ldots, p-1$ of degree $p$ and distinct form $E$.

By functoriality of VBMS the isogenies $\lambda_i$ induce morphisms $\lambda_i^*\colon \mathrm{H}^\sharp_{E_i} \to \mathrm{H}^\sharp_{E'}$. On the other hand $E'$ has CM by $\mathcal{O}_{pc}$. Possibly enlarging $R'$ even more to the ring of integers of a finite extension of $L$, wich we'll still denote $L$ and its ring of integers will still be denoted $R'$, we have two embeddings $\tau$, $\overline{\tau}\colon \mathcal{O}_{pc} \to R'$. We define $H_{E',\tau} \oplus H_{E',\overline{\tau}} \subset \mathrm{H}_{E'}$ as in §3.3 and

$$\widetilde{\mathrm{H}}^\sharp_{E'} := H^\sharp_{E',\tau} \oplus H^\sharp_{E',\overline{\tau}} := \delta_{E'}\mathrm{H}_{E',\tau} \oplus \delta^p_{E'}\mathrm{H}_{E',\overline{\tau}} \subset H^\sharp_{E'}.$$

**Lemma 5.1.** *The image of* $\mathrm{H}^\sharp_{E_i}$ *via* $\lambda_i^*$ *is contained in* $\widetilde{\mathrm{H}}^\sharp_{E'}$.

*Proof.* Notice that $\lambda_i^*$ induces an isomorphism $\Omega_{E_i} \cong \Omega_{E'} = H^\sharp_{E',\tau}$. thus it suffices to prove that the map induced on the quotients $\mathrm{H}^\sharp_{E_i}/\Omega_{E_i} \to \mathrm{H}^\sharp_{E_i}/\Omega_{E'}$ factors through $H^\sharp_{E',\overline{\tau}}$.

Recall that $\mathrm{H}^\sharp_{E_i}/\Omega_{E_i} = \delta^p_{E_i}\omega^\vee_{E_i}$ and $\mathrm{H}^\sharp_{E_i}/\Omega_{E'} = \delta^p_{E'}\omega^\vee_{E'}$ and by construction the considered map between them is induced by the map $\big((\lambda_i^\vee)^*\big)^\vee\colon \omega^\vee_{E_i} \to \omega^\vee_{E'}$, the $R$-dual to the map of differentials $\omega_{E'} \to \omega_{E_i}$ defined by pull-back via the dual isogeny $\lambda_i^\vee$. Recall from Lemma 3.3 that

(i) $\mathrm{Hdg}(E')\omega^\vee_{E'} \subset \mathrm{H}_{E',\overline{\tau}} \subset \omega^\vee_{E'}$;

(ii) $\lambda_i^*\big(\omega^\vee_{E_i}\big) \subset p\mathrm{Hdg}(E_i)^{-1}\omega^\vee_{E'}$.

Thus it suffices to show that $\delta^p_{E_i}p\mathrm{Hdg}(E_i)^{-1}\omega^\vee_{E'} \subset \mathrm{Hdg}(E')\delta^p_{E'}\omega^\vee_{E'}$. This amounts to prove that $\mathrm{val}_p(\delta_{E_i}p) \geq \mathrm{val}_p(\delta^{2p-1}_{E'})$ (we use that $\mathrm{val}_p(\mathrm{Hdg}(E_i)) = (p-1)\mathrm{val}_p(\delta_{E_i})$ and likewise for $E'$). Since $\mathrm{val}_p(\delta_{E'}) = p\mathrm{val}_p(\delta_{E_i})$, this amounts to show that $\big((2p-1)p-1\big)\mathrm{val}_p(\delta_{E_i}) \leq 1$. But $\mathrm{val}_p(\delta_{E_i}) = \frac{1}{p(p+1)(p-1)}$ by Lemma 3.1. Hence we need to show that $2p^2 - p - 1 \leq p^3 - p$ or equivalently $2p^2 - 1 \leq p^3$ which is true for any prime $p$.

□

By the formalism of VBMS we get that $\mathbb{W}_{E',k+2\nu}$ contains a submodule $\mathbb{W}_{k+2\nu,R'}\big(\widetilde{\mathrm{H}}^{\sharp}_{E'},s\big)$ whose filtration is canonically split. In particular it has a canonical $\mathcal{O}_{pc}$-equivariant projection

$$\Psi_{E'}\colon \mathbb{W}_{k+2\nu}\big(\widetilde{\mathrm{H}}^{\sharp}_{E'},s\big)\longrightarrow \mathfrak{w}_{E'}^{k+2\nu}.$$

Thanks to Lemma 5.1 the image of $\lambda_i^*\colon \mathbb{W}_{E_i,k+2\nu}\to\mathbb{W}_{E',k+2\nu}$ factors through $\mathbb{W}_{k+2\nu}\big(\widetilde{\mathrm{H}}^{\sharp}_{E'},s\big)$ so that we get a natural projection

$$\Psi_i\big(\lambda_i^*\circ\varphi_{x_i}^*\big((\nabla_k)^{w(\chi)}(F^{[p]})\big)\big)\in p^{-a}\mathfrak{w}_{E'}^{k+2\nu}\subset\mathfrak{w}_{E'}^{k+2\nu}\big[p^{-1}\big].$$

We recall that the constant $a$ was defined in Remark 4.14. Let $\omega'$ be a generator of $\Omega_{E'/R'}$ reducing to $s':=\mathrm{dlog}(P')\big(\mathrm{mod}\ p\underline{\delta}(E')^{-p}\big)$. In particular, $(\omega')^{k+2\nu}$ is a generator of $\mathfrak{w}_{E',R'}^{k+2\nu}$ that provides an identification of $v_{\omega'}\colon\mathfrak{w}_{E',R'}^{k+2\nu}\cong R'$. Moreover $\omega:=(\lambda^\vee)^*(\omega')$ is a generator of $\omega_E\otimes_R L$, where we recall that $L$ denotes the fraction field of $R'$.

**Remark 5.2.** At this point we see the reason why we need to work on $\mathcal{X}(pN,p^2)$ instead of $\mathcal{X}(N,p^2)$: the need to choose a natural, functorial generator of $\Omega_{E'/R'}$ (and not of $\omega_{E'/R'}$) forces us to choose a trivialization of $\big(D'[p]\big)^\vee$. We recall that $D'[p]$ is the canonical subgroup of $E'$.

**Definition 5.3.** We define

$$\delta_k^\nu(F^{[p]})(E/R',\psi_N,D,P,\omega):=-\sum_{i=1}^{p-1}v_{\omega'}\circ\Psi_i\Big(\lambda_i^*\circ\varphi_{x_i}^*\big((\nabla_k)^\nu(F^{[p]})\big)\Big)\in p^{-a}R'\subset L.$$

**Remark 5.4.** Let $\Omega^{\mathrm{can}}_{E'/R'}\subset\Omega_{E'/R'}$ be the $\mathbb{Z}_p^*(1+p\underline{\delta}(E')^{-p}R')$-torsor of sections arising via $(\lambda^\vee)^*$ from sections of $\Omega_{E'/R'}$, reducing to $s'\mathbb{F}_p^*$ modulo $p\underline{\delta}(E')^{-p}$.

Let $D'$ and $D''\subset E[p^2]$ be two cyclic subgroups of oder $p^2$. Let $(\lambda')^\vee\colon E\to E'=E/D'[p]$ and $(\lambda'')^\vee\colon E\to E''=E/D''[p]$ be the two corresponding cyclic isogenies of degree $p$, as above. Since $\big(\mathcal{O}_c\otimes\mathbb{Z}_p\big)^*$ acts transitively on the set of subgroups of $E$ of order $p^2$, there exists $a\in\widehat{\mathcal{O}}_c^*$ such that $[a](D')=D''$ so that $\alpha'=\alpha''\circ[a]$ and hence

$$\Omega^{\mathrm{can}}_{E'/R'}=[a]^*\big(\Omega^{\mathrm{can}}_{E''/R'}\big)=a^{-1}\cdot\Omega^{\mathrm{can}}_{E''/R'}.$$

## 5.3 Definition of the $p$-adic $L$-function in the inert case.

Let us recall the working assumptions of §2.3 with $p$ inert oin $K$. We fix an elliptic curve $A_0$ with CM by $\mathcal{O}_c\subset\mathcal{O}_K$, which was denoted $E$ in the previous section, seen over the ring of integers $R$ of a finite extension $L$ of $\mathbb{Q}_p$ ($R$ was denoted $R'$ in the previous section) and a cyclic subgroup $D\subset A_0[p^2]$ of order $p^2$. These data define: $A':=A_0/C$ with $C=D[p]$ and $C':=D/C$. We have also chosen $D',P'$ where $D'$ is a subgroup scheme of order $p^2$ of $A'$ which is generically cyclic and such that $D'[p]=C'$ and $P'$ is a trivialization of $\big(D'[p]\big)^\vee$. Let $\lambda\colon A_0\to A'$ be the projection and take $\omega'$ a generator of the $\mathbb{Z}_p^*(1+p\underline{\delta}(A')^{-p}R$-torsor $\Omega^{\mathrm{can}}_{A'/R}$ of Remark 5.4. Let $\omega:=\big(\lambda^\vee\big)^*(\omega')$.

As in §3, for every $\mathfrak{a}\in K^{(\mathfrak{N}p^{2c})}\backslash\mathbb{A}_K^{(\mathfrak{N}p^{2c})}/H^{c,p^2\mathfrak{N}}$ we set

$$\mathfrak{a}*(A_0/R,\psi_N,D,P,\omega)=\big(\mathfrak{a}*(A_0,\psi_N,D,P),\iota_p(\mathfrak{a}_p)\omega\big);$$

notice that if we write $(A_{\mathfrak{a}}, \psi_{N,\mathfrak{a}}, D_{\mathfrak{a}}, P_{\mathfrak{a}})$ for $\mathfrak{a}*(A_0, \psi_N, D, P)$ then $\iota_p(\mathfrak{a}_p)\omega \in \Omega^{\mathrm{can}}_{(A')_{\mathfrak{a}}/R}$ by Remark 5.4.

We set $m(p^2c, \mathfrak{N}) := |K^{(\mathfrak{N}p^2c)}\backslash \mathbb{A}_K^{(\mathfrak{N}p^2c)}/H^{c,p^2\mathfrak{N}}|$ and $m(c) = |K^{(c)}\backslash \mathbb{A}_K^{(c)}/H^{c,1}| = |\mathrm{Pic}(\mathcal{O}_c)|$. Then for every $\chi \in \widehat{\Sigma}^{(2)}$ with weight $w(\chi) = \nu$ we define

**Definition 5.5.**

$$L_p(F, \chi^{-1}) := \frac{p^a m(c)}{m(p^2c, \mathfrak{N})} \sum_{\mathfrak{a} \in K^{(\mathfrak{N}p^2c)}\backslash \mathbb{A}_K^{(\mathfrak{N}p^2c)}/H^{c,p^2\mathfrak{N}}} \hat{\Pi}_F(\chi)\Big(\mathfrak{a}, \big(\mathfrak{a} * (A_0/R, t_0, D, P)\big)\Big) :=$$

$$:= \frac{p^a m(c)}{m(p^2c, \mathfrak{N})} \sum_{\mathfrak{a} \in K^{(\mathfrak{N}p^2c)}\backslash \mathbb{A}_K^{(\mathfrak{N}p^2c)}/H^{c,p^2\mathfrak{N}}} \chi_\nu^{-1}(\mathfrak{a}) \delta_k^\nu\big(F^{[p]}\big)\big(\mathfrak{a} * (A_0/R, t_0, D, P, \omega_0)\big),$$

where the evaluation of $(\nabla_k)^\nu(F^{[p]})$ at $\mathfrak{a} * (A_0/R', t_0, D, P, \omega_0)$ is done using Definition 5.3 and the constant $a \in \mathbb{N}$ is the one defined in Remark 4.14.

**Remark 5.6.** The formula above is well posed, namely if we multiply $\mathfrak{a}$ by an element $r \in \widehat{\mathcal{O}}^*_{K,p}$ lying in $(\mathbb{Z}/p^2\mathbb{Z})^*$ modulo $p^2$, then one has $\chi_\nu^{-1}(\mathfrak{a})(\nabla_k)^\nu\big(f^{[p]}\big)\big(\mathfrak{a} * (E/R', \psi_N, D, P, \omega)\big) = \chi_\nu^{-1}(r\mathfrak{a})(\nabla_k)^\nu\big(f^{[p]}\big)\big((r\mathfrak{a}) * (E/R, \psi_N, D, P, \omega)\big)$. Indeed

$$\big((r\mathfrak{a}) * (E/R, \psi_N, D, P, \omega)\big) = \big(\mathfrak{a} * (E/R, \psi_N, D, P, \iota_p(r_p)\omega)\big)$$

so that $(\nabla_k)^\nu\big(f^{[p]}\big)\big((r\mathfrak{a})*(E/R, \psi_N, D, P, \omega)\big) = (k+2\nu)\big(\iota_p(r)\big)(\nabla_k)^\nu\big(f^{[p]}\big)\big(\mathfrak{a}*(E/R, \psi_N, D, P, \omega)\big)$. The conclusion follows as $\chi_\nu(r) = (k+2\nu)\big(\iota_p(r)\big)$ by construction.

## 5.4 Interpolation properties in the case $p$ is inert in $K$.

In this section we show that the values $L_p(F, \chi^{-1})$, for $\chi \in \Sigma^{(2)}_{cc}(\mathfrak{N})$ can be related to the classical values $L_{\mathrm{alg}}(F, \chi^{-1})$, in the case $p$ is inert on $K$. We denote by $a_p$ the $T_p$-eigenvalue of $F$, i.e., $T_p(F) = a_p F$. Assume also that $\chi \in \Sigma^{(2)}_{cc}(\mathfrak{N})$, i.e., $\chi$ is a classical algebraic Hecke character of $K$ and let $m = w(\chi) \in \mathbb{N}$ (i.e. the infinite type of $\chi$ is $(k+m, -m)$.) Then, using Lemma 4.15, we can intepret $F^{[p]}$ as a classical modular form of level $\Gamma_1(N) \cap \Gamma_0(p^2)$ whose inverse image under $p_1$ can be seen as as an element, also denoted $F^{[p]}$, of $\mathrm{H}^0\big(\mathcal{X}(pN, p^2)_L, \mathrm{Sym}^k(\mathrm{H}_{\mathbb{E}}\big)$.

As $(\nabla_k)^m(F^{[p]}) \in \mathrm{H}^0\big(\mathcal{X}(pN, p^2), \mathrm{Sym}^{k+2m}(\mathrm{H}_{\mathbb{E}}\big)$, i.e. it is a classical (i.e. global) object, the evaluation of $(\nabla_k)^m(F^{[p]})$ at $(\mathfrak{a} * (A_0/L, (\psi_N)_L, D_{0,L}, P_L, \omega)$ can be done directly by splitting the Hodge filtration of $\mathrm{Sym}^{k+2m}(\mathrm{H}_{\mathfrak{a}*A_0})$ using the action of the CM field $K$. In this way we can relate $L_p(F, \chi^{-1})$ to $L_{\mathrm{alg}}(F, \chi^{-1})$ and the goal of this section is to make the relation precise so that the values $L_p(F, \chi^{-1})$, for $\chi \in \Sigma^{(2)}_{cc}(\mathfrak{N})$, can be seen as classical arithmetic numbers which modulo certain $p$-adic and complex periods are algebraic and which have been $p$-adically interpolated by $L_p(F, \ -)$. We prove in Proposition 5.11 that, under the assumption that $\mathcal{O}_c^* = \{\pm 1\}$, we have for every $\chi \in \Sigma^{(2)}_{cc}(\mathfrak{N})$ with infinity type $(k+m, -m)$, $m \geq 0$ :

$$L_p(F, \chi^{-1}) = \frac{p^k \mathcal{E}_p(a_p, \chi)}{\Omega_p(k, m)} L_{\mathrm{alg}}(F, \chi^{-1})$$

with $\Omega_p(k,m) = \frac{(\Omega'_p)^{k+2m}}{\widetilde{\Omega}_p^m} \in R$ such that $\dfrac{pk}{p^2-1} + \dfrac{2m}{p(p^2-1)} \leq v_p\big(\Omega_p(k,m)\big) \leq \dfrac{pk+2m}{p^2-1}$.

There are two correcting factors appearing in this formula:

(1) the "$p$-adic period"-type factor denoted $\Omega_p(k,m) := \dfrac{(\Omega'_p)^{k+2m}}{\widetilde{\Omega}_p^m}$, where the two $p$-adic periods $\Omega'_p$ and $\widetilde{\Omega}_p$ appear as follows:

i) $\Omega'_p$ appears as $L_p(F,\chi^{-1})$ is calculated using the section $\omega$ of the invariant differentials $\omega_{\mathfrak{a}*A_0}$, for $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{cp^2})$, which is not a generator of that $R$-module, while $L_{\mathrm{alg}}(F,\chi^{-1})$ is calculated using a generator of $\omega_{\mathfrak{a}*A_0}$.

ii) $\widetilde{\Omega}_p$ comes from the fact that in order to define $L_p(F,\chi^{-1})$ we work on $\mathcal{X}(pN,p_L^2)$ while $L_{\mathrm{alg}}(F,\chi^{-1})$ is defined evaluating sections of $\mathrm{Sym}^{k+2m}\mathrm{H}_{\mathbb{E}}$ at CM points of $\mathcal{X}_1(N)$. This makes a difference as in the iteration of the Gauss-Manin connection $\nabla_k$ the Kähler differentials of specific formal models of $\mathcal{X}(pN,p^2)$, resp. $\mathcal{X}_1(N)$ appear;

(2) "the Euler like" factor $\mathcal{E}_p(a_p,\chi)$ appears because in the first value one uses $F^{[p]}$ while in the second one uses $F$;

### 5.4.1 Values at classical Hecke characters.

Let $(E/R,\psi_N,D,P,\omega) := (\mathfrak{a}*(A_0/R,\psi_N,D_0,P_0,\omega)$ and let $\chi$ be a classical algebraic Hecke character of $K$ of infinite type $(k+m,-m)$. Then:

**Lemma 5.7.** *The value given in Definition 5.3 coincides with* $p^{k+2m}\theta^m_{\mathrm{Hodge}}(F^{[p]})(E/L,(\psi_N)_L,D_L,P_L,\omega)$.

*Proof.* We let $x_L := [E/L,(\psi_N)_L,D_L,P_L]$, $x'_L := [E'/L,(\psi_N)_L,D'_L,P'_L]$ and $x_{i,L} := [E_i/L,(\psi_N)_L,D_{i,L},P_{i,L}]$ be the corresponding moduli points of $\mathcal{X}(pN,p^2)_L$ viewed as morphisms $\varphi_{x_L}, \varphi_{x'_L}, \varphi_{x_{i,L}} : \mathrm{Spec}(L) \to \mathcal{X}_1(pN,p^2)$. Since $U(F^{[p]}) = 0$, working on $X(pN,p^2)_L$ we have

$$0 = \varphi^*_{x'_L}\Big(U\big(\nabla_k^m(F^{[p]})\big)\Big) = \lambda^*\varphi^*_{x_L}\big((\nabla_k^m(F^{[p]})\big) + \sum_{i=1}^{p-1}\lambda_i^*(\varphi^*_{x_{i,L}}(\nabla_k^m(F^{[p]}))\big),$$

therefore, indeed in this case we have

$$\lambda^*\varphi^*_{x_L}\big((\nabla_k^m(F^{[p]})\big) = -\sum_{i=1}^{p-1}\lambda_i^*(\varphi^*_{x_{i,L}}(\nabla_k^m(F^{[p]}))\big),$$

in $\mathrm{Sym}^{k+2m}\big(\mathrm{H}_{E'}\big)_L$.

As recalled in the introduction, it is proven in [K3] that the operator $\theta^m_{\mathrm{Hodge}}$ coincides with the composite of $\nabla_k^m$ and the projection $\mathrm{Sym}^{k+2m}\big(\mathrm{H}_{E'}\big)_L \to \big(\omega_{E'}^{k+2m}\big)_L$ onto the eigenspace on which the imaginary field $K$ acts via the CM type to the $k+2m$-th power (and similarly for $E$ and the $E_i$'s). Since the isogenies $\lambda$ and $\lambda_i$ commute with the (rational) action of $K$ we get the equality

$$\lambda^*\varphi^*_{x_L}\big(\theta^m_{\mathrm{Hodge}}(F^{[p]})\big) = -\sum_{i=1}^{p-1}\lambda_i^*(\varphi^*_{x_{i,L}}(\theta^m_{\mathrm{Hodge}}(F^{[p]}))\big)$$

in $\left(\omega_{E'}^{k+2m}\right)_L$. Taking the generator $\omega'$ of $\left(\omega_{E'}\right)_L$ as in Definition 5.3 to define a trivialization $v_{\omega'}\colon \left(\omega_{E'}^{k+2m}\right)_L \cong L$, we conclude that the value $\delta_k^{\nu}(F^{[p]})(E/L,(\psi_N)_L,D_L,P_L,\omega)$ of Definition 5.3 coincides with

$$\delta_k^{\nu}(F^{[p]})(E/R,\psi_N,D,\omega) = v_{\omega'}\big(\lambda^*\varphi_{x_L}^*\big(\theta_{\mathrm{Hodge}}^m(F^{[p]})\big)\big).$$

The isogenies $\lambda\colon E'\to E$ and $\lambda^{\vee}\colon E\to E'$ define morphisms on differentials and provide the following commutative diagram:

$$\begin{array}{ccccc} \omega_E^{k+2m}\otimes_R L & \xrightarrow{\lambda^*} & \omega_{E'}^{k+2m}\otimes_R L & \xrightarrow{(\lambda^{\vee})^*} & \omega_E^{k+2m}\otimes_R L \\ \downarrow v_{\frac{\omega}{p}} & & \downarrow v_{\omega'} & & \downarrow v_{\omega} \\ L & = & L & = & L. \end{array}$$

Recall that $\omega=(\lambda^{\vee})^*(\omega')$ and that $\lambda^{\vee}\circ\lambda$ is multiplication by $p$. In particular the induced map $(\lambda^{\vee})^*\circ\lambda^*\colon \omega_E^{k+2m}\to\omega_E^{k+2m}$ is multilication by $p^{k+2m}$. Then

$$v_{\omega'}\big(\lambda^*\varphi_x^*\big(\theta_{\mathrm{Hodge}}^m(F^{[p]})\big)\big) = v_{\frac{\omega}{p}}\big(\varphi_x^*\big(\theta_{\mathrm{Hodge}}^m(F^{[p]})\big)\big) = p^{k+2m}v_{\omega}\big(\varphi_x^*\big(\theta_{\mathrm{Hodge}}^m(F^{[p]})\big)\big).$$

The conclusion follows.

□

Next we compare the values $\theta_{\mathrm{Hodge}}^m(F)(E/L,(\psi_N)_L,\omega)$ and $\theta_{\mathrm{Hodge}}^m(F)(E/L,(\psi_N)_L,D_L,P_L,\omega)$: the first is defined by applying $\nabla^m$ to $F$ directly on $\mathcal{X}_1(N)_L$ and is related to the value of $L_{\mathrm{alg}}(F,\chi^{-1})$, the second is defined by applying $\nabla^m$ to the pull-back of $F$ to $\mathcal{X}(pN,p^2)_L$ and is related to the value of $L_p(F,\chi^{-1})$ .

Suppose now that $L=\mathbb{Q}_p(\zeta_N,\zeta_p)$ i.e. it is the extension of $\mathbb{Q}_p$ generated by a primitive $N$-th root of unity $\zeta_N$ and a primitive $p$-th root of unity $\zeta_p$; denote by $\pi=\zeta_p-1$ a uniformizer of $L$ and by $R$ its ring of integers. We consider the modular curve $X_1(N)$ over the ring of integers of $L$ and the open $Y_1(N)\subset X_1(N)$ obtained by removing the cusps. Denote by $\mathcal{E}\to Y_1(N)$ the universal elliptic curve. Set

$$\tau\colon Y(pN,p^2)\longrightarrow Y_1(N)$$

to be the normalization of the moduli problem over $L$ classifying elliptic curves, with $\Gamma_1(N)$-level structure, a cyclic subgroup $D$ of order $p^2$ and a generator of the Cartier dual $D[p]^{\vee}$ of $D[p]$. We have the following commutative diagram of vector bundles with connections:

$$\begin{array}{ccc} \mathrm{Sym}^k\mathrm{H}_{\mathcal{E}} & \xrightarrow{\nabla} & \mathrm{Sym}^k\mathrm{H}_{\mathcal{E}}\otimes\Omega^1_{Y_1(N)} \\ \tau^*\downarrow & & \downarrow\tau^*\otimes d\tau \\ \tau^*\big(\mathrm{Sym}^k\mathrm{H}_{\mathcal{E}}\big) & \xrightarrow{\nabla} & \tau^*\big(\mathrm{Sym}^k\mathrm{H}_{\mathcal{E}}\big)\otimes\Omega^1_{Y(pN,p^2)}. \end{array}$$

We recall that we have the Kodaira-Spencer isomorphism $\omega_{\mathcal{E}}^2\cong\Omega^1_{Y_1(N)}$ which is used to identify $\mathrm{Sym}^k\mathrm{H}_{\mathcal{E}}\otimes\Omega^1_{Y_1(N)}\subset\mathrm{Sym}^{k+2}\mathrm{H}_{\mathcal{E}}$ and allows us to iterate $\nabla_k$. On the other hand $\theta_{\mathrm{Hodge}}^m(F^{[p]})(E/L,\psi_N,D,\omega)$ is computed in Theorem 4.13 and Lemma 5.7 iterating the Gauss-Manin connection over $\mathcal{X}(pN,p^2)$. By GAGA the same calculation can be carried out over $Y(pN,p^2)_L$. Thus we need to study the map

$$d\tau\colon \Omega^1_{Y_1(N)}\to\Omega^1_{Y(pN,p^2)}.$$

Let $x \in Y_1(N)(R)$ be the $R$-valued point defined by $(E/R, \psi_N)$ and let $y \in Y(pN, p^2)(R)$ be an $R$-valued point such that $\pi(y) = x$. The pull back $\Omega_x := x^*\big(\Omega^1_{Y_1(N)}\big)$ coincides with $\omega_E^2$ as Kodaira-Spencer provides an isomorphim $\omega_{\mathcal{E}}^2 \cong \Omega^1_{Y_1(N)}$. Consider $\Omega_y$ to be the torsion free part of $y^*\big(\Omega^1_{Y(pN,p^2)}\big)$. The natural morphism $d\tau\colon \Omega^1_{Y_1(N)} \to \Omega^1_{Y_1(N,p^2)}$ defines a morphism $\Omega_x \to \Omega_y$ which is an isomorphism after inverting $p$. Thus $\Omega_x$ and $\Omega_y$ are free $R$-modules of rank 1. Let $\widetilde{E} := E/D$, we denote its Néron model by $\widetilde{E}/R$ and consider the ideal $\mathrm{Hdg}(\widetilde{E}/R) \subset R$. We denote by $p^2\Omega_y\Omega_x^{-1}$ the $R$-fractional ideal of $L$ of elements $t \in L$ such that $t\Omega_x \subset p^2\Omega_y$.

**Lemma 5.8.** *We have the following relations on $p$-adic valuations*

$$2pv_p\big(\mathrm{Hdg}(\widetilde{E})\big) \leq v_p\big(p^2\Omega_y\Omega_x^{-1}\big) \leq 2(p+1)v_p\big(\mathrm{Hdg}(\widetilde{E})\big)$$

*and, for a suitable generator $\widetilde{\Omega}_p$ of $p^2\Omega_y\Omega_x^{-1}$, we have*

$$\theta^m_{\mathrm{Hodge}}(F)(E/R, \psi_N, \omega) = \frac{p^{2m}}{\widetilde{\Omega}_p^m}\theta^m_{\mathrm{Hodge}}(F)(E/R, \psi_N, D, \omega).$$

*Proof.* The second statement follows from the first. Let $D \subset \mathcal{E}[p^2]$ be the universal cyclic subgroup of order $p^2$ and $P$ the universal trivialization of $(D[p])^\vee$ over $Y(pN, p^2)_L$. We have a morphism $\rho\colon Y(pN, p^2) \to Y_1(N)$ defined over $L$ by sending $(\mathcal{E}, t_N, D, P)$ to $\widetilde{\mathcal{E}} := \mathcal{E}/D$ with level $N$ structure induced by the isogeny $\alpha\colon \mathcal{E} \to \widetilde{\mathcal{E}}$. Finally we define an involution $w\colon Y(pN, p^2) \to Y(pN, p^2)$, given over $L$ by sending $(\mathcal{E}, t_N, D, P)$ to $\widetilde{\mathcal{E}} := \mathcal{E}/D$, with level subgroup $\widetilde{D} := \mathcal{E}[p^2]/D$ of order $p^2$ and level $N$ structure induced by the isogeny $\alpha\colon \mathcal{E} \to \widetilde{\mathcal{E}}$. Notice that $\widetilde{D} = D^\vee$ so that a trivialization $P$ of $D^\vee[p] = D[p]^\vee$ induces an isomorphisom $\mathbb{Z}/p\mathbb{Z} \cong \widetilde{D}[p]$ and, dualizing, an isomorphism $\widetilde{D}[p]^\vee \cong \mu_p \cong \mathbb{Z}/p\mathbb{Z}$ i.e. a trivialization $\widetilde{P}$ of $(\widetilde{D}[p])^\vee$ (recall that we work over a field $L$ in which we have a chosen primitive $p$-th root of unity that allows to identify $\mu_p$ and $\mathbb{Z}/p\mathbb{Z}$). Then $\rho \circ w = \tau$ and the dual isogeny $\varphi := \alpha^\vee\colon \widetilde{\mathcal{E}} \to \mathcal{E}$ is the quotient of $\widetilde{\mathcal{E}}$ by $\widetilde{\mathcal{D}}$. Summarizing we have the commutative diagram:

$$\begin{array}{ccc} Y(pN, p^2) & \xrightarrow{w} & Y(pN, p^2) \\ {\scriptstyle\tau}\searrow & & \swarrow{\scriptstyle\rho} \\ & Y_1(N) & \end{array}$$

The reason to write $\tau$ as $\rho \circ w$ is that, for the point $y_L \in Y(pN, p^2)(L)$ defined above, the elliptic curve $E/D$ has canonical subgroup of order $p^2$ (see Lemma 3.1) which coincides with $E[p^2]/D$ so that $w(y_L)$ is an $L$-valued point of a neighborhood of the ordinary locus of $Y(pN, p^2)_L$ where the geometry is easier to understand. We consider the following commutative diagram:

$$\begin{array}{ccc} \omega_{\mathcal{E}} & \xrightarrow{\varphi^*} & \omega_{\widetilde{\mathcal{E}}} \\ \downarrow & & \downarrow \\ \omega_{\mathcal{E}}^\vee \otimes \Omega^1_{Y_1(N)} & \xrightarrow{(\alpha^*)^\vee \otimes d\rho} & \omega_{\widetilde{\mathcal{E}}} \otimes \Omega^1_{Y(pN,p^2)} \end{array}$$

where the vertical maps are induced by the Kodaira-Spencer morphisms for $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ respectively. Since $\alpha \circ \varphi$ is multiplication by $p^2$ then $\varphi^* = p^2(\alpha^*)^{-1}$ and $\varphi^* \otimes (\alpha^*)^{-1}\colon \omega_{\mathcal{E}}^2 \to \omega_{\widetilde{\mathcal{E}}}^2$ is $p^2(\alpha^*)^{-2}$.

In conclusion, we get

$$\begin{array}{ccccc} \omega_{\mathcal{E}}^2 & \xrightarrow{p^2(\alpha^*)^{-2}} & \omega_{\widetilde{\mathcal{E}}}^2 & & \\ \wr\downarrow & & \downarrow & & \\ \Omega^1_{Y_1(N)} & \xrightarrow{d\rho} & \Omega^1_{Y(pN,p^2)} & \xrightarrow{dw} & \Omega^1_{Y(pN,p^2)}; \end{array}$$

where $dw$ is an isomorphism and the composite $dw \circ d\rho\colon \Omega^1_{Y_1(N)} \to \Omega^1_{Y(N,p^2)}$ is the map $d\tau$ we want to understand.

Given a positive integer $r$ such that $p(p+1) \geq (p-1)r > p+1$ let $\mathfrak{Y}_r \to \mathfrak{Y}$ be the (normalization of) the formal open subscheme in the formal admissible blow-up of $\mathfrak{Y}$, the $p$-adic formal scheme associated to $Y_1(N)$, with respect to the sheaf of ideals $(\pi, \mathrm{Hdg}^r)$, where this ideal is generated by $\mathrm{Hdg}^r$. Let $\widetilde{\mathcal{E}}$ be the universal elliptic curve over $\mathfrak{Y}$. Over $\mathfrak{Y}_r$ it admits a canonical subgroup $H$ of level $p^2$. Let $\mathfrak{IG}_{1,r} \to \mathfrak{Y}_r$ be the normalization of the first layer of the Igusa tower that on generic fibers classify generators of $H[p]^\vee$; write $\nu\colon \mathfrak{IG}_{1,r} \to \mathfrak{Y}$ for the induced map. Then the morphism $\nu$ factors via a morphism $\mathfrak{IG}_{1,r} \to \mathfrak{Y}(pN,p^2)$ (where $\mathfrak{Y}(pN,p^2)$ is the $p$-adic formal scheme associated to $Y(pN,p^2)$). Moreover, $w(y)$ defines an $R$-valued point of $\mathfrak{IG}_{1,r}$. Consider the map of differentials $\nu^*\big(\Omega^1_{\mathfrak{Y}}\big) \subset \Omega^1_{\mathfrak{IG}_{1,r}}$; it is generically an isomorphism.

*Claim:* $\Omega^1_{\mathfrak{IG}_{1,r}} \subset \frac{1}{\mathrm{Hdg}(\widetilde{\mathcal{E}})^2}\nu^*\big(\Omega^1_{\mathfrak{Y}}\big)$.

This is a local statement. Let $\mathfrak{U} = \mathrm{Spf}(A) \subset \mathfrak{Y}$ be an open formal subscheme. Its inverse image on $\mathfrak{Y}_r$ is the formal spectrum of $B = A\langle t\rangle/\big(z^r t - \pi\big)$ (which is already normal), with $z$ a generator of $\mathrm{Hdg}(\widetilde{\mathcal{E}})$ over $\mathfrak{U}$, and $\Omega^1_B = \big(B \otimes_A \Omega^1_A \oplus Bdt\big)/\big(Bz^{r-1}(rtdz + zdt)\big)$ which maps to $\big(B \otimes_A \Omega^1_A \oplus Bdt\big)/\big(B(rtdz + zdt)\big)$ which is contained in $z^{-1} \cdot B \otimes_A \Omega^1_A$. On the other hand the inverse image of $\mathfrak{U}$ in $\mathfrak{IG}_{1,r}$ is the formal spectrum of $C = B[s]/(s^{p-1} - z)$ by [AIPHS, Lemme 3.4] so that $\Omega^1_{C/B} = Cds/C\big((p-1)s^{p-2}ds\big)$ which is killed by mutplication by $z$. Putting these two computations together the claim follows. Notice that Kodaira-Spencer defines an isomorphism $\omega^2_{\widetilde{\mathcal{E}}} \cong \Omega^1_{\mathfrak{Y}}$. In conclusion we have the commutative diagram

$$\begin{array}{ccccccc} \omega_{\mathcal{E}}^2 & \xrightarrow{p^2(\alpha^*)^{-2}} & \omega^2_{\widetilde{\mathcal{E}}}|_{\mathfrak{Y}_r} & & & & \\ \wr\downarrow & & \downarrow & & & & \\ \Omega^1_{\mathfrak{Y}} & \xrightarrow{d\rho} & \Omega^1_{\mathfrak{IG}_{1,r}} & \longrightarrow & \frac{1}{\mathrm{Hdg}(\widetilde{\mathcal{E}})^2} \cdot \nu^*\big(\Omega^1_{\mathfrak{Y}}\big) & \cong & \frac{1}{\mathrm{Hdg}(\widetilde{\mathcal{E}})^2} \cdot \omega^2_{\widetilde{\mathcal{E}}}|_{\mathfrak{IG}_{1,r}}. \end{array}$$

As $\varphi$ on $\mathfrak{Y}_r$ is the quotient by the canonical subgroup of order $p^2$, it coincides with Frobenius squared modulo $p\mathrm{Hdg}(\widetilde{\mathcal{E}})^{-(p+1)}$ and hence $\alpha$ is Verschiebung squared modulo the same ideal. Then the image of $\alpha^*\colon \omega_{\widetilde{\mathcal{E}}} \to \omega_{\mathcal{E}}$ is $\mathrm{Hdg}(\widetilde{\mathcal{E}})^{p+1}\omega_{\mathcal{E}}$; see also the proof of Lemma 3.3 for more details. We now take pull backs via the $R$-valued points $x$ and $w(y)$. We then have the commutative diagram

$$\begin{array}{ccccccc} \omega_E^2 & \xrightarrow{\sim} & \frac{p^2}{\mathrm{Hdg}(\widetilde{\mathcal{E}})^{2(p+1)}} \cdot \omega^2_{\widetilde{E}} & \subset & \omega^2_{\widetilde{E}} & & \\ \wr\downarrow & & \downarrow & & \downarrow & & \\ \Omega_x & \xrightarrow{d\rho} & \frac{p^2}{\mathrm{Hdg}(\widetilde{\mathcal{E}})^{2(p+1)}} \cdot \Omega_{w(y)} & \subset & \Omega_{w(y)} & \longrightarrow & \frac{1}{\mathrm{Hdg}(\widetilde{\mathcal{E}})^2} \cdot \omega^2_{\widetilde{E}}. \end{array}$$

and, using the identification $\Omega_y \cong \Omega_{w(y)}$ defined by the involution $w$, the claim follows.

□

### 5.4.2 The factor $\mathcal{E}(a_p,\chi)$ in the inert case.

Let $D_0\subset A_0$ be a cyclic subgroup of order $p^2$, let $t_0$ be a generator of $A_0[\mathfrak{N}]$ and let $\omega_0$ be a generator of the invariant differentials of $A_0$. Let $\widetilde{A}_0:=A_0/D_0$; it is an elliptic curve admitting canonical subgroup of level $p^2$. Finally set

$$\mathcal{E}_p(a_p,\chi):=1-\frac{(p-1)a_p^2}{\chi(p)(p+1)}-\frac{1}{p^2}.$$

**Lemma 5.9.** *We have*

$$\frac{m(c)}{m(p^2c,\mathfrak{N})}\sum_{\mathfrak{a}\in K^{(\mathfrak{N}p^2c)}\backslash\mathbb{A}_K^{(\mathfrak{N}p^2c)}/H^{c,p^2\mathfrak{N}}}\chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a}*(A_0,t_0,D_0,\omega_0)\big)=\frac{\widetilde{\Omega}_p^j}{p^{2j}}\mathcal{E}_p(a_p,\chi)L_{\mathrm{alg}}(F,\chi^{-1})$$

*with $\widetilde{\Omega}_p\in R$ an element of valuation $2pv_p\big(\mathrm{Hdg}(\widetilde{A}_0)\big)\leq v_p\big(\widetilde{\Omega}_p\big)\leq(2p+2)v_p\big(\mathrm{Hdg}(\widetilde{A}_0)\big)$.*

*Proof.* Since $\chi$ is of type $(c,\mathfrak{N},\epsilon)$, arguing as in Remark 5.6 one computes that the quantity $\chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a}*(A_0,t_0,D_0,\omega_0)\big)$ depends only on the class of $\mathfrak{a}$ in the quotient $\mathrm{Pic}(\mathcal{O}_{p^2c})\cong K^{(p^2c)}\backslash\mathbb{A}_K^{(p^2c)}/H^{p^2a}$ so that we are left to compute

$$\frac{m(c)}{m(p^2c)}\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{p^2c})}\chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a}*(A_0,t_0,D_0,\omega_0)\big).$$

Given a positive integer $d$ coprime to $c$, let $\rho\colon\mathrm{Pic}(\mathcal{O}_{dc})\to\mathrm{Pic}(\mathcal{O}_c)$ given by $\mathbb{L}\mapsto\mathbb{L}\otimes_{\mathcal{O}_{dc}}\mathcal{O}_c$ and set $h_d=|\mathrm{Ker}(\rho)|$ so that $m(dc)=|\mathrm{Pic}(\mathcal{O}_{dc})|=h_d|\mathrm{Pic}(\mathcal{O}_c)|=h_dm(c)$. We have an exact sequence

$$0\to\mathcal{O}_{cd}^*\to\mathcal{O}_c^*\to(\mathcal{O}_c/d\mathcal{O}_c)^*/(\mathcal{O}_{dc}/d\mathcal{O}_{cd})^*\to\mathrm{Ker}(\rho)\to0$$

so that if $h'_d=|(\mathcal{O}_c/d\mathcal{O}_c)^*/(\mathcal{O}_{dc}/d\mathcal{O}_{cd})^*|$ and $h''_d=|\mathcal{O}_c^*/\mathcal{O}_{cd}^*|$ then $h_d=h'_d/h''_d$. Our assumption implies that $h_d=h'_d$. We consider two cases:

1) $p$ inert and $d=p^2$ and then $h_{p^2}=h'_{p^2}=\frac{(p^2-1)p^2}{(p-1)p}=(p+1)p$. We denote by $\rho_{p^2}\colon\mathrm{Pic}(\mathcal{O}_{p^2c})\to\mathrm{Pic}(\mathcal{O}_c)$ the base change map.

2) $p$ inert and $d=p$ and then $h_p=h'_p=\frac{p^2-1}{p-1}=p+1$. We denote by $\rho_p\colon\mathrm{Pic}(\mathcal{O}_{pc})\to\mathrm{Pic}(\mathcal{O}_c)$ the base change map.

Recall that $A$ is an elliptic curve with full CM by $\mathcal{O}_K$ and we fix cyclic isogenies $\phi_c\colon A\to A_c=A_0$ and $\phi_{dc}\colon A_c\to A_{dc}$ of degree $c$ and $d$ respectively. We fix a basis element $\omega$ of the invariant differentials of $A$ and we let $\omega_c$ and $\omega_{dc}$ be the elements of the invariant differentials of $A_c$, resp. $A_{dc}$ such that $\omega=\phi_c^*(\omega_c)=\phi_{dc}^*(\omega_{dc})$.

Denote by $C$ the kernel of $\phi_{dc}$. Let $V_d\colon X\big(\Gamma_0(dc)\cap\Gamma_1(N)\big)\to X\big(\Gamma_0(c)\cap\Gamma_1(N)\big)$ be the operator sending $(E,H_{dc},\psi_N)$ to $(E/H_d,\psi_N)$ (here $H_d$ is the cyclic subgroup of order $d$ in $H_{dc}$). Then

$$V_d\big(\mathfrak{b}*(A_c,t_c,\omega_c,C)\big)=\big(\mathfrak{b}*(A_{dc},\frac{1}{d}t_{dc},d\omega_{dc})\big)$$

for every $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O}_{dc})$. Recall from Lemma 4.15 that $F^{[p]} = \big(1 - a_p V + \epsilon(p) p^{k-1} V^2\big)(F)$. Since $\nabla \circ V = pV \circ \nabla$, we get for every $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{p^2c})$:

$$\theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a} * (A_c, t_c, \omega_c, H_{p^2})\big) = \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_c, t_c, \omega_c)\big) +$$

$$-p^j a_p \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{pc}, \frac{1}{p} t_{pc}, p\omega_{pc})\big) +$$

$$+\epsilon(p) p^{2j+k-1} \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{p^2c}, \frac{1}{p^2} t_{p^2c}, p^2 \omega_{p^2c})\big) =$$

$$= \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_c, t_c, \omega_c)\big) - p^j a_p \theta^j_{\mathrm{Hodge}}(F)\big((p^{-1}\mathfrak{a}) * (A_{pc}, t_{pc}, \omega_{pc})\big) +$$

$$+\epsilon(p) p^{2j+k-1} \theta^j_{\mathrm{Hodge}}(F)\big((p^{-2}\mathfrak{a}) * (A_{p^2c}, t_{p^2c}, \omega_{p^2c})\big).$$

Hence

$$\chi_j^{-1}(\mathfrak{a}) \theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a} * (A_c, t_c, \omega_c, H_{p^2})\big) = \chi_j^{-1}(\mathfrak{a}) \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_c, t_c, \omega_c)\big) +$$

$$-\chi_j^{-1}(p^{-1}\mathfrak{a}) p^{-j} \chi^{-1}(p) a_p \theta^j_{\mathrm{Hodge}}(F)\big((p^{-1}\mathfrak{a}) * (A_{pc}, t_{pc}, \omega_{pc})\big) +$$

$$+\chi_j^{-1}(p^{-2}\mathfrak{a}) \chi^{-1}(p^2) \epsilon(p) p^{-2j+k-1} \theta^j_{\mathrm{Hodge}}(F)\big((p^{-2}\mathfrak{a}) * (A_{p^2c}, t_{p^2c}, \omega_{p^2c})\big)$$

and

$$\frac{m(c)}{m(p^2c, \mathfrak{N})} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{p^2c})} \chi_j^{-1}(\mathfrak{a}) \theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a} * (A_c, t_c, \omega_c, H_{p^2})\big) =$$

$$= \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a}) \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_c, t_c, \omega_c)\big) +$$

$$-\frac{p^{-j} \chi^{-1}(p) a_p}{h_p} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{pc})} \chi_j^{-1}(\mathfrak{a}) \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{pc}, t_{pc}, \omega_{pc})\big) +$$

$$+\frac{\chi^{-1}(p^2) \epsilon(p) p^{-2j+k-1}}{h_{p^2}} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{p^2c})} \chi_j^{-1}(\mathfrak{a}) \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{p^2c}, t_{p^2c}, \omega_{p^2c})\big).$$

As $\theta_{\mathrm{Hodge}} \circ T_p = p^{-1} T_p \circ \theta_{\mathrm{Hodge}}$, for $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O}_c)$ we have

$$a_p p^j \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) = \big(\theta^j_{\mathrm{Hodge}}(F) | T_p\big)\big(\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) =$$

$$= \frac{1}{p} \sum_{\mathfrak{a} \in \rho_p^{-1}(\mathfrak{b})} \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{pc}, t_{pc}, \omega_{pc})\big).$$

Hence

$$\sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{pc})} \chi_j^{-1}(\mathfrak{a}) \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{pc}, t_{pc}, \omega_{pc})\big) =$$

$$= p^{j+1} a_p \Big( \sum_{\mathfrak{b} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{b}) \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big)\Big).$$

Analogously, for $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O}_c)$, we compute

$$(p^{2j}a_p^2 - (p+1)\epsilon(p)p^{k+2j-2})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) =$$
$$= \big(\theta^j_{\mathrm{Hodge}}(F)\big)|\big(T_p^2 - \frac{p+1}{p^2}R_p\big)\big(\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) =$$
$$= \frac{1}{p^2}\sum_{\mathfrak{a}\in\rho_{p^2}^{-1}(\mathfrak{b})} \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{p^2c}, t_{p^2c}, \omega_{p^2c})\big)$$

where $R_p$ is the quotient of multiplication by $p$. The factor $\frac{p+1}{p^2}R_p$ accounts for the fact that $T_p^2(A,t,\omega)$ is defined as the correspondence given by taking all cyclic subgroups of order $p^2$ of $A$ *and* $p+1$-times the $p$-torsion of $A$. Hence,

$$\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{p^2c})} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{p^2c}, t_{p^2c}, \omega_{p^2c})\big)) =$$
$$= (a_p^2p^{2j+2} - (p+1)\epsilon(p)p^{k+2j})\Big(\sum_{\mathfrak{b}\in\mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_c, t_c, \omega_c)\big)\Big).$$

Putting everything together, we get

$$\frac{m(c)}{m(p^2c,\mathfrak{N})}\sum_{\mathfrak{a}\in K^{(\mathfrak{N}p^{2c})}\backslash\mathbb{A}_K^{(\mathfrak{N}p^{2c})}/H^{c,p^2\mathfrak{N}}} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a} * (A_0, t_0, D_0, \omega_0)\big) =$$

$$= \big(1 - \frac{pa_p^2}{\chi(p)h_p} + \frac{\epsilon(p)p^{k+1}(a_p^2 - (p+1)\epsilon(p)p^{k-2})}{\chi(p^2)h_{p^2}}\big)\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_0, t_0, D_0, \omega_0)\big).$$

Since $\chi(p) = \epsilon(p)p^k$, $h_p = p+1$ and $h_{p^2} = p(p+1)$ we compute

$$1 - \frac{pa_p^2}{\chi(p)(p+1)} + \frac{\epsilon(p)p^k(a_p^2 - (p+1)\epsilon(p)p^{k-2})}{\chi(p^2)(p+1)} = \mathcal{E}_p(a_p, \chi).$$

Using Lemma 5.8 to compare $\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_0, t_0, D_0, \omega_0)\big)$ and $\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big)$ and the fact that $L_{\mathrm{alg}}(F, \chi^{-1}) = \sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big)$, the claim follows.

□

### 5.4.3 The $p$-adic periods $\Omega_p(k,m)$ in the inert case.

Let $D_0 \subset A_0$ be a cyclic subgroup of order $p^2$, let $t_0$ be a generator of $A_0[\mathfrak{N}]$ and let $\omega_0$ be a generator of the invariant differentials of $A_0$. Let $A_0' := A_0/D[p]$ and $\widetilde{A}_0 := A_0/D_0$. We have isogenies $\lambda\colon A_0 \to A_0'$, $\beta\colon A_0' \to \widetilde{A}_0$ and $\alpha := \beta \circ \lambda\colon A_0 \to \widetilde{A}_0$. We have fixed a generator $\omega'$ of the $\mathbb{Z}_p^*(1 + p\underline{\delta}(A_0')^{-p}R)$-torsor $\Omega^{\mathrm{can}}_{A_0'/R}$ of Remark 5.4 and we have defined $\omega := \big(\lambda^\vee\big)^*(\omega')$. Then $\omega = \Omega_p'\omega_0$ for a non-zero element $\Omega_p' \in R$.

**Lemma 5.10.** *The $p$-adic valuation of $\Omega_p'$ is $\frac{p}{p^2-1}$.*

*Proof.* It follows from Lemma 3.1 that $A_0'$ has $A_0[p]/D[p]$ as canonical subgroup. The dual isogeny $\lambda^\vee\colon A_0' \to A_0$ is the quotient of the canonical subgroup so that it coincides with Frobenius modulo $p/\mathrm{Hdg}(A_0')$. Hence $\lambda$ coincides with Vershiebung modulo $p/\mathrm{Hdg}(A_0')$ so that the image of $\lambda^*\colon \omega_{A_0'} \to \omega_{A_0}$ is $\mathrm{Hdg}(A_0')\omega_{A_0}$. Recall that $\mathrm{Hdg}(A_0')$ has $p$-adic valuation $\frac{1}{p+1}$ by Lemma 3.1. As explained in §4.1 the element $\omega'$ is a generator of $\underline{\delta}(A_0')\omega_{A_0'} = \mathrm{Hdg}(A_0')^{\frac{1}{p-1}}\omega_{A_0'}$. We conclude that the $p$-adic valuation of $\Omega_p'$ is $(1+\frac{1}{p-1})v_p\big(\mathrm{Hdg}(A_0')\big) = \frac{p}{p^2-1}$ as claimed.

□

Our main result on interpolation properties is:

**Proposition 5.11.** *Assume that $\mathcal{O}_c^* = \{\pm 1\}$. Then, for every $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ with infinity type $(k+m,-m)$, $m \geq 0$ we have:*

$$L_p(F,\chi^{-1}) = \frac{p^k\mathcal{E}_p(a_p,\chi)}{\Omega_p(k,m)}L_{\mathrm{alg}}(F,\chi^{-1})$$

*with $\Omega_p(k,m) = \frac{(\Omega_p')^{k+2m}}{\widetilde{\Omega}_p^m}$ where $\widetilde{\Omega}_p \in R$ is defined in Lemma 5.9. In particular we have*

$$\frac{pk}{p^2-1} + \frac{2m}{p(p^2-1)} \leq v_p\big(\Omega_p(k,m)\big) \leq \frac{pk+2m}{p^2-1}.$$

*Proof.* It follows from Lemma 3.1 that $\mathrm{Hdg}(\widetilde{A}_0)$ has $p$-adic valuation $1/(p(p+1))$. The Proposition follows then from Lemma 5.7, Lemma 5.9 and Lemma 5.10.

□

If $F$ is the Eisenstein series $E_{k,\epsilon}$ we have $a_p = 1 + p^{k-1}\epsilon(p)$ and the interpolation formula becomes:

**Corollary 5.12.** *Assume that $\mathcal{O}_K^* = \{\pm 1\}$. We then have*

$$L_p\big(E_{k,\epsilon},\chi^{-1}\big) = \frac{p^k\mathcal{E}_p\big(1+p^{k-1}\epsilon(p),\chi\big)}{\Omega_p(k,m)}L_{\mathrm{alg}}\big(E_{k,\epsilon},\chi^{-1}\big).$$

# 6 The case: $p$ is ramified in $K$.

In this section we assume the hypothesis of §2.3 and we furher assume that $p$ is ramified in the quadratic imaginary field $K$. Denote by $\mathfrak{P}$ the unique prime ideal of $\mathcal{O}_K$ over $p$ and by $K_{\mathfrak{P}}$ the totally ramified extension of $\mathbb{Q}_p$ of degree 2 given by the $\mathfrak{P}$-adic completion of $K$.

In this section we define the $p$-adic $L$-function associated to a classical eigenform $F$ of weight $k \geq 2$ (as in §2.3) twisted by algebraic Hecke characters of $K$. Let us remark that as in the case when $p$ is inert in $K$ the main problem to solve is the following: let $\chi \in \widehat{\Sigma}^{(2)}$ with weight $w(\chi) = \nu$ and let $(E/R,t)$ be a pair consisting of an elliptic curve with CM by $\mathcal{O}_c \subset \mathcal{O}_K$ seen over the ring of integers $R$ of a finite extension $L$ of $\mathbb{Q}_p$ and a level $\Gamma_1(\mathfrak{N})$-structure. The isomorphism class of this pair can be seen as an $R$-point $x_E$ of $\mathfrak{X}_1(N)$. We choose a basis $\omega$ of $\omega_E$ over $R$ and wish to evaluate $(\nabla_k)^\nu(F^{[p]})(E/R,t,\omega)$.

In the inert case $E[p^\infty]$ had full CM by the ring of integers of the $p$-adic completion of $K$ which was the unramified extension of degree 2 of $\mathbb{Q}_p$ and therefore the elliptic curve $E$ did not have a canonical subgroup. In the ramified case the situation is formally different, i.e., the completion $K_{\mathfrak{P}}$ of $K$ at $\mathfrak{P}$ is totally ramified over $\mathbb{Q}_p$ and accordinag to Lemma 3.2 $E[\mathfrak{P}]$ is the canonical subgroup of $E/R$ of order $p$ and moreover $\mathrm{val}_p\big(\mathrm{Hdg}(x_E)\big) = 1/2$. Therefore this valuation is too large to apply Corollary 4.9 and we can't directly evaluate $(\nabla_k)^\nu(F^{[p]})$ at the point $x_E$. We will skirt the issue in a way similar to the one adopted in the inert case.

## 6.1 Evaluation at CM-points.

The situation is very similar to the one in the inert case. In the notations of the previous sectio, we choose a subgroup $D \subset E[p^2]$, cyclic of order $p^2$ over $L$ such that $D[p] \cap E[\mathfrak{P}] = \{0\}$. Recall that $E[\mathfrak{P}] = \mathrm{H}$ is the canonical subgroup of $E$. Set $C := D[p]$ and $E' := E/C$; the subgroup $E[p]/C = \mathrm{H}'$ is the canonical subgroup of $E'$ and $C' := D/C \subset E'[p]$ is a subgroup of order $p$ such that $C' \cap \mathrm{H}' = \{0\}$. Let $D' \subset E'[p^2]$ be a subgroup of order $p^2$, cyclic over $L$, such that $D'[p] = C'$. Denote the other subgroups of $E'$ of order $p$ (i.e. the ones different from $C'$) by: $\mathrm{H}'$, $C_1$, $C_2$, ..., $C_{p-1}$. Set $E_i := E'/C_i$ for $i = 1, \ldots, p-1$ and notice that $\big(E'/\mathrm{H}', (D' + \mathrm{H}')/\mathrm{H}'\big) \cong \big(E, D\big)$. We also choose a trivialization $P'$ of $\big(D'[p]\big)^\vee$ over maybe a finite extension of $L$, which we still denote $L$. This trivialization determines natural, compatible trivializations $P$ of $\big(D[p]\big)^\vee$ and $P_i$ of $\big(D_i[p]\big)^\vee$ for all $i = 1, 2, ..., p-1$. Consider the $L$-valued points of $\mathcal{X}(pN, p^2)$ given by $x_L = (E/L, \psi_{N,L}, D_L, P_L)$, $x'_L = (E'/L, \psi'_{N,L}, D'_L, P'_L)$ and for $1 \leq i \leq p-1$ by $x_{i,L} := (E_i/L, (\psi_N)_{i,L}, D_{i,L}, P_{i,L})$ with $D_{i,L}$ the image of $D'_L$ in $E_i$. Then as in the inert case we have $U(x'_L) = \{x_L, x_{1,L}, ..., x_{p-1,L}\}$ (here $U$ is the $U_p$-correspondence on $\mathcal{X}(pN, p^2)$).

By lemma 3.2 for each $1 \leq i \leq p-1$ we have $\mathrm{val}_p\big(\mathrm{Hdg}(E_i)\big) = \frac{1}{2p^2}$ so that $\pi(x_i) := (E_i, (\psi_N)_i, D_i[p] = \mathrm{H}_i)$ defines an $R$-valued point $\varphi_{\pi(x_i)} \colon \mathrm{Spf}(R) \to \mathfrak{X}^{(\infty)}_{2p^2}$. In particular, all the points $x_{i,L}$ belong to $\mathcal{X}(pN, p^2)_{p+2,\neq 0}$ and extend canonically to sections of its formal model $\mathfrak{X}(pN, p^2)_{p+2,\neq 0}$ over $\mathrm{Spf}(R)$.

Therefore we can define:

$$\varphi^*_{x_i}\big((\nabla_k)^\nu(F^{[p]})\big) \in \varphi_{x_i}\big(\mathbb{W}^{(\infty)}_{k+2\nu}\big)\big(\mathrm{Spf}(R)\big) \cong \mathbb{W}_{E_i, k+2\nu}.$$

## 6.2 Splitting the Hodge filtration.

With the notations of the previous section we notice that $E'$ has CM by $\mathcal{O}_{pc}$. We have isogenies $\lambda \colon E' \to E$ and $p-1$ quotients $\lambda_i \colon E' \to E_i$ for $i = 1, \ldots, p-1$ of degree $p$ and distinct form $E$.

By functoriality of VBMS the isogenies $\lambda_i$ induce morphisms $\lambda_i^* \colon \mathrm{H}^\sharp_{E_i} \to \mathrm{H}^\sharp_{E'}$. We define $H_{E',\tau} \oplus H_{E',\overline{\tau}} \subset \mathrm{H}_{E'}$ as in §3.3 and

$$\widetilde{\mathrm{H}}^\sharp_{E'} := H^\sharp_{E',\tau} \oplus H^\sharp_{E',\overline{\tau}} := \delta_{E'}\mathrm{H}_{E',\tau} \oplus \delta^p_{E'}\mathrm{H}_{E',\overline{\tau}} \subset H^\sharp_{E'}.$$

We then have the following analogue of 5.1

**Lemma 6.1.** *The image of* $\mathrm{H}^\sharp_{E_i}$ *via* $\lambda_i^*$ *is contained in* $\widetilde{\mathrm{H}}^\sharp_{E'}$.

*Proof.* The proof proceeds as the proof of Lemma 5.1. In this case Lemma 3.3 states that

(i) $\mathrm{Hdg}(E')\mathrm{diff}_{K/\mathbb{Q}}\omega_{E'}^{\vee} \subset \mathrm{H}_{E',\bar{\tau}} \subset \omega_{E'}^{\vee}$;

(ii) $\lambda_i^*\big(\omega_{E_i}^{\vee}\big) \subset p\mathrm{Hdg}(E_i)^{-1}\omega_{E'}^{\vee}$.

As in loc. cit. it suffices to show that $\delta_{E_i}^p p\mathrm{Hdg}(E_i)^{-1}\omega_{E'}^{\vee} \subset \mathrm{Hd}(E')\mathrm{diff}_{K/\mathbb{Q}}\delta_{E'}^p\omega_{E'}^{\vee}$. Since $\mathrm{diff}_{K/\mathbb{Q}} = \mathfrak{P}$, then $\mathrm{val}_p(\mathrm{diff}_{K/\mathbb{Q}}) = 1/2$ and we need to prove that $1/2 + \mathrm{val}_p(\delta_{E_i}p) \geq \mathrm{val}_p(\delta_{E'}^{2p-1})$ or equivalently that $\big((2p-1)p-1\big)\mathrm{val}_p(\delta_{E_i}) \leq 1/2$. But $\mathrm{val}_p(\delta_{E_i}) = \frac{1}{2p^2(p-1)}$ by Lemma 3.2. Hence we need to show that $2p^2 - p - 1 \leq p^3 - p^2$ or equivalently $3p^2 - p - 1 \leq p^3$ and this is true for any prime $p$.

□

Thanks to Lemma 6.1 we deduce that $\lambda_i^*\colon \mathbb{W}_{E_i,k+2\nu} \to \mathbb{W}_{E',k+2\nu}$ factors through $\mathbb{W}_{k+2\nu}\big(\widetilde{\mathrm{H}}_{E'}^{\sharp}, s\big)$. For the latter we have a canonical splitting $\Psi_i$ of the Hodge filtration $\mathfrak{w}_{E'}^{k+2\nu}\big[p^{-1}\big] \subset \mathbb{W}_{k+2\nu}\big(\widetilde{\mathrm{H}}_{E'}^{\sharp}, s\big)$. In particualr, we get

$$\Psi_i\big(\lambda_i^* \circ \varphi_{x_i}^*\big((\nabla_k)^{w(\chi)}(F^{[p]})\big)\big) \in p^{-a}\mathfrak{w}_{E'}^{k+2\nu} \subset \mathfrak{w}_{E'}^{k+2\nu}\big[p^{-1}\big],$$

where $a$ is a constant introduced in Remark §4.13. Let $\omega'$ be a generator of $\Omega_{E'/R}$ reducing to $s' := \mathrm{dlog}(P')\big(\mathrm{mod}\ p\underline{\delta}(E')^{-p}\big)$. In particular, $(\omega')^{k+2\nu}$ is a generator of $\mathfrak{w}_{E',R}^{k+2\nu}$ defining a trivialization $v_{\omega'}\colon \mathfrak{w}_{E',R}^{k+2\nu} \cong R$. We denote by $\omega := (\lambda^{\vee})^*(\omega')$, which is a generator of $\omega_E \otimes \mathbb{Q}$. Then,

**Definition 6.2.** We define

$$\delta_k^{\nu}(F^{[p]})(E/R, \psi_N, D, \omega) := -\sum_{i=1}^{p-1} v_{\omega'} \circ \Psi_i\Big(\alpha_i^* \circ \varphi_{x_i}^*\big((\nabla_k)^{\nu}(F^{[p]})\big)\Big) \in p^{-a}R \subset R[p^{-1}] = L$$

## 6.3 Definition of the $p$-adic $L$-function in the ramified case.

Let $F$ be a classical eigenform of level $\Gamma_1(N)$, weight $k \geq 2$ and character $\epsilon$ as in §2.3. For every $\chi \in \widehat{\Sigma}^{(2)}$ with weight $w(\chi) = \nu \in W(\mathbb{Q}_p)$, we define the $p$-adic $L$-function $L_p(F, \chi^{-1})$ by the same formula as in the inert case:

**Definition 6.3.**

$$L_p(F, \chi^{-1}) := \frac{p^{-a}m(c)}{m(p^2c, \mathfrak{N})} \sum_{\mathfrak{a} \in K^{(\mathfrak{N}pc)}\backslash \mathbb{A}_K^{(\mathfrak{N}pc)}/H^{c,p\mathfrak{N}}} \hat{\Pi}_F(\chi)\Big(\mathfrak{a}, \big(\mathfrak{a} * (A_0/R, t_0, D, P, \omega_0)\big)\Big)$$

$$:= \frac{p^{-a}m(c)}{m(p^2c, \mathfrak{N})} \sum_{\mathfrak{a} \in K^{(\mathfrak{N}pc)}\backslash \mathbb{A}_K^{(\mathfrak{N}pc)}/H^{c,p\mathfrak{N}}} \chi_{\nu}^{-1}(\mathfrak{a})\delta_k^{\nu}(F^{[p]})\big(\mathfrak{a} * (A_0, t_0, D, \omega_0)\big),$$

where the pair $(A_0, t_0)$ from the first section of the article was denoted $(E, t)$ since the beginning of this section, $D$ is a cyclic subgroup of $A_0[p^2]$ of order $p^2$ such that $D[p] \cap A_0[\mathfrak{P}] = \{0\}$ and the evaluation at $\mathfrak{a} * (A_0, t_0, D, P, \omega_0)$ is defined via Definition 6.2.

## 6.4 Interpolation properties in the case $p$ is ramified in $K$.

Let now $\chi \in \Sigma_{cc}^{(2)}(\mathfrak{N})$ be an algebraic Hecke character with weight $w(\chi) = j$ (i.e. the infinity type of $\chi$ is $(k+j, -j)$ with $j \in \mathbb{N}$). As in the inert case, we wish to relate the value $L_p(F, \chi^{-1})$ to the classical value $L_{\mathrm{alg}}(F, \chi^{-1})$. Recall that $F$ is an eigenform of weight $k \geq 2$, level $\Gamma_1(N)$, nebentypus $\epsilon$. Let $a_p$ be the $T_p$-eigenvalue of $F$. Then:

**Proposition 6.4.** *Assume that $\mathcal{O}_c^* = \{\pm 1\}$. Then*

$$L_p\big(F, \chi^{-1}\big) = \frac{p^k \mathcal{E}_p(a_p, \chi)}{\Omega_p(k,m)} L_{\mathrm{alg}}(F, \chi^{-1})$$

*wiith $\Omega_p(k,m) = \dfrac{(\Omega_p')^{k+2m}}{\widetilde{\Omega}_p^m}$ where $\widetilde{\Omega}_p \in R$ is defined in Lemma 6.6, $\Omega_p'$ is defined in Lemma 6.7 and $\mathcal{E}_p(a_p, \chi)$ is defined in §6.4.1. Furthermore we have*

$$\frac{k}{2(p-1)} + \frac{m}{p^2(p-1)} \leq v_p\big(\Omega_p(k,m)\big) \leq \frac{pk+2m}{2p(p-1)}.$$

If $F$ is the Eisenstein series $E_{k,\epsilon}$ of level $\Gamma_1(N)$, weight $k \geq 2$ and character $\epsilon$, then $a_p = 1 + p^{k-1}\epsilon(p)$ and we have:

**Corollary 6.5.** *Assume that $\mathcal{O}_K^* = \{\pm 1\}$. We have the interpolation formula*

$$L_p\big(E_{k,\epsilon}, \chi^{-1}\big) = \frac{p^k \mathcal{E}_p\big(1 + p^{k-1}\epsilon(p), \chi\big)}{\Omega_p(k,m)} L_{\mathrm{alg}}\big(K, \chi^{-1}\big).$$

### 6.4.1 The factor $\mathcal{E}(a_p, \chi)$ and the $p$-adic periods $\widetilde{\Omega}_p$ in the ramified case.

We set

$$\mathcal{E}_p(a_p, \chi) := 1 - \frac{1}{p^2} + \big(1 - \frac{1}{p^3}\big)\frac{a_p}{\chi(\mathfrak{P})p} - \big(1 - \frac{1}{p}\big)\frac{a_p^2}{\chi(p)}.$$

We recall that $\omega_0$ denoted the generator of the invariant differentials of $A_0$ used to compute $L_{\mathrm{alg}}(F, \chi^{-1})$. We let $\widetilde{A}_0$ be the quotient $A_0/D$.

**Lemma 6.6.** *With the above notations we have*

$$\frac{m(c)}{m(p^2c, \mathfrak{N})} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{cp^2})} \chi_j^{-1}(\mathfrak{a}) \theta_{\mathrm{Hodge}}^j(F^{[p]})\big(\mathfrak{a} * (A_0, t_0, D_0, \omega_0)\big) = \mathcal{E}_p(a_p, \chi)\frac{\widetilde{\Omega}_p^j}{p^{2j}} L_{\mathrm{alg}}(F, \chi^{-1})$$

*with $\widetilde{\Omega}_p \in R$ an element of valuation $2pv_p\big(\mathrm{Hdg}(\widetilde{A}_0)\big) \leq v_p\big(\widetilde{\Omega}_p\big) \leq (2p+2)v_p\big(\mathrm{Hdg}(\widetilde{A}_0)\big)$.*

*Proof.* We proceed as in the inert case by seeing $F^{[p]} \in \mathrm{H}^0(X(N, p^2), \omega_{\mathbb{E}}^k)$ and we denote also by $F^{[p]}$ its inverse image to a global section over $X(pN, p^2)$. In the ramified case we have one notable difference. Let $\mathfrak{P}$ be the prime ideal of $\mathcal{O}_K$ over $p$. Given the elliptic curve $A_0$ with CM by $\mathcal{O}_c$ among the $p+1$-subgroups of order $p$ of $A_0[p]$ we have one subgroup, namely $A_0[\mathfrak{P}]$,

which is stable under the action of $\mathcal{O}_c$ and we have $p$ which constitute one single orbit for the action of $\mathcal{O}_c$. Similalry $\mathcal{O}_c$ defines two orbits on the set of subgroups $D$ of order $p^2$, cyclic over $L$: one is given by the subgroups such that $D[p] = A_0[\mathfrak{P}]$ and the other by those subgroups such that $D[p] \cap A_0[\mathfrak{P}] = \{0\}$. Fix $D$ in one of the latter orbits.

Recall the conventions of lemma 5.9. Given a positive integer $d$ coprime to $c$ we set $m(dc) = |\mathrm{Pic}(\mathcal{O}_{dc})| = h_d|\mathrm{Pic}(\mathcal{O}_c)| = h_d m(c)$. For $p$ ramified we have:

1) $h_{p^2} = h'_{p^2} = \frac{(p-1)p^3}{(p-1)p} = p^2$.

2) $h_p = h'_p = \frac{(p-1)p}{p-1} = p$.

In our case we start with an elliptic curve $A$ with CM by $\mathcal{O}_K$ and we fix cyclic isogenies $\phi_c\colon A \to A_c =: A_0$ and $\phi_{dc}\colon A_c \to A_{dc}$ of degree $c$ and $d = p$, $p^2$ respectively, with $\phi_{pc}$ defined by the quotient $A_0/D[p]$ and $\phi_{p^2c}$ defined by the quotient $A_0/D$. We fix a basis element $\omega$ of the invariant differentials of $A$ and we let $\omega_c$ and $\omega_{dc}$ be the elements of the invariant differentials of $A_c$, resp. $A_{dc}$ such that $\omega = \phi_c^*(\omega_c) = \phi_{dc}^*(\omega_{dc})$. Similalry give a $\Gamma_1(N)$-level structure on $A$ we let $t_c$, resp. $t_{dc}$ be the induced level $\Gamma_1(N)$-level structure on $A_c$, resp. $A_{dc}$. Recall that $F^{[p]} = \big(1 - a_p V + \epsilon(p)p^{k-1}V^2\big)(F)$. For every $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{p^2c})$ it follows as in loc. cit.:

$$\theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a} * (A_c, t_c, \omega_c, D)\big) = \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_c, t_c, \omega_c)\big) +$$

$$-p^j a_p \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{pc}, \frac{1}{p}t_{pc}, p\omega_{pc})\big) +$$

$$+\epsilon(p)p^{2j+k-1}\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{p^2c}, \frac{1}{p^2}t_{p^2c}, p^2\omega_{p^2c})\big) =$$

$$= \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_c, t_c, \omega_c)\big) - p^j a_p \theta^j_{\mathrm{Hodge}}(F)\big((p^{-1}\mathfrak{a}) * (A_{pc}, t_{pc}, \omega_{pc})\big) +$$

$$+\epsilon(p)p^{2j+k-1}\theta^j_{\mathrm{Hodge}}(F)\big((p^{-2}\mathfrak{a}) * (A_{p^2c}, t_{p^2c}, \omega_{p^2c})\big).$$

Hence

$$\frac{m(c)}{m(p^2c, \mathfrak{N})} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{p^2c})} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a} * (A_c, t_c, \omega_c, D)\big) =$$

$$= \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_c, t_c, \omega_c)\big) +$$

$$-\frac{p^{-j}\chi^{-1}(p)a_p}{h_p} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{pc})} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{pc}, t_{pc}, \omega_{pc})\big) +$$

$$+\frac{\chi^{-1}(p^2)\epsilon(p)p^{-2j+k-1}}{h_{p^2}} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{p^2c})} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{p^2c}, t_{p^2c}, \omega_{p^2c})\big).$$

Denote by $A_{\mathfrak{P}}$ the quotient $A_0/A_0[\mathfrak{P}]$, by $t_{\mathfrak{P}}$ the induced $\Gamma_(N)$-level structure and by $\omega_{\mathfrak{P}}$ the invariant differential on $A_{\mathfrak{P}}$ whose pull-back to $A_0$ is $\omega_0$. For every $\mathfrak{a}$ in $\mathrm{Pic}(\mathcal{O}_{pc})$ we have

$$\mathfrak{a} * (A_{\mathfrak{P}}, t_{\mathfrak{P}}, \omega_{\mathfrak{P}}) = (\mathfrak{P}\mathfrak{a}) * (A_c/R, t_c, \omega_c).$$

As $\theta_{\mathrm{Hodge}} \circ T_p = p^{-1}T_p \circ \theta_{\mathrm{Hodge}}$, for every $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O}_c)$ we have

$$a_p p^j \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) = \big(\theta^j_{\mathrm{Hodge}}(F)|T_p\big)\big(\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) =$$

$$= \frac{1}{p} \sum_{\mathfrak{a} \in \rho_p^{-1}(\mathfrak{b})} \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{pc}, t_{pc}, \omega_{pc})\big) + \frac{1}{p}\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{\mathfrak{P}}, t_{\mathfrak{P}}, \omega_{\mathfrak{P}})\big).$$

We conclude that

$$\frac{p^{-j}\chi^{-1}(p)a_p}{h_p} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{pc})} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{pc}, t_{pc}, \omega_{pc})\big) =$$

$$= \frac{p^{-j}\chi^{-1}(p)a_p}{p}\big(a_p p^{j+1} - \chi_j(\mathfrak{P})\big) \sum_{\mathfrak{b} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{b})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) =$$

$$= \big(a_p^2\chi^{-1}(p) - a_p\chi^{-1}(\mathfrak{P})p^{-1}\big) \sum_{\mathfrak{b} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{b})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big).$$

Similalry, the correspondence $T_p^2(A_c, t_c, \omega_c\big)$ is defined by taking all subgroups $D$ of order $p^2$ of $A$ such that $D[p] \cap A[\mathfrak{P}] = \{0\}$, $p+1$-times the $p$-torsion of $A$ and all cyclic subgroups $D$ of order $p^2$ of $A$ such that $D[p] = A[\mathfrak{P}]$. Thus, denoting by $R_p$ the quotient of multiplication by $p$ and for $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O}_c)$, we compute

$$\frac{1}{p^2} \sum_{\mathfrak{a} \in \rho_{p^2}^{-1}(\mathfrak{b})} \theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_{p^2c}, t_{p^2c}, \omega_{p^2c})\big) =$$

$$= \big(\theta^j_{\mathrm{Hodge}}(F)|T_p^2\big)\big(\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) - \big(\theta^j_{\mathrm{Hodge}}(F)|\frac{p+1}{p^2}R_p\big)\big(\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) +$$

$$-\frac{1}{p}\ \big(\theta^j_{\mathrm{Hodge}}(F)|T_p\big)\big(\mathfrak{b} * (A_{\mathfrak{P}}, t_{\mathfrak{P}}, \omega_{\mathfrak{P}})\big) + \frac{1}{p^2}\big(\theta^j_{\mathrm{Hodge}}(F)|R_p\big)\big(\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big)$$

$$= \big(p^{2j}a_p^2 - \epsilon(p)p^{k+2j-1}\big)\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{b} * (A_c, t_c, \omega_c)\big) - p^{j-1}a_p\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{P}\mathfrak{b} * (A_c, t_c, \omega_c)\big).$$

Since

$$\frac{\chi^{-1}(p^2)\epsilon(p)p^{-2j+k-1}}{p^2}\ \big(p^{2j+2}a_p^2 - \epsilon(p)p^{k+2j+1} - p^{j-1}a_p\chi_j(\mathfrak{P})\big) =$$

$$= \chi^{-1}(p^2)\epsilon(p)p^{k-4}\big(p^3a_p^2 - \epsilon(p)p^{k+2} - a_p\chi(\mathfrak{P})\big),$$

putting everything together we get the formula

$$\frac{m(c)}{m(p^2c, \mathfrak{N})} \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_{p^2c})} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F^{[p]})\big(\mathfrak{a} * (A_c, t_c, \omega_c, D)\big) =$$

$$= (*) \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_j^{-1}(\mathfrak{a})\theta^j_{\mathrm{Hodge}}(F)\big(\mathfrak{a} * (A_c, t_c, \omega_c)\big)$$

and, using that $\chi(p) = \epsilon(p)p^k$,

$$(*) = 1 - a_p^2\chi^{-1}(p) + a_p\chi^{-1}(\mathfrak{P})p^{-1} + \chi^{-1}(p^2)\epsilon(p)p^{k-4}\big(p^3a_p^2 - \epsilon(p)p^{k+2} - a_p\chi(\mathfrak{P})\big) =$$

$$1 - \frac{1}{p^2} + \big(1 - \frac{1}{p^3}\big)\frac{a_p}{\chi(\mathfrak{P})p} - \big(1 - \frac{1}{p}\big)\frac{a_p^2}{\chi(p)} = \mathcal{E}_p(a_p, \chi).$$

The conclusion, with the appearance of the factor $\widetilde{\Omega}_p$, follows as in the proof of Lemma 5.9.

□

### 6.4.2 The $p$-adic periods $\Omega'_p$ in the ramified case.

Let $D_0 \subset A_0$ be a cyclic subgroup of order $p^2$ such that $D_0 \cap A_0[\mathfrak{P}] = \{0\}$, let $t_0$ be a generator of $A_0[\mathfrak{N}]$ and let $\omega_0$ be a generator of the invariant differentials of $A_0$. Let $A'_0 := A_0/D[p]$ and $\widetilde{A}_0 := A_0/D_0$. We have an isogeny $\lambda\colon A_0 \to A'_0$. Having chosen a generator $\omega'$ of $\Omega^{\mathrm{can}}_{A'_0/R}$, we defined $\omega := \big(\lambda^\vee\big)^*(\omega')$. We define $\Omega'_p$ by the equality: $\omega = \Omega'_p\omega_0$ and we have:

**Lemma 6.7.** *The $p$-adic valuation of $\Omega'_p$ is $\frac{1}{2(p-1)}$.*

*Proof.* Proceed as in the proof of Lemma 5.10 using that the $p$-adic valuation of $\mathrm{Hdg}(A'_0) = \frac{1}{2p}$ thanks to Lemma 3.2.

□

Finally we can prove Proposition 6.4. It follows from Lemma 3.2 that $\mathrm{Hdg}(\widetilde{A}_0)$ has $p$-adic valuation $1/(2p^2)$. The Proposition follows then from Lemma 6.6 and Lemma 6.7.

# 7 Special values $L_p(F, \chi^{-1})$, for $\chi \in \Sigma^{(1)}_{cc}(\mathfrak{N})$.

In this section we give formulae for the special values of our $p$-adic $L$-functions at algebraic Hecke characters in $\Sigma^{(1)}_{cc}(\mathfrak{N})$. In the case $p$ is split in $K$, these formulae have been called "$p$-adic Gross-Zagier" formulae in [BDP], if $F$ is a cuspform and they have been called "Kronecker limit formulae" in [K2], if $F$ is an Eisenstein series.

In the next two subsections we prove such formulae in the cases in which $p$ is inert, respectively ramified in $K$.

## 7.1 The $p$-adic Gross-Zagier formulae.

We work under the assumptions of §2.3 where $F = f$ is a normalized new cuspform for $\Gamma_1(N)$ of weight $k \geq 2$ and character $\epsilon$. We recall that the set $\Sigma_{cc}(\mathfrak{N})$ of central critical algebraic Hecke characters of finite type $(c, \mathfrak{N}, \epsilon)$ decomposes as $\Sigma_{cc}(\mathfrak{N}) = \Sigma^{(1)}_{cc}(\mathfrak{N}) \cup \Sigma^{(2)}_{cc}(\mathfrak{N})$ where

- $\Sigma^{(1)}_{cc}(\mathfrak{N})$ is the subset of characters of $\Sigma_{cc}(\mathfrak{N})$ having infinity type $(k-1-j, 1+j)$, with $0 \leq j \leq k-2$

and

- $\Sigma^{(2)}_{cc}(\mathfrak{N})$ is the subset of characters of $\Sigma_{cc}(\mathfrak{N})$ having infinity type $(k+j, -j)$ for $j \geq 0$.

The goal of the present section is to prove the following:

**Theorem 7.1.** *Let $\chi \in \Sigma^{(1)}_{cc}(\mathfrak{N})$ be a character of infinity type $(k-1-j, 1+j)$ with $0 \leq j \leq r := k-2$. Assume that $\mathcal{O}_c^* = \{\pm 1\}$. Then*

$$L_p(f, \chi^{-1}) = \frac{p^{-k}\mathcal{E}_p(a_p, \chi)}{\Omega_p(-k, 2(j+1))} \left( \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \frac{c^{-j}}{j!} \chi^{-1}(\mathfrak{a}) \mathbf{N}(\mathfrak{a}) \mathrm{AJ}_F(\Delta_{\varphi_\mathfrak{a}\varphi_0}) \big(\omega_f \wedge \omega_A^j \wedge \eta_A^{r-j}\big) \right).$$

*Here and elsewhere $\omega_f$ is $f$ seen as a section of $\mathrm{H}^1_{\mathrm{dR}}\big(X_1(N)^{\mathrm{an}}, \mathrm{Sym}^k(\mathrm{H}_{\mathbb{E}})\big)$.*

We explain the notation. The pair $(A, t_A)$ denotes an elliptic curve with CM by $\mathcal{O}_K$ and $\Gamma_1(\mathfrak{N})$-level structure, $\omega_A$ is a generator of the invariant differentials of $A$ and $\eta_A$ is an element of $\mathrm{H}^1_{\mathrm{dR}}(A)$ such that $\langle \omega_A, \eta_A \rangle = 1$ via the Poincaré pairing. Morever, $\varphi_0 \colon A \to A_0$ is a cyclic isogeny of degree $c$ so that $A_0$ has CM by $\mathcal{O}_c$. For every $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)$ we have an isogeny $\varphi_{\mathfrak{a}} \colon A_0 \to \mathfrak{a} * A_0$ and $\mathrm{AJ}_F(\Delta_{\varphi_{\mathfrak{a}}\varphi_0})$ is the $p$-adic Abel-Jacobi map of a generalised Heegner cycle constructed in [BDP, §2] over the modular point $\varphi_{\mathfrak{a}} \circ \varphi_0 \colon A \to \mathfrak{a} * A_0$. Finally $\Omega_p(-k, 2(j+1)) = (\Omega'_p)^{2(j+1)-k}/\widetilde{\Omega}_p^{j+1}$ is the $p$-adic period defined in Proposition 5.11 , respectively Proposition 6.4 when $p$ is inert, respectively ramified in $K$.

The rest of the section is devoted to the proof of Theorem 7.1 As explained in [BDP, §5.3] the characters in $\Sigma^{(1)}_{cc}(\mathfrak{N})$ can be realized in the completion $\widehat{\Sigma}^{(2)}$ of $\Sigma^{(2)}_{cc}(\mathfrak{N})$ defined in §2.2, i.e. they are $p$-adic limits of characters in $\Sigma^{(2)}_{cc}(\mathfrak{N})$. In particular, evaluating the $p$-dic $L$-function at $\chi^{-1}$ we obtain:

$$L_p(f, \chi^{-1}) = \frac{m(c)}{m(p^2 c, \mathfrak{N})} \Big( \sum_{\mathfrak{a}} \chi^{-1}_{-1-j}(\mathfrak{a}) \delta_k^{-1-j} \big( f^{[p]} \big) \big( \mathfrak{a} * (A_0, t_0, D, \omega) \big) \Big)$$

The evaluation at the point $\mathfrak{a} * (A_0, t_0, D, \omega)$ is done using Definition 5.3 in the inert case, respectively Definition 6.2 in the ramified case.

Following [BDP] we explain how one can compute the Abel-Jacobi image of the generalized Heegner cycle in terms of a Coleman primitive of our modular form $f$. Let us denote by $\mathcal{X}$ the open analytic subspace of the modular curve $X_1(N)$ defined in §4.1. It is a wide open rigid subspace of $X_1(N)^{\mathrm{an}}$ and consists of the points $(E, t)$ such that $E$ has a canonical subgroup of order $p$. Let us denote by $\Phi$ the Frobenius on $\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}, \mathrm{Sym}^r \mathrm{H}_E\big)$ and by $P(X) \in \mathbb{Q}_p[X]$ a polynomial such that: $P(\Phi)([\omega_f]) = 0$ and $P(\Phi)$ defines an automorphism of $\mathrm{H}^0_{\mathrm{dR}}(\mathcal{X}, \mathrm{Sym}^r(\mathrm{H}_E)^{\mathrm{loc}})$; we recall that $[\omega_f]$ denotes the cohomology class of $f$ in $\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}, \mathrm{Sym}^r \mathrm{H}_E\big)$, and $\mathrm{Sym}^r \mathrm{H}^{\mathrm{loc}}_E$ denotes the sheaf of locally analytic sections (i.e. analytic in every residue class of $X_1(N)^{\mathrm{an}}$) of the sheaf $\mathrm{Sym}^r \mathrm{H}_E$. Recall from [Co, §11] that we have a Coleman primitive $G$ of $f$: this is a section $G$ over $X_1(N)^{\mathrm{an}}$ of $\mathrm{Sym}^r \mathrm{H}^{\mathrm{loc}}_E$ such that $\nabla(G) = f$ and such that $P(\Phi)(G|_{\mathcal{X}})$ is an analytic section of $\mathrm{Sym}^r \mathrm{H}_E$ on some overconvergent neighbourhood $\mathcal{X}' \subset \mathcal{X}$ of the ordinary locus in $X_1(N)^{\mathrm{an}}$. It then follows that $G$ is unique up to a horizontal section of $\mathrm{Sym}^r \mathrm{H}_E$ on $\mathcal{X}'$, i.e., it is unique if $r > 0$ and unique up to a constant if $r = 0$.

This is related to $\mathrm{AJ}_F(\Delta_{\varphi_{\mathfrak{a}}\varphi_0})(\omega_f \wedge \omega_A^j \eta_A^{r-j})$ as follows. The isogeny $\varphi \colon A \to A_0$ and the $\Gamma_1(\mathfrak{N})$-level structure $t_A$ on $A$ induces a $\Gamma_1(\mathfrak{N})$-level structure $t_0$ on $A_0$. The element $\eta_A$ defines a generator $\omega_A$ of the invariant differentials such that $\langle \omega_A, \eta_A \rangle = 1$ via the Poincaré pairing. Thus we get an invariant differential $\omega_0$ on $A_0$ such that $\varphi_0^*(\omega_0) = \omega_A$. Coleman's primitive $G$, being defined over $X_1(N)^{\mathrm{an}}$, can be evaluated at the points $\mathfrak{a} * (A_0, t_0)$ and, using the CM action, we can decompose $\mathrm{Sym}^{k-2}(\mathrm{H}_{\mathfrak{a} * A_0}) = \oplus_{i=0}^{k-2} \mathrm{Sym}^{k-2}(\mathrm{H}_{\mathfrak{a} * A_0})_{\tau^{r-i}\overline{\tau}^i}$ into eigenspaces for the action of $K$. Then we can decompose $G\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big) = \sum_{i=1}^{k-2} (-1)^i G_i\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big) \omega_{\mathfrak{a}}^{r-j} \eta_{\mathfrak{a}}^j$, where $\omega_{\mathfrak{a}}, \eta_{\mathfrak{a}}$ is a basis of $\mathrm{Sym}^r \mathrm{H}_{\mathfrak{a} * A_0}$ adapted to the $K$-decomposition such that $\omega_{\mathfrak{a}}$ corresponds to $\omega_0$.

**Lemma 7.2.** [BDP, Lemma 3.22]. *We have* $\mathrm{AJ}_F(\Delta_{\varphi_{\mathfrak{a}}\varphi_0})(\omega_f \wedge \omega_A^j \eta_A^{r-j}) = c^j \mathbf{N}(\mathfrak{a})^{-1} G_j\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big)$.

Thus, in order to prove Theorem 7.1 we are left to prove the formula:

$$L_p(f,\chi^{-1})=\frac{\widetilde{\Omega}_p^{j+1}\mathcal{E}_p(a_p,\chi)}{(\Omega'_p)^{2(j+1)-k}p^k}\left(\frac{1}{j!}\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}\chi_{-1-j}^{-1}(\mathfrak{a})G_j\big(\mathfrak{a}*(A_0,t_0,\omega_0)\big)\right). \tag{2}$$

We'll first prove some properties of the Coleman primitives. Let $f$ be our classical cuspidal eigenform of weight $k\geq 2$ and level $\Gamma_1(N)$. Let $P(X)$ be a polynomial with the property (1) $P(\Phi)([\omega_f])=0$ and (2) $P(\Phi)$ defines an automorphism of $\mathcal{H}^0:=\mathrm{H}^0_{\mathrm{dR}}\big(\mathcal{X}',\mathrm{Sym}^r(\mathrm{H}_{\mathcal{E}})^{\mathrm{loc}}\big)$ over a strict neighbourhood $\mathcal{X}'$ of the ordinary locus where we have a lift $\Phi$ of Frobenius. Then

**Lemma 7.3.** *Let $G$ and $G'$ be two Coleman primitives of $f$ such that $P(\Phi)(G)$ and $P(\Phi)(G')$ are analytic over a strict neighbourhood $\mathcal{X}'$ of the ordinary locus. Then we have:*

*a) If $G(q)=G'(q)$ then $G=G'$ (where $G(q)$ and $G'(q)$ denote the $q$-expansions of $G$ and $G'$ respectevely).*

*b) If $T\subset\mathcal{X}'$ is an admissible open and $G|_T=G'|_T$ then $G=G'$.*

*Proof.* We set $F:=G-G'$, then $\nabla(F)=\nabla(G)-\nabla(G')=f|_{\mathcal{X}'}-f|_{\mathcal{X}'}=0$, therefore $F\in\mathcal{H}^0$. We also have: $P(\Phi)(F)=P(\Phi)(G)-P(\Phi)(G')\in\mathrm{H}^0\big(\mathcal{X}',\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})\big)$, i.e., $P(\Phi)(F)$ is an analytic section of $\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})$.

a) Let now $F(q)=G(q)-G'(q)$. Then $P(\Phi)(F)(q)=P(\Phi)(G)(g)-P(\Phi)(G')(q)=0$ and as $P(\Phi)(F)$ is an analytic section of $\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})$, we have $P(\Phi)(F)=0$. But by property 2) above $P(\Phi)$ is an automorphism of $\mathcal{H}^0$, therefore $F=0$, i.e., $G=G'$.

b) Similarly we have $P(\Phi)(F)|_{P(\Phi)^{-1}(T)}=0$ and as $P(\Phi)^{-1}(T)$ is an open of $\mathcal{X}'$ we have $P(\Phi)(F)=0$ by analytic continuation which implies, as above, $F=0$, i.e., $G=G'$. □

We define $G^{[p]}:=G|(1-VU)=G|\big(1-T_p\circ V+\frac{1}{p}[p]V^2\big)$, viewed as a locally analytic section of $\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})$ on $\mathcal{X}(pN,p^2)$ (the rigid analytic space associated to the curve $X(pN,p^2)$ defined in §4.2.1). Let $G_r^{[p]}$ be the locally analytic global section of $\omega_E^{-r}$ over $\mathcal{X}(pN,p^2)$ given by

$$G_r^{[p]}:=(-1)^r\times\text{ the image of }G^{[p]}\text{ via the quotient map }\mathrm{Sym}^r(\mathrm{H}_E)\to\omega_E^{-r}.$$

**Remark 7.4.** 1) The sign in the above definition is there so that the restriction of $G_r^{[p]}$ to the ordinary locus agrees with the section denoted $G_r^\flat$ in [BDP], proposition 3.24.

2) In [BDP] the objects $G|(UV-VU)$ and $f|(UV-VU)$ are denoted by $G^\flat$ and respectively $f^\flat$ but we prefer to follow the later notations of [DR1] of $G^{[p]}$ and respectively $f^{[p]}$ for the same objects.

Recall also from §4.2.3 that we have a sheaf $\mathbb{W}_{-r}$ over the rigid open subspace $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$ of the modular curve $\mathcal{X}(pN,p^2)$ with a connection $\nabla^j\colon\mathbb{W}_{-r}\to\mathbb{W}_{-r+2j}$ . Moreover we have an inclusion $\omega_E^{-r}=\mathrm{Fil}_0\mathbb{W}_{-r}\subset\mathbb{W}_{-r}$. Furthermore, as $-1-j\in\mathbb{Z}$ can be seen as a weight in $W(\mathbb{Q}_p)$ then $\nabla^{-1-j}(f^{[p]})$ was constructed in Theorem 4.13 as a section of $\mathbb{W}_{r-2j}$ over $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$.

**Proposition 7.5.** *(1) The section $G^{[p]}$ is analytic on $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$ and $\nabla(G^{[p]})=f^{[p]}$;*

*(2) $G_r^{[p]}$ is an analytic section of $\omega_E^{-r}$ on $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$ and we have $\nabla^{r-j}G_r^{[p]}=r!\nabla^{-1-j}\big(f^{[p]}\big)$ as sections of $\mathbb{W}_{r-2j}$ over $\mathcal{X}(pN,p^2)_{p+2,\neq 0}$.*

*Proof.* Using the isomorphism $\mathcal{X}(pN,p^2)_{p+2,\neq 0}\cong \amalg_{j=1}^{p-1}\mathcal{IG}_{1,p+2}$ of Lemma 4.12 and the definition of $\nabla^{-1-j}$ it suffices to prove both statements on each of the components $\mathcal{IG}_{1,p+2}$.

(1) Following the proof of Proposition 4.16 we have that $G^{[p]}$ is, on each of the components $\mathcal{IG}_{1,p+2}$, the pull-back of $G^{[p]}=G|(1-VU)$ viewed as a section over the strict neighbourhood $\mathcal{X}_{p+2}$ of the ordinary locus in $X_1(N)$ (see section 8 for the definition of the operators $V$, $U$ on the sections of our locally analytic sheaves). Thus it suffices to prove (1) over $\mathcal{X}_{p+2}$. Write $f^{[p]}=f|(1-VU)=f|(1-a_pV+\epsilon(p)p^{k-1}V^2)$, where the last polynomial of degree 2 in $V$ can be written as a polynomial $P$ for a Frobenius lift $\Phi$ on $\mathcal{H}^1:=\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X},\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})\big)$. Moreover we have $P(\Phi)([\omega_f])=[f^{[p]}]=0$ as $\mathcal{H}^1$ is finite dimensional and in a finite dimensional vector space if $UV=1$ then $VU=1$ as well. Also $P(\Phi)=(1-VU)$ is an isomorphism on $\mathcal{H}^0$ by the calculations of lemma 11.1 of [Co], i.e., the operator $(1-VU)=P(\Phi)$ is one of the polynomials in Frobenius which can be used to define the Coleman primitives. Therefore, $G^{[p]}$ is an analytic section of $\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})$ on a strict neighbourhood $\mathcal{X}'\subset\mathcal{X}$ of the ordinary locus of $X_1(N)^{\mathrm{an}}$. Secondly, as both $\nabla(G^{[p]})$ and $f^{[p]}$ are analytic overconvergent sections of $\mathrm{Sym}^r\mathrm{H}_{\mathbb{E}}$ which by Proposition 8.1 agree on the admissible open $Y^{\mathrm{ord}}$, the ordinary locus of $\mathcal{X}_1(N)$ minus the residue classes of the cusps. Therefore they are equal on $\mathcal{X}'$. And thirdly, as the cohomology class $[f^{[p]}]=0$ in $\mathcal{H}^1:=\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X},\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})\big)$ and as $\mathcal{X}$ is a wide open analytic space, i.e. a Stein space so that $\mathcal{H}^1=\dfrac{\mathrm{H}^0\big(\mathcal{X},\mathrm{Sym}^{r+2}\mathrm{H}_{\mathbb{E}}\big)}{\nabla\Big(\mathrm{H}^0\big(\mathcal{X},\mathrm{Sym}^r\mathrm{H}_{\mathbb{E}}\big)\Big)}$, there is a section $G'\in\mathrm{H}^0\big(\mathcal{X},\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})\big)$, unique up to horizontal section of the sheaf $\mathrm{Sym}^r\mathrm{H}_{\mathbb{E}}$ such that $f^{[p]}=\nabla(G')$. Choose $G'$ such that $G'|_{\mathcal{X}'}=G^{[p]}$. It follows that $G^{[p]}$ can be analytically extended to $\mathcal{X}$ (by $G'$).

(2) It follows from (1) that $G_r^{[p]}$ is an overconvergent modular form defined on each component $\mathcal{IG}_{1,p+2}$ as it is a quotient of $G^{[p]}$. We first check the second statement for $j=0$, on $q$-expansions. We write $G^{[p]}(q)=\sum_{j=0}^r g_j^{[p]}(q)V_{r,j}$ according to the canonical basis of $\mathrm{Sym}^r(\mathrm{H}_E)$ at the cusp. In this case [AI, Thm. 4.3] gives the expression of the $q$-expansion of

$$\nabla^{-1-r}\big(f^{[p]}\big):=\partial^{-1-r}\big(f^{[p]}(q)\big)V_{-r,0}+\sum_{j\geq 1}\binom{-1-r}{j}\prod_{i=0}^{j-1}(-i)\partial^{-r-1-j}\big(f^{[p]}(q)\big)V_{-r,j}=$$

$$=\partial^{-r-1}\big(f^{[p]}(q)\big)V_{-r,0}.$$

On the other hand $r!\partial^{-r-1}\big(f^{[p]}(q)\big)=g_r^{[p]}(q)$ (see [BDP, Prop. 3.24] or the proof of Proposition 7.7) and the claim follows as $G_r^{[p]}(q)=g_r^{[p]}(q)V_{-r,0}$ . This implies that $r!\nabla^{-1}(f^{[p]})=\nabla^rG_r^{[p]}$. Therefore for every $0<j\leq r$ we have $r!\nabla^{-1-j}(f^{[p]})=\nabla^{-j}\big(\nabla^{-1}(f^{[p]})\big)=\nabla^{-j}\big(\nabla^r(G_r^{[p]})\big)=\nabla^{r-j}(G_r^{[p]})$. □

**Remark 7.6.** As observed in proposition 7.5 the element $G^{[p]}$ is an analytic section of $\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})$ on $\mathcal{X}$. Its pullback to $X(N,p^2)^{\mathrm{an}}$ extends to a global, locally analytic section of the same sheaf which is analytic on the inverse image of $\mathcal{X}$. This follows as $f^{[p]}$ is a classical modular form on $X(N,p^2)^{\mathrm{an}}$ and $G$ is a global, locally analytic section on $X_1(N)^{\mathrm{an}}$.

We fix an $\mathfrak{a}\in\mathbb{A}_K^{(\mathfrak{p}pa)}$ and denote by $(B,t_B,D_B,P_B,\omega_B):=\mathfrak{a}*(A_0,t_0,D,P,\omega)$ so that $x_0:=(B,t_B,D_B,P_B)\in\mathcal{X}(pN,p^2)$. Similarly consider the usual quadruple $\big(B',t_{B'},D_{B'},P_{B'},\omega_{B'}\big)$, where $B':=B/D_B[p]$, $D_{B'}$ is a subgroup scheme of $B'[p^2]$ of order $p^2$, generically cyclic such that

$D_{B'}[p] = D_B/D_B[p]$ and denote $x'$ the point corresponding to $(B', t_{B'}, D_{B'}, P_{B'}) \in \mathcal{X}(pN, p^2)$. For each $1 \leq i \leq p-1$ consider the quadruple $(B_i, t_i, D_i, P_i, \omega_i)$ defined by $B_i := B'/C_i$, for $i = 1, 2, \ldots, p-1$, where $C_1, C_2, \ldots, C_{p-1}$ are the subgroups of $B'$ of order $p$ distinct from $C' := D_B/D_B[p]$ and the canonical subgroup of $B'$. We let $x_i$ denote the point $(B_i, t_i, D_i, P_i) \in X(pN, p^2)^{\mathrm{an}}$ .

We recall that $G^{[p]} \in \mathrm{H}^0\big(\mathcal{X}, \mathrm{Sym}^r(\mathrm{H}_E)\big)$ and as $\nabla_k^{-1}(f^{[p]}) = G^{[p]}$, then $G^{[p]}$ can be evaluated at $(B_i, t_i, D_i)$ for every $i = 1, 2, \ldots, p-1$. Let $\lambda_i \colon B' \to B_i$ denote the isogeny defining $B_i$, then $\lambda_i^*(G^{[p]}(x_i)) \in \mathrm{Sym}^r \mathrm{H}^1_{\mathrm{dR}}(B')$.

For every $i = 1, 2, \ldots, p-1$ we decompose $\lambda_i^*\big(G^{[p]}(B_i, t_i, D_i)\big)$ in $\mathrm{Sym}^r \mathrm{H}^1_{\mathrm{dR}}(B')$ according to the $K$-action as follows (we remind the reader that in this section $\mathrm{Sym}^r \mathrm{H}^1_{\mathrm{dR}}(B')$ is seen as a finite dimensional vector space over a finite extension of $\mathbb{Q}_p$ with an action of $K$, so the decomposition is a full decomposition)

$$\lambda_i^*\Big(G^{[p]}(B_i, t_i, D_i)\Big) = \sum_{j=0}^{r} (-1)^j G^{[p]}_{i,j}(B_i, t_i, D_i, \omega_i) \omega_i^{r-j} \eta_i^j,$$

where $\omega_i$ is the invariant differential such that $\lambda_i^*(\omega_i)$ is the pull-back of $\omega_0$ via the isogeny $B' \to B$, dual to the quotient $B \to B' = B/D_B[p]$, and $\eta_i$ is an element of $\mathrm{H}^1_{\mathrm{dR}}(B_i)$ such that the pair $\omega_i, \eta_i$ is a basis compatible with the $K$ decomposition.

As $G^{[p]}|U = G|\big(U(1 - VU)\big) = 0$ and as we have $U(x') = \{x_0, x_1, \ldots, x_{p-1}\}$, the following equality holds:

$$p^{2j-r}(\Omega'_p)^{r-2j} G_j^{[p]}\big(\mathfrak{a} * (A_0, t_0, D, \omega_0)\big) = p^{2j-r} G_j^{[p]}\big(\mathfrak{a} * (A_0, t_0, D, \omega)\big) = -\sum_{i=1}^{p-1} G^{[p]}_{i,j}(B_i, t_i, D_i, \omega_i).$$

(As explained in the proof of Lemma 5.7 the differential on $A_0$ corresponding to the differentials $\omega_i$ via the $U$ correspondence is $\omega/p$; this explains the appearence of the factor $p^{2j-r}$. The extra power $\Omega'_p$ comes from the expression $\omega = \Omega'_p \omega_0$ of Lemma 5.10 and Lemma 6.7 expressing the differential $\omega$ on $A_0$ used to compute the $p$-adic $L$-function and the chosen generator $\omega_0$ of the invariant differentials of $A_0$)

**Proposition 7.7.** *We have*

$$\delta_k^{-1-j}\big(f^{[p]}\big)\big(\mathfrak{a} * (A_0, t_0, D, \omega)\big) = -\frac{1}{j!} \sum_{i=1}^{p-1} G^{[p]}_{i,j}(B_i, t_i, D_i, \omega_i) = \frac{p^{2j-r}(\Omega'_p)^{r-2j}}{j!} G_j^{[p]}\big(\mathfrak{a} * (A_0, t_0, D, \omega_0)\big).$$

*Proof.* By definition $\delta_k^{-1-j}\big(f^{[p]}\big)\big(\mathfrak{a} * (A_0, t_0, D, \omega)\big) = -\sum_{i=1}^{p-1} \delta_k^{-1-j}\big(f^{[p]}\big)(B_i, t_i, D_i, P_i, \omega_i)$. Thanks to Proposition 7.5 we have

$$\delta_k^{-1-j}\big(f^{[p]}\big)(B_i, t_i, D_i, P_i, \omega_i) = \frac{1}{r!}\big(\nabla^{r-j} G^{[p]}_{i,r}\big)_0(B_i, t_i, D_i, \omega_i),$$

where $\big(\nabla^{r-j} G^{[p]}_{i,r}\big)_0\big(B_i, t_i, D_i, \omega_i\big)$ is the 0-th component of $\big(\nabla^{r-j} G^{[p]}_{i,r}\big)\big(B_i, t_i, D_i, \omega_i\big)$ for the action of $K$.

Let $\mathcal{D}_i$ be the analytic generic fiber associated to the formal completion of the modular curve $X(pN, p^2)$ over $R$ at the $R$-valued point $x_i$ defined by $(B_i, t_i, D_i, P_i)$. We also denote by $\mathcal{D}'_i$ the

analytic generic fiber associated to the formal completion of the modular curve $X_1(N)$ at the point $x_i' := (B_i, t_i) \in \mathcal{X}$. As $x_i'$ is a smooth point of the special fiber of $X_1(N)$, then $\mathcal{D}_i'$ is a wide open disk. The forgetful functor gives a finite map of adic spaces $\phi_i \colon \mathcal{D}_i \to \mathcal{D}_i'$ which sends $x_i \to x_i'$.

We first work over $\mathcal{D}_i'$, namely we denote $\mathrm{H}_{\mathcal{D}_i'} := \mathrm{H}_{\mathbb{E}}|_{\mathcal{D}_i'}$. We obviously have $\mathrm{H}_{\mathcal{D}_i'} \cong \mathrm{H}_{x_i'} \otimes \mathcal{O}_{\mathcal{D}_i'}$ so that $\mathrm{H}_{x_i'} = \mathrm{H}^1_{\mathrm{dR}}(B_i)$ is isomorphic to the space of horizontal sections for the Gauss-Manin connection $\nabla$ on $\mathrm{H}_{\mathcal{D}_i'}$. Let us fix a basis $\omega_i$, $\eta_i$ respecting the $K$-decomposition of $\mathrm{H}_{x_i'} = \mathrm{H}_{B_i}$.The Hodge filtration of $\mathrm{H}_{\mathcal{D}_i'}$ is generated by an element $\omega_i' := \omega_i \otimes 1 + b_i(\eta_i \otimes 1)$ for some global section $b_i \in \mathcal{O}_{\mathcal{D}_i'}$ such that $b_i(x_i) = 0$. Then $\omega_i'$ and $\eta_i' := \eta_i \otimes 1$ provide a basis for of $\mathrm{H}_{\mathcal{D}_i'}$. The Kodaira-Spencer isomorphism KS identifies $\omega_{\mathbb{E}}^{\otimes 2}|_{\mathcal{D}_i'} \cong \Omega^1_{\mathcal{D}_i'}$. Since $\nabla\big(\omega_i'\big) = (\eta_i \otimes 1) \otimes db_i$ then KS is defined by $db_i$ with respect to the basis $\omega_i \otimes 1$ and $\eta_i \otimes 1$ so that $db_i$ is a basis element of $\Omega^1_{\mathcal{D}_i'}$. We let $\partial_i$ be the derivation of $\mathcal{O}_{\mathcal{D}_i'}$ dual to $db_i$ and compute the derivation $\nabla(\partial_i) \colon \mathrm{H}_{\mathcal{D}_i'} \to \mathrm{H}_{\mathcal{D}_i'}$ given by the Gauss-Manin connection contracted by $\partial_i$; we have $\nabla(\partial_i)(\omega_i') = \eta_i'$ and $\nabla(\partial_i)(\eta_i') = 0$. We write $\mathrm{Sym}^r\big(\mathrm{H}_{\mathbb{E}}|_{\mathcal{D}_i'}\big) = \oplus_{j=0}^r \mathcal{O}_{\mathcal{D}_i'}(\omega_i')^j(\eta_i')^{r-j}$. If we specialize at $x_i'$ this decomposition induces the decomposition of $\mathrm{Sym}^r \mathrm{H}_{x_i'}$ into eigensapces for the $K$-action. Write $G^{[p]}|_{\mathcal{D}_i'} = \sum_{j=0}^r (-1)^j g_{i,j}(\omega_i')^{r-j}(\eta_i')^j$ and $f^{[p]}|_{\mathcal{D}_i'} = h(\omega_i')^k$. Notice that the $g_{i,j}$'s, $G^{[p]}$ and $h$ are only defined and analytic on an annulus of $\mathcal{D}_i'$ close to its boundary, annulus which contains $x_i'$ as $G^{[p]}$ and $f^{[p]}$ are only an overconvergent section of $\mathrm{Sym}^r(\mathrm{H}_{\mathbb{E}})$ respectively an overconvergent modular form on $\mathcal{X}$. Then, restricting to the annulus where these sections are defined we have: $\nabla(\partial_i)(G^{[p]}|_{\mathcal{D}_i'}) = f^{[p]}|_{\mathcal{D}_i'}$ implies that $\partial_i(g_{i,0}) = h$ and $\partial_i(g_{i,j}) - (r-j+1)g_{i,j-1} = 0$ for $1 \leq j \leq r$. This implies that, for all $0 \leq j \leq r$, and on the annulus where all the sections are defined (in particular at $x_i'$) we have $\Big(\nabla^{r-j} G_r^{[p]}\Big)_0 = \partial_i^{r-j}(g_{i,r}) = (r-j)! g_{i,j}$. Evaluating at $x_i'$ we conclude that

$$\big(\nabla^{r-j} G_r^{[p]}\big)_0(B_i, t_i, \omega_i) = \big(\nabla^{r-j} G_r^{[p]}\big)_0(x_i') = (r-j)! g_{i,j}(x_i') = (r-j)! G_{i,j}^{[p]}(B_i, t_i, \omega_i).$$

Now we pull-back these equalities to $\mathcal{D}_i$ via the tamely ramified map $\phi_i \colon \mathcal{D}_i \to \mathcal{D}_i'$ and we evaluate at $x_i$. We obtain the claimed equality

$$\delta_k^{-1-j}\big(f^{[p]}\big)(B_i, t_i, D_i, P_i, \omega_i) = \frac{1}{j!} G_{i,j}^{[p]}(B_i, t_i, D_i, \omega_i).$$

□

In particular, $L_p(f, \chi^{-1}) = (*)\Big(\frac{p^{2j-r}(\Omega_p')^{r-2j}}{j!} \sum_{\mathfrak{a}} \chi_{-1}^{-1-j}(\mathfrak{a}) G_j^{[p]}\big(\mathfrak{a} * (A_0, t_0, D, \omega_0)\big)\Big)$. In order to prove (2) and conclude the proof of Thorem 7.1, it remains to show the following:

**Lemma 7.8.** *Assume that $\mathcal{O}_c^* = \{\pm 1\}$. We then have*

$$\mathcal{E}_p(a_p, \chi) \frac{\widetilde{\Omega}_p^{j+1}}{p^{2(j+1)}} \left( \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi_{-1-j}^{-1}(\mathfrak{a}) G_j\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big) \right) = (*) \sum_{\mathfrak{a}} \chi_{-1-j}^{-1}(\mathfrak{a}) G_j^{[p]}\big(\mathfrak{a} * (A_0, t_0, D, \omega_0)\big)$$

*where $\widetilde{\Omega}_p$ is the p-adic period introduced in in the inert case and in Lemma 5.9 and in Lemma 6.6 in the ramified case.*

*Proof.* This computation is the same as in the proof of Proposition 5.11 in the inert case and as in the proof of Proposition 6.4 in the ramified case. We provide the proof in the inert case, leaving to the reader the proof in the ramified case. In the inert case we have

$$(*)\sum_{\mathfrak{a}}\chi_{-1-j}^{-1}(\mathfrak{a})G_j^{[p]}\big(\mathfrak{a}*(A_0,t_0,D,\omega_0)\big)=\frac{1}{p(p+1)}\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{p^2c})}\chi_{-1-j}^{-1}(\mathfrak{a})G_j^{[p]}\big(\mathfrak{a}*(A_0,t_0,D,\omega_0)\big).$$

This is the sum of three terms $S_1$, $S_2$ and $S_3$ corresponding to the three terms $G^{[p]}:=\big(1-T_pV+\frac{[p]}{p}V^2\big)(G)$. The first term is $S_1=\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}\chi_{-1-j}^{-1}(\mathfrak{a})G_j\big(\mathfrak{a}*(A_0,t_0,D,\omega_0)\big)$. The second is $-\dfrac{a_p}{p^{1+j}(p+1)}$ times the term

$$\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{pc})}\chi_{-1-j}^{-1}(\mathfrak{a})V(G_j)\big(\mathfrak{a}*(A_0,t_0,D,\omega_0)\big)=\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{pc})}\chi_{-1-j}^{-1}(\mathfrak{a})G_j\big(\mathfrak{a}*(A_{pc},\frac{1}{p}t_{pc},p\omega_{pc})\big)$$

where $\varphi_p\colon A_0\to A_{pc}$ is the cyclic isogeny defined by modding out by $D[p]$ and $\varphi_p^*(\omega_{pc})=\omega_0$. Since $\chi_{-1-j}^{-1}(\mathfrak{a})G_j\big(\mathfrak{a}*(A_{pc},\frac{1}{p}t_{pc},p\omega_{pc})\big)=\chi_{-1-j}^{-1}(p)\chi_{-1}^{-1}(p^{-1}\mathfrak{a})G_j\big(p^{-1}\mathfrak{a}*(A_{pc},t_{pc},\omega_{pc})\big)$ and $\chi_{-1-j}^{-1}(p)=\dfrac{p^{2+2j}}{\chi(p)}$, this sum coincides with

$$\frac{p^{2+2j}}{\chi(p)}\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{pc})}\chi_{-1-j}^{-1}(\mathfrak{a})G_j\big(\mathfrak{a}*(A_{pc},t_{pc},\omega_{pc})\big).$$

Using that $T_p(G_j)\big(\mathfrak{a}*(A_0,t_0,\omega_0)\big)=a_pp^{-1-j}G_j\big(\mathfrak{a}*(A_0,t_0,\omega_0)\big)$ poven in Lemma 7.9 and the map $\rho\colon\mathrm{Pic}(\mathcal{O}_{pc}\to\mathrm{Pic}(\mathcal{O}_c)$ we compute that for every $\mathfrak{b}\in\mathrm{Pic}(\mathcal{O}_c)$ we have

$$p^{-1}\sum_{\mathfrak{a}\in\rho^{-1}(\mathfrak{b})}G_j\big(\mathfrak{a}*(A_{pc},t_{pc},\omega_{pc})\big)=T_p(G_j)\big(\mathfrak{b}*(A_0,t_0,\omega_0)\big)=a_pp^{-1-j}G_j\big(\mathfrak{b}*(A_0,t_0,\omega_0)\big)$$

and the second term becomes

$$S_2:=-\frac{pa_p^2}{\chi(p)(p+1)}\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}\chi_{-1}^{-1}(\mathfrak{a})G_j\big(\mathfrak{a}*(A_0,t_0,\omega_0)\big)$$

(notice that the correcting factor is the same as in the analogous formula in the proof of Proposition 5.11). The third term $S_3$ is $\dfrac{\epsilon(p)p^{r-1-2j}}{p(p+1)}$ times

$$\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{p^2c})}\chi_{-1-j}^{-1}(\mathfrak{a})G_j\big(\mathfrak{a}*(A_{p^2c},\frac{1}{p^2}t_{p^2c},p^2\omega_{p^2c})\big)=\chi_{-1-j}^{-1}(p^2)\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{p^2c})}\chi_{-1-j}^{-1}(\mathfrak{a})G_j\big(\mathfrak{a}*(A_{p^2c},t_{p^2c},\omega_{p^2c})\big)$$

where $\varphi_{p^2}\colon A_0\to A_{p^2c}$ is the cyclic isogeny defined by modding out by $D$ and $\varphi_{p^2}^*(\omega_{p^2c})=\omega_0$. Using that $T_p^2(G_j)\big(\mathfrak{a}*(A_0,t_0,\omega_0)\big)=a_p^2p^{-2-2j}G_j\big(\mathfrak{a}*(A_0,t_0,\omega_0)\big)$ and the map $\rho\colon\mathrm{Pic}(\mathcal{O}_{p'^2c}\to\mathrm{Pic}(\mathcal{O}_c)$ we compute that for every $\mathfrak{b}\in\mathrm{Pic}(\mathcal{O}_c)$ we have

$$p^{-2}\sum_{\mathfrak{a}\in\rho^{`,-1}(\mathfrak{b})}G_j\big(\mathfrak{a}*(A_{p^2c},t_{p^2c},\omega_{p^2c})\big)=\left((T_p^2-\frac{p+1}{p^2}R_p)(G_j)\right)\big(\mathfrak{b}*(A_0,t_0,\omega_0)\big)=$$

$$= \big(\frac{a_p^2}{p^{2+2j}} - \frac{p+1}{p^2}\epsilon(p)p^{r-2j}\big) G_j\big(\mathfrak{b} * (A_0, t_0, \omega_0)\big).$$

As $\chi_{-1-j}^{-1}(p^2) = \frac{p^{4j+4}}{\chi(p^2)}$ and $\chi(p) = \epsilon(p)p^{r+2}$, we have

$$S_3 = \frac{\epsilon(p)p^{r-1-2j}}{p(p+1)} \frac{p^{2j+4}}{\chi(p)\epsilon(p)p^{r+2}} \big(a_p^2 - (p+1)\epsilon(p)p^r\big) \sum_{\mathfrak{a}\in \mathrm{Pic}(\mathcal{O}_c)} G_j\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big) =$$

$$= \big(\frac{a_p^2}{(p+1)\chi(p)} - \frac{1}{p^2}\big) \sum_{\mathfrak{a}\in \mathrm{Pic}(\mathcal{O}_c)} G_j\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big),$$

(again as in in the proof of Proposition 5.11). Here $G_j\big(\mathfrak{a}*(A_0, t_0, \omega_0)\big)$ stands abusively for $G_j\big(\mathfrak{a}*(A_0, t_0, D_0, \omega_0)\big)$ and the difference between the two is accounted for by the factor $\big(\frac{\widetilde{\Omega}_p}{p}\big)^{2(j+1)}$.

In fact, from the equation $\nabla(G) = f$ one deduces, arguing as in the proof of Proposition 7.7, the decomposition $G|_{\mathcal{D}'} = \sum_{j=0}^r (-1)^j g_j'(\omega')^{r-j}(\eta')^j$ for the wide open $\mathcal{D}'$ of $X_1(N)$ defined by the residue disk at $x' := \mathfrak{a} * (A_0, t_0)$; here $\omega' = \mathfrak{a} * \omega_0$. If $f = h(\omega')^k$ we get $\partial(g_0') = h$ and $\partial(g_j') - (r-j+1)g_{j-1}' = 0$ for $1 \le j \le r$, where $\partial$ is the derivation defined by $\omega'$ via Kodaira-Spencer as in loc. cit. Consider now the point $x := \mathfrak{a} * (A_0, t_0, D_0, s)$ of $X_1(pN, p^2)$. As explained in Lemma 5.8 in the inert case and in Lemma 6.6 in the ramified case, there is a discrepancy between the differentials of $X_1(N)$ at $x'$ and of $X(pN, p^2)$ at $x$ given by $\frac{\widetilde{\Omega}_p}{p^2}$. One concludes that $\frac{\widetilde{\Omega}_p^{j+1}}{p^{2(j+1)}} G_j\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big) = G_j\big(\mathfrak{a} * (A_0, t_0, D_0, s, \omega_0)\big)$. Summing the three terms $S_1 + S_2 + S_3$ we get the claim. □

**Lemma 7.9.** *We have* $T_p(G_j)\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big) = a_p p^{-1-j} G_j\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big)$.

*Proof.* One has $T_p \circ \nabla = p\nabla \circ T_p$ on sections of $\mathrm{Sym}^r(\mathrm{H}_{\mathcal{E}})$ by Proposition 8.1. In particular $\frac{p}{a_p} T_p(G) = G$ thanks to Lemma 7.3 as they are both Coleman's primitive of $f$ and they have the same $q$-expasions arguing as in [BDP, Proposition 3.24]. Hence, $T_p(G) = a_p p^{-1} G$. The correspondence $T_p$ on $\mathrm{Sym}^r \mathrm{H}_E$ preserves the Hodge filtration and induces on the $i$-th graded piece for the Hodge filtration $\mathrm{gr}_i \mathrm{Sym}^r \mathrm{H}_E$, via the identification with $\omega_E^{r-2i}$, the operator $p^i$ times the action of $T_p$ on $\omega_E^{k-2i}$; this can be checked on $q$-expansions, cf. [Ur, Prop. 3.3.7]. In our case, the decomposition of $G\big(\mathfrak{a} * (A_0, t_0)\big) = \sum_i (-1)^i G_i\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big)\omega_{\mathfrak{a}}^{r-i}\eta_{\mathfrak{a}}^i$ is compatible with the Hodge filtration namely $\sum_{i=0}^j (-1)^i G_i\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big)\omega_{\mathfrak{a}}^{r-i}\eta_{\mathfrak{a}}^i \in \mathrm{Fil}_j \mathrm{Sym}^r \mathrm{H}_{\mathfrak{a}*A_0}$. Using that $T_p(G) = a_p p^{-1} G$, the claim follows.

□

## 7.2 The Kronecker limit formula.

We next prove in our example the analogue of the Kronecker limit formulae [K2, §10] (more specifically the account in [BCDDPR, Thm. 1.3]) for $p$ non-split in $K$, proceeding as in §7.1. We take $k = 2$ and $\epsilon$ an even, non trivial character.

Let $u_\epsilon$ be a modular unit, namely an element $u_\epsilon \in \mathrm{H}^0\big(Y_1(N), \mathcal{O}_{Y_1(N)}^*\big)$ on the open modular curve $Y_1(N)$, such that

$$\mathrm{dlog} u_\epsilon = \frac{du_\epsilon}{u_\epsilon} = E_{2,\epsilon}.$$

We provide a formula for the value $L_p(E_{2,\epsilon},\chi^{-1})$, where $\chi$ is an algebraic Hecke character of infinity type $(1,1)$ and finite type $(\mathfrak{N},\epsilon)$. Let us remark that the Hecke character $\chi$ is a $p$-adic limit of characters in $\Sigma^{(2)}_{cc}(\mathfrak{N})$, i.e. $\chi\in\widehat{\Sigma}^{(2)}$ and so $L_p(E_{2,\epsilon},\chi^{-1})$ makes sense. Then,

**Proposition 7.10.** *Assume that $\mathcal{O}_K^*=\{\pm 1\}$. We have the following equality*

$$L_p(E_{2,\epsilon},\chi^{-1})=\frac{\widetilde{\Omega}_p}{p^2}\mathcal{E}_p\big(1+\epsilon(p)p,\epsilon^{-1}\big)\cdot\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_K)}\frac{\mathbf{N}(\mathfrak{a})}{\epsilon_{\mathfrak{N}}(\mathfrak{a})N}\log_p(u_\epsilon)\big(\mathfrak{a}*(A_0,t_0,\omega_0)\big).$$

### 7.2.1 On the $p$-adic logarithm.

Let $X$ be a wide open disk or annulus in $\mathbb{P}^1_{\mathbb{C}_p}$ and $g\in\mathcal{O}_X(X)^*$. We wish to study the Coleman integral of the differential form $\omega_g:=\mathrm{dlog}(g)=\dfrac{dg}{g}$ on $X$. Let us denote by $t$ a parameter of $X$. We work over $\mathbb{C}_p$ and denote by $\mathbb{F}$ its residue field.

With the notations above we have the following

**Lemma 7.11.** *a) If $X$ is a disk and $g\in\mathcal{O}_X(X)^*$ then there is $a\in\mathbb{C}_p^*$, $h\in\mathcal{O}_X(X)$ with the property $|h(x)|_X<1$ for all $x\in X$ such that $g=a(1+h)$.*

*b) If $X$ is an annulus and $g\in\mathcal{O}_X(X)^*$ then there are: $a\in\mathbb{C}_p^*$, $h\in\mathcal{O}_X(X)$ with the property $|h(x)|_X<1$ for all $x\in X$ and $n\in\mathbb{Z}$ such that $g=at^n(1+h)$ and $\mathrm{Res}_t(\omega_g)=n$.*

*Proof.* This lemma is probably welknown but we'll sketch the proof of a) for the convenience of the reader and leave she/he to think about b). As the power series expansion of $g$ does not change if we restrict to a smaller disk, it is enough to prove the lemma for all affinoid disks contained in $X$, i.e. it is enough to prove it for $X=\{x\in\mathbb{P}^1_{\mathbb{C}_p}\mid |x|\leq 1\}$. On $\mathcal{O}_X$ then we have the norm $|g|_X:=\sup_{x\in X}|g(x)|$, which satisfies the maximum modulus principle, i.e. the norm of $g$ can be calculated on the annulus $Y:=\{x\in X\quad |x|=1\}\subset X$. As $\overline{X}=\mathrm{Spec}(\mathbb{F}[t])$ is irreducible the norm $|\ |_X$ is multiplicative. Let $c\in\mathbb{C}_p^*$ be such that $|cg|_X=1$, then $|(cg)^{-1}|_X=1$ i.e. $\overline{cg}\in\big(\mathbb{F}[t]\big)^*=\mathbb{F}^*$. Let $b\in\mathcal{O}^*_{\mathbb{C}_p}$ be such that $\overline{bcg}=1$. Then clearly if we set $a=bc$ and $h=a^{-1}g-1$ we have $g=a(1+h)$ with $h$ satisfying the desired property. □

Let us denote by $\log_p\colon\mathbb{C}_p^*\to\mathbb{C}_p$ the locally analytic homomorphism uniquely determined by the properties: i) $\log_p(p)=0$ and ii) If $x\in\mathbb{C}_p^*$ is such that $|x|<1$ then $\log_p(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{(x-1)^n}{n}$. Then $d\big(\log_p(z)\big)=\dfrac{dz}{z}$.

Let us now remark that if $X$ is a wide open disk and $g\in\mathcal{O}_X(X)^*$, using lemma 7.11, the function $G:=\log_p(a)+\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{h^n}{n}\in\mathcal{O}_X(X)$ and it satisfies $dG=\omega_g:=\dfrac{dg}{g}$. We'll use the notation $G:=\log_p(g)$.

If $X$ is an annulus with parameter $t$ let us denote by $T:=\log_p(t)$ a new variable such that $dT=\mathrm{dlog}(t)=\dfrac{dt}{t}$. Let $\mathcal{O}_{\log}(X):=\mathcal{O}_X(X)[T]$. Let $g\in\mathcal{O}_X(X)^*$ be such that $\mathrm{Res}_t(\omega_g)=n$. Then the function $G\in\mathcal{O}_{\log}(X)$ defined by:

$$G:=\log_p(a)+nT+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{h^n}{n}$$

satisfies $dG = \omega_g := \dfrac{dg}{g}$. We'll use the notation $G := \log_p(g) \in \mathcal{O}_{\log}(X)$.

We remark that we have the following rigidity property: if $V \subset X$ is a non-void admissible open subspace and $G$, $G' \in \mathcal{O}_{\log}(X)$ are such that $G|_V = G'|_V$ that $G = G'$.

### 7.2.2 The proof of Proposition 7.10

We now come back to our modular unit $u_\epsilon$. Let $G := \log_p(u_\epsilon)$ denote the locally analytic function on $\mathcal{Y} := \big(\widehat{Y}_1(N)\big)^{\mathrm{an}}$ which is $\log_p(u_\epsilon)|_X \in \mathcal{O}_X(X)$ for every residue class $X$ of a point in $\mathcal{Y}$. Here $\mathcal{Y}$ is the rigid generic fiber of the formal scheme $\widehat{Y}_1(N)$, i.e. the rigid space which is the complement in $X_1(N)^{\mathrm{an}}$ of the residue classes of all the cusps. Then $G$ is a Coleman primitive of $\dfrac{du_\epsilon}{u_\epsilon} = E_{2,\epsilon}$ on $\mathcal{Y}$, uniquely defined and which satisfies the rigidity principle stated at the end of the previous section. Proposition 7.5 and Proposition 7.7 with $r = j = 0$ imply that

$$L_p(E_{2,\epsilon}, \chi^{-1}) = \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_K)} \epsilon_{\mathfrak{N}}(\mathfrak{a})^{-1} \mathbf{N}(\mathfrak{a}) N^{-1} \delta_2^{-1}\big(E_{2,\epsilon}^{[p]}\big)\big(\mathfrak{a} * (A_0, t_0, \omega)\big) =$$

$$= \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_K)} \epsilon_{\mathfrak{N}}(\mathfrak{a})^{-1} \mathbf{N}(\mathfrak{a}) N^{-1} G^{[p]}\big(\mathfrak{a} * (A_0, t_0, \omega)\big).$$

Lemma 7.8 implies that the latter value coincides with

$$\frac{\widetilde{\Omega}_p}{p^2} \mathcal{E}_p\big(1 + \epsilon(p)p, \epsilon_{\mathfrak{N}}^{-1}\big) \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_K)} \epsilon_{\mathfrak{N}}(\mathfrak{a})^{-1} \mathbf{N}(\mathfrak{a}) N^{-1} \log_p(u_\epsilon)\big(\mathfrak{a} * (A_0, t_0, \omega_0)\big),$$

as let us recall that the point $\mathfrak{a} * (A_0, t_0)$ being supersingular is not in the residue class of any cusp. The claim follows.

# 8 Appendix: $\nabla$, $U$ and $V$.

Let $\widehat{Y}^{\mathrm{ord}}$ be the formal open subscheme of $\widehat{X}_1(N)$ corresponding the ordinary locus (with the cusps removed). Let $\mathbb{E} \to \widehat{Y}^{\mathrm{ord}}$ be the universal ellitpic curve. Denote by $\varphi\colon \mathbb{E} \to \mathbb{E}'$ the quotient by the canonical subgroup of order $p$ and by $\varphi^\vee\colon \mathbb{E}' \to \mathbb{E}$ the dual isogeny. We have a unique morphism $\Phi\colon \widehat{Y}^{\mathrm{ord}} \to \widehat{Y}^{\mathrm{ord}}$ such that the pull-back of $\mathbb{E}$ is $\mathbb{E}'$. It is a finite and flat mophism of degree $p$. Let $r \geq 0$ be an integer and denote by $\mathcal{F}_r$ either the sheaf $\mathrm{Sym}^r \mathrm{H}_{\mathbb{E}}$ or the sheaf $\mathrm{Sym}^r \mathrm{H}_{\mathbb{E}}^{\mathrm{loc}}$ of locally analytic sections of the first sheaf as defined in section §7.1.

*The $V$ operator:* The operator $V\colon \mathcal{F}_r \to \mathcal{F}_r$ is defined by the following rational map on $\mathrm{H}_E$

$$V(\gamma) := \big((\varphi^\vee)^*\big)^{-1}\big(\Phi^*(\gamma)\big) = \frac{\varphi^*}{p}\big(\Phi^*(\gamma)\big);$$

here $\gamma \in \mathrm{H}_{\mathbb{E}}$ and $\Phi^*(\gamma)$ is viewed as an element of $\mathrm{H}_{\mathbb{E}'} \cong \Phi^*(\mathrm{H}_{\mathbb{E}})$. The map $((\varphi^\vee)^*\colon \mathrm{H}_{\mathbb{E}} \to \mathrm{H}_{\mathbb{E}'}$ is the pull-back via $\varphi^\vee$ and similarly $\varphi^*\colon \mathrm{H}_{\mathbb{E}'} \to \mathrm{H}_{\mathbb{E}}$ is induced by $\varphi$.

*The U operator:* The operator $U\colon \Phi_*\big(\mathcal{F}_r\big) \to \mathcal{F}_r$ is defined by the rational map on $\mathrm{H}_{\mathbb{E}}$ given by the composite

$$U := \frac{1}{p}\mathrm{Tr}_{\Phi} \circ (\varphi^{\vee})^*$$

where $(\varphi^{\vee})^*\colon \Phi_*\big(\mathrm{H}_{\mathbb{E}}\big) \to \Phi_*\big(\Phi^*\mathrm{H}_{\mathbb{E}}\big)$ is defined by pull-back via $\varphi^{\vee}$ and $\mathrm{Tr}_{\Phi}\colon \Phi_*\big(\Phi^*\mathrm{H}_{\mathbb{E}}\big) \to \mathrm{H}_{\mathbb{E}}$ is the trace map (as coherent sheaves) via the finite and flat map $\Phi$. Notice that $U \circ V = \mathrm{Id}$.

**Proposition 8.1.** *Let $r \in \mathbb{N}$ and consider the connection $\nabla\colon \mathcal{F}_r \to \mathcal{F}_{r+2}$. We have the formulae*

$$\nabla \circ V = pV \circ \nabla \qquad \textit{and} \qquad \nabla \circ U = \frac{1}{p}U \circ \nabla.$$

*In particular if $\nabla(G) = f$ then $\nabla(G^{[p]}) = f^{[p]}$ where $(\ -\ )^{[p]}$ stands for the p-depletion operator $(1 - V \circ U)$.*

*Proof.* The compatibility of $\nabla$ with the $p$-depletion clearly follows from the commutation formula for $\nabla$ with $V$ and $U$ respectively.

The isogeny $\varphi\colon \mathbb{E} \to \mathbb{E}'$ composed with the projection $\pi\colon \mathbb{E}' = \mathbb{E} \times^{\Phi}_{\widehat{Y}^{\mathrm{ord}}} \widehat{Y}^{\mathrm{ord}} \to \mathbb{E}$ induces the map $\eta := \varphi^* \circ \Phi^*\colon H_{\mathbb{E}} \to H_{\mathbb{E}}$. Let $d\Phi\colon \Omega^1_{\widehat{Y}^{\mathrm{ord}}} \to \Omega^1_{\widehat{Y}^{\mathrm{ord}}}$ be the map induced by pull-back via $\Phi$. By functoriality we get a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_{\mathbb{E}} & \stackrel{\nabla}{\to} & \mathrm{H}_{\mathbb{E}} \otimes \Omega^1_{\widehat{Y}^{\mathrm{ord}}} \\ \eta \downarrow & & \downarrow \eta \otimes d\Phi \\ \mathrm{H}_{\mathbb{E}} & \stackrel{\nabla}{\to} & \mathrm{H}_{\mathbb{E}} \otimes \Omega^1_{\widehat{Y}^{\mathrm{ord}}}. \end{array}$$

Recall that the Kodaira-Spencer isomorphism $\omega^2_{\mathbb{E}} \cong \Omega^1_{\widehat{Y}^{\mathrm{ord}}}$ is defined by restricting $\nabla$ to $\omega_{\mathbb{E}}$ and projecting onto $\omega^{\vee}_{\mathbb{E}} \otimes \Omega^1_{\widehat{Y}^{\mathrm{ord}}}$. The commutative diagram above implies that the map $d\Phi\colon \Omega^1_{\widehat{Y}^{\mathrm{ord}}} \to \Omega^1_{\widehat{Y}^{\mathrm{ord}}}$ induces the map

$$pV = p\big((\varphi^{\vee})^*\big)^{-2} \circ \Phi^* = \big(\varphi^* \otimes (\varphi^{\vee})^*)^{-1}\big) \circ \Phi^*\colon \omega^2_{\mathbb{E}} \to \omega^2_{\mathbb{E}}.$$

Indeed $\eta$ is $\varphi^*$ on $\omega_{\mathbb{E}}$ and $(\varphi^{\vee})^*$ on $\omega^{\vee}_{\mathbb{E}}$. Since by construction $\eta = pV$, we conclude that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_{\mathbb{E}} & \stackrel{\nabla}{\to} & \mathrm{H}_{\mathbb{E}} \otimes \omega^2_{\mathbb{E}} \\ V \downarrow & & \downarrow V \otimes (pV) \\ \mathrm{H}_{\mathbb{E}} & \stackrel{\nabla}{\to} & \mathrm{H}_{\mathbb{E}} \otimes \omega^2_{\mathbb{E}}. \end{array}$$

Passing to $\mathcal{F}_r$ we get the statement on the $V$-operator.

Next we study the $U$ operator. The map $(\varphi^{\vee})^*$, induced by $\varphi^{\vee}\colon \mathbb{E}' \to \mathbb{E}$, is compaible with the Gauss-Manin connection by functoriality. On the other hand $\Phi^*\big(\Omega^1_{\widehat{Y}^{\mathrm{ord}}}\big) = p\Omega^1_{\widehat{Y}^{\mathrm{ord}}}$ (as can be seen using Serre-Tate coordinates) so that $\frac{1}{p}\Phi_* \circ \Phi^*\big(\Omega^1_{\widehat{Y}^{\mathrm{ord}}}\big) = \Phi_*\big(\Omega^1_{\widehat{Y}^{\mathrm{ord}}}\big)$. We then have the commutative diagram

$$\begin{array}{ccc} \Phi_*(\mathrm{H}_{\mathbb{E}}) & \stackrel{\nabla}{\to} & \Phi_*((\mathrm{H}_{\mathbb{E}}) \otimes \Phi_*\big(\Omega^1_{\widehat{Y}_{\mathrm{ord}}}\big)[p^{-1}] \\ (\varphi^\vee)^* \downarrow & & \downarrow (\varphi^\vee)^* \otimes \mathrm{Id} \\ \Phi_*(\mathrm{H}_{\mathbb{E}'}) & \stackrel{\nabla}{\to} & \Phi_*(\mathrm{H}_{\mathbb{E}'}) \otimes \Big(\Phi_* \circ \Phi^*\big(\Omega^1_{\widehat{Y}_{\mathrm{ord}}}\big)\Big)[p^{-1}] \\ \mathrm{Tr}_\Phi \downarrow & & \downarrow \mathrm{Tr}_\Phi \\ \mathrm{H}_{\mathbb{E}} & \stackrel{\nabla}{\to} & \mathrm{H}_{\mathbb{E}} \otimes \Omega^1_{\widehat{Y}_{\mathrm{ord}}}[p^{-1}]. \end{array}$$

(here we use $\Phi_*(\mathrm{H}_{\mathbb{E}'}) = \Phi_* \circ \Phi^*(\mathrm{H}_{\mathbb{E}})$) Using this commutative diagram and the Kodiara-Spencer isomorphism, we deduce that the map

$$\Phi_*\big(\Omega^1_{\widehat{Y}_{\mathrm{ord}}}\big)[p^{-1}] = \Phi_* \circ \Phi^*\big(\Omega^1_{\widehat{Y}_{\mathrm{ord}}}\big)[p^{-1}] \stackrel{\mathrm{Tr}_\Phi}{\to} \Omega^1_{\widehat{Y}_{\mathrm{ord}}}[p^{-1}]$$

coincides with the rational map

$$\rho := \mathrm{Tr}_\Phi \circ \big((\varphi^\vee)^* \otimes (\varphi^*)^{-1}\big) \colon \Phi_*\big(\omega^2_{\mathbb{E}}\big) \to \Phi_* \circ \Phi^*\big(\omega^2_{\mathbb{E}}\big) \to \omega^2_{\mathbb{E}}.$$

As $(\varphi^*)^{-1} = p^{-1}(\varphi^\vee)^*$ then $\rho = \frac{1}{p}U$. We conclude that the following diagram commutes:

$$\begin{array}{ccc} \Phi_*(\mathrm{H}_{\mathbb{E}}) & \stackrel{\nabla}{\to} & \Phi_*(\mathrm{H}_{\mathbb{E}}) \otimes \Phi_*(\omega^2_{\mathbb{E}}) \\ U \downarrow & & \downarrow U \otimes (p^{-1}U) \\ \mathrm{H}_{\mathbb{E}} & \stackrel{\nabla}{\to} & \mathrm{H}_{\mathbb{E}} \otimes \omega^2_{\mathbb{E}}. \end{array}$$

Passing to $\mathcal{F}_r$ we get the claim for the $U$ operator.

□

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