Deformation of torsors under monogenic group schemes

F. Andreatta & C. Gasbarri

9 October 2014

Abstract. We show that one can always deform torsors over smooth curves under finite and commutative group schemes under the assumption that their Lie algebras have dimension less or equal to 1 and that the torsor does not arise from a proper subgroup. We apply this result to the study of a stack classifying p-covers of curves.

1 Introduction.

Let R be a complete local ring with residue field k of positive characteristic p. Let G be a finite, flat and of finite presentation, commutative group scheme over R. Let X_k be a smooth curve over k i. e., a smooth k–scheme of dimension 1, and let $Y_k \to X_k$ be a G_k–torsor over X_k. One may ask whether there exist a lifting X of X_k over R and a G–torsor $Y \to X$ deforming $Y_k \to X_k$. If G is étale, the answer is well known to be positive. Indeed, for *every lifting* X of X_k the problem of deforming $Y_k \to X_k$ to a G–torsor over X admits a unique solution. As the following example shows, if G is not étale and X is *a fixed* lifting of X_k, the problem of lifting G_k–torsors might not have a solution. We suppose that R is a dvr of characteristic p and $G = \alpha_p$. Then, given a family $\mathcal{E} \to \mathrm{Spec}(R)$ of elliptic curves with ordinary generic fiber and supersingular special fiber $\mathcal{E}_k$, any non–trivial G_k–torsor over $\mathcal{E}_k$ *can not* be extended to a G–torsor over $\mathcal{E}$. In examples 3.4 and 3.5, we show that, given a dvr R of unequal characteristic, there exist curves X over R and α_p–torsors (resp. μ_{p^n}–torsors for any $n \in \mathbf{N}$) over its special fiber which cannot be lifted to torsors over X under any group scheme deforming α_p (resp. μ_{p^n}). In the other direction it is easy to construct examples for which the lifting problem has more than one solution. In this paper we prove the following:

1.1 Theorem. *Assume that* $\mathrm{Lie}\, G_k$ *is of dimension* ≤ 1 *and that* Y_k *does not arise as the push–forward of a torsor over* X_k *under a proper subgroup scheme of* G_k*. Then, there exist a smooth formal curve* X *over* R *and a* G*–torsor* $Y \to X$ *whose special fiber is the* G_k*–torsor* $Y_k \to X_k$*.*

The proof relies on the fact that G_k-torsors over X_k are defined by isogenies on the jacobian of X_k. This allows to translate the deformation theory of torsors, using the cotangent complex (see §2), into a rather explicit computation which is performed in §3. Here the fact that we are working with curves turns out to be an essential ingredient.

Using Theorem 1.1 we obtain fine information on the structure of the stack classifying p-covers of curves. When p is a unit, the structure of such spaces is well understood. Their reduction modulo p is more elusive. The main difficulty is to understand the possible specializations of a p-cyclic cover of smooth projective curves $Y \to X$ defined over a

2000 *Mathematics Subject Classification.* Primary 11G20, Secondary 14G20,14H30.
Key words: Lifting of torsors, Finite flat group schemes, algebraic curves, cotangent complex.

complete discrete valuation field with residue field of characteristic p. The semistable reduction theorem for curves allows to suppose that *either* the reduction of X *or* the reduction of Y is a semistable curve. If one wants that the reduction of the cover is again a finite morphism with a group scheme acting, in general one cannot obtain that both the reductions of X and Y are semistable.

In [AR] the authors approach the problem imposing the semistability of Y and considering torsors $Y \to X$ under group schemes over X (not necessarily defined over the base). They define a Deligne-Mumford stack over $\mathbf{Z}$, which after inverting p classifies p-cyclic covers of curves of given genus, and has the property of being proper over $\mathbf{Z}$. The main difficulty in this approach is to understand the singularity of such stack.

In this paper we impose the semistability of X, following the approach of [AOV,§5]. We introduce in §4 a stack $\mathrm{CC}_{g,p}$ which associates to a scheme S triples (X, G, Y) where X is a smooth projective curve of genus $g \geq 2$ over S (a 1-pointed smooth projective curves for $g = 1$), G is a finite locally free group scheme over S of rank p and Y is a G–torsor over X. This is the analogue of the stack introduced in [AOV] with two notable differences: we allow only smooth curves and not semistable ones as in loc. cit. but we do not confine ourselves to the case of linearly reductive, finite flat group schemes G as in loc. cit. This is an important feature as, for example, we allow torsors under the group scheme α_p in characteristic p which appear as specializations of p-covers. The advantage of limiting ourself to group schemes of order p is that we have the theory of Oort–Tate at our disposal which provides a simple description of the Artin stack $\mathcal{G}_p$ of finite and locally free group schemes of order p. The stack $\mathrm{CC}_{g,p}$ has natural forgetful morphisms $\mathrm{pr}_G \colon \mathrm{CC}_{g,p} \to \mathcal{G}_p$, sending (X, G, Y) to G, and $\mathrm{pr}_{\mathcal{M}} \colon \mathrm{CC}_{g,p} \to \mathcal{M}_g$, to the stack of smooth curves of genus g if $g \geq 2$ (of 1-pointed smooth projective curves if $g = 1$), sending (X, G, Y) to X. The morphism $\mathrm{CC}_{g,p} \to \mathcal{M}_g \times \mathcal{G}_p$ is representable and $\mathrm{pr}_{\mathcal{M}}$ is proper; see §4. Examples show, see §3, that $\mathrm{pr}_{\mathcal{M}}$ is *not* smooth. As an application of Theorem 1.1 we further obtain:

1.2 Corollary. *The map* pr_G *is formally smooth.*

As a corollary we obtain that $\mathrm{CC}_{g,p}$ is a regular Artin stack flat over $\mathbb{Z}$ whose fiber over p is a simple normal crossing divisor. This provides a strengthening of [AOV, Thm. 5.1] in our setting. As our deformation theory heavily relies on the deformation theory of abelian varieties (via the theory of Jacobians), we do not see a straightforward generalization of our methods to the case of semistable curves. It would be an interesting problem to extend our theory to that case and especially to compare our stack to the one introduced in [AR].

1.3 *Acknowledgments.* We would like to thank M. Raynaud who suggested the problem to us and for his encouragement. He convinced us of the validity of Theorem 1.1 which we initially proved only for the special class of monogenic group schemes considered in [AG]. We thank Matthieu Romagny, Dajano Tossici and the referee for feed back and suggestions that hopefully improved the exposition. The first author was partially supported by the Italian grant Prin 2010/2011. The second author was supported by the USIAS and FRIAS.

2 Deformation of torsors via isogenies of abelian schemes.

Let $j\colon S \to S'$ be a closed immersion of affine schemes defined by a square zero ideal $\mathcal{J}$. Consider a smooth morphism $X' \to S'$ and let $g\colon X \to S$ be the base change via j. Let $G' \to S'$ be a group scheme over S', commutative, flat and of finite presentation. Assume that the base change $G := G' \times_{S'} S$ over S is the kernel of a faithfully flat morphism $\alpha\colon A \to B$ of smooth groups schemes over S so that we have the exact sequence

$$0 \longrightarrow G \xrightarrow{\iota} A \xrightarrow{\alpha} B \longrightarrow 0.$$

Suppose we are given a G-torsor $Y \to X$ (for the fpqc topology on X). We study the obstruction theory to deform Y to a G'-torsor over X' using the theory of the equivariant cotangent complex elaborated by Illusie. It turns out that in the case that G is the kernel of a morphism as above various simplifications take place.

We denote by $Z := \iota_*(Y)$, the A-torsor over X defined by push-out of Y via ι. We write $W := \alpha_*(Z)$ for the B-torsor over X given by push-out of Z via α. Using that W admits a canonical section, the following easy lemma shows how to recover the original G-torsor from the morphism of A-schemes $Z \to W$. As this recipe is needed in §2.6 we prefer to state it explicitly.

2.1 Lemma. *The category of G–torsors over X is equivalent to the category of A–torsors Z over X with a section $\gamma\colon X \to W$ of the B-torsor $W := \alpha_*(Z)$.*

Proof: We refer to [AG, Prop. 4.4] for a proof. We only sketch the main steps.

$\Longleftarrow$ Let (Z, γ) be an A–torsor with a section of W over X. Define the associated G–torsor Y as the fibred product of $Z \to W$ and $\gamma\colon X \to W$ over W.

$\Longrightarrow$ Let $Y \to X$ be a G–torsor and let $Z = \iota_*(Y)$. It is an A-torsor and Y is naturally a closed G-equivariant subscheme of Z. The torsor $W = \alpha_*(Z)$ and the natural map $Z \to W$ identifies W with Z/G. Then, W is endowed with the section $\gamma\colon X \cong Y/G \to W = Z/G$.

$\square$

In the theory of the cotangent complex the invariant differentials of a torsor for the action of a group scheme appear. We describe the structure of such module.

2.2 *Invariant differentials.* Let H be a flat group scheme of finite presentation over S. In our case it could be G or A or B.

Let $h\colon Z \to S$ be an S–scheme and let $g\colon U \to Z$ be a morphism of S–schemes. Let $f\colon T \to U$ be a scheme over U endowed with an action of H. Such action provides a commutative diagram

$$\begin{array}{ccc} H \times_S T & \xrightarrow{m} & T \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{f} & U, \end{array}$$

where pr_2 is the projection on the second factor and m defines the action of H on T. Define *the $\mathcal{O}_U$–module of invariant differentials* $\Omega^{\mathrm{inv}}_{T/U/Z}$ as the $\mathcal{O}_U$-submodule of $f_*\big(\Omega^1_{T/Z}\big)$ given on an open subscheme $V \subset U$ by

$$\Omega^{\mathrm{inv}}_{T/U/Z}(V) := \Big\{\omega \in \Omega^1_{T/Z}\big(f^{-1}(V)\big) | pr_2^*(\omega) = m^*(\omega)\Big\}.$$

We write $\Omega^{\mathrm{inv}}_{T/U}$ for $\Omega^{\mathrm{inv}}_{T/U/U}$. For example, if $U = S$ and $T = H$ is the trivial H-torsor, $\Omega^{\mathrm{inv}}_{H/S}$ is the $\mathcal{O}_S$–module of invariant differentials on H. If $W \subseteq H$ is an open subset containing the zero section and $I \subset \mathcal{O}_W$ is the ideal sheaf defining the zero section as a closed subscheme of W, then $\Omega^{\mathrm{inv}}_{H/S} \cong I/I^2$.

Observe that an H–equivariant map $T_1 \to T_2$ of U-schemes gives rise, functorially, to a map of sheaves $\Omega^{\mathrm{inv}}_{T_2/U/Z} \to \Omega^{\mathrm{inv}}_{T_1//U/Z}$.

2.3 Proposition. *If $f: T \to U$ is an H-torsor, then*

(a) *the map of $\mathcal{O}_T$–modules $f^*(\Omega^{\mathrm{inv}}_{T/U}) \to \Omega^1_{T/U}$, deduced by adjunction from the inclusion $\Omega^{\mathrm{inv}}_{T/U} \to f_*(\Omega^1_{T/U})$, is an isomorphism;*

(b) *there is a unique isomorphism of $\mathcal{O}_U$–modules*

$$t: \Omega^{\mathrm{inv}}_{T/U} \xrightarrow{\sim} g^*\big(\Omega^{\mathrm{inv}}_{h^*H/Z}\big)$$

satisfying the following property. Let $q: Y \to U$ be a flat morphism and let $\sigma: Y \to T_Y := T \times_U Y$ be a section. The section σ defines a trivialization of T_Y as H–torsor so that $\Omega^{\mathrm{inv}}_{T_Y/Y}$ is isomorphic to the pull–back of $\Omega^{\mathrm{inv}}_{H/S}$ to Y. Then, such isomorphism coincides with $q^*(t)$;

(c) *let $\alpha: H \to G$ be a morphism of group schemes flat and of finite presentation over S. Denote by $W := \alpha_*(T)$ is the G-torsor given by push-out of T via α. Write $\rho: T \to W$ for the associated map of U–schemes. The induced morphism $d\rho^{\mathrm{inv}}: \Omega^{\mathrm{inv}}_{W/U} \to \Omega^{\mathrm{inv}}_{T/U}$ is identified, via the isomorphism in (b), with the base change via g^* of the morphism $d\alpha^{\mathrm{inv}}: \Omega^{\mathrm{inv}}_{G/S} \to \Omega^{\mathrm{inv}}_{H/S}$.*

Proof: (a) and (c) It suffices to verify the claims passing to an fpqc covering of U. In particular, we may assume that $T \to U$ is the trivial H–torsor. The statements are obvious in this case.

(b) Assume first that $T \to U$ is the trivial torsor and let $\sigma: U \to T$ be a section. The choice of σ defines an isomorphism $\rho: H \times_S U \xrightarrow{\sim} T$ given by $(a, x) \mapsto a \cdot \sigma(x)$. Using ρ and (a), we deduce an isomorphism $t_\rho: \Omega^{\mathrm{inv}}_{T/U} \cong g^*(\Omega^{\mathrm{inv}}_{H/S})$. Choose a different section σ' and let $\rho': H \times_S U \xrightarrow{\sim} T$ be the induced isomorphism. Then, $\sigma' = \alpha \cdot \sigma$ for some $\alpha: U \to H$ and $\rho^{-1} \circ \rho': H \times_S U \xrightarrow{\sim} H \times_S U$ is $\big(\alpha(x) + a, x\big)$. Such map induces the identity on $\Omega^{\mathrm{inv}}_{H \times_S U/U}$. Hence, $t_{\rho'} \circ t_\rho^{-1}$ is the identity i. e., $t_{\rho'} = t_\rho$. Hence, t_ρ does not depend on the choice of the section σ.

In the general case, let $R \to U$ be an fpqc cover of U such that there exists a section $\sigma: R \to T \times_U R$. We then get an isomorphism t of the pull–back of $\Omega^{\mathrm{inv}}_{T/U}$ and $g^*(\Omega^{\mathrm{inv}}_{H/S})$ to T. The pull–back t_1 (resp. t_2) of t via the first (resp. second) projection $R \times_U R \to R$ is the isomorphism of the pull back of $\Omega^{\mathrm{inv}}_{T/U}$ and $g^*(\Omega^{\mathrm{inv}}_{H/S})$ to $R \times_U R$ defined by the pull–back of the section σ. In particular, by the previous discussion $t_1 = t_2$ so that t satisfies a descent datum relative to $R \to U$. In particular, t descends and it defines the sought isomorphism. $\square$

2.4 *The cotangent complex.* We start by reviewing the theory of deformations of torsors using Illusie's theory of the equivariant cotangent complex. See [Il, §VII.2.4] and also [We, §1].

First of all we recall that, since we fixed a deformation X' of X over S', the set of isomorphism classes of deformations X'' of X over S' are classified by $\mathrm{Ext}^1\big(\Omega^1_{X/S}, \mathcal{J}\mathcal{O}_X\big)$. If X'' is such a deformation, we denote by $\Psi_{X'}(X'')$ the class in $\mathrm{Ext}^1\big(\Omega^1_{X/S}, \mathcal{J}\mathcal{O}_X\big)$ associated to it.

Under the assumption that $G \to S$ is commutative, flat and of finite presentation and $f\colon Y \to X$ is a G–torsor, one constructs:

(1) a perfect complex $\ell'_{Y/X}$ of $\mathcal{O}_X$-modules of perfect amplitude $[-1,0]$, called the *co-Lie complex* associated to the G-torsor $Y \to X$, see [Il, §VII, §§2.4.2];

(2) an extension class $\mathrm{at}(Y/X/S) \in \mathrm{Ext}^1\big(\ell'_{Y/X}, \Omega^1_{X/S}\big)$ of $\mathcal{O}_X$-modules, called the *Atiyah class* of the G-torsor $Y \to X$, see [Il, §VII, (2.4.2.11)] with the following properties.

Let $S \subset S'$ be a closed immersion of affine schemes defined by a square zero ideal $\mathcal{J}$ which we then view as a $\mathcal{O}_S$-module. Let

$$\delta'\colon \mathrm{Ext}^1\big(\Omega^1_{X/S}, \mathcal{J}\mathcal{O}_X\big) \longrightarrow \mathrm{Ext}^2\big(\ell'_{Y/X}, \mathcal{J}\mathcal{O}_X\big)$$

be the map defined by the long exact sequence of Ext-groups associated to the Atiyah class. Fix a group scheme $G' \to S'$ commutative, flat and of finite presentation over S' deforming $G \to S$.

2.5 Proposition. [Il, §VII, Theorem 2.4.4] *Let $X' \to S'$ be a flat deformation of $X \to S$.*

i. *There exists an element $\Theta\big(Y, X', G'\big) \in \mathrm{Ext}^2\big(\ell'_{Y/X}, \mathcal{J}\mathcal{O}_X\big)$ which vanishes if and only if the G-torsor $Y \to X$ can be lifted to a G'-torsor $Y' \to X'$;*

ii. *if $\Theta\big(Y, X', G'\big) = 0$, the set of isomorphism classes of G'-torsors deforming the G-torsor $Y \to X$ over X' is a principal homogeneous space under $\mathrm{Ext}^1\big(\ell'_{Y/X}, \mathcal{J}\mathcal{O}_X\big)$;*

iii. *if X'' is the deformation of X to S' defined by $\Psi_{X'}(X'') \in \mathrm{Ext}^1\big(\Omega^1_{X/S}, \mathcal{J}\mathcal{O}_X\big)$, then $\Theta\big(Y, X'', G'\big) = \Theta\big(Y, X', G'\big) + \delta'\big(\Psi_{X'}(X'')\big)$.*

Strictly speaking in [Il] one fixes the group scheme $G' \to S'$ as above, a G'-torsor $Y \to X$ and a closed immersion $X \subset X'$, viewed as a morphism of S'-schemes and defined by a square 0 ideal. One then studies the problem of deforming $Y \to X$ to a G'-torsor $Y' \to X'$. In our case the closed immersion $X \subset X'$ arises from the closed immersion $S \subset S'$ defined by the ideal $\mathcal{J}$.

2.6 *An explicit description.* Recall that G is the kernel of a faithfully flat map of smooth group schemes:

$$0 \longrightarrow G \xrightarrow{\iota} A \xrightarrow{\alpha} B \longrightarrow 0.$$

Under this assumption the co-Lie complex and the Atiyah class constructed by Illusie become very explicit.

Set Z to be the A-torsor $\iota_*(Y)$ over X and W to be the B-torsor $\alpha_*(Z)=Z/G$ over X. The induced map $\rho\colon Z\to W$ provides a morphism of invariant differentials $\Omega^{\rm inv}_{W/X}\to\Omega^{\rm inv}_{Z/X}$. Define $\ell_{Y/X}$ to be the complex of $\mathcal{O}_X$-modules concentrated in degrees -1 and 0 given by

$$\ell_{Y/X}:=\Big[0\to\Omega^{\rm inv}_{W/X}\to\Omega^{\rm inv}_{Z/X}\to 0\Big].$$

Denote by $d\alpha^{\rm inv}\colon\Omega^{\rm inv}_{B/S}\to\Omega^{\rm inv}_{A/S}$ the induced map on invariant differentials associated to $\alpha\colon A\to B$. Due to Proposition 2.3 we can identify $\ell_{Y/X}$ with the complex $g^*\big(d\alpha^{\rm inv}\big)$

$$\ell_{Y/X}\cong\Big[0\to g^*\big(\Omega^{\rm inv}_{B/S}\big)\to g^*\big(\Omega^{\rm inv}_{A/S}\big)\to 0\Big].$$

Let $s\colon Z\to X$ be the structural morphism. Consider as in §2.2 the $\mathcal{O}_X$-module $\Omega^{\rm inv}_{Z/X/S}$. The natural map $\Omega^1_{Z/S}\to\Omega^1_{Z/X}$ induces a map $\Omega^{\rm inv}_{Z/X/S}\to\Omega^{\rm inv}_{Z/X}$. The map $\Omega^1_{X/S}\to s_*\big(\Omega^1_{Z/S}\big)$ factors via $\Omega^{\rm inv}_{Z/X/S}$ so that we have a sequence

$$0\longrightarrow\Omega^1_{X/S}\longrightarrow\Omega^{\rm inv}_{Z/X/S}\longrightarrow\Omega^{\rm inv}_{Z/X}\longrightarrow 0. \tag{2.6.1}$$

2.7 Lemma. *The sequence (2.6.1) is exact.*

Proof: Assume first that Z is the trivial A-torsor so that $Z\cong X\times_S A$. The pull-back via the 0-section of A defines a left splitting of the sequence (2.6.1). If $q\colon Z=X\times_S A\to A$ denotes the second projection, the pull back via q induces a map $q^*\big(\Omega^{\rm inv}_{A/S}\big)\to\Omega^{\rm inv}_{Z/X/S}$. As $q^*\big(\Omega^{\rm inv}_{A/S}\big)$ is identified with $\Omega^{\rm inv}_{Z/X}$ by Proposition 2.3, this defines a right splitting of the sequence (2.6.1).

As A and X are smooth over S and we have $Z\cong X\times_S A$, the sequence of differentials $0\to s^*\big(\Omega^1_{X/S}\big)\to\Omega^1_{Z/S}\to\Omega^1_{Z/X}\to 0$ is exact. Thus the kernel of $\Omega^{\rm inv}_{Z/X/S}\to\Omega^{\rm inv}_{Z/X}$ is the $\mathcal{O}_X$-module of A-invariant differentials $s_*\big(s^*\big(\Omega^1_{X/S}\big)\big)^{\rm inv}$. Exactness in the middle in (2.6.1) follows if we show that the natural map $\Omega^1_{X/S}\to s_*\big(s^*\big(\Omega^1_{X/S}\big)\big)^{\rm inv}$ is an isomorphism. As $\Omega^1_{X/S}$ is finite and locally free as $\mathcal{O}_X$-module, it is enough to prove this for $\mathcal{O}_X$ instead of $\Omega^1_{X/S}$. As $X\to S$ is flat, it suffices to prove that, denoting by $\pi\colon A\to S$ the natural morphism, $\mathcal{O}_S\to\pi_*(\mathcal{O}_A)^{\rm inv}$ is an isomorphism and this is clear. Thus (2.6.1) is exact in this case.

As Z is locally trivial for the fpqc topology on X we deduce that this sequence is locally exact and, hence, exact. $\square$

As W is the trivial torsor with a canonical section σ by Lemma 2.1, we deduce that the exact sequence (2.6.1) for $\Omega^{\rm inv}_{W/X/S}$ splits as a direct sum

$$\Omega^{\rm inv}_{W/X/S}=\Omega^1_{X/S}\oplus\Omega^{\rm inv}_{W/X}.$$

The induced map $\rho\colon Z \to W$ provides a morphism of invariant differentials $\Omega^{\mathrm{inv}}_{W/X/S} \to \Omega^{\mathrm{inv}}_{Z/X/S}$. Composing with the inclusion $\Omega^{\mathrm{inv}}_{W/X} \subset \Omega^{\mathrm{inv}}_{W/X/S}$ we get a complex of $\mathcal{O}_X$-modules concentrated in degrees -1 and 0, denoted $\ell_{Y/S}$,

$$\ell_{Y/S} := \left[0 \to \Omega^{\mathrm{inv}}_{W/X} \to \Omega^{\mathrm{inv}}_{Z/X/S} \to 0\right].$$

Due to Lemma 2.7 the natural morphism of complexes $\ell_{Y/S} \to \ell_{Y/X}$ has kernel equal to $\Omega^1_{X/S}$, viewed as a complex concentrated in degree 0 and as a subsheaf of $\Omega^{\mathrm{inv}}_{Z/X/S}$, so that we get an extension of complexes

$$0 \longrightarrow \Omega^1_{X/S} \longrightarrow \ell_{Y/S} \longrightarrow \ell_{Y/X} \longrightarrow 0. \tag{2.7.1}$$

2.8 *Functoriality.* Let $h\colon G \to H$ be a faithfully flat morphism of groups schemes over S and denote by K the kernel of h. Then $\alpha\colon A \to B$ factors via the quotient map $\pi\colon A \to A' := A/K$ and H is the kernel of the induced morphism $\alpha'\colon A' \to B$. Denote by $\iota'\colon H \to A'$ the induced inclusion. Let $Q := h_*(Y)$ be the push-forward of Y. Denote as in §2.6 by Z' the A'-torsor $\iota'_*(Y)$ over X and by W' to the B-torsor $\alpha'_*(Z)$ over X. Then $Z' = \pi_*(Z)$ and $W' = W$ and we have a commutative diagram

$$\begin{array}{ccc} Z & \to & W \\ \downarrow & & \downarrow \Vert \\ Z' & \to & W'. \end{array} \tag{2.8.1}$$

The construction in §2.6 provides a map of complexes of $\mathcal{O}_X$-modules $\ell_{Q/X} \to \ell_{Y/X}$. Then

2.9 Lemma. *The extension $\ell_{Q/S}$, see (2.7.1), is obtained from the extension $\ell_{Y/X}$ by pull–back via $\ell_{Q/X} \to \ell_{Y/X}$.*

Proof: It follows from the construction that the diagram (2.8.1) provides a map $\ell_{Q/S} \to \ell_{Y/S}$ which induces the given map $\ell_{Q/X} \to \ell_{Y/X}$ and is the identity on $\Omega^1_{X/S}$. The lemma follows. □

2.10 Proposition. *We have an isomorphism of complexes of $\mathcal{O}_X$-modules $\ell'_{Y/X} \cong \ell_{Y/X}$ such that the Atiyah class $\mathrm{at}(Y/X/S)$ is given by the extension (2.7.1). In particular, $\ell'_{Y/X}$ and $\mathrm{at}(Y/X/S)$ commute with arbitrary base change $T \to S$.*

Proof: We first study $\ell_{Y/X}$. The closed immersion $\iota\colon G \to A$ defines a closed immersion $\iota\colon Y \subset Z := \iota_*(Y)$ and the latter is a smooth scheme over X. The closed immersion is defined by an ideal I which, thanks to Lemma 2.1, is the inverse image via $\rho\colon Z \to W := \alpha_*(Z)$ of the ideal J defining the zero section of the trivial torsor W. Consider the complex of $\mathcal{O}_Y[G]$-modules given by

$$L^G_{Y/X} := \left[0 \longrightarrow I/I^2 \longrightarrow \iota^*\left(\Omega^1_{Z/X}\right) \longrightarrow 0\right],$$

introduced in [Il, §VII.2.4.2]. We observe that $I/I^2 \cong \rho^*(J/J^2) \cong f^*\big(\Omega^{\mathrm{inv}}_{W/X}\big)$. It follows from Proposition 2.3 that $\Omega^1_{Z/X} \cong s^*\big(\Omega^{\mathrm{inv}}_{Z/X}\big)$ where $s\colon Z \to X$ is the structural morphism. Hence, $\iota^*\big(\Omega^1_{Z/X}\big) \cong f^*\big(\Omega^{\mathrm{inv}}_{Z/X}\big)$. We can then rewrite the complex

$$L^G_{Y/X} \cong \big[0 \longrightarrow f^*\big(\Omega^{\mathrm{inv}}_{W/X}\big) \longrightarrow f^*\big(\Omega^{\mathrm{inv}}_{Z/X}\big) \longrightarrow 0\big].$$

In [Il, §VII, (2.4.2.8)'] the co-Lie complex, denoted here $\ell'_{Y/X}$, is defined as $R\epsilon_* f^G_* L^G_{Y/X}$ where ϵ is the morphism from the fpqc topos of X to the Zariski topos of X and f^G_* is the morphism from the topos of G-sheaves over Y to the fpqc topos over X associating to a sheaf L the G-invariants of $f_*(L)$. The adjoint $f^{G,*}$ associates to a sheaf L on X the sheaf $f^*(L)$ with the induced G-action. As $\epsilon^*\big(L^G_{Y/X}\big) = f^{G,*}\epsilon^*\big(\ell_{Y/X}\big)$ we get by adjunction a morphism of complexes $\ell_{Y/X} \to \ell'_{Y/X}$. To prove that it is an isomorphism we may base change with a faithfully flat map $V \to X$ and as both complexes commute with flat base change, we may assume that $Y \to X$ admits a section s. It then follows from [Il, §VII, (2.4.2.8)"] that $\ell'_{Y/X} \cong Ls^* L^G_{Y/X}$ which coincides with $\ell_{Y/X}$ as wanted.

Secondly, we prove that the exact sequence $0 \to \Omega^1_{X/S} \to \ell_{Y/S} \to \ell_{Y/X} \longrightarrow 0$ coincides with the Atiyah class $\mathrm{at}(Y/X/S)$ introduced by Illusie in [Il, §VII, (2.4.2.6)]. The proof proceeds as before. As $X \to S$ is smooth and $Z \to S$ is smooth, we have a complex $L^G_{Y/S}$ given by

$$L^G_{Y/S} := \big[0 \longrightarrow f^*\big(\Omega^{\mathrm{inv}}_{W/X}\big) \longrightarrow f^*\big(\Omega^{\mathrm{inv}}_{Z/X/S}\big) \longrightarrow 0\big];$$

see [Il, §VII, §2.2.5]. Following [Il, §VII, (2.4.2.7)'] one defines $\ell'_{Y/S}$ as $R\epsilon_* f^G_* L^G_{Y/S}$ with a natural map to $\ell'_{Y/X}$. As above, one has a natural morphism $\ell_{Y/S} \to \ell'_{Y/S}$, by adjunction, which is compatible with the map to $\ell_{Y/X}$ via the identification $\ell_{Y/X} \cong \ell'_{Y/X}$. To prove that it is an isomorphism one takes a base change via a faithfully flat morphism $W \to X$ and reduces to the case that the torsor $Y \to X$ admits a section s. Thanks to [Il, §VII, (2.4.2.7)"] we have $\ell'_{Y/S} \cong Ls^* L^G_{Y/S}$ and this is $\ell_{Y/S}$ as wanted.

We prove the last claim. Let $T \to S$ be an arbitrary morphism. Using the theory of the equivariant cotangent complex we have complexes ℓ'_{Y_T/X_T} and the Atiyah extension class $\mathrm{at}(Y_T/X_T/T)$, where the subscript T denotes the base change to T. By the first claim of the Proposition they admit explicit description in terms of the resolution $0 \to G_T \to A_T \to B_T \to 0$, e.g., $\ell'_{Y_T/X_T} \cong \ell_{Y_T/X_T}$ and $\mathrm{at}(Y_T/X_T/T)$ admits a description similar to (2.7.1).

By definition of the complex $\ell_{Y/X}$ and the fact that the invariant differentials $\Omega^{\mathrm{inv}}_{W/X}$ and $\Omega^{\mathrm{inv}}_{Z/X}$ commute with arbitrary base change, one deduces that ℓ_{Y_T/X_T} is obtained from $\ell_{Y/X}$ by base change. Similarly, using (2.6.1) one concludes that also $\Omega^{\mathrm{inv}}_{Z/X/S}$ commutes with base change. It then follows from the construction of the extension (2.7.1) that the extension class $\mathrm{at}(Y_T/X_T/T)$ is obtained from $\mathrm{at}(Y/X/S)$ by base change via $T \to S$. $\square$

Using the exact sequence of complexes $0 \to g^*\big(\Omega^{\mathrm{inv}}_{A/S}\big) \to \ell_{Y/X} \to g^*\big(\Omega^{\mathrm{inv}}_{B/S}\big)[1] \to 0$ and the fact that $\Omega^{\mathrm{inv}}_{A/S}$ and $\Omega^{\mathrm{inv}}_{B/S}$ are locally free $\mathcal{O}_S$-modules, we get that

$$\mathrm{Ext}^i\big(\ell_{Y/X}, \mathcal{O}_X\big) \cong \mathrm{H}^i\big(X, \ell^\vee_{Y/X} \otimes_{\mathcal{O}_S} \mathcal{O}_X\big).$$

Write $\operatorname{Lie} A := \Gamma\big(S, \Omega^{\mathrm{inv},\vee}_{A/S}\big)$ and $\operatorname{Lie} B := \Gamma\big(S, \Omega^{\mathrm{inv},\vee}_{B/S}\big)$. We then get an exact sequence

$$0 \longrightarrow \frac{\operatorname{Lie} B}{\operatorname{Lie} A} \otimes \mathrm{H}^{i-1}\big(X, \mathcal{O}_X\big) \longrightarrow \operatorname{Ext}^i\big(\ell_{Y/X}, \mathcal{O}_X\big) \longrightarrow \operatorname{Lie} A \otimes \mathrm{H}^i\big(X, \mathcal{O}_X\big)$$
$$\longrightarrow \operatorname{Lie} B \otimes \mathrm{H}^i\big(X, \mathcal{O}_X\big),$$

where the tensor product is taken over $\Gamma(S, \mathcal{O}_S)$.

In particular, let us *assume* that R is a complete local ring with maximal ideal $\mathfrak{m}$ and with residue field k, that $S = \operatorname{Spec}(R/\mathfrak{m}^n)$ and $S' = \operatorname{Spec}(R/\mathfrak{m}^{n+1})$ so that $\mathcal{J} = \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a k-vector space. Assume that the special fiber G_k of G is the kernel of a faithfully flat map of smooth group schemes $\alpha_k \colon A_k \to B_k$ (not necessarily obtained from a map of group schemes over S). Denote $\mathbb{T}_{X_k/k} := \operatorname{Hom}(\Omega^1_{X_k/k}, \mathcal{O}_{X_k})$. The extension $\operatorname{at}(X_k/Y_k/k)$

$$0 \longrightarrow \Omega^1_{X_k/k} \longrightarrow \ell_{Y_k/k} \longrightarrow \ell_{Y_k/X_k} \longrightarrow 0$$

and the complex $\ell_{Y_k/X_k} = [0 \to g^*\big(\Omega^{\mathrm{inv}}_{B_k/k}\big) \to g^*\big(\Omega^{\mathrm{inv}}_{A_k/k}\big) \to 0]$ provide a map

$$\gamma \colon \mathrm{H}^{-1}(\ell_{Y_k/X_k}) = g^*\big(\operatorname{Ker}(\Omega^{\mathrm{inv}}_{B_k/k} \to \Omega^{\mathrm{inv}}_{A_k/k})\big) \longrightarrow \Omega^1_{X_k/k}.$$

The following corollary summarizes what we proved so far. We recall the notations for reader convenience. We denote by j the closed immersion $j \colon S \hookrightarrow S'$. We have a smooth morphism $X' \to S'$ of relative dimension one and we denote by $g \colon X \to S$ its base change via j. Let $G' \to S'$ be a commutative, flat and of finite presentation group scheme over S'. We suppose also that the base change $G := G' \times_{S'} S$ is the kernel of a faithfully flat morphism $\alpha \colon A \to B$ of smooth groups schemes over S.

2.11 Corollary. *Under the hypotheses above we have:*

i. *There exists an element* $\Theta\big(Y, X', G'\big) \in \frac{\operatorname{Lie} B_k}{\operatorname{Lie} A_k} \otimes \mathrm{H}^1\big(X_k, \mathcal{O}_{X_k}\big) \otimes_k \mathcal{J}$ *which vanishes if and only if the G-torsor $Y \to X$ can be lifted to a G'-torsor $Y' \to X'$;*

ii. *if $\Theta\big(Y, X', G'\big) = 0$, the set of isomorphism classes of G-torsors deforming $Y \to X$ is a principal homogeneous space under* $\operatorname{Ext}^1\big(\ell_{Y_k/X_k}, \mathcal{O}_{X_k}\big) \otimes_k \mathcal{J}$ *and* $\operatorname{Ext}^1\big(\ell_{Y_k/X_k}, \mathcal{O}_{X_k}\big)$ *is an extension of* $\operatorname{Ker}(\operatorname{Lie} B_k \to \operatorname{Lie} A_k) \otimes \mathrm{H}^1\big(X_k, \mathcal{O}_{X_k}\big)$ *by* $\frac{\operatorname{Lie} B_k}{\operatorname{Lie} A_k} \otimes \mathrm{H}^0\big(X_k, \mathcal{O}_{X_k}\big)$*;*

iii. *if X'' is the deformation of X to S' defined by* $\Psi_{X'}(X'') \in \mathrm{H}^1\big(X_k, \mathbb{T}_{X_k/k}\big) \otimes_k \mathcal{J}$*, then* $\Theta\big(Y, X'', G'\big) = \Theta\big(Y, X', G'\big) + (\delta \otimes 1)\big(\Psi_{X'}(X'')\big)$ *with*

$$\delta = \delta_{Y_k/X_k} \colon \mathrm{H}^1\big(X_k, \mathbb{T}_{X_k/k}\big) \longrightarrow \frac{\operatorname{Lie} B_k}{\operatorname{Lie} A_k} \otimes \mathrm{H}^1\big(X_k, \mathcal{O}_{X_k}\big)$$

given by taking $\mathrm{H}^1(X_k, \_)$ of the dual of γ.

Proof: Note that $\operatorname{Ext}^i\big(\ell_{Y/X}, \mathcal{J}\mathcal{O}_X\big) \cong \operatorname{Ext}^i\big((\ell_{Y/X})|_{X_k}, \mathcal{O}_{X_k}\big) \otimes_k \mathcal{J}$ and similarly for $\Omega^1_{X/S}$ in place of $\ell_{Y/X}$. As we are assuming that G is the kernel of a morphism of smooth S-group schemes, Proposition 2.10 implies that the complex $(\ell_{Y/X})|_{X_k}$ coincides with ℓ_{Y_k/X_k} and the Atiyah extension class $\operatorname{at}(Y/X/S) \otimes_R k$ coincides with $\operatorname{at}(Y_k/X_k/k)$. Both the complex ℓ_{Y_k/X_k} and the Atiyah class $\operatorname{at}(Y_k/X_k/k)$ can be computed using the resolution $0 \to G_k \to$

$A_k \to B_k \to 0$ over k thanks to Proposition 2.10. In particular, $\mathrm{Ext}^i\big(\ell_{Y/X}, \mathcal{J}\mathcal{O}_X\big) \cong \mathrm{Ext}^i\big(\ell_{Y_k/X_k}, \mathcal{O}_{X_k}\big) \otimes_k \mathcal{J}$ and, via these identifications, the map $\delta'\colon \mathrm{Ext}^1\big(\Omega^1_{X/S}, \mathcal{J}\mathcal{O}_X\big) \longrightarrow \mathrm{Ext}^2\big(\ell'_{Y/X}, \mathcal{J}\mathcal{O}_X\big)$ of Proposition 2.5 is obtained by tensoring the map δ of Claim (iii) with $\otimes_k \mathcal{J}$.

The claim follows using these identifications, Propositions 2.5 and 2.10 and the discussion following Proposition 2.10. □

The exact sequence $0 \to G_k \to A_k \to B_k \to 0$ induces a morphism $B_k(X_k) \to \mathrm{H}^1(X_k, G_k)$. Given a morphism $\sigma\colon X_k \to B_k$ the associated G_k-torsor is defined as the fiber product of σ and $\alpha\colon A_k \to B_k$.

2.12 Lemma. *Assume that the G_k-torsor $Y_k \to X_k$ is obtained from a morphism $\sigma\colon X_k \to B_k$. Then, the map $\gamma\colon g^*\big(\mathrm{Ker}(\Omega^{\mathrm{inv}}_{B_k/k} \to \Omega^{\mathrm{inv}}_{A_k/k})\big) \longrightarrow \Omega^1_{X_k/k}$ is induced by the map $d\sigma\colon \sigma^*(\Omega^1_{B_k/k}) \to \Omega^1_{X_k/k}$.*

Proof: The A_k-torsor Z_k defined by $\iota_*(Y_k)$ is A_k by construction so that $\Omega^{\mathrm{inv}}_{Z_k/X_k/k} = \Omega^1_{X_k/k} \oplus \Omega^{\mathrm{inv}}_{A_k/k}$. As $W_k = \alpha_*(Z_k) = B_k$ then $\Omega^{\mathrm{inv}}_{W_k/X_k/k} = \Omega^1_{X_k/k} \oplus \Omega^{\mathrm{inv}}_{B_k/k}$. The map $\rho\colon A_k \to B_k$ is the morphism α so that the map $d\rho^{\mathrm{inv}}\colon \Omega^{\mathrm{inv}}_{W_k/X_k/k} \to \Omega^{\mathrm{inv}}_{Z_k/X_k/k}$ is the identity on $\Omega^1_{X_k/k}$ and is induced by $d\alpha$ on the second factor. The map γ is the composite of the morphism $\Omega^{\mathrm{inv}}_{B_k/k} \to \Omega^{\mathrm{inv}}_{W_k/X_k/k}$, provided by the section σ and sending $\omega \mapsto \big(d\sigma(\omega), \omega\big)$, composed with with $d\rho^{\mathrm{inv}}$ and the projection onto $\Omega^1_{X_k/k}$. This coincides with $d\sigma$ as claimed. □

2.13 *Some Examples.* We compute the map γ in the following examples.

Relation with Jacobians: Let G_k be a finite and flat commutative group scheme. Let $q\colon G_k^\vee \to \mathrm{Pic}^0(X_k/k)$ be a closed immersion. Due to [Mi, Prop. III.4.16] we have an isomorphism $\mathrm{H}^1(X_k, G_k) \cong \mathrm{Hom}\big(G_k^\vee, \mathrm{Pic}^0(X_k/k)\big)$. Thus, associated to q we have a G_k–torsor $Y_k \to X_k$. Write B_k for the Albanese variety of X_k i.e., for the dual abelian variety $\mathrm{Pic}^0(X_k/k)^\vee$. Let $A_k^\vee$ be the abelian variety $\mathrm{Pic}^0(X_k/k)/G_k^\vee$. The dual of the quotient map $B_k^\vee \to A_k^\vee$ defines an isogeny $\alpha\colon A_k \to B_k$ with kernel $\iota\colon G_k \to A_k$. Assume there exists a k–valued point of X_k and let $\sigma\colon X_k \to B_k$ be the associated Albanese map. Then, the G_k–torsor Y_k arises as the fibre product of $\sigma\colon X_k \to B_k$ and of $\alpha\colon A_k \to B_k$. Thanks to Lemma 2.12, in this case the map

$$\gamma\colon g^*\left(\mathrm{Ker}\left(\Omega^{\mathrm{inv}}_{B_k/k} \to \Omega^{\mathrm{inv}}_{A_k/k}\right)\right) \longrightarrow \Omega^1_{X_k/k}$$

is described in terms of the map on differentials defined by σ.

μ_p-torsors: Take $A_k = \mathbf{G}_{m,k} = \mathrm{Spec} k[z, z^{-1}]$, $B_k = \mathbf{G}_{m,k}$ and α the multiplication by p map. Let $Y_k \to X_k$ be a μ_p-torsor induced by a map $\sigma\colon X_k \to \mathbf{G}_{m,k}$ i.e., by an invertible element $a \in \Gamma(X_k, \mathcal{O}_{X_k})$ so that $\mathcal{O}_{Y_k} = \mathcal{O}_{X_k}[T]/(T^p - a)$. The map γ is given in this case by the map

$$d\sigma\colon k\frac{dz}{z} = \sigma^*(\Omega^{\mathrm{inv}}_{\mathbf{G}_{m,k}/k}) \longrightarrow \Omega^1_{X_k/k}, \qquad dz/z \mapsto da/a.$$

Thus $\gamma(dz/z)$ is the differential form defined in [Mi].

α_p-torsors: Assume that k is of positive characteristic p. Take $A_k = \mathbf{G}_{a,k} = \mathrm{Spec} k[z]$, $B_k = \mathbf{G}_{a,k}$ and α the Frobenius map. Let $Y_k \to X_k$ be an α_p-torsor induced by a map $\sigma\colon X_k \to \mathbf{G}_{a,k}$ i.e., by an element $a \in \Gamma(X_k, \mathcal{O}_{X_k})$ so that $\mathcal{O}_{Y_k} = \mathcal{O}_{X_k}[T]/(T^p - a)$. The map γ is given in this case by the map

$$d\sigma\colon k dz = \sigma^*(\Omega^{\mathrm{inv}}_{\mathbf{G}_{a,k}/k}) \longrightarrow \Omega^1_{X_k/k}, \qquad dz \mapsto da.$$

Thus $\gamma(dz)$ is the differential form defined in [Mi].

3 Proof of the main Theorem.

3.1 *Proof of the main theorem.* The notation is as in the statement of Theorem 1.1.

We will denote by $\mathfrak{m}$ the maximal ideal of R. If n is a natural number, we will denote by R_n the ring $R/\mathfrak{m}^n$ and, if $X \to S$ is a scheme over R, we will denote by X_n the base change of X to $\mathrm{Spec}(R_n)$; in particular $S_n = \mathrm{Spec}(R_n)$.

If G_k is étale, the proof is obvious since $Y_k \to X_k$ is étale and, given a deformation X of X_k to R, it can be deformed uniquely to a G–torsor $Y \to X$. Thus, we may assume that $\mathrm{Lie}\, G_k$ has dimension 1. By a classical theorem of M. Raynaud [BBM, §3.1.1], we can find two abelian schemes A and B over R and an exact sequence of R group schemes

$$0 \longrightarrow G \overset{\iota}{\longrightarrow} A \overset{\alpha}{\longrightarrow} B \longrightarrow 0.$$

By induction on n we construct a flat scheme $X_n \to S_n$ and a G–torsor $Y_n \to X_n$ such that (1) $X_1 = X_k, Y_1 = Y_k$ as G–torsor, (2) $X_n \cong X_{n+1} \times_{S_{n+1}} S_n$ and $Y_n \cong Y_{n+1} \times_{S_{n+1}} S_n$ as G–torsor over X_n. For $n = 1$ there is nothing to prove. Assume we have constructed $Y_n \to X_n$. Since X_n is a smooth curve, it can be deformed to a smooth curve $X_{n+1} \to S_{n+1}$. Due to Corollary 2.11, to prove that there exist a lifting $X'_{n+1} \to S_{n+1}$ of X_n and a G–torsor $Y_{n+1} \to X'_{n+1}$ deforming $Y_n \to X_n$, it suffices to show that the map

$$\delta_{Y_k/X_k}\colon \mathrm{H}^1\big(X_k, \mathbb{T}_{X_k/k}\big) \longrightarrow \big(\mathrm{Lie}\, B_k/\mathrm{Lie}\, A_k\big) \underset{k}{\otimes} \mathrm{H}^1\big(X_k, \mathcal{O}_{X_k}\big) \tag{3.1.1}$$

is surjective. Passing to an algebraic closure of k we may assume that k is algebraically closed and that X_k is irreducible. In particular, X_k admits a k–valued point. Since k is algebraically closed, G_k is the product $G_k \cong G_k^0 \times G_k^{\mathrm{et}}$ of its connected component at the identity G_k^0 and its étale part G_k^{et}. For $i \in \mathbf{N}$ denote by $F^i\colon G_k^0 \to G_k^{0,(p^i)}$ the i-th iterate of the Frobenius morphism and set $G_k^0[F^i]$ its kernel. Since G_k^0 is a finite and connected group scheme it is annihilated by some power of F. Let $N \in \mathbf{N}$ be such that $G_k^0[F^N] = G_k^0$ and $G_k^0[F^{N-1}]$ is strictly contained in G_k^0. Since $\mathrm{Lie}\, G_k^0 = \mathrm{Lie}\, G_k$ is a k-vector space of dimension 1 by assumption, G_k^0 is not trivial, proving that such N exists.

Let $T_k \to G_k$ be the subgroup $G_k^0[F^{N-1}] \times G_k^{\mathrm{et}}$. As $\mathrm{Lie}\, G_k^0 = \mathrm{Lie}\, G_k$ is a k-vector space of dimension 1, then $G_k^0 \cong \mathrm{Spec} k[T]/\big(T^{p^N}\big)$ as a scheme so that the quotient $H_k := G_k/T_k$ is isomorphic to $\mathrm{Spec} k[S]/(S^p)$. In particular, H_k is local, of rank p. Set $A'_k := A_k/T_k$ and $B'_k := A'_k/H_k = B_k$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & G_k & \to & A_k & \to & B_k & \to 0 \\ & \downarrow & & \downarrow & & \big\| & \\ 0 \to & H_k & \to & A'_k & \to & B'_k & \to 0 \end{array}$$

The kernel of the map $\mathrm{Lie}\, A_k \to \mathrm{Lie}\, B_k$ is $\mathrm{Lie}\, G_k$ which is of dimension 1 as k-vector space. As $\mathrm{Lie}\, A_k$ and $\mathrm{Lie}\, B_k$ have the same dimension, the quotient $\mathrm{Lie}\, B_k/\mathrm{Lie}\, A_k$ has dimension 1. Similarly as $\mathrm{Lie}\, H_k$ is of dimension 1, also $\mathrm{Lie}\, B'_k/\mathrm{Lie}\, A'_k$ is a k-vector space of dimension 1. Since the map $\mathrm{Lie}\, B_k \to \mathrm{Lie}\, B'_k$ is an isomorphism, the induced map $\mathrm{Lie}\, B_k/\mathrm{Lie}\, A_k \to \mathrm{Lie}\, B'_k/\mathrm{Lie}\, A'_k$ is surjective. As both are 1-dimensional k-vector spaces, it is an isomorphism. By assumption the pushforward Q_k of Y_k via $G_k \to H_k$ is non–trivial. It follows from Lemma 2.9 and Proposition 2.10 that the Atiyah extension class $\mathrm{at}(Q_k/X_k/k)$ is obtained from $\mathrm{at}(Y_k/X_k/k)$ so that $\delta_{Y_k/X_k} = \delta_{Q_k/X_k}$ via the isomorphism $\mathrm{Lie}\, B_k/\mathrm{Lie}\, A_k \cong \mathrm{Lie}\, B'_k/\mathrm{Lie}\, A'_k$.

Thus, we may replace G_k with H_k and we are reduced to prove that δ_{Y_k/X_k} is surjective in the case that G_k is local, of rank p. Since k is algebraically closed, we have two cases $G_k \cong \mu_p$ or $G_k \cong \alpha_p$.

It follows from Corollary 2.11 that for the computation of $\delta = \delta_{Y_k/X_k}$ it does not matter which resolution of G_k one takes.

Case 1: $G_k = \mu_p$.

Then we take $A_k = \mathbf{G}_{m,k} \to \mathbf{G}_{m,k} = B_k$ given by raising to the p-th power. There exists a covering by open affine subschemes $U_i := \mathrm{Spec}(C_i)$ of X_k such $Y_k|_{U_i} = \mathrm{Spec}\big(C_i[T_i]/(T_i^p - a_i)\big)$ for a suitable $a_i \in C_i^*$. Furthermore, $T_i = g_{ij}T_j$ for a suitable cocycle $g_{ij} \in \Gamma\big(U_i \cap U_j, \mathcal{O}_{X_k}^*\big)$. Due to §2.13 the map $\gamma\colon \Omega^{\mathrm{inv}}_{B_k/k} \otimes \mathcal{O}_{X_k} \to \Omega^1_{X_k/k}$ (the connection of γ and δ is given in Corollary 2.11) is defined over U_i by the differential $\omega_i := \frac{da_i}{a_i}$. If we denote by $j_i\colon U_i \hookrightarrow X_k$ the inclusion, the differential ω_i is trivial if and only if $j_i^*(a_i)$ is a p–th power for some (equivalently any) i i. e., if and only if Y_k is the trivial μ_p–torsor which is not the case by assumption.

Case 2: $G_k = \alpha_p$.

In this case $A_k \to B_k$ is given by Frobenius $\mathbf{G}_{a,k} \to \mathbf{G}_{a,k}$. There exists a covering by open affines $U_i := \mathrm{Spec}(C_i)$ of X_k such $Y_k|_{U_i} = \mathrm{Spec}\big(C_i[T_i]/(T_i^p - a_i)\big)$ for a suitable $a_i \in C_i$. The gluing is given by $T_i = T_j + g_{ij}$ for a suitable cocycle $g_{ij} \in \Gamma\big(U_i \cap U_j, \mathcal{O}_{X_k}\big)$. Due to §2.13 the map $\gamma\colon \Omega^{\mathrm{inv}}_{B_k/S_k} \otimes \mathcal{O}_{X_k} \to \Omega^1_{X_k/k}$ is defined over U_i by $\omega_i := da_i$. This is trivial if and only if $j_i^*(a_i)$ is a p–th power for some (equivalently any) i i. e., if and only if Y_k is the trivial α_p–torsor which is not the case by assumption.

In both cases we proved that the map γ is injective so, by Serre duality, the map (3.1.1) is surjective as claimed. □

3.2 Corollary. *Assume that R is a dvr of characteristic 0, that the group scheme G is connected and that $\mathrm{Lie}\, G_k$ is of dimension ≤ 1. Then, there exist a finite extension of dvr $R \subseteq R'$, a smooth formal curve X' over R' and a G–torsor $Y \to X'$ whose fiber over the residue field of R' coincides with the base change of $Y_k \to X_k$.*

Proof: If X_k is affine, then $\mathrm{H}^1\big(X_k, \mathcal{O}_{X_k}\big) = 0$ so that $\Theta(Y_n, X_{n+1}, G) = 0$ and the conclusion follows from Corollary 2.11. Assume that X_k is projective. Possibly passing to an unramified extension $R \subset R'$ we may further assume that X_k is geometrically irreducible and admits a k–valued point. Let $g_k\colon G_k^\vee \to \mathrm{Pic}^0(X_k/k)$ be the homomorphism associated to the G_k–torsor $Y_k \to X_k$ as in §2.13. Let $H_k^\vee \subset \mathrm{Pic}^0(X_k/k)$ be the image of g_k. Dualizing the quotient map $G_k^\vee \to H_k^\vee$ we get a closed immersion $h_k\colon H_k \to G_k$ and Y_k arises by push–forward via h_k of an H_k–torsor $Z_k \to X_k$. By Lemma 3.3 the subgroup scheme $H_k \subset G_k$ can be lifted to a subgroup scheme $H \subset G \times_R R'$ possibly after an extension of dvr's $R \subset R'$. Let k' be the residue field of R'. By Theorem 1.1 the $H_{k'}$–torsor $Z_{k'} \to X_{k'}$, obtained by base change of $Z_k \to X_k$ via $k \to k'$, can be lifted to an H–torsor $Z \to X'$. Let $Y \subset X$ be the G–torsor defined by push–forward of $Z \to X$ via the closed immersion $H \subset G \times_R R'$. By construction it lifts the base change of $Y_k \to X_k$ to k' as wanted. $\square$

3.3 Lemma. *Assume that R is a dvr of characteristic 0, that G is connected and that $\mathrm{Lie}\, G_k$ is of dimension 1. Let $H_k \subset G_k$ be a subgroup scheme. Then, there exists a finite extension of dvr $R \subset R'$ and a subgroup scheme $H \subset G \times_R R'$ lifting the base change of $H_k \subset G_k$ to the residue field of R'.*

Proof: Denote by K the fractions field of R. Possibly passing to a finite extension of R we may assume that G_K is a constant group scheme. Let $G_{i,K} \subset G_{2,K} \subset \ldots \subset G_{N,K} = G_K$ be a tower of subgroups with cyclic quotients of prime order (necessarily of order p). Let G_i be the schematic closure of $G_{i,K}$ in G. It is a finite and flat over R by construction. It is connected and commutative since G is, it is a closed normal subgroup scheme of G_{i+1} and the quotient G_{i+1}/G_i is of order p. In particular, since $\mathrm{Lie}\, G_k$ is 1–dimensional, $G_{i,k}$ is the kernel $G_k\big[F^i\big]$ of the i–th iterated of Frobenius $F^i\colon G_k \to G_k^{(p^i)}$ on G_k. In particular $\mathrm{Lie}\, G_{i,k}$ is also 1–dimensional. Since the Hopf algebra underlying G_k is monogenic because $\mathrm{Lie}\, G_k$ has dimension 1, these are the only subgroup schemes of G_k so that $H_k = G_{i,k}$ for some i. $\square$

3.4 *Example.* Let R be a dvr of unequal characteristic with field of fractions K and perfect residue field k. Let $\widetilde{X}$ be a smooth and projective curve over R such that the Jacobian of X_k contains α_{p^n} as a closed subgroup scheme and such that the p^n–torsion of the Jacobian of $\widetilde{X}_K$ is rational over K. For example, for $n = 1$, any prime to p cover of a supersingular elliptic curve has this property.

The number of non–isomorphic geometric cyclic covers of order p of $\widetilde{X}_K$ is bounded by a constant c depending only on p and the genus g of $\widetilde{X}_K$ (but independent of K). Indeed $\mathrm{Pic}^0\big(\widetilde{X}_K/K\big)[p] \cong \mathbb{F}_p^{2g}$ by assumption so that, due to [Mi, Prop. III.4.16], the non-trivial torsors over $\widetilde{X}_K$ under a group scheme of order p are $\mathbf{Z}/p\mathbf{Z}$-torsors and are in bijection with the non-zero elements of $\mathrm{Hom}(\mathrm{F}_p, \mathbb{F}_p^{2g})$. Since the relative Jacobian $\mathrm{Pic}^0(\widetilde{X}/R)$ is proper, the $\mathbf{Z}/p\mathbf{Z}$-torsors of $\widetilde{X}_K$ extend *uniquely* to torsors over $\widetilde{X}$ under suitable finite and flat group schemes of order p over R (take the Zariski closure of the corresponding subgroup of $\mathrm{Pic}^0(\widetilde{X}_K/K)$ in $\mathrm{Pic}^0(\widetilde{X}/R)$).

The α_p–torsors over X_k are in one to one correspondence with the morphisms from α_p to $\mathrm{Pic}^0(X_k/k)$; it follows from our assumptions that X_k has at least as many non–trivial and non–isomorphic α_p–torsors as the cardinality of $\mathrm{Aut}(\alpha_p) \cong k^*$. In particular, if such cardinality is strictly bigger than c not every α_p–torsor can be lifted.

For example, one can take a curve whose special fiber has superspecial Jacobian e.g. the Fermat curve of degree $p+1$ or the curve $y^2 = x^p - x$ (cf. [Ek]). Other examples are provided by curves contained in the product of two non–isogenous elliptic curves over $\mathbb{Q}$ such that there exist infinitely many primes where both elliptic curves have supersingular reduction.

3.5 *Another example.* Let X_k be an ordinary, smooth and projective curve of genus g with $1 \leq g \leq 3$ over an algebraically closed field k of characteristic p. If $g = 3$, we suppose that X_k is not hyperelliptic. Let $Y_k \to X_k$ be a non trivial μ_{p^n}–torsor. This is defined by a closed subgroup scheme $\iota: \mathbf{Z}/p^n\mathbf{Z} \hookrightarrow J$, where J is the Jacobian of X_k. Let R be a complete dvr of unequal characteristic and with residue field k. We will show that there exist lifts $\widetilde{X}$ of X_k over R such that ι can not be lifted to a subgroup scheme isomorphic to $\mathbf{Z}/p^n\mathbf{Z}$ of the Jacobian of $\widetilde{X}$ relative to R. This implies that $Y_k \to X_k$ can not be lifted to a μ_{p^n}–torsor over $\widetilde{X}$. We do not know if similar examples exist for curves of higher genus.

Let $j: J \to A$ be the isogeny with kernel $\iota(\mathbf{Z}/p^n\mathbf{Z})$. Note that the formal universal deformation space $\mathrm{Def}(A)$ is canonically isomorphic to the formal universal deformation space $\mathrm{Def}(J, \iota)$ of J with the subgroup scheme $\iota(\mathbf{Z}/p^n\mathbf{Z})$. From Serre–Tate theory ([Ka] Theorem 2.1) we deduce the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Def}(J,\iota) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{Z}_p}(T_p(A) \otimes_{\mathbf{Z}_p} T_p(A^\vee), \widehat{\mathbf{G}}_m) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle j \otimes (j^\vee)^{-1}} \\
\mathrm{Def}(J) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{Z}_p}(T_p(J) \otimes_{\mathbf{Z}_p} T_p(J^\vee), \widehat{\mathbf{G}}_m);
\end{array}$$

where f is the map associated to the forgetful functor, $T_p(\_)$ is the p–adic Tate module and we use the fact that the dual map $j^\vee: T_p(A^\vee) \to T_p(J^\vee)$ is an isomorphism. We fix suitable bases of the various $T_P(\_)$'s in such a way that $\mathrm{Def}(J) \simeq M_{g\times g}(\widehat{\mathbf{G}}_m)$ and $\mathrm{Def}(J,\iota) \simeq M_{g\times g}(\widehat{\mathbf{G}}_m)$ (where $M_{g\times g}(\_)$ is the group of $g \times g$ matrices), the deformation subspace of J as principally polarized abelian variety corresponds to the symmetric matrices and the map $j \otimes (j^\vee)^{-1}$ is obtained by raising the entries of the first column to the p^n–th power.

Let $\widetilde{J}$ be a deformation of J as principally polarized abelian variety corresponding to a symmetric matrix having at least one entry in the first column which is not a p^n–th power. Then, as proven in [OS], $\widetilde{J}$ is the relative Jacobian of a curve $\widetilde{X}$ over R over which the μ_{p^n}–torsor Y_k can not be lifted.

4 An application to the theory moduli of p–covers of curves.

Let R be a dvr with residue field k of positive characteristic p and fraction field K. We denote by $\mathcal{G}_p \to \mathrm{Spec}(R)$ the *Artin Stack* of finite and flat group schemes of order p over

R. An explicit description of this stack is given in the paper [OT]. The stack $\mathcal{G}_p$ is regular and the fiber over the closed point of the structural morphism $\mathcal{G}_p \to \mathrm{Spec}(R)$ is a simple normal crossing divisor.

Let g be a non negative integer. A smooth p torsor of genus g over a base scheme S is a triple (X, Y, G) where:

i) $X \to S$ is a smooth family of curves of genus g;

ii) $G \to S$ is a group scheme of order p over S;

iii) $Y \to X$ is a G–torsor.

The smooth p torsors of genus g define a category $\mathrm{CC}_{p,g}$ fibered over $\mathrm{Spec}(\mathbb{Z})$. There are evident forgetful functors

$$\mathrm{pr}_{\mathcal{M}} \colon \mathrm{CC}_{g,p} \longrightarrow \mathcal{M}_g, \qquad (X, Y, G) \mapsto X$$

and

$$\mathrm{pr}_G \colon \mathrm{CC}_{p,g} \longrightarrow \mathcal{G}_p, \qquad (X, Y, G) \mapsto G.$$

The fiber of $\mathrm{pr}_{\mathcal{M}} \times \mathrm{pr}_G \colon \mathrm{CC}_{g,p} \to \mathcal{M}_g \times \mathcal{G}_p$ over an S-valued point (X, G) of $\mathcal{M}_g \times \mathcal{G}_p$ consists of all G-torsors $Y \to X$. By a theorem of Raynaud, see [Mi, Prop. III.4.16], these correspond to homomorphisms of S-group schemes $\mathrm{Hom}_S\big(G^\vee, \mathrm{Pic}^0(X/S)[p]\big)$. As $X \to S$ is a smooth curve, then $\mathrm{Pic}^0(X/S)$ is an abelian scheme and the kernel $\mathrm{Pic}^0(X/S)[p]$ of multiplication by p is a finite and locally free S-group scheme. It follows that $\mathrm{pr}_{\mathcal{M}} \times \mathrm{pr}_G$ is representable and that $\mathrm{pr}_{\mathcal{M}}$ satisfies the valuative criterion of properness. We can apply the main theorem 1.1 in this context to obtain Corollary 1.2:

4.1 Corollary. *With the notations as above, the morphism $\mathrm{pr}_G \colon \mathrm{CC}_{p,g} \to \mathcal{G}_p$ is formally smooth.*

Proof: We recall that a morphism of artinian local rings $h \colon B \to B'$ is said to be a *small extension* if it is surjective and $\ker(h)$ has length 1 as B module. This means that $\ker(h)^2 = 0$ and $\ker(h) = (x)$ with $x \cdot \mathfrak{m} = 0$ where $\mathfrak{m} \subset B$ is the maximal ideal. An equivalent definition of formally smooth morphism is the following: for every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(B') & \xrightarrow{f'} & \mathrm{CC}_{p,g} \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_G} \\ \mathrm{Spec}(B) & \xrightarrow{f} & \mathcal{G}_p \end{array}$$

where h is a small extension, there exist a morphism $f \colon \mathrm{Spec}(B) \to \mathrm{CC}_{p,g}$ making the diagram commutative.

Thus, to conclude we need to prove the following: let $h \colon B \to B'$ be a small extension and let $G \to \mathrm{Spec}(B)$ a flat group scheme of order p. Let G' be the base change of G to $\mathrm{Spec}(B')$. Let $X' \to \mathrm{Spec}(B')$ be a smooth projective curve of genus g over B' and $Y' \to X'$ a G'–torsor. Then there exists a smooth projective curve $X \to \mathrm{Spec}(B)$ and a G–torsor $Y \to X$ which extends $Y' \to X'$. The proof of Theorem 1.1 applies in this situation and the conclusion follows. $\square$

From the explicit description of $\mathcal{G}_p$ we thus obtain:

4.2 Corollary. *The category* $\mathrm{CC}_{p,g}$ *is a regular Artin stack flat over* $\mathbb{Z}$ *with fibre over the prime ideal* (p) *which is a simple normal crossing divisor.*

References.

[AOV] D. Abramovich, M. Olsson, A. Vistoli: *Twisted stable maps to tame Artin stacks.* J. Algebraic Geom. **20** (2011), 399-477.

[AR] D. Abramovich, M. Romagny: *Moduli of Galois p-covers in mixed characteristics.* Algebra Number Theory **6** (2012), 757-780.

[AG] F. Andreatta, C. Gasbarri: *Torsors under some group schemes of order* p^n. J. of Algebra **318** (2007), 1057–1067.

[BBM] P. Berthelot, L. Breen, W. Messing: *Théorie de Dieudonné cristalline. II.* Lecture Notes in Math. **930** (1982).

[Ek] T. Ekedhal: *On supersingular curves and abelian varieties*, Math. Scand. **60** (1987), 151–178.

[Il] L. Illusie: *Complexe cotangent et déformations II*, Lecture Notes in Math. **283** (1972).

[Ka] N. Katz: *Serre–Tate local moduli*, in "Surfaces algébriques", J. Giraud, L. Illusie, M. Raynaud eds. Lecture Notes in Math. **868** (1981), 138–202.

[Mi] J. Milne: *Étale cohomology*, Princeton University Press (1980).

[OS] F. Oort, J. Steenbrink: *The local Torelli problem for algebraic curves*, Journées de Géométrie Algébriques d'Angers, Juillet 1979 (1980), 157–204.

[OT] J. Tate. F. Oort; *Group schemes of prime order.* Ann. Sci. École Norm. Sup. (4) 3 1970 121.

[We] S. Wewers: *Formal deformation of curves with group scheme action.* Ann. Inst. Fourier **55** (2005), 1105–1165.

F. Andreatta: Dipartimento di Matematica "Federigo Enriques" Via C. Saldini, 50 – 20133 Milano (I). e-mail: *fabrizio.andreatta@unimi.it*

C. Gasbarri: IRMA - Université de Strasbourg, 4, rue René Descartes, - 67000 Strasbourg (F). e-mail: *gasbarri@math.unistra.fr*