EFFECTIVE STABILITY FOR REALISTIC PHYSICAL SYSTEMS

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Abstract. The problem of the stability of an elliptic equilibrium of a Hamiltonian system is considered in the framework of Nekhoroshev like theory. In particular, it is shown how a careful evaluation of some constants by computer allows to obtain realistic results when considering specific physical systems.

1. Introduction

The present talk is concerned with a new insight on the stability problem, in the light of Nekhoroshev–like theory for Hamiltonian systems. Precisely, I will consider a neighbourhood of an elliptic equilibrium, with the aim of giving a quantitative estimate of the time needed in order to reveal a significant deviation of the dynamical evolution of the system from that of an approximated linear system.

The general framework can be briefly described as follows. One considers a canonical system with Hamiltonian

$$H(p,q,\varepsilon) = h(p) + \varepsilon f(p,q) \ , \tag{1.1}$$

where $p \in \mathcal{G} \subset \mathbf{R}^n$ are action variables, with $\mathcal{G}$ an open set, $q \in \mathbf{T}^n$ are angle variables, and ε is a small real parameter. All the functions will be assumed to be analytic. According to Poincaré, this the general problem of dynamics.[1]

The case of an elliptic equilibrium can be considered as a particular case of the system (1.1). Indeed, in such a case the Hamiltonian can be typically given the form

$$H(x,y) = H_0(x,y) + f(x,y) \ , \quad (x,y) \in \mathbf{R}^{2n} \tag{1.2}$$

where

$$H_0(x,y) = \frac{1}{2}\sum_l \omega_l (x_l^2 + y_l^2) \tag{1.3}$$

is the Hamiltonian of a system of uncoupled harmonic oscillators with frequencies $\omega \equiv (\omega_1,\ldots,\omega_n) \in \mathbf{R}^n$, and $f(x,y)$ is an analytic function whose power series expansion in a suitable neighbourhood of the origin starts with terms of order at least 3. If one is interested in studying the dynamical evolution of orbits with initial point in a small neighbourhood of the origin, then the size, ϱ say, of that neighbourhood plays the role of the perturbation parameter ε in (1.1).

For $\varepsilon = 0$ the evolution of the system (1.1) is well known: the actions $p_1, \ldots, p_n$ are first integrals, and the flow in phase space is given by

$$p(t) = p^{(0)} \quad , \quad q(t) = \omega(p^{(0)})t + q^{(0)}$$

where $p^{(0)}, q^{(0)}$ are the initial data, and $\omega(p^{(0)}) = \frac{\partial h}{\partial p}(p^{(0)})$ are the unperturbed frequencies of the system. The same holds for the system (1.2), the action variables being $p_l = (x_l^2 + y_l^2)/2$, $(1 \le l \le n)$, and the frequencies being constant. The phase space, in both cases, is foliated into invariant tori $p(t) = p^{(0)}$, parameterized by the initial value of the action variables, and the motion on each torus is either quasi periodic or periodic according to the existence of resonance relations among the frequencies ω, namely relations of the form $k \cdot \omega = 0$ with $0 \ne k \in \mathbf{Z}^n$.

Such a plain picture is no longer valid for the perturbed system, namely the one with $\varepsilon \ne 0$ (or $f(x, y) = 0$). The strongest result for such a system came in 1954 with the celebrated work of Kolmogorov[2], who succeeded in proving that tori with strongly nonresonant frequencies are not destroyed by the perturbation, but are simply distorted. By “strongly nonresonant frequencies” it is meant frequencies satisfying a condition like

$$|k \cdot \omega| \ge \gamma |k|^{-\tau} ,$$

with $|k| = |k_1| + \ldots + |k_n|$, for some constants $\gamma > 0$ and $\tau > n-1$. The set of preserved invariant tori is a nowhere dense set of large measure. This can be interpreted as a stability result “in measure”, in the sense that the evolution is known for the majority of the initial conditions.

The most unpleasant aspect of the Kolmogorov's theorem is that the set of invariant tori is not an open set, and that, for $n > 2$, the complement of the set of invariant tori is connected. Thus, an orbit starting in the gaps between tori can in principle diffuse through the gaps, so that the values of the action variables can significantly change during the time evolution of the system. Such a phenomenon has been named “Arnold diffusion”. Although the existence of such a diffusion has not been rigorously proven till now, nevertheless the existing examples suggest that it can well be a generic phenomenon.

Now, as was realized already by Moser[3] and Littlewood[4], and next fully stated by Nekhoroshev[5], it is interesting to investigate *how much time* is needed in order to observe a significant change in the actions of the system. Indeed, following the Nekhoroshev's formulation, one has

$$|p(t) - p(0)| < \varepsilon^b \text{ for } |t| \le T_* \exp\left(\frac{\varepsilon_*}{\varepsilon}\right)^a \quad , \tag{1.4}$$

with positive constants T_*, ε_*, a and b. The exponential growth of the time with $1/\varepsilon$ is the relevant part of the result, since a reasonable decrease of the perturbation can make the time so large to exceed any realistic lifetime of the physical system considered. Moreover, such a result holds for an open set of initial data, so that it may be considered as most suitable, for physical applications, than the Kolmogorov's theorem.

However, despite their mathematical beauty, these theories can hardly be applied to real physical models. The reason is that in all these theorems there appears a threshold, ε_c say, in the sense that for a perturbation larger than ε_c no conclusions can be drawn. Now, the purely analytic estimates of the value of this threshold, as well as of the constants T_*, ε_* and a in the exponential estimate (1.4), give usually ridiculously small values.

The aim of my talk is now twofold. First, I shall illustrate the application of Nekhoroshev like theory to the case of an elliptic equilibrium, namely to a Hamiltonian of the type (1.2); this will be given via a particularly simple method which avoids almost all the technical difficulties. The second goal is to show that, by combining the theory with a numerical computation of the relevant quantities by computer, one can reach realistic estimates of the threshold, which can be hopefully applied to a real system.

As a significant example, I shall illustrate the results in the case of the Lagrangian equilibrium points of the restricted problem of three bodies. Although this example looks quite far from the problem of the dynamics of particles in accelerators, I think that the problems are strictly connected by the fact that one is actually considering the stability of some periodic orbit. Indeed, the essential difference concerns essentially the choice of a natural time unit, since the dynamical evolution of a beam in an accelerator is considerably faster than that of the solar system. But, as I will show, such a difference is not relevant, due to the exponential behaviour of the stability time, according to (1.4).

2. Theoretical scheme

I shall consider here the simplest case of a Hamiltonian

$$H(x,y) = H_0(x,y) + H_1(x,y) \ , \quad (x,y) \in \mathbf{R}^{2n} \ , \tag{2.1}$$

with

$$H_0 = \frac{1}{2}\sum_l \omega_l (x_l^2 + y_l^2) \tag{2.2}$$

and H_1 a homogeneous polynomial of degree 3 in x, y. The general case of an analytic Hamiltonian with an infinite power series expansion does not actually introduce additional difficulties, the extension being just a technical matter. The interested reader will find the complete treatment in ref. [6].

I will also need a restriction on the frequencies. Precisely, I will require that the frequencies be nonresonant, i.e., $k \cdot \omega = 0$ for $k \in \mathbf{Z}^n$ implies $k = 0$. This is the most unpleasant restriction; however, the existence of resonances does not actually affect the development of the theory, since one simply has to use a more complex approach, using the normal form theory. Such a kind of study can be found, for example, in ref. [7].

Starting with the formal aspect of the theory, one tries to build up a set of n independent first integrals of the system (2.1), looking for functions $\Phi_1, \ldots, \Phi_n$ of the form

$$\Phi^{(l)}(x,y) = I_l(x,y) + \Phi_1^{(l)}(x,y) + \ldots \ , \quad 1 \le l \le n \ , \tag{2.3}$$

where

$$I_l(x,y) = \frac{1}{2}(x_l^2 + y_l^2) \tag{2.4}$$

are the actions of the harmonic oscillators, and $\Phi_s^{(l)}$, for $s \ge 1$, are homogeneous polynomials of degree $s+2$. To this end, one substitutes the explicit expansions (2.1) and (2.3) in the well known equation $\{H, \Phi\} = 0$ for a first integral (here, $\{\cdot,\cdot\}$ denotes the Poisson bracket), and gets the infinite system of equations

$$\begin{aligned} L_{H_0} I_l &= 0 \\ L_{H_0}\Phi_1^{(l)} &= -\{H_1, I_l\} \\ &\vdots \\ L_{H_0}\Phi_s^{(l)} &= -\{H_1, \Phi_{s-1}^{(l)}\} \ . \end{aligned} \tag{2.5}$$

Here, the linear operator $L_{H_0}\cdot = \{H_0, \cdot\}$ has been introduced.

The formal aspect of the system (2.5) is well known. By the canonical transformation to complex variables ξ, η

$$x_l = \frac{1}{\sqrt{2}}(\xi_l + i\eta_l) \ , \quad y_l = \frac{i}{\sqrt{2}}(\xi_l - i\eta_l) \ , \quad 1 \le l \le n \ , \tag{2.6}$$

the unperturbed Hamiltonian H_0 is given the form

$$H_0(\xi,\eta) = i\sum_l \omega_l \xi_l \eta_l \ , \tag{2.7}$$

so that the operator L_{H_0} takes a diagonal form, while the order of the homogeneous polynomials is preserved. Denote now by $\Psi_s^{(l)}$ the r.h.s. of eq. (2.5) for $s \ge 1$, and write it in the general form

$$\Psi_s^{(l)} = \sum_{j,k} c_{jk} \xi^j \eta^k \ ,$$

where $j = (j_1, \ldots, j_n)$ and $k = (k_1, \ldots, k_n)$ are integer vectors with non negative components and satisfying $|j+k| = s$, and $\xi^j \eta^k = \xi_1^{j_1} \cdot \ldots \cdot \xi_n^{j_n} \eta_1^{j_1} \cdot \ldots \cdot \eta_n^{j_n}$, while the coefficients c_{jk} are known. Assuming for $\Phi_s^{(l)}$ the same form with unknown coefficients, one has then the formal solution

$$\Phi_s^{(l)} = \sum_{j,k} \frac{c_{jk}}{i(j-k)\cdot\omega} \xi^j \eta^k \ . \tag{2.8}$$

The presence of the so called *small denominators* $(j-k)\cdot\omega$ raises a consistency problem: one has to ensure that the coefficient c_{jk} of the known function $\Psi_s^{(l)}$ vanishes for all j,k for which $(j-k)\cdot\omega = 0$. This is a minor problem in the nonresonant case, since then one can prove that $\Psi_s^{(l)}$ does not contain terms $\xi^j\eta^j$, and for this reason the nonresonance condition has been requested. Thus, one can solve the system (2.5), at least formally, at any desired order.

Let me now come to the quantitative aspect. The main ingredients to be introduced are the *domains* and a suitable *norm* on the homogeneous polynomials.

For what concerns the domains, the most natural choice is to take into account the fact that the actions $I_1,\ldots,I_n$ of the harmonic oscillators are the first approximations of the formal first integrals. This suggests to introduce the domain

$$\mathcal{D}_{\varrho R} = \bigl\{(x,y)\in\mathbf{R}^{2n} \;:\; x_l^2+y_l^2 \le \varrho^2 R_l^2\;,\; 1\le l\le n\bigr\}\;, \tag{2.9}$$

where $R\equiv(R_1,\ldots,R_n)$ is a vector of positive real numbers, and ϱ a nonnegative real parameter. The choice of $(R_1,\ldots,R_n)$ is quite arbitrary, so that they can be adapted to the initial data.

Coming now to the norm, writing a homogeneous polynomial of order s in the general form $f(x,y)=\sum_{j,k} f_{jk}x^j y^k$, one can define

$$\|f\|_R = \sum_{j,k} |f_{j,k}| R^{j+k} \tag{2.10}$$

(with the short notation $R^{j+k} = R_1^{j_1+k_1}\cdot\ldots\cdot R_n^{j_n+k_n}$). With these definitions, it is immediately seen that one has the bound

$$|f(x,y)| \le \varrho^s\,\|f\|_R \;\text{ for } (x,y)\in\mathcal{D}_{\varrho R}\;;$$

thus, the analytic setting so introduced allows to estimate the size of a homogeneous polynomial, and so also, if needed, the convergence properties of a power series.

With the analytic settings above, the formal scheme of construction of the first integrals can be made quantitative. To this end, we use the following estimates.

i. The transformation (2.6) to complex variables changes the norm of a homogeneous polynomial of degree s at most by a factor $2^{s/2}$; the same holds for the inverse of (2.6).

ii. The Poisson bracket between two homogeneous polynomials f_s and g_r of degree s and r respectively is estimated by

$$\|\{f_s,g_r\}\|_R \le sr\Lambda^2\|f_s\|_R\|g_r\|_R\;, \tag{2.11}$$

with $\Lambda = (\min_l R_l)^{-1}$.

iii. Given a sequence $\{\alpha_s\}_{s\ge1}$ of positive numbers such that

$$|k\cdot\omega| \ge \alpha_s \;\text{ for } 0<|k|\le s+2\;,\quad k\in\mathbf{Z}^n\;, \tag{2.12}$$

the solution (2.8) of the equation (2.5) is estimated by

$$\|\Phi_s^{(l)}\|_R \le \frac{1}{\alpha_s}\|\Psi_s^{(l)}\|_R\;. \tag{2.13}$$

By ii. and iii. one finds the recursive formula

$$\|\Psi_s^{(l)}\|_R \le \frac{3(s+1)\Lambda^2}{\alpha_{s-1}} \|\Psi_s^{(l)}\|_R \|H_1\|_R \ , \tag{2.14}$$

which holds in complex variables. To complete the quantitative estimates one has to recall that at the very beginning of the procedure the Hamiltonian must be expressed in complex variables ξ, η, and that at the very end the formal integral must be expressed again in real variables (x, y); thus, the property i. must be used in order to estimate the norms. The final estimate is

$$\|\Phi_s^{(l)}\|_R \le 12E\,(12\Lambda^2 E)^{s-1} \frac{(s+1)!}{\prod_{l=1}^{s-1} \alpha_l} \ , \quad s \ge 1 \ . \tag{2.15}$$

Such an estimate does not allow to prove the convergence of the series for $\Phi^{(l)}$; on the other hand, it is well known that the perturbation series are generally non convergent. However, one can truncate the series expansion to an arbitrary order r, and use these truncated series in order to investigate the dynamics of the system.

3. Use of the truncated first integrals

A first approach consists in using the analytic estimates (2.15) without actually computing the power expansion of the first integrals. Recalling that the harmonic actions $I_1, \ldots, I_n$ are first approximation to such integrals, one looks for a bound on $I_l(t)$ for a time large enough. To this end, denoting by

$$\Phi^{(l,r)} = I_l + \Phi_1^{(l)} \cdots + \Phi_r^{(l)} \ , \quad 1 \le l \le r \ ,$$

the first integrals truncated at an arbitrary order $r \ge 1$, one uses the trivial inequality

$$|I_l(t) - I_l(0)| \le \left| I_l(t) - \Phi^{(l,r)}(t) \right| + \left| \Phi^{(l,r)}(t) - \Phi^{(l,r)}(0) \right| + \left| \Phi^{(l,r)}(0) - I_l(0) \right| \ . \tag{3.1}$$

This elementary inequality is enough to obtain a qualitative description of the dynamics.

To simplify the discussion, let me put for a moment $r = 1$, and assume moreover that the $\Phi^{(l,1)}$'s are exact first integrals (this is generally false, of course, but helps to interpret the results). Then, it is easily seen that the neighbourhood of the origin is foliated into invariant n–dimensional tori, which are deformations of the tori $I_l =$ const of the unperturbed system. Thus, if we observe the dynamical evolution of the harmonic actions $I_l(t)$, we find that they are no more constant, but undergo a quasi periodic variation, due to the deformation of the invariant tori. The size of such a variation can be estimated, since it is nothing but $\Phi^{(l,1)} - I_l = \Phi_1^{(l)}$ (recall that $\Phi^{(l,1)}$ is assumed to be an exact first integral). Indeed, it is clear that in a domain $\mathcal{D}_{\varrho R}$ one has the *a priori* estimate $|\Phi_1^{(l)}(x, y)| \le \tilde{C}\varrho^3$, with some constant $\tilde{C}$. On the other hand, it is also clear that the deformation manifests itself in a quite short time, since the time derivative $\dot{I}_l = \{H_1, I_l\}$ turns out to be of order ϱ^3. Such a situation is illustrated in fig. 1–(a): the harmonic action $I_l(t)$ exhibits a quasi periodic oscillation, but it is confined in the strip $I_{l,\min} \le I_l(t) \le I_{l,\max}$, whose width can be estimated.

Let us now take into account the fact that $\Phi^{(l,1)}$ is not an exact first integral, still letting $r = 1$. The time derivative $\dot\Phi^{(l,1)}$ is a homogeneous polynomial of order 4, so that in a domain $\mathcal{D}_{\varrho R}$ it can be bounded by $C_1\varrho^4$, with a suitable constant C_1. Thus, for small ϱ, the variation in time of $\Phi^{(l,1)}$ is much slower than that of the harmonic action I_l, but it cannot be bounded *a priori*, so that in the worst case it can be linear in time. This is illustrated in fig. 1–(b).

The same arguments apply to a generic r. The main contribution to the variation of the harmonic actions I_l is due to the *deformation* $\Phi^{(l,r)} - I_l$, which, using the estimates of the previous sections, can be bounded in a domain $\mathcal{D}_{\varrho R}$ by

$$
|\left(\Phi^{(l,r)} - I_l\right)(x,y)| \le \delta_r^{(l)}(\varrho) \ , \tag{3.2}
$$

where

$$
\delta_r^{(l)}(\varrho) = \sum_{j=1}^{r} \varrho^{j+2}\|\Phi_j^{(l)}\|_R
$$

can be explicitly estimated, and turns out to be of order ϱ^3; such a contribution is fast, but bounded. Superimposed to the deformation, there is a slow evolution of $\Phi^{(l,r)}$, whose time derivative is now

$$
\dot\Phi^{(l,r)} = \{H_1, \Phi_r^{(l)}\} \ ,
$$

namely, a homogeneous polynomial of order $r+3$. This will be referred to as the *noise*. Still using the estimates of the previous section, in a domain $\mathcal{D}_{\varrho R}$ one has the bound

$$
|\dot\Phi^{(l,r)}| \le \mathcal{R}^{(l)}(\varrho) \ , \tag{3.3}
$$

where

$$
\mathcal{R}^{(l)}(\varrho) = \varrho^{r+3}\|\{H_1, \Phi_r^{(l)}\}\|_R
$$

can be explicitly computed, and turns out to be bounded by $C_r\varrho^{r+3}$ with a suitable constant C_r. The relevant fact here is that, according to our estimates, the constant C_r increases with r at least as fast as $r!$, so that some care must be taken in estimating the effect of the noise.

4. Estimate of the stability time

Let me now illustrate how the arguments above can be made quantitative. Let us proceed as follows. Assume that the initial point $(x(0), y(0))$ of an orbit lies in a domain $\mathcal{D}_{\varrho_0 R}$, so that one has $I_l(0) \le \varrho_0^2 R_l^2/2$, and ask $(x(t), y(t)) \in \mathcal{D}_{\varrho R}$ with $\varrho > \varrho_0$ for a finite time interval $|t| \le T$; this is guaranteed provided one has $I_l(t) \le \varrho^2 R_l^2/2$. Using (3.1) and (3.2) one has

$$
|I_l(t) - I_l(0)| \le \delta_r(\varrho_0) + \delta_r(\varrho) + \left|\Phi^{(l,r)}(t) - \Phi^{(l,r)}(0)\right| \ ,
$$

so that one can define

$$\Delta_r^{(l)}(\varrho_0,\varrho) = \frac{1}{2}R_l^2(\varrho_2 - \varrho_0^2) - \delta_r(\varrho_0) - \delta_r(\varrho)$$

as the maximum allowed effect for the noise. This gives a first consistency condition on r, ϱ_0 and ϱ, namely

$$\Delta_r^{(l)}(\varrho_0,\varrho) > 0 \ . \tag{4.1}$$

If this condition is satisfied, then, using (3.3), one has that the orbit is confined to $\Delta_{\varrho R}$ at least for

$$|t| \le \tau_r(\varrho_0,\varrho) = \min_l \frac{\Delta_r^{(l)}(\varrho_0,\varrho)}{\mathcal{R}_r^{(l)}(\varrho)} \ . \tag{4.2}$$

Such an estimate is still unsatisfactory, since the stability time contains the arbitrary parameter r, while it would be more interesting to give an estimate depending only on the size of the initial domain $\mathcal{D}_{\varrho_0 R}$. To this end, it is natural to look for the best possible estimate by defining the stability time $T(\varrho_0)$ as

$$T(\varrho_0) = \max_r \sup_\varrho \tau_r(\varrho_0,\varrho) \ , \tag{4.3}$$

where the $\max_r \sup_\varrho$ must be taken over r and ϱ satisfying the consistency condition (1.3).

Let me stress again that all the quantities $\delta_r^{(l)}(\varrho)$, $\Delta_r^{(l)}(\varrho_0,\varrho)$ and $\mathcal{R}_r^{(l)}(\varrho)$ can be explicitly computed provided some estimate on the norms $\|\Phi_1^{(l)}\|_R,\ldots,\|\Phi_r^{(l)}\|_R$ of the expansions of the first integrals is known. Thus, the stability time $T(\varrho_0)$ can be actually estimated by computer. However, it is interesting to check the relevance of the estimate by investigating the analytic dependence of T on ϱ_0, based on the recursive estimates of sect. 2.

Looking in particular at the estimate (2.15), one first needs some analytic form of the sequence $\{\alpha_s\}_{s>0}$ which bounds the small denominators, as requested by (2.12). The usual choice in perturbation theory is

$$\alpha_s = \gamma(s+2)^{-\tau} \tag{4.4}$$

with some constants $\gamma > 0$ and $\tau \ge 0$. Indeed, it is well known that the relative measure of the ω's not satisfying (2.12) is at most proportional to γ, so that it can be made arbitrarily small. With this choice, it is an easy matter to get

$$\begin{aligned} &\delta_r^{(l)}(\varrho) < C_1\varrho^3 \\ &\mathcal{R}_r^{(l)}(\varrho) < C_2(r!)^{1+\tau}\left(\frac{\varrho}{\varrho_*}\right)^r \varrho^3 \end{aligned} \tag{4.5}$$

with suitable constants C_1, C_2 and ϱ_* (see ref. [6] for details). To simplify the discussion, let us now put (arbitrarily)

$$\Delta_r^{(l)}(\varrho_0,\varrho) = C_1\varrho^3 \ , \tag{4.6}$$

namely, allow the effect of the noise to become as large as that of the deformation. Then, by (4.2) we get

$$\tau_r(\varrho_0, \varrho) \geq \frac{A}{(r!)^{1+\tau}} \left(\frac{\varrho}{\varrho_*}\right)^r , \tag{4.7}$$

with a suitable constant A. Since a relation between ϱ_0 and ϱ is implicitly established by (4.6), we just have to optimize $\tau_r(\varrho_0, \varrho)$ against r. From (4.7) it is easily found that the optimal choice for r is

$$r_{\rm opt} = \left(\frac{\varrho}{\varrho_*}\right)^{1/(1+\tau)} .$$

Substituting the this value in (4.7) one gets the final estimate of the stability time

$$T(\varrho) = T_* \exp\left(\frac{\varrho}{\varrho_*}\right)^{1/(1+\tau)} , \tag{4.8}$$

with a constant T_*. This is the essential content of the Nekhoroshev theorem when applied to the case of an elliptic equilibrium.

5. An example: the Lagrangian equilibrium points of the restricted problem of three bodies

In order to illustrate the possible physical relevance of the theory above, let me consider the case of the triangular equilibrium solutions of the restricted problem of three bodies.

To briefly recall the model, we consider two mass points (the *primaries*) of mass μ and $1-\mu$ respectively, uniformly rotating on a Keplerian circular orbit around their common center of mass. The problem is to study the motion of a third point of negligible mass (the *planetoid*) under the influence of the gravitational field of the primaries, assuming that the motion of the primaries is not influenced by the presence of the planetoid.

One usually considers a reference frame uniformly rotating with the primaries, so that the position of the primaries in fixed in that frame. It was known to Lagrange that in such a reference frame the Newton equation for the restricted problem of three bodies admit five equilibrium solutions. Three of these solutions lie on the straight line joining the primaries (the *collinear* solutions); the remaining equilibrium points, usually denoted by L_4 and L_5 respectively, are so located that the primaries and the point L_4 (L_5 respectively) are the vertices of an equilateral triangle (the *triangular* solutions).

Concerning the stability, it is well known that the linear approximation of the Newton equations in the neighbourhood of the equilibria always gives instability for the collinear points, while the triangular points are stable for $0 < \mu < \mu_0 = 0.03852089604551\ldots$. However, the problem of the stability of the triangular points with respect to the full (i.e., nonlinear) system is a very hard problem, which has been solved only recently for the planar case (i.e., the planetoid is constrained to move on the orbital plane of the primaries), but is still unsolved in the spatial case.

In the neighbourhood of the point L_4 the Hamiltonian can be expanded in power series as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y + \frac{x^2}{8} - \frac{5}{8}y^2 - axy + \frac{1}{2}(p_z^2 + z^2) + H_1(x,y) + H_2(x,y) + \dots \ ,$$

where $a = -3\sqrt{3}(1-2\mu)/4$, and H_1, $H_2, \dots$ are known homogeneous polynomials of degree $3, 4, \dots$ respectively. By a linear change of coordinates the second order Hamiltonian takes the form

$$H_0 = \sum_{l=1}^{3} \frac{\omega_l}{2}(x_l^2 + y_l^2) \ ,$$

with ω_1, ω_2 and ω_3 roots of the characteristic equation

$$(\omega^4 - \omega^2 + \frac{27}{16} - a^2) \cdot (\omega^2 - 1) = 0 \ .$$

The sign of the ω's is determined by the linear transformation itself, and it turns out that one has $\omega_1 > 0$, $\omega_2 < 0$ and $\omega_3 = 1$, so that the linear approximation is not sufficient to decide about the stability of the nonlinear system. However, the Hamiltonian has the general form (1.2), so that the theory developed in the previous sections can be applied. (In fact, the Hamiltonian is not truncated at degree three, nor it does satisfy the reversibility condition discussed in sect. 2, but this is just a minor technical problem, which can be easily overcome; so, let me skip these details.)

A particularly interesting case is that of the Sun–Jupiter system. Indeed, it is known that some 20 asteroids are close enough to the point L_4 that the their motion could be a libration around that point. A detailed study was performed by Celletti and Giorgilli in ref. [9], and I plan to illustrate here the main result. The value of the parameter μ in this case can be assumed to be $\mu = 0.95387536 \times 10^{-3}$; the frequencies then turn out to be $\omega_1 \simeq 9.9675752552 \times 10^{-1}$, $\omega_2 \simeq -8.0463875837 \times 10^{-2}$ and $\omega_3 = 1$. The expansion of the Hamiltonian and of the formal first integrals has been performed by computer up to $r = 20$, namely up to terms of degree 22. Such an expansion has been used to compute the norms used in estimating the stability time $T(\varrho_0)$, according to the procedure outlined in sect. 3. The final result is illustrated in fig. 2. The four different curves correspond to different choices of the maximal truncation order r, namely $r = 2, 8, 14, 20$ respectively (namely, working with polynomials of degree $4, 10, 16$ and 22 respectively); this gives an idea of how the results improve by increasing the truncation order.

The most striking feature of the graph is the sudden jump of $T(\varrho_0)$ a little below $\varrho_0 = 3 \times 10^{-4}$. This is more evident from the enlargement of fig. 3: it is clear that there is a sharp threshold close to $\varrho_0 = 2.88248 \times 10^{-4}$, where the estimated stability time jumps from 1 to more than 10^4. For smaller values of ϱ_0 the function $T(\varrho_0)$ increases more regularly, roughly as a power of $1/\varrho_0$. Moreover, the value of this threshold looks well defined even at quite low orders, while the estimate of the time could still improve by further increasing r. Such a fast increase is the most relevant result, since it allows to ensure stability for exceedingly large times.

In order to understand the physical relevance, one can remark that, in our units, the revolution of Jupiter takes 2π time units, so that the estimated age of the universe is of order 10^{10}. If now one asks for stability over a time interval $T = 10^{10}$, one gets that the initial point should lie inside a domain $\mathcal{D}_{\varrho_0 R}$ with $\varrho_0 \simeq 2.58397 \times 10^{-4}$. In physical coordinates, this corresponds roughly to a distance from the point L_4 of about 4×10^{-5} times the distance of Jupiter from the Sun; one has then, roughly speaking, a stability neighbourhood of size 3×10^4 Km. Although not so large (and certainly not optimal), this is a realistic result.

A last remark concerns the possible relevance of such a result for accelerators. As said in the introduction, the naive impression is that this problem is quite different from the problems of stability in Celestial Mechanics, because one has to do with definitely different time scales. However, a more careful consideration shows that this is not the case. Indeed, looking at fig. 2, one sees that increasing the required stability time from 10^{10} to, say, 10^{20} just requires a decrease of the initial size ϱ_0 by a factor less than 2. Thus, a reasonable restriction of the initial domain can well compensate a very large factor in the natural time scale of the phenomenon,

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