RIGOROUS RESULTS ON THE POWER EXPANSIONS FOR THE INTEGRALS OF A HAMILTONIAN SYSTEM NEAR AN ELLIPTIC EQUILIBRIUM POINT

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Abstract. The classical problem of the direct construction of integrals for a Hamiltonian system in the neighbourhood of an elliptic equilibrium point is revisited in the light of the rigorous Nekhoroshev's like theory. It is shown how the results about stability over exponentially large times can be recovered in a simple and effective way, at least in the nonresonant case, and in fact even more conveniently than with the usual indirect method involving normalizing canonical transformations. An application is also made to the problem of the freezing of the harmonic actions in classical models.

Sunto. Si ridiscute, alla luce dei risultati rigorosi del tipo di Nekhoroshev, il problema classico della costruzione diretta di integrali primi nell'intorno di un punto di equilibrio ellittico di un sistema Hamiltoniano. Si mostra come, almeno per quanto riguarda il caso non risonante, si possano riottenere in modo semplice e diretto i risultati relativi alla stabilità su tempi esponenzialmente lunghi. Il metodo qui esposto risulta, di fatto, più comodo rispetto al metodo usuale consistente nell'introdurre trasformazioni canoniche che pongono il sistema in forma normale. Si dà infine un'applicazione dei risultati al problema del congelamento delle azioni armoniche in modelli classici.

1. Introduction

The study of a Hamiltonian system in the neighbourhood of an elliptic equilibrium point can typically be reduced, as is well known, to the study of the Hamiltonian

$$H(x,y) = \sum_{s\geq 0} H_s(x,y) \ , \tag{1.1}$$

where

$$H_0(x,y) = \sum_{l=1}^{n} \frac{\omega_l}{2}(x_l^2 + y_l^2) \ , \tag{1.2}$$

and $H_s(x,y)$, for $s \geq 1$, is a homogeneous polynomial of degree $s+2$ in the canonical variables $(x,y) \in \mathbf{R}^{2n}$; the harmonic frequencies $(\omega_1,\ldots,\omega_n) \equiv \omega \in \mathbf{R}^n$ are assumed to be all different from zero, and the Hamiltonian is assumed to be analytic in some neighbourhood of the origin of $\mathbf{R}^{2n}$. This is of interest in many fields of mathematical physics and astronomy, mainly in connection with the problem of the stability of the equilibrium.

Indeed, while the stability problem is easily solved if all the frequencies are positive (or negative), since in such case the equilibrium point is a local minimum (or maximum), no straightforward solution exists instead if the frequencies have different signs. For example, if one considers the triangular Lagrangian equilibrium points of the restricted spatial three body problem, such equilibrium is a saddle point of the Hamiltonian, and the discussion about the stability must take into account the nonlinear terms.

A more complex problem, which is met even when the equilibrium is proven to be stable, is that of finding concrete estimates for finite times, in the spirit of perturbation theory. This is of interest for example in connection with the freezing of the harmonic actions in classical mechanical models.

The formal solution of such problems is a classical topic: the system can be proven to be formally integrable if the frequencies are nonresonant, i.e. if the condition $k \cdot \omega \neq 0$ for $0 \neq k \in \mathbf{Z}^n$ is satisfied. Two methods are usually found in the literature. The first one, going back to Birkhoff[1], consists in performing a formal canonical transformation $(x,y) \to (x',y')$ which gives the Hamiltonian a normal form, in the sense that the transformed Hamiltonian $H'(x',y')$ is a function of the n quantities $I'_l = \frac{1}{2}({x'_l}^2 + {y'_l}^2)$ only, so that the system is immediately seen to be formally integrable. The second method, used by Whittaker[2] and Cherry[3], and which I will call the direct one, tries to build n independent formal integrals, i.e. formal power series $\Phi^{(l)} = I_l + \sum_{s\geq 1} \Phi_s^{(l)}$ with $I_l = \frac{1}{2}(x_l^2 + y_l^2)$, by recursively solving the equation $\{H, \Phi^{(l)}\} = 0$, where $\{\cdot,\cdot\}$ is the Poisson bracket; no canonical transformation is introduced, and one always works with the original variables (x,y). The normal form method is the most common one, and has the advantage of being easily generalized to the resonant case[4]. The direct method has the disadvantage of requiring an explicit proof that the equation $\{H, \Phi^{(l)}\} = 0$ can be consistently solved to all orders. In fact, although a method of solution was proposed by Cherry and explicitly used by Contopoulos[5], the consistency was directly proven only with some restriction on the Hamiltonian, namely in the so called reversible case[6]. This will be discussed below in some more detail.

All these results are formal, in the sense that they rely on power series developments which are generally known to be divergent[7]. Rigorous results proving the existence of orbits which do not leave a neighbourhood of the origin can be given instead in the framework of KAM theory, which guarantees that many n–dimensional invariant tori of the unperturbed system H_0 are preserved. However, such invariant tori do not fill an open region, so that the possibility of the so called Arnold diffusion cannot be excluded, except for the two dimensional case.

An alternative approach in getting rigorous theorems consists in looking for results which are valid only over a finite time interval, but give an effective bound on the Arnold diffusion. Such kind of results, which were already investigated by Moser[8] and Nekhoroshev[9], have been proven to be able to give effective bounds in practical applications[10]. Roughly speaking, one shows that the system admits a number of approximate integrals whose change in time can be controlled to be small over an exceedingly large time interval. In particular, for a system like (1.1) the approximate integrals are just the formal power series expansions discussed above, truncated at a suitable order.

Such results are usually obtained via the normal form methods, by looking for a normal form of the Hamiltonian up to a finite order. In particular, ref. [10] contains the general treatment of the Hamiltonian (1.1) by such methods. However, the direct method, although being less general, allows to get similar results in a much simpler way, so that it is more convenient for a general introduction to such kind of problems. It then seemed to be worthwhile to develop here a rigorous perturbation scheme on the basis of the direct method. As the reader will see, the technical difficulties are substantially reduced in comparison with those of the normal form methods. In particular, the computation is quite trivial if one considers the case of a polynomial Hamiltonian, for example just the one with $H_s(x, y) = 0$ for $s > 1$. Such case, which is particularly instructive, could be easily discussed by adapting the general treatment given below.

Before entering the technical scheme, let me briefly illustrate the results. Consider the Hamiltonian (1.1), and give initial values $I_{l,0}$, $1 \leq l \leq n$ to the harmonic actions $I_l = \frac{1}{2}(x_l^2 + y_l^2)$; this corresponds to considering a given distribution of the harmonic energies of the oscillators, without taking care of the phases. The question is then whether such initial distribution is qualitatively maintained or not during the evolution of the system. More precisely, having fixed $I_l(0) = I_{l,0} > 0$, one looks for a bound like $I_{l,\min} < I_l(t) < I_{l,\max}$, with $0 < I_{l,\min} < I_{l,0} < I_{l,\max}$, over a time interval $|t| < T$, where T is a possibly large (or even infinite) time. A weaker result, which is enough in order to answer the question of the stability for finite large times, is obtained if one renounces to keep $I_l(t)$ away from zero, and only looks for a bound $I_l(t) < I_{l,\max}$. In such case, the initial values of the actions are not required to be all different from zero.

In order to handle such problem, one makes the restriction, which is usual in classical perturbation theory, that the harmonic frequencies satisfy the nonresonance condition

$$|k \cdot \omega| \geq \gamma \, |k|^{-\tau} \quad , \quad k \in \mathbf{Z}^n \tag{1.3}$$

with real constants $\gamma > 0$ and $\tau \geq 0$ (it is known[11] that this is true for a set of ω's of large measure if $\tau > n-1$ and γ is small enough), and builds approximate integrals $\Phi^{(l,r)} = I_l + \sum_{s=1}^{r} \Phi_s^{(l)}$ which are polynomials of degree $r+2$; precisely, one determines $\Phi^{(l,r)}$ by requiring $\dot{\Phi}^{(l,r)} = \mathcal{R}^{(l,r)}$, namely $\{H, \Phi^{(l,r)}\} = \mathcal{R}^{(l,r)}$, where $\mathcal{R}^{(l,r)}$ are power series whose lowest degree term is at least of degree $r+3$. If one could forget $\mathcal{R}^{(l,r)}(x,y)$, i.e. if the $\Phi^{(l,r)}$ were exact integrals, then the change in time of the harmonic energies I_l would be estimated as a *deformation* due to the difference $\left|\Phi^{(l,r)} - I_l\right|$ between the integrals and their harmonic approximations; such deformation turns out to be of order ϱ^3, with $\varrho \simeq \max_l \sqrt{2I_{l,0}}$. Thus, $\mathcal{R}^{(l,r)}$ plays the role of the source of a kind of additional *noise* which causes the approximate integrals $\Phi^{(l,r)}$ to slowly change. Such change can be estimated via the known size of $\mathcal{R}^{(l,r)}$, which turns out to have a bound like $c_r \varrho^{r+3}$ with a certain coefficient c_r. If one allows the change $\left|\Phi^{(l,r)}(t) - \Phi^{(l,r)}(0)\right|$ to be of the same order ϱ^3 of the deformation, then the limits $I_{l,\min}$ and $I_{l,\max}$ can be computed, and the bound above on $I_l(t)$ thus holds at least up to a time $T_r \sim c_r^{-1} \varrho^{-r}$.

Now, in agreement with general considerations on the divergence of the series expansions of Hamiltonian perturbation theory, the estimated value of c_r is found to increase as fast as $\varrho_*^{-r}(r!)^{\tau+1}$, ϱ_* being a constant, so that the limit $r \to \infty$ cannot be reached. Thus, one has a result strongly dependent on the truncation order r of the approximate integrals, which is arbitrary, and appears as an extraneous element introduced by the perturbation scheme. This cannot of course be avoided if one plans to actually compute explicit expressions for the approximated integrals at a given order, but can be removed if one simply looks for a bound like the one indicated above. To this end, r can be chosen in such a way that the noise $\mathcal{R}^{(l,r)}$ is close to a minimum, so that the correspondingly estimated stability time T is as large as possible. Looking at the bound above, one sees that r can be chosen larger and larger when ϱ, i.e. the harmonic energy, is smaller and smaller. The natural choice turns out to be $r = r_{\rm opt} \sim (\varrho_*/\varrho)^{\frac{1}{\tau+1}}$, and the estimated stability time, which is now a function of ϱ, turns out to be $T \sim \exp(\varrho_*/\varrho)^{\frac{1}{\tau+1}}$, i.e to exponentially increase when the energy of the system approaches zero.

The aim of the present note is to develop such perturbation scheme, and to give explicit estimates of all constants entering the theory. The paper is organized as follows. In sect. 2 the classical formal scheme is briefly recalled. In sect. 3 the scheme is made rigorous by giving explicit bounds on the size of the deformation and of the the noise at an arbitrary truncation order r. In sect. 4 the exponential estimates are obtained, and the main theorem is stated. Sect. 5 is devoted to the application to the problems of stability and of freezing of the harmonic actions over exponentially large times.

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2. The formal perturbation scheme

Let me first recall how the problem of building formal integrals for the Hamiltonian (1.1) is classically faced. One introduces the linear spaces Π_s whose elements are the homogeneous polynomials of order s in the canonical variables (x, y), and the linear operator $L_{H_0} : \Pi_s \to \Pi_s$ defined by $L_{H_0}\cdot = \{H_0, \cdot\}$. The Hamiltonian is then characterized by a sequence of elements $H_s \in \Pi_{s+2}$, and the equation $\{H, \Phi^{(l)}\} = 0$ for a formal integral $\Phi^{(l)} = I_l + \sum_{s\geq 1} \Phi_s^{(l)}$, with $I_l = \frac{1}{2}(x_l^2 + y_l^2)$ and $\Phi_s^{(l)} \in \Pi_{s+2}$, turns out to be equivalent to the infinite recursive system of equations $L_{H_0} I_l = 0$, and

$$L_{H_0}\Phi_s^{(l)} = \Psi_s^{(l)} \ , \quad s \geq 1 \ , \tag{2.1}$$

where $\Psi_s^{(l)} \in \Pi_{s+2}$ is known, being given by

$$\begin{aligned} \Psi_1^{(l)} &= \{I_l, H_1\} \\ \Psi_s^{(l)} &= \sum_{j=1}^{s-1} \{\Phi_j^{(l)}, H_{s-j}\} + \{I_l, H_s\} \ , \quad s > 1 \ . \end{aligned} \tag{2.2}$$

The operator L_{H_0} can be diagonalized via the usual linear canonical transformation to complex variables ξ, η

$$x_l = \frac{1}{\sqrt{2}}(\xi_l + i\eta_l) \ , \quad y_l = \frac{i}{\sqrt{2}}(\xi_l - i\eta_l) \ , \quad 1 \leq l \leq n \ . \tag{2.3}$$

Indeed, the unperturbed Hamiltonian H_0 takes in the new variables the form (omitting a prime in the new Hamiltonian)

$$H_0(\xi, \eta) = i \sum_{l=1}^{n} \omega_l \xi_l \eta_l \ ; \tag{2.4}$$

so, writing $f \in \Pi_s$ in complex variables as

$$f(\xi, \eta) = \sum_{|j+k|=s} f_{jk} \xi^j \eta^k \tag{2.5}$$

with $|j + k| = \sum_{l=1}^{n} |j_l + k_l|$, one has

$$L_{H_0} f = i \sum_{j,k} (k - j) \cdot \omega f_{jk} \xi^j \eta^k \ . \tag{2.6}$$

This shows that eq. (2.1) can be solved provided the known polynomial $\Psi_s^{(l)}$ contains no monomial $\xi^j \eta^k$ such that $(k - j) \cdot \omega = 0$. This is the consistency problem referred to in sect. 1. If this is the case, then the solution of eq. (2.1) is determined up to an arbitrary term $\tilde{\Phi}_s^{(l)}$ satisfying $L_{H_0}\tilde{\Phi}_s^{(l)} = 0$.

The consistency problem is easily solved if one assumes that the harmonic frequencies ω of the unperturbed Hamiltonian H_0 are nonresonant, i.e. $k \cdot \omega = 0$ with $k \in \mathbf{Z}^n$ implies $k = 0$, and that the Hamiltonian $H(x, y)$ is even in the momenta (or reversible), i.e. one has $H(x, -y) = H(x, y)$. For self consistency, I give here a sketch lines of the rather simple proof, which can be found in ref. 6.

The proof is based on the fact that the critical monomials in such case all have the form $\xi^j \eta^j$, so that in real variables they are functions of $I_1, \ldots, I_n$, which in turn are even functions of the momenta. Indeed, write any polynomial $f \in \Pi_s$, for any s, as a sum $f = f_+ + f_-$, where f_+ and f_- are even and odd respectively in the momenta, i.e. $f_+(x, -y) = f_+(x, y)$ and $f_-(x, -y) = -f_-(x, y)$; such decomposition is clearly unique. Moreover, it is an easy matter to check that the Poisson bracket between functions of the same parity gives an odd function, and that the Poisson bracket between functions of different parity gives an even function. Using these simple remarks, one proves by induction that if $\Phi^{(l)}$ has been determined up to order $s - 1$ as an even function of the momenta (which is true for $s = 1$), then $\Psi_s^{(l)}$ is an odd function, so that it cannot contain critical monomials, and so $\Phi_s^{(l)}$ is necessarily an even function. The solution can be made unique by simply choosing $\tilde{\Phi}_s^{(l)} = 0$.

A proof for the general nonresonant case, when the assumption $H(x, -y) = H(x, y)$ is removed, could be given indirectly by making use of the existence of the formal integrals, which in turn can be proven by the normal form method. The resonant case unfortunately does not allow so simple a treatment: the existence of formal integrals similar to the ones introduced above can be indirectly proven via the normal form method, but in the direct construction the arbitrary term $\tilde{\Phi}_s^{(l)}$ in the solution must be carefully determined in order to ensure the consistency at higher orders. An attempt to make rigorous such a procedure led in fact to establishing a close connection between the direct method and the normal form method[12].

Thus, let's proceed by only considering the nonresonant case.

3. The rigorous perturbation scheme

In this section the formal scheme discussed in sect. 2 is made rigorous by estimating the sizes of the deformation and of the noise, as illustrated in the introduction. To this end, the recursive formal scheme is translated into a sequence of recursive bounds on each term of the formal integrals, and finally all the contributions are added up to produce the final sizes of the functions. Here, the procedure is briefly illustrated, and the result is summarized below in proposition 3.1. The proofs are deferred to the appendix, so that the reader not interested in technical details can skip them by only reading the present section. It seems however to be appropriate to point out at least how simpler appears to be the treatment given here with respect to the more common one involving recursive canonical transformations.

The sizes of the various polynomials entering the perturbation scheme are estimated by introducing the following norm on the linear spaces Π_s: having fixed $R = (R_1, \ldots, R_n) \in \mathbf{R}^n_+$, i.e. real positive numbers $R_1, \ldots, R_n$, the norm of a polynomial $f \in \Pi_s$, $f = \sum_{j,k} f_{jk} x^j y^k$ is defined as

$$\|f\|_R = \sum_{j,k} |f_{jk}|\, R^{j+k} \ . \tag{3.1}$$

Use will also be made of the quantity

$$\Lambda = (\min_l R_l)^{-1} \ . \tag{3.2}$$

The small denominators arising from the solution of (2.1) according to the formal scheme of sect. 2 are assumed to be bounded from below by a nonincreasing sequence $\{\alpha_s\}_{s\geq 1}$ of positive real numbers satisfying the diophantine–like condition

$$|k\cdot\omega| \geq \alpha_s \ \text{for}\ k \in \mathbf{Z}^n \ , \quad 0 < |k| \leq s+2 \ . \tag{3.3}$$

A few technical lemmas allow to control how the norms are propagated through the recursive solution of (2.1), and, for the truncated integrals $\Phi^{(l,r)} = I_l + \sum_{s=1}^r \Phi_s^{(l)}$, explicit bounds on the norms $\|\Phi_s^{(l)}\|_R$ are produced.

For what concerns the noise $\mathcal{R}^{(r,l)} = \{H, \Phi^{(l,r)}\}$ referred to in the introduction, one finds the explicit expression

$$\mathcal{R}^{(r,l)} = \sum_{s>r} Q_s^{(l)} \ , \tag{3.4}$$

with $Q_s^{(l)} \in \Pi_{s+2}$ defined by

$$Q_s^{(l)} = \sum_{j=1}^{r} \{\Phi_j^{(l)}, H_{s-j}\} + \{I_l, H_s\} \ , \quad s > r \ , \tag{3.5}$$

and the norms $\|Q_s^{(l)}\|_R$ are explicitly estimated.

Finally, one considers the neighbourhood of the origin

$$\Delta_{\varrho R} = \left\{ (x,y) \in \mathbf{R}^n : (x_l^2 + y_l^2)^{1/2} \leq \varrho R_l, 1 \leq l \leq n \right\} \ , \tag{3.6}$$

where ϱ is a positive real number. Then, the size of a given polynomial $f \in \Pi_s$ can be estimated in $\Delta_{\varrho R}$ by $|f(x,y)| \leq \|f\|_R\, \varrho^s$. This allows to estimate the deformation and the noise for the truncated integrals $\Phi^{(l,r)}(x,y)$ in the domain $\Delta_{\varrho R}$ by the following

Proposition 3.1: *Consider the Hamiltonian system $H(x,y) = \sum_{s\geq 0} H_s$, where $H_0 = \sum_{l=1}^n \frac{\omega_l}{2}(x_l^2 + y_l^2)$ and $H_s \in \Pi_{s+2}$, and assume that for a given $R \in \mathbf{R}^n_+$ there exist real constants $h \geq 0$ and $E > 0$ such that $\|H_s\|_R \leq h^{s-1} E$ for $s \geq 1$; assume moreover that the harmonic frequencies ω satisfy the nonresonance condition $|k\cdot\omega| \geq \alpha_s$ for $k \in \mathbf{Z}^n$ and $0 < |k| \leq s+2$, where $\{\alpha_s\}_{s\geq 1}$ is a nonincreasing sequence of positive real numbers.*
Then for any integer $r > 0$ there exist n truncated integrals $\Phi^{(l,r)} = I_l + \sum_{s=1}^r \Phi_s^{(l)}$,

$1 \le l \le n$, with $I_l = \frac{1}{2}(x_l^2 + y_l^2)$, such that $\dot\Phi^{(l,r)} = \{H, \Phi^{(l,r)}\}$ is a power series starting with terms of degree at least $r+3$. Moreover, for $(x,y) \in \Delta_{\varrho R}$ defined by (3.6) and for $\varrho < 1/h$ one has the bounds

$$\left|\left(\Phi^{(l,r)} - I_l\right)(x,y)\right| < \frac{24E}{\alpha_1}\varrho^3\left[1 - (\sigma_r\varrho)^r\right](1 - \sigma_r\varrho)^{-1} \tag{3.7}$$

$$\left|\dot\Phi^{(l,r)}(x,y)\right| < C_r\varrho^{r+3}(1 - h\varrho)^{-2} \ , \tag{3.8}$$

where

$$\begin{aligned} \sigma_1 &= 1 \\ \sigma_r &= \left(12\Lambda^2 E + \frac{8}{9}h\alpha_1\right)\left[\frac{(r+1)!}{\prod_{l=2}^r \alpha_l}\right]^{\frac{1}{r-1}} \quad , \quad r > 1 \\ C_r &= 8E\left(12\Lambda^2 E + \frac{8}{9}h\alpha_1\right)^r \frac{(r+2)!}{\prod_{l=1}^r \alpha_l} \quad , \quad r \ge 1 \ . \end{aligned} \tag{3.9}$$

The proof is deferred to the appendix.

4. Exponential estimates

The aim of this section is to remove the arbitrary truncation order r from the perturbative theory, thus obtaining results which depend on the smallness of the domain $\Delta_{\varrho R}$ where the behaviour of the system is considered. To this end, one has to assume a more explicit expression for the sequence $\{\alpha_s\}_{s\ge1}$. According to the usual perturbation scheme, assume a diophantine bound as in (1.3), namely put

$$\alpha_s = \gamma(s+2)^{-\tau} \ , \quad s \ge 1 \tag{4.1}$$

with suitable real constants $\gamma > 0$ and $\tau \ge 0$. By introducing then the constant

$$\varrho_* = \frac{\gamma}{4}\left(3\Lambda^2 E + \frac{2\gamma h}{3^{\tau+2}}\right)^{-1} \tag{4.2}$$

one can write (3.8) as

$$\left|\dot\Phi^{(l,r)}(x,y)\right| < 2^{3-\tau}E\varrho^3\left(\frac{\varrho}{\varrho_*}\right)^r\left[(r+2)!\right]^{\tau+1}(1 - h\varrho)^{-2} \ . \tag{4.3}$$

Look now for a value $r_{\rm opt}$ such that the estimate above is close to a minimum. By minimizing $(\varrho/\varrho_*)^r[(r+2)!]^{\tau+1}$ one is led to take as $r_{\rm opt}$ the integer satisfying

$$\left(\frac{\varrho_*}{\varrho}\right)^{\frac{1}{\tau+1}} - 1 < r_{\rm opt} + 2 \le \left(\frac{\varrho_*}{\varrho}\right)^{\frac{1}{\tau+1}} \tag{4.4}$$

The perturbative theory is then useful only if $r_{\rm opt} \ge 1$, i.e. if $\varrho \le 3^{-(\tau+1)}\varrho_*$. Such choice of the truncation order allows to prove the main result of the present note.

Theorem 4.1: *Consider the canonical system with Hamiltonian* $H(x,y) = \sum_{s\geq 0} H_s$, *where* $H_0 = \sum_{l=1}^{n} \frac{\omega_l}{2}(x_l^2 + y_l^2)$ *and* $H_s \in \Pi_{s+2}$, *and assume that for a given* $R \in \mathbf{R}_+^n$ *there exist real constants* $h \geq 0$ *and* $E > 0$ *such that* $\|H_s\|_R \leq h^{s-1}E$ *for* $s \geq 1$; *assume moreover that the harmonic frequencies* ω *satisfy the nonresonance condition* $|k\cdot\omega| > \gamma\,|k|^{-\tau}$ *for* $k \in \mathbf{Z}^n$, *with real constants* $\gamma > 0$ *and* $\tau \geq 0$.
Then for any $\varrho \leq 3^{-(\tau+1)}\varrho_*$, *with* ϱ_* *given by (4.2), there exist* n *approximate integrals* $\Phi^{(l)}(x,y)$ *such that for* $(x,y) \in \Delta_{\varrho R}$, *defined by (3.6), one has*

$$\left|\left(\Phi^{(l)} - I_l\right)(x,y)\right| < \frac{8\cdot 3^{\tau}}{\Lambda^2\varrho_*}\varrho^3 \tag{4.5}$$

$$\left|\dot\Phi^{(l)}(x,y)\right| < 3\cdot 2^4\left(\frac{e^2}{2}\right)^{\tau+1} E\varrho_*^3\left(\frac{\varrho}{\varrho_*}\right)^{\frac{1}{2}}\exp\left[-(\tau+1)\left(\frac{\varrho_*}{\varrho}\right)^{\frac{1}{\tau+1}}\right]\ . \tag{4.6}$$

Proof. Consider first (4.6). The condition $\varrho \leq 3^{-(\tau+1)}\varrho_*$ with the explicit value of ϱ_* gives $\varrho < \frac{3}{8h}$, so that in (4.3) one gets $(1-h\varrho)^{-2} < 3$; then (4.6) is found by substituting $r_{\rm opt}$ for r, and using the known inequality $s! < s^{s+\frac{1}{2}}e^{-(s-1)}$. In order to get (4.5), first check that $\left(1-(\sigma_{r_{\rm opt}}\varrho)^{r_{\rm opt}}\right)\left(1-\sigma_{r_{\rm opt}}\right) < 4$. This is trivial for $r=1$; for $r \geq 2$ it is somehow tricky: using (3.9), (4.1), (4.2) and (4.4) compute (here r stands for $r_{\rm opt}$)

$$\varrho\sigma_r = \left(\frac{\varrho}{\varrho_*}\right)\left[(r+1)!\left(\frac{(r+2)!}{6}\right)^{\tau}\right]^{\frac{1}{r-1}} \leq \frac{[(r+1)!]^{\frac{1}{r-1}}}{r+2}\left[\frac{(r+2)!}{6(r+2)^{r-1}}\right]^{\frac{\tau}{r-1}}\ .$$

For $\tau \geq 0$ and integer $r \geq 1$ this is clearly a nonincreasing function of τ, so that one has $[1-(\sigma_r\varrho)^r]\,(1-\sigma_r\varrho)^{-1} < (1-\psi_r^r)(1-\psi_r)^{-1}$, with $\psi_r = \frac{[(r+1)!]^{\frac{1}{r-1}}}{r+2}$; this in turn is a decreasing function of r, and one easily computes $\psi_2 = \frac{3}{2}$, $\psi_3 < 1$, $\psi_4 < 1$ and $\psi_r < \frac{3}{4}$ for $r \geq 5$, so that the statement is directly checked for $r \leq 4$, while for $r \geq 5$ one gets $(1-\psi_r^r)(1-\psi_r)^{-1} < (1-\psi_r)^{-1} < 4$. Substitute now $\alpha_1 = 3^{-\tau}\gamma$, and use the inequality $\frac{12E}{\gamma} < \frac{1}{\Lambda^2\varrho_*}$, which follows from the definition (4.2) of ϱ_*, and (4.5) is found. *Q.E.D.*

5. Stability of the equilibrium and freezing of the harmonic actions

Let me now come to the problem of the stability of the equilibrium. To simplify the discussion, suppose for a moment that $\Phi^{(1)},\ldots,\Phi^{(n)}$ are exact integrals: this, by the way, can be true for particular systems. According to the usual stability theory, given a domain $\Delta_{\varrho R}$ look for a domain $\Delta_{\varrho_0 R}$ such that an orbit starting from a point $(x_0,y_0) \in \Delta_{\varrho_0 R}$ is confined to $\Delta_{\varrho R}$ for all times. To this end, use can be made of the fact that $(x,y) \in \Delta_{\varrho R}$ if and only if $I_l(x,y) \leq I_{l,\max}$ for $1 \leq l \leq n$, with $I_{l,\max} = \varrho^2 R_l^2/2$. The change in time of $I_l(t)$ can be estimated by

$$|I_l(t) - I_l(0)| \leq |I_l(t) - \Phi^{(l)}(t)| + |\Phi^{(l)}(t) - \Phi^{(l)}(0)| + |\Phi^{(l)}(0) - I_l(0)|\ , \tag{5.1}$$

and, since $\Phi^{(l)}$ is supposed to be an exact integral, one can use the estimate (4.5) of the deformation in $\Delta_{\varrho R}$, and get

$$|I_l(t) - I_l(0)| \le \frac{16\cdot 3^\tau}{\Lambda^2 \varrho_*}\varrho^3 = \frac{32\cdot 3^\tau \varrho}{\Lambda^2 R_l^2 \varrho_*} I_{l,\max}\ . \tag{5.2}$$

One has then $I_l(t) < I_{l,\max}$ provided $I_l(0) \le \left(1 - \frac{32\cdot 3^\tau \varrho}{\Lambda^2 R_l^2 \varrho_*}\right) I_{l,\max}$ for $1 \le l \le n$, and this gives the condition $\varrho \le \frac{\varrho_*}{32\cdot 3^\tau}$. If this is the case, then the orbit is confined to the domain $\Delta_{\varrho R}$ for all times. Notice that the condition above on ϱ is consistent with the weaker one $\varrho \le 3^{-(\tau+1)}\varrho_*$ required by theorem 4.1.

A stronger result, namely the freezing of the harmonic actions, is obtained if the parameters $R_1,\ldots,R_n$ entering the definition of the norm are chosen to fit the initial data by putting $I_l(0) = \left(1 - \frac{32\cdot 3^\tau \varrho}{\Lambda^2 R_l^2 \varrho_*}\right) I_{l,\max}$, and a lower bound to $I_l(t)$ is imposed in order to bound the harmonic actions away from zero. Still using the estimate (5.2), and requiring $I_l(0) > \frac{32\cdot 3^\tau \varrho}{\Lambda^2 R_l^2 \varrho_*} I_{l,\max}$ for $1 \le l \le n$, one gets the bound $I_{l,\min} < I_l(t) < I_{l,\max}$, with $I_{l,\min} = \left(1 - \frac{64\cdot 3^\tau \varrho}{\Lambda^2 R_l^2 \varrho_*}\right) I_{l,\max}$, for all times, provided the condition $\varrho \le \frac{\varrho_*}{64\cdot 3^\tau}$ is satisfied. The projection of the orbit on the x_l, y_l plane is then confined to an annulus. Notice that the actual change of the actions takes a short time, because the time derivative $\dot I_l$ is of order ϱ^3.

Unfortunately, the nice picture above does not apply in general, due to the fact that the truncated integrals $\Phi^{(1)},\ldots,\Phi^{(n)}$ are not exactly constant. However, according to theorem 4.1, their time derivatives $\dot\Phi^{(l)}$ can be made very small, so that the picture above, with a small relaxation of the limits, turns out to be valid at least over very large times. Indeed, allow the change $|\Phi^{(l)}(t) - \Phi^{(l)}(0)|$ due to the noise to be of the same order of the deformation, say $\mu\frac{16\cdot 3^\tau}{\Lambda^2 \varrho_*}\varrho^3$ with some positive μ, by determining the stability time in such a way that $\left|\dot\Phi^{(l)}\right| T < \mu\frac{16\cdot 3^\tau}{\Lambda^2 \varrho_*}\varrho^3$. One has then the following

Proposition 5.1: *Consider the Hamiltonian system of theorem 4.1, with the same convergence and nonresonance hypotheses. Let μ be any positive real number, and define the constants*

$$\begin{aligned}
\varrho_* &= \frac{\gamma}{4}\left(3\Lambda^2 E + \frac{2\gamma h}{3^{\tau+2}}\right)^{-1} \\
T &= \frac{\mu}{9\Lambda^2 E \varrho_*}\left(\frac{6}{e^2}\right)^{\tau+1}\left(\frac{\varrho}{\varrho_*}\right)^{5/2}\exp\left[(\tau+1)\left(\frac{\varrho_*}{\varrho}\right)^{\frac{1}{\tau+1}}\right] \\
I_{l,\max} &= \frac{1}{2}\varrho^2 R_l^2 \\
I_{l,\min} &= \left[1 - \frac{64\cdot 3^\tau \varrho}{\Lambda^2 R_l^2 \varrho_*}(1+\mu)\right] I_{l,\max}\ , \quad 1 \le l \le n\ .
\end{aligned} \tag{5.3}$$

Then:

i. for any $\varrho < [32 \cdot 3^\tau (1+\mu)]^{-1} \varrho_*$ *and any initial data satisfying*

$$I_l(0) \le \left[1 - \frac{32 \cdot 3^\tau \varrho}{\Lambda^2 R_l^2 \varrho_*}(1+\mu)\right] I_{l,\max}$$

one has $I_l(t) < I_{l,\max}$ *for* $|t| < T$*;*

ii. for any $\varrho < [64 \cdot 3^\tau (1+\mu)]^{-1} \varrho_*$ *and any initial data satisfying*

$$I_l(0) = \left[1 - \frac{32 \cdot 3^\tau \varrho}{\Lambda^2 R_l^2 \varrho_*}(1+\mu)\right] I_{l,\max}$$

one has $I_{l,\min} < I_l(t) < I_{l,\max}$ *for* $|t| < T$*.*

Proof. The condition $\left|\dot\Phi^{(l)}\right| T < \mu \frac{16 \cdot 3^\tau}{\Lambda^2 \varrho_*} \varrho^3$ and the estimate (4.6) for $\dot\Phi^{(l)}$ immediately give T. Next, still using (5.1), one gets

$$|I_l(t) - I_l(0)| \le \frac{16 \cdot 3^\tau}{\Lambda^2 \varrho_*}(1+\mu)\varrho^3 = \frac{32 \cdot 3^\tau \varrho}{\Lambda^2 R_l^2 \varrho_*}(1+\mu) I_{l,\max} \ ,$$

to be used instead of (5.2). The statement then follows by the same computations as above. *Q.E.D.*

Appendix

Technical lemmas and proof of proposition 3.1

This rather technical appendix is devoted to the proof of proposition 3.1. The proof depends on several technical lemmas. First, the definition of the norm is used in order to estimate the transformation (2.3) to complex variables and the Poisson brackets.

Lemma A.1: *The norm of the transformed polynomial* $f'(\xi,\eta)$ *of* $f(x,y) \in \Pi_s$ *under the canonical transformation (2.3) is bounded by*

$$\|f'\|_R \le 2^{s/2} \|f\|_R \ . \qquad (A.1)$$

Conversely, the norm of the transformed polynomial $g(x,y)$ *of* $g'(\xi,\eta) \in \Pi_s$ *under the inverse of the canonical transformation (2.3) is bounded by*

$$\|g\|_R \le 2^{s/2} \|g'\|_R \ . \qquad (A.2)$$

Proof. Take $f = \sum_{|j+k|=s} f_{jk} x^j y^k$, and perform the substitution

$$\begin{aligned} f' &= \sum_{j,k} i^{|k|} 2^{-\frac{|j+k|}{2}} f_{jk} (\xi + i\eta)^j (\xi - i\eta)^k \\ &= \sum_{j,k} i^{|k|} 2^{-\frac{|j+k|}{2}} f_{jk} \prod_{l=1}^{n} \sum_{s=0}^{j_l} i^{j_l - s} \binom{j_l}{s} \sum_{r=0}^{k_l} i^{k_l - r} \binom{k_l}{r} \xi_l^{s+r} \eta_l^{j_l + k_l - s - r} \ , \end{aligned}$$

so that $f' \in \Pi_s$, and the definition of the norm gives

$$\begin{aligned} \|f'\|_R &\le \sum_{j,k} 2^{-\frac{|j+k|}{2}} |f_{jk}| R^{j+k} \prod_{l=1}^{n} \sum_{s=0}^{j_l} \binom{j_l}{s} \sum_{r=0}^{k_l} \binom{k_l}{r} \\ &= \sum_{j,k} 2^{\frac{|j+k|}{2}} |f_{jk}| R^{j+k} \\ &= 2^{s/2} \sum_{j,k} |f_{jk}| R^{j+k} \ , \end{aligned}$$

and (A.1) follows. The statement about the inverse transformation is proven by essentially the same computation. *Q.E.D.*

Lemma A.2: *For given polynomials $f \in \Pi_s$ and $f' \in \Pi_{s'}$ the norm of the Poisson bracket $\{f, f'\}$ is bounded by*

$$\|\{f, f'\}\|_R \le ss'\Lambda^2 \|f\|_R \|f'\|_R \ , \tag{A.3}$$

with Λ given by (3.2).

Proof. Compute

$$\{f, f'\} = \sum_{j,k,j',k'} f_{jk} f'_{j'k'} x^{j+j'} y^{k+k'} \sum_{l=1}^{n} \frac{j_l k'_l - j'_l k_l}{x_l y_l} \ ,$$

and use the definition of the norm to get

$$\begin{aligned} \|\{f, f'\}\|_R &\le \sum_{j,k,j',k'} |f_{jk}| \left|f'_{j'k'}\right| R^{j+j'+k+k'} \sum_{l=1}^{n} \frac{j_l k'_l + j'_l k_l}{R_l^2} \\ &\le ss'\Lambda^2 \Bigl(\sum_{j,k} |f_{jk}| R^{j+k}\Bigr) \Bigl(\sum_{j',k'} \left|f'_{j'k'}\right| R^{j'+k'}\Bigr) \ , \end{aligned}$$

so that (A.3) immediately follows. Here, the definition (3.2) of Λ and the trivial inequality $\sum_{l=1}^{n} (j_l k'_l + j'_l k_l) \le s \sum_{l=1}^{n} (j'_l + k'_l) = ss'$ have been used. In complex variables ξ, η the computation is obviously the same. *Q.E.D.*

Lemma A.3: *For a function $f \in \Pi_s$ the norm of the Poisson bracket $\{I_l, f\}$ in real or complex variables is bounded by*

$$\|\{I_l, f\}\|_R \le s \|f\|_R \ . \tag{A.4}$$

Proof. Recalling that in real variables it is $I_l = \frac{1}{2}(x_l^2 + y_l^2)$, compute

$$\{I_l, f\} = \sum_{j,k} \left(k_l \frac{x_l}{y_l} - j_l \frac{y_l}{x_l} \right) f_{jk} x^j y^k \ ,$$

so that

$$\|\{I_l, f\}\|_R \le \sum_{j,k} (k_l + j_l) |f_{jk}| R^{j+k} \le s \sum_{j,k} |f_{jk}| R^{j+k} \ ,$$

and (A.4) follows. In complex variables it is $I_l = i\xi_l\eta_l$, and one computes

$$\{I_l, f\} = i \sum_{j,k} (k_l - j_l) f_{jk} \xi^j \eta^k \ ,$$

so that (A.4) immediately follows. *Q.E.D.*

Use now the fact that $\{\alpha_s\}_{s\ge 1}$ is a nonincreasing sequence of positive real numbers satisfying (3.3). Then, by solving eq (2.1) as illustrated in sect. 2, without adding any arbitrary term $\tilde{\Phi}_s^{(l)}$, one has the following

Lemma A.4: *The norm of the unique solution $\Phi_s^{(l)}$ of eq. (2.1) is bounded in complex variables by*

$$\left\|\Phi_s^{(l)}\right\|_R \le \frac{1}{\alpha_s} \left\|\Psi_s^{(l)}\right\|_R \ . \tag{A.5}$$

Proof. Recalling the explicit expression (2.6) of the operator L_{H_0} in complex variables, write $\Psi_s^{(l)} = \sum_{j,k} c_{jk}\xi^j\eta^k$ with known coefficients c_{jk}, and $\Phi_s^{(l)} = \sum_{j,k} d_{jk}\xi^j\eta^k$ with unknown coefficients d_{jk} to be determined. Then one immediately finds $d_{jk} = -i\frac{c_{jk}}{(k-j)\cdot\omega}$, so that

$$\left\|\Phi_s^{(l)}\right\|_R = \sum_{j,k} |d_{jk}| R^{j+k} \le \frac{1}{\alpha_s} \sum_{j,k} |c_{jk}| R^{j+k} \ ,$$

and (A.5) follows. *Q.E.D.*

The lemmas above allow to control how the norms are propagated through the procedure of building the formal integrals $\Phi^{(l)}$ by recursively solving the system (2.1). Use now the convergence hypothesis on the Hamiltonian, namely the hypothesis that there exist real constants $h \ge 0$ and $E > 0$ such that $\|H_s\|_R \le h^{s-1}E$ for $s \ge 1$. Then one has the following

Lemma A.5: *The norms of the polynomials $\Psi_s^{(l)}$ and $\Phi_s^{(l)}$ generated by recursively solving the system (2.1) are bounded for $s \ge 1$ by*

$$\left\|\Psi_s^{(l)}\right\|_R \le A_s \ , \quad \left\|\Phi_s^{(l)}\right\|_R \le \frac{A_s}{\alpha_s} \ , \tag{A.6}$$

where

$$\begin{aligned} A_1 &= 24E \\ A_s &= 12E\left(12\Lambda^2 E + \frac{8}{9}h\alpha_1\right)^{s-1} \frac{(s+1)!}{\prod_{l=1}^{s-1}\alpha_l} \ , \quad s \ge 1 \ . \end{aligned} \tag{A.7}$$

Proof. First, transform the Hamiltonian to complex variables, so that the convergence hypothesis above, by lemma A.1, becomes

$$(A.8)\qquad \|H_s\|_R \le \beta^{s-1}\mathcal{E}\ ,\quad \beta = 2^{1/2}h\ ,\quad \mathcal{E} = 2^{3/2}E\ .$$

Look now for a sequence $\{B_s\}_{s\ge1}$ of positive real numbers such that $\left\|\Phi_s^{(l)}\right\|_R \le B_s$. By lemma A.3 one clearly has $\left\|\Psi_1^{(l)}\right\|_R \le 3\mathcal{E}$, so that one can take $B_1 = 3\mathcal{E}/\alpha_1$, and, assuming that B_j has been determined for $1\le j<s$, by lemmas A.2 and A.3 one has

$$(A.9)\qquad \left\|\Psi_s^{(l)}\right\|_R \le \sum_{j=1}^{s-1}(j+2)(s-j+2)\Lambda^2\beta^{s-j-1}B_j\mathcal{E} + (s+2)\beta^{s-1}\mathcal{E}\ ,$$

so that one can recursively define

$$(A.10)\qquad B_s = \frac{\Lambda^2\mathcal{E}}{\alpha_s}\sum_{j=1}^{s-1}(j+2)(s-j+2)\beta^{s-j-1}B_j + (s+2)\beta^{s-1}\frac{\mathcal{E}}{\alpha_s}\ .$$

For $s\ge3$, write the r.h.s. of (A.9) as

$$\begin{aligned}3(s+1)\Lambda^2\mathcal{E}B_{s-1} + \Lambda^2\mathcal{E}\beta\sum_{j=1}^{s-2}(j+2)(s-j+2)\beta^{s-j-2}B_j + (s+2)\beta^{s-1}\mathcal{E}&\\ \le \left[3(s+1)\Lambda^2\mathcal{E} + \frac{4}{3}\beta\alpha_{s-1}\right]B_{s-1}&\\ < (s+1)\left(3\Lambda^2\mathcal{E} + \frac{4}{9}\beta\alpha_1\right)B_{s-1}\ ;&\end{aligned}$$

the first inequality is obtained by keeping the first term and using $s-j+2 \le \frac{4}{3}(s-j+1)$ for $s\ge2$ and $0\le j\le s-2$ in order to compare the terms in the sum with B_{s-1} as given by (A.10); the second one by simply using the fact that $\{\alpha_s\}_{s\ge1}$ is a nonincreasing sequence. The same inequality is easily checked to be valid also for $s=2$. Then, recalling that $\left\|\Psi_s^{(l)}\right\|_R \le 3\mathcal{E}$ and $B_1 = 3\mathcal{E}/\alpha_1$, one gets, for $s>1$,

$$\left\|\Psi_s^{(l)}\right\|_R < \frac{3}{2}\mathcal{E}\left(3\Lambda^2\mathcal{E} + \frac{4}{9}\beta\alpha_1\right)^{s-1}\frac{(s+1)!}{\prod_{l=1}^{s-1}\alpha_l}$$

and

$$B_s < \frac{3}{2}\mathcal{E}\left(3\Lambda^2\mathcal{E} + \frac{4}{9}\beta\alpha_1\right)^{s-1}\frac{(s+1)!}{\prod_{l=1}^{s}\alpha_l}\ .$$

Such inequalities have been obtained in complex variables, and the change back to real variables is taken into account by substituting the values of β and $\mathcal{E}$ given by (A.8) and by multiplying by a factor $2^{s/2+1}$. This immediately gives (A.6) and (A.7). *Q.E.D.*

Consider now the explicit expression (3.4) of the noise $\mathcal{R}^{(r,l)}$ for the truncated integrals. The norms of the polynomials $Q_s^{(l)}$ are then estimated by the following

Lemma A.6: *The norms of the polynomials $Q_s^{(l)}$ defined by (3.5) are bounded by*

$$\left\| Q_s^{(l)} \right\|_R \le \frac{2}{3}(s-r)h^{s-r-1}A_{r+1} \ , \quad s>r \tag{A.11}$$

with A_{r+1} defined by (A.7).

Proof. By (3.5) one immediately sees that $\left\| Q_s^{(l)} \right\|_R \le D_s$, with

$$D_s = \sum_{j=1}^{r} (j+2)(s-j+2)h^{s-j-1} \left\| \Phi_j^{(l)} \right\|_R E + (s+2)h^{s-1}E \ .$$

Since $Q_{r+1}^{(l)} = \Psi_{r+1}^{(l)}$ as given by (2.2), D_{r+1} can be estimated by lemma A.5, giving $D_{r+1} \le A_{r+1}$. Using now the inequality $\frac{s-j+2}{r-j+3} \le \frac{2}{3}(s-r)$ for $r \ge 1$ and $0 \le j \le r \le s-2$, one has

$$D_s \le \frac{2}{3}(s-r)h^{s-r-1}D_{r+1} \ ,$$

and this immediately gives (A.11). *Q.E.D.*

Finally, the estimates obtained in lemmas A.5 and A.6 are used in order to achieve the

Proof of proposition 3.1. By the definition of the norm and of the domain $\Delta_{\varrho R}$ one has

$$\left| \left(\Phi^{(l,r)} - I_l \right)(x,y) \right| < \sum_{s=1}^{r} \left\| \Phi_s^{(l)} \right\|_R \varrho^{s+2} \ .$$

Using now (A.6) and (A.7) one gets, for $1 \le s \le r$,

$$\left\| \Phi_s^{(l)} \right\|_R \le \frac{24E}{\alpha_1} \sigma_r^{s-1} \ ,$$

with σ_r defined by (3.9), so that one has

$$\left| \left(\Phi^{(l,r)} - I_l \right)(x,y) \right| < \frac{24E}{\alpha_1} \varrho^3 \sum_{s=1}^{r} (\sigma_r \varrho)^{s-1} \ ,$$

and (3.7) immediately follows. Recalling now that $\dot{\Phi}^{(l,r)} = \mathcal{R}^{(r,l)}$, by (3.4), (3.5) and lemma A.6 one has

$$\begin{aligned} \left| \dot{\Phi}^{(l,r)}(x,y) \right| &\le \sum_{s>r} \left\| Q_s^{(l)} \right\|_R \varrho^{s+2} \\ &< \frac{2}{3} A_{r+1} \varrho^{r+3} \sum_{s>r} (s-r)(h\varrho)^{s-r-1} \ , \end{aligned}$$

with A_{r+1} defined by (A.7), so that (3.8) follows. *Q.E.D.*

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ERRATUM: The proof of lemma A.6 contains an error. The text below is intended to give a correct proof.

Proof of lemma A.6. By (3.5), lemmas A.2 and A.3 and the hypothesis $\|H_s\|_R \leq h^{s-1}E$ one immediately sees that $\|Q_s^{(l)}\|_R \leq D_s$, with

$$D_s = \sum_{j=1}^{r} (j+2)(s-j+2)\Lambda^2 h^{s-j-1} \|\Phi_j^{(l)}\|_R E + (s+2)h^{s-1}E \ .$$

`(a factor` Λ^2 `is missing in the paper)`
Write explicitly the expression for $s = r+1$, namely

$$D_{r+1} = \sum_{j=1}^{r} (j+2)(r-j+3)\Lambda^2 h^{r-j} \|\Phi_j^{(l)}\|_R E + (r+3)h^r E \ ,$$

and compare it with the general one above, considering $s \geq r+2$ and remarking that both expressions contain the same number of term (the sum always stops at $j = r$, since the integrals are truncated at order r). A term by term comparison, using also the inequality

$$\frac{s-j+2}{r-j+3} \leq \frac{2}{3}(s-r) \quad \text{for } 0 \leq j \leq r \leq s-2 \ ,$$

gives

$$D_s \leq \frac{2}{3}(s-r)h^{s-r-1} \left[\sum_{j=1}^{r} (j+2)(r-j+3)\Lambda^2 h^{r-j} \|\Phi_j^{(l)}\|_R E + (r+3)h^r E \right] \ ,$$

i.e.,

$$D_s \leq \frac{2}{3}(s-r)h^{s-r-1} D_{r+1} \ .$$

The inequality above is easily checked as follows:

$$\begin{aligned}\frac{s-j+2}{r-j+3} &= \frac{s-r}{r-j+3} + \frac{r-j+2}{r-j+3}\\ &= \frac{s-r}{r-j+3} + 1 - \frac{1}{r-j+3}\\ &= (s-r)\left[\frac{1}{r-j+3} + \frac{1}{s-r} - \frac{1}{(s-r)(r-j+3)}\right]\\ &\le (s-r)\left[\frac{1}{3} + \frac{1}{2} - \frac{1}{6}\right]\\ &= \frac{2}{3}(s-r)\end{aligned}$$

(in the fourth line the conditions $j \le r \le s-2$ have been used). Thus, the proof of the statement for $s \ge r+2$ requires only to prove that $D_{r+1} \le A_{r+1}$, while the stronger estimate $D_{r+1} \le 2A_{r+1}/3$ is required for $s = r+1$.

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Here begins the hidden part of the proof
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Looking at the proof of lemma A.5, one sees that in complex variables one has

$$\|\Phi_s^{(l)}\|_R \le B_s \ ,$$

where B_s is recursively defined by (A.10), starting with $B_1 = 3\mathcal{E}/\alpha_1$. Transforming to real variables, by lemma A.1 one gets the estimate

$$(B.1) \qquad \|\Phi_s^{(l)}\|_R \le 2^{(s+2)/2} B_s \ ;$$

this is the estimate to be used in place of the weaker estimate in the statement of lemma A.5. Compute (see comments at the end of the formula)

$$\begin{aligned}D_{r+1} &\le \sum_{j=1}^{r} (j+2)(r-j+3)\Lambda^2 2^{(j+2)/2} B_j h^{r-j} E + (r+3)h^r E\\ &= \sum_{j=1}^{r} (j+2)(r-j+3)\Lambda^2 2^{(2j-r-1)/2} B_j \beta^{r-j}\mathcal{E} + 2^{-(r+3)/2}(r+3)\beta^r\mathcal{E}\\ &\le 2^{(r-1)/2}\alpha_{r+1}\cdot\left[\frac{\Lambda^2\mathcal{E}}{\alpha_{r+1}}\sum_{j=1}^{r}(j+2)(r-j+3)B_j\beta^{r-j} + \frac{\mathcal{E}}{\alpha_{r+1}}(r+3)\beta^r\right]\\ &= 2^{(r-1)/2}\alpha_{r+1}B_{r+1} \ .\end{aligned}$$

Here are the details, on a line by line basis. Line 1: just substitute the estimate (B.1). Line 2: replace $h = 2^{-1/2}\beta$ and $E = 2^{-3/2}\mathcal{E}$ as given by (A.8). Line 3: factor out $2^{(r-1)/2}\alpha_{r+1}$ and trow away all the remaining inverse powers of 2, so that the bracketed expression turns out to coincide with B_{r+1} as given by (A.10); this gives the last line.

Using now the estimate for B_s given in lemma A.5, namely

$$B_s < \frac{3}{2}\mathcal{E}\left(3\Lambda^2\mathcal{E} + \frac{4}{9}\beta\alpha_1\right)^{s-1} \frac{(s+1)!}{\prod_{l=1}^{s}\alpha_l} \ ,$$

put $s = r+1$, and compute (see comments at the end of the formula)

$$\begin{aligned} D_{r+1} &\leq 2^{(r-1)/2}\alpha_{r+1}\cdot\frac{3}{2}\mathcal{E}\left(3\Lambda^2\mathcal{E} + \frac{4}{9}\beta\alpha_1\right)^{r} \frac{(r+2)!}{\prod_{l=1}^{r+1}\alpha_l} \\ &= 2^{(r-1)/2}\cdot\frac{3}{2}2^{3/2}E\left(3\Lambda^2 2^{3/2}E + \frac{4}{9}2^{1/2}h\alpha_1\right)^{r} \frac{(r+2)!}{\prod_{l=1}^{r}\alpha_l} \\ &= 3E\left(12\Lambda^2 E + \frac{8}{9}h\alpha_1\right)^{r} \frac{(r+2)!}{\prod_{l=1}^{r}\alpha_l} \\ &= \frac{1}{4}A_{r+1} \ . \end{aligned}$$

Here are the details, on a line by line basis. Line 1: just substitute the estimate above for B_{r+1}. Line 2: substitute $\beta = 2^{1/2}h$ and $\mathcal{E} = 2^{3/2}E$ as given by (A.8), and simplify α_{r+1} with the same term in the product. Line 3: move the factor $2^{r/2}$ inside the parentheses, so that the expression of A_{r+1} as given by (A.7) is recovered; this gives the last line.

Thus, the estimate $D_{r+1} < 2A_{r+1}/3$ is achieved, and this concludes the proof of the lemma.