LYAPUNOV CHARACTERISTIC EXPONENTS FOR SMOOTH DYNAMICAL SYSTEMS AND FOR HAMILTONIAN SYSTEMS; A METHOD FOR COMPUTING ALL OF THEM.

PART 2: NUMERICAL APPLICATION

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SOMMARIO. Questo articolo, insieme con il precedente (Parte I: Teoria, pubblicato in questa stessa rivista) è inteso a fornire un metodo esplicito per il calcolo di tutti gli esponenti caratteristici di Lyapunov per un sistema dinamico. Dopo la teoria generale sui tali espensori sviluppata nella prima parte, qui si illustra il metodo di calcolo (Capitolo A) e si danno esempi numerici per applicazioni di varietà in sé e per sistemi Hamiltoniani (Capitolo B).

SUMMARY. The present paper, together with the previous one (Part I: Theory, published in this journal) is intended to give an explicit method for computing all Lyapunov Characteristic Exponents of a dynamical system. After the general theory on such exponents developed in the first part, in the present paper the computational method is described (Chapter A) and some numerical examples for mappings on manifolds and for Hamiltonian systems are given (Chapter B).

1. Introduction.

1.1. Lyapunov Characteristic Exponents (briefly LCEs) are of particular interest in the study of stochasticity properties of dynamical systems, being essentially related to the exponential divergence of nearby orbits; one then often meets with the problem of their explicit numerical computation. The method for computing the maximal one was already given in ref [1] (see also [2] and [3]), while the method for computing all of them, based on the possibility of computing LCEs of order greater than one which are related to the exponential increase of volumes, was announced in the brief note [4], where a numerical application was also given.

In the paper [5], which constitutes the first part of the present work, the general theory of LCEs was recalled and in particular the relation between the exponents of order one and those of greater order, which is at the basis of the numerical method for the computation of all LCEs, was given. Now, in this second part, we illustrate the computational method (Chapter A) and give some numerical examples for mappings on manifolds and for Hamiltonian systems (Chapter B). Preleminarily, in Sec. 2, some general problems on the reliability of numerical computations are discussed.

1.2. The present Part II is largely independent of Part I, and we hope that it can be readable even without a detailed knowledge of the latter. In order to concentrate on the essential problems, we will consider the case of a diffeomorphism $T$ of a differentiable compact connected manifold $M$ onto itself. Indeed the case of a differentiable flow $\{T^t\}$ essentially reduces to the previous one because, as already remarked in Sec. 1.4.4 (1), the LCEs of a flow $\{T^t\}$ are the same as those of the mapping $T^1$. However, by the method described below one can also consider differentiable mappings more general than diffeomorphisms (see Sec. 1.3.2). $Z^+$ will denote the nonnegative integers and $\ln$ the natural logarithm.

2. Some problems on numerical studies of stochasticity.

2.1. In the present section for convenience we will use the following simplified notation: having fixed $x \in M$ we denote $x_t = T^t(x)$ (in particular $x_0 = x$) and $E_t = T_{T^t}M$.

A first general problem concerns the numerical errors, which in principle could be particular relevant in our case, because we are interested in computing limits when $t \to \infty$. Precisely, given a point $x \in M$, the «true» orbit $\{x_t\}_{t \in Z^+}$ is defined; analogously, given a vector $v \in E_0$, the sequence of evolved vectors $\{dT^t:v\}_{t \in Z^+}$, where $dT^t:v \in E_t$, is also defined.

In principle, these sequences are computed by iterations, using the properties $T^t = T \circ T^{t-1}$ and $dT^t = dT \circ dT^{t-1}$. Actually, due to unavoidable errors, one computes sequences $\{x_t\}_{t \in Z^+}$ and $\{v_t\}_{t \in Z^+}$ which are only some approximations to $\{x_{t}\}_{t \in Z^+}$ and $\{dT^t:v\}_{t \in Z^+}$ respectively; precisely, one can only control that the error be small at each iteration step, while nothing is known a priori for large $t$. The problem of the relations between true orbits $\{x_t\}$ and «pseudoorbits» $\{x_t\}$ was studied by means of numerical experiments in [6]. From the results reported there and inspired by the so-called Anosov-Bowen theorem it appears that, if the orbit has a sufficiently stochastic behaviour and if the precision of the computation at each step is high enough, the time averages of continuous functions on $M$ computed along a pseudoorbit are practically reliable. One can conjecture that the same

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(1) Sec. 1.4.4 means Sec. 4.4 of Part I; analogously, for example, Theorem 1.5 will denote Theorem 5 of Part I.

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happens for what concerns the computation of LCEs, although nothing is known at the moment.

2.2. Another important problem is related to the role played by the \( T \)-invariant Borel probability measure \( \mu \) on \( M \) appearing in Theorem 1.5 if one takes into consideration the effect of numerical errors. This problem too has been studied in an experimental way in the slightly different context of computations of time averages in [6, 7]. The conclusion there drawn is that, if \( T \) admits a Liouville measure \( \mu_L \), this one appears to play a privileged role in numerical computations, at least if the system has sufficiently strong stochastic properties. One can then again conjecture that the same should happen in the computations of LCEs.

Now, many dynamical systems of interest are non conservative and thus do not admit a Liouville measure; such are for example the systems studied by Lorenz (see [8, 9]) and Hénon [10].

In particular, the maximal LCEs for the Lorenz and the Hénon models have been numerically computed in [11] and [12] respectively.

The problem of a privileged measure for numerical experiments on such systems, in the sense explained in [6, 7], is also a delicate one, which is completely open to our knowledge (see however [13, 14]).

2.3. Finally, another problem is related to the circumstance that \( T \) might not be explicitly defined on the manifold \( M \); for example, \( M \) could be a compact component \( \Gamma_E \) of a surface of constant energy, with \( \dim \Gamma_E = 2n - 1 \), and the flow be defined by differential equations on an open subset \( U \) of \( \mathbb{R}^{2n} \). Then, due to numerical errors, the pseudo-orbit will not lie exactly on \( M \) and the computed vectors \( \{v_i\} \) will not be tangent to \( M \), even if one had initially \( x \in \mathcal{E} M \) and \( v \) tangent to \( M \). In such case, this difficulty is overcome in virtue of Corollary 1.9, which insures that the LCEs computed for the flow on \( U \) are the same as those for the induced flow on \( M \), apart from a vanishing one.

A. THE NUMERICAL METHOD.

3. The method for the maximal exponent.

3.1. Neglecting all the difficulties related to the general problems discussed in Sec. 2, we come now to the specific problems that one encounters in the numerical application of Theorem 1.5.

Let \( M \) be a compact connected \( m \)-dimensional Riemannian manifold of class \( C^1 \), and \( T \) a diffeomorphism of \( M \) into itself. One wants to compute the LCEs \( \chi_1(x), \ldots, \chi_m(x) \) as defined in Sec. 1.2. Given \( m \) initial vectors \( v_1, \ldots, v_m \in E_0 \), chosen at random in the usual sense of the term, one has thus to evaluate, according to Theorem 1.5, the quantities

\[
\chi_1(x) = \lim_{t \to +\infty} \frac{1}{t} \ln \text{Vol}^p(dT^t_1v_1, \ldots, dT^t_nv_p) =
\]

for \( p = 1, \ldots, m \), where \( v_1, \ldots, v_p \) denotes the open parallelepiped defined by the vectors \( v_1, \ldots, v_p \in E_0 \) and \( \text{Vol}^p \) denotes the \( p \)-dimensional (Riemannian) volume in the tangent spaces to \( M \); in particular, \( \text{Vol}^1([v_1]) = \|v_1\| \), where \( \|\cdot\| \) denotes the norm induced in \( E_0 \) by the Riemannian metric on \( M \).

However, the direct application of this formula meets with two difficulties. Precisely, if \( v \) is chosen at random one has in general that \( \|dT^t_1v\| \) increases exponentially with increasing \( t \), so that it rapidly exceeds the possibilities of ordinary numerical computations; this is indeed the case if \( \chi_1(x) > 0 \), in virtue of Theorem 1.5. Furthermore, it occurs that, if \( v_1 \) and \( v_2 \) are chosen at random, the angle between the directions of \( dT^t_1v_1 \) and \( dT^t_2v_2 \) in general becomes rapidly so small that again the possibilities of ordinary numerical computations are exceeded; indeed this can be easily shown as a consequence of Theorem 1.5 if one has \( \chi_1(x) > \chi_2(x) \).

3.2. In the computation of the maximal LCE \( \chi_1 \) only the first difficulty occurs, and it is overcome by a well known procedure that takes profit of the linearity and of the composition law \( dT^t_1dT^s_2v = dT^t_1(dT^s_2v) \) for the tangent mappings \( \{dT^t_x\} \). Roughly speaking, having fixed a number \( s \) of iterations, one replaces after each \( s \) iterations the evolved tangent vector \( dT^s_1v \) by another vector having the same direction and a fixed norm, for example norm \( 1 \). More precisely, having fixed \( s \geq 1 \), one can choose at random \( v \in E_0 \) with \( \|v\| = 1 \) and define and compute recursively

\[
w_0 = v \in E_0 \quad \alpha_k = \frac{\|dT^s_1v\|}{\|dT^s_2w_k-1\|} \quad w_k = \frac{dT^s_2w_k-1}{\alpha_k} \in E_k,
\]

for \( k \geq 1 \). One then easily sees that

\[
\|dT^s_1v\| = \alpha_1 \ldots \alpha_k
\]

for \( k \geq 1 \), so that obviously one has

\[
\chi_1 = \lim_{k \to +\infty} \frac{1}{k} \sum_{i=1}^{k} \ln \alpha_i.
\]

Now, if \( s \) is not too large, all of the quantities \( \alpha_i(i \geq 1) \) are uniformly bounded and no difficulty arises. A minor variant of this method is presented in [1] (see also [2]).

4. The method for all exponents.

4.1. The second difficulty occurs when at least two vectors are involved, i.e. already in the computation of \( \chi_p(x) \), and in general in the computation of \( \chi_p(x) \) for \( 2 < p \leq m \), where the evaluation of \( \text{Vol}^p([dT^t_1v_1, \ldots, dT^t_nv_p]) \) is required. Indeed, if the vectors \( v_1, \ldots, v_p \) are chosen at random, the angles between the directions of \( dT^t_1v_1 \) and \( dT^t_2v_2 \) (\( i \neq f \)) become in general too small for numerical computations. This difficulty can be overcome on the basis of the following simple remark: an invertible linear mapping, as our tangent mapping is, maps linear \( p \)-dimensional subspaces
onto linear subspaces of the same dimension, and the 'rate of growth' of $p$-dimensional volumes under the action of any linear mapping $A$, i.e.

$$\frac{\text{Vol}^p(A(U))}{\text{Vol}^p(U)}$$

where $U$ is an open set in the considered $p$-dimensional subspace, does not depend on the choice of the open set $U$. Indeed, for a given linear mapping this is just an intrinsic quantity of the subspace involved.

Thus one can replace after each $s$ iterations all the $p$ evolved tangent vectors by a set of $p$ other vectors spanning the same $p$-dimensional subspace and suitably chosen, for example $p$ vectors which form an orthonormal set. More precisely, one can choose at random the vectors $v_1, \ldots, v_p \in \mathbb{E}_0$ with $(v_i, v_j) = \delta_{ij}$, where $(,)$ is the scalar product induced in $\mathbb{E}_0$ by the Riemannian metric on $M$ and, as usual

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then one defines recursively

$$w_0^{(i)} = v_i \in \mathbb{E}_0, \quad j = 1, \ldots, p,$$

$$\beta_k^{(p)} = \text{Vol}^p \{dT_{x(k-1)s}^p w_0^{(1)}, \ldots, dT_{x(k-1)s}^p w_0^{(p)}\},$$

where $\{w_0^{(k)}\}_{1 \leq k \leq p}$ is an arbitrary orthonormalization of

$$\{dT_{x(k-1)s}^p w_0^{(j)}\}_{1 \leq j \leq p}, \quad \text{and } w_0^{(k)} \in \mathbb{E}_{ks}, \quad \text{for } k \geq 1.$$

One then easily sees that

$$\frac{\text{Vol}^p([dT^p_{x(k-1)s} w_0^{(1)}], \ldots, dT^p_{x(k-1)s} w_0^{(p)})}{\text{Vol}^p(U)} = \beta_k^{(p)} \cdots \beta_1^{(p)}.$$  \hfill (2)

In such way, by relation (1) one can compute $X_1(x) + \ldots + X_p(x)$ for $1 \leq p \leq m$. The computation of $X_1(x), \ldots, X_p(x)$ thus requires the computation of suitable $p$-dimensional volumes for all $1 \leq p \leq m$.

4.2. As a matter of fact, a particular choice of the orthonormalization procedure allows one to evaluate all such volumes at once by computing the evolution of just $m$ vectors. Indeed, for any $k \geq 1$, choose the Gram-Schmidt orthonormalization procedure defined by the recursive formulae

$$\alpha_k^{(1)} = \|dT_{x(k-1)s}^p w_0^{(1)}\|$$

$$w_0^{(1)} = \frac{dT_{x(k-1)s}^p w_0^{(1)}}{\alpha_k^{(1)}}$$

and

$$\alpha_k^{(f)} = \|dT_{x(k-1)s}^p w_0^{(f)} - \sum_{i=1}^{f-1} (w_0^{(i)}, dT_{x(k-1)s}^p w_0^{(f)}) w_0^{(i)}\|$$

$$w_0^{(f)} = \frac{dT_{x(k-1)s}^p w_0^{(f)} - \sum_{i=1}^{f-1} (w_0^{(i)}, dT_{x(k-1)s}^p w_0^{(f)}) w_0^{(i)}}{\alpha_k^{(f)}}$$

for $f = 2, \ldots, p$.

One then easily sees that

$$\beta_k^{(p)} = \alpha_k^{(1)} \cdots \alpha_k^{(p)},$$

so that, by (2),

$$\text{Vol}^p([dT^p_{x(k-1)s} v_1, \ldots, dT^p_{x(k-1)s} v_p]) =$$

$$= (\alpha_1^{(1)} \cdots \alpha_k^{(1)}) \cdots (\alpha_k^{(p)} \cdots \alpha_k^{(p)}).$$  \hfill (3)

where $1 \leq p \leq m$, and all the required volumes are thus given as a byproduct of the orthonormalization procedure.

In such way one also obtains a direct formula for $x_1(x), \ldots, x_m(x)$; indeed by (1) and (3) one immediately gets

$$x_p(x) = \lim_{k \to \infty} \frac{1}{ks} \sum_{i=1}^{p} \ln \alpha_i^{(p)}$$

for $1 \leq p \leq m$. This is the relation that was actually applied in our computations.

A completely analogous algorithm also holds for the computation of all LCEs $x_1(x), \ldots, x_m(x)$ of the family $dT_{x}^p$ where $x \in M$ and $dT_{x}^p : T_{x}N \to T_{x}N$ (see Sec. 1.3).

B. NUMERICAL TESTS AND EXAMPLES.

5. Generalities.

5.1. According to what explained in the previous section, we then compute the quantities

$$x_p^{(k,s)}(x) = \frac{1}{ks} \sum_{i=1}^{p} \ln \alpha_i^{(p)}$$

for $1 \leq p \leq m$ and expect that, as $k$ increases, $x_p^{(k,s)}(x)$ approaches a sufficiently well defined limit to be identified with the LCE $x_p(x)$.

From the very definition of $x_p^{(k,s)}(x)$ it follows that it depends on $k$ and $s$ only thrhrough their product $r = ks$. Thus we will write in the following $x_p^{(r)}(x)$ instead of $x_p^{(k,s)}(x)$.

Moreover we may expect that:

1) $\lim_{r \to \infty} x_p^{(r)}(x)$ is independent of the choice of the initial vectors $v_1, \ldots, v_p$ if they are chosen at random, and that such limit is equal to $x_p(x)$ for $1 \leq p \leq m$;

2) $\lim_{r \to \infty} \sum_{p=1}^{m} x_p^{(r)}(x) = 0$

if $T$ admits a Liouville measure (see Corollary 1.7);

3) $\lim_{r \to \infty} (x_1^{(r)}(x) + x_m^{(r)} + x_1^{(r)}(x) = 0$

for all $1 \leq p \leq m$, if $T$ is a symplectic diffeomorphism (see Theorem 1.8; $m$ is necessarily even in such case); this implies in particular property 2).

These properties provide significative tests for the consistency of our numerical computations. Actually we will check properties 1) and 3).

First, in Sec. 6, we consider two simple algebraic automor-
phisms of the torus $T^6$, where the LCEs are known analyti-
cally and can thus be compared with our numerical results.
Then, in Sec. 7, we report the results of the computation of all
LCEs on two examples of symplectic nonlinear dif-
feomorphisms of the tori $T^4$ and $T^6$ respectively, studied
previously by C. Froeschlé and J.-P. Scheidecker [15-21].
Finally, in Sec. 8 we report the results on an Hamiltonian
system with three degrees of freedom of interest in astro-
nomy (see [3]). All the computations were performed on
a CDC-CY76, with a precision of 14 digits.

6. Algebraic automorphisms of the torus $T^6$ and $T^2$.

6.1. Consider the algebraic automorphisms of the torus
$T^6$ defined by the matrices

$$
A = \begin{pmatrix}
2 & 5 & 0 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 4 & 5 & 0 & 0 \\
0 & 1 & 3 & 4 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
$$

For such automorphisms it is easily seen that all the
LCEs are constant on $T^6$, being given by the logarithms
of the moduli of the eigenvalues $\lambda_1, \ldots, \lambda_6$ of the correspond-
ing matrices. The eigenvalues of both matrices are real
and positive with moreover $\lambda_1 = \lambda_6^{-1}, \lambda_2 = \lambda_5^{-1}$ and $\lambda_3 = \lambda_4^{-1}$,
where $\lambda_1 \geq \ldots \geq \lambda_6$. For matrix $A$ one has

$$
\lambda_1 = 5 + \sqrt{24}, \quad \lambda_2 = 4 + \sqrt{15}, \quad \lambda_3 = \frac{3 + \sqrt{5}}{2};
$$

for matrix $B$ one has

$$
\lambda_1 = \frac{11 + \sqrt{117}}{2}, \quad \lambda_2 = 4 + \sqrt{15}, \quad \lambda_3 = 1.
$$

In the particular case of algebraic automorphisms of the
torus, the computation scheme described above essential-
ly trivializes, because the knowledge of an orbit is irrelevant,
as $dT^i_t$ is independent of $x$.

Coming now to the numerical results, in Table I we report,
for matrices $A$ and $B$ respectively, the values of $x_p^p$ at several
values of $r$ (with $s = 2$), and correspondingly the exact values
of $x_p = \ln \lambda_p$ for $1 \leq p \leq 6$. As one sees, the agreement is
excellent; in particular, notice that $x_3^p$ and $x_4^p$ appear to tend
to zero with increasing $r$ in case of matrix $B$, as expected.

6.2. In this subsection we intend to illustrate on a very
simple example a fact already discussed in Sec. 1.2.3 and
related to the role of the random choice of the initial vectors
in our method. Precisely we want to illustrate how it occurs
that, even if one happened to know exactly the subspaces
$L_1, \ldots, L_s$ described in Sec. 1.2.1, one would not in general
be able to compute the LCEs different from the maximal
one.

Consider the well known Anosov automorphism of the

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torus $T^2$ defined by the matrix

$$C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The eigenvectors are

$$e_1 = \left(-1, \frac{\sqrt{5} + 1}{2}\right), \quad e_2 = \left(1, \frac{\sqrt{5} + 1}{2}\right)$$

with eigenvalues

$$\lambda_1 = \frac{3 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{3 - \sqrt{5}}{2}$$

respectively; correspondingly one has $x_1 = \ln \lambda_1 \approx 0.96242$ and $x_2 = \ln \lambda_2 = -x_1$.

The problem is then that of the random choice of the initial vector $v$ in the computation of $x_1$. In this connection we made what a priori is the worst choice, i.e. the choice $v = e_2$ as far as allowed by the precision available in the computer. More precisely, we performed two computations, one with 14 digits (single precision) and the other one with 28 digits (double precision); the graphs for $x_1^{(r)}$ as a function of $r$ (with $s = 5$) in semilogarithmic scale for both cases are reported in Fig. 1. As one sees, up to a certain number of iterations ($r = 15$ for single precision) one has with excellent accuracy $x_1^{(r)} = x_2$, but then $x_1^{(r)}$ 'relaxes' to the value of $x_1$, and an excellent agreement is obtained at $r = 5000$.

The double precision calculation does not change substantially the picture.

This indicates, on the other hand, that the choice of the initial vector is really unessential for the limit of $x_1^{(r)}$ as $r$ increases: the unavoidable computational errors produce in a sense themselves the good random choice.

7. Symplectic diffeomorphisms of $T^2$ and $T^3$.

7.1. We then come to the class of mappings studied by C. Froeschlé and J. P. Scheidecker [15–21]. Given functions $f_1, \ldots, f_n$, $g \in C^1(T^1)$ and real numbers $a_1, \ldots, a_n, b$, define the diffeomorphism $T_n : T^{2n} \to T^{2n}$ by the formula

$$(q_1, p_1, \ldots, q_n, p_n) \mapsto (q_1 + a_1 f_1(p_1 + q_1) + bg \sum_{i=1}^{n} (p_i + q_i), p_1 + q_1) \pmod{1},$$

$$(q_2, p_2, \ldots, q_n, p_n) \mapsto (q_2 + a_2 f_2(p_2 + q_2) + bg \sum_{i=1}^{n} (p_i + q_i), p_2 + q_2) \pmod{1},$$

$$\vdots$$

$$(q_n, p_n, \ldots, q_n, p_n) \mapsto (q_n + a_n f_n(p_n + q_n) + bg \sum_{i=1}^{n} (p_i + q_i), p_n + q_n) \pmod{1},$$

$T_n$ is easily seen to be symplectic. Indeed, it is not difficult to recognize that one can write $T_n = T_n^{(2)} \circ T_n^{(1)}$, with $T_n^{(1)} : (q_1, p_1, \ldots, q_n, p_n) \mapsto (q_1, q_1 + p_1, \ldots, q_n, q_n + p_n) \pmod{1}$, and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Showing the independence of the computed maximal LCE from the initial vector. Even if the (worst) choice is made, after a sufficiently large number of iterations the computed value relaxes to the value of $x_1 \approx 0.96242$. The number of iterations required for relaxation increases with the accuracy of the computation: dots, single precision (14 digits); triangles, double precision (28 digits).}
\end{figure}
For $n > 1$ and $f_1(x) = \ldots = f_n(x) = g(x) = \sin(2\pi x)$, these mappings were extensively studied by C. Froeschlé and J.-P. Scheidecker in order to investigate numerically the stochastic behaviour in conservative dynamical systems. As in the present paper, one of their main motivations was to find a quantitative characterization of stochasticity by numerical means.

7.2. As in refs. [15-21], in our computations we considered the particular case with $a_1 = a_2 = \ldots = a_n = a$ and $f_1(x) = \ldots = f_n(x) = g(x) = \sin(2\pi x)$; we restricted ourselves to the cases $n = 2$ and $n = 3$.

All the properties expected were always well satisfied. As an example, we report here in detail the results for $n = 3$, $a = -1.3$, $b = 0.1$, initial point $(2.02,0.0,2.1,0.0,2.002)$ and a certain choice of the orthonormal initial vectors.

Table II. Values of $x_{p}^{(n)} (p = 1, \ldots , 6)$ for increasing $r$, for the symplectic diffeomorphism of the torus $T^6$ described in Sec. 6.4, with $a = -1.3$ and $b = 0.1$. This is a reproduction of the original computer output. $N$ stand for $r$, and columns 1, \ldots, 6 give the values of $x_{p}^{(n)}$, $\ldots$, $x_{6}^{(n)}$. NIT denotes $n$, MRANF is a parameter determining the choice of the initial vectors, and $X$ is the initial point.

<table>
<thead>
<tr>
<th>N</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>0.4121611</td>
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<td>0.2398909</td>
<td>-0.679061</td>
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<tr>
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<td>-0.102208</td>
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<tr>
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<tr>
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<td>-0.1034656</td>
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<tr>
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<td>-0.2557816</td>
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<td>-0.2974266</td>
</tr>
<tr>
<td>80</td>
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<td>-0.2216008</td>
<td>-0.3127366</td>
</tr>
<tr>
<td>90</td>
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<td>0.1807809</td>
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<td>-0.2069888</td>
<td>-0.2916094</td>
</tr>
<tr>
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<td>-0.2616998</td>
<td>-0.296094</td>
</tr>
<tr>
<td>150</td>
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<td>0.1879344</td>
<td>0.1782684</td>
<td>-0.1361945</td>
<td>-0.2616998</td>
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<td>0.1845344</td>
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<td>-0.2216008</td>
<td>-0.3127366</td>
</tr>
<tr>
<td>300</td>
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<td>0.1807809</td>
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<td>350</td>
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<td>-0.2616998</td>
<td>-0.296094</td>
</tr>
</tbody>
</table>

Table II. Values of $x_{p}^{(n)} (p = 1, \ldots , 6)$ for increasing $r$, for the symplectic diffeomorphism of the torus $T^6$ described in Sec. 6.4, with $a = -1.3$ and $b = 0.1$. This is a reproduction of the original computer output. $N$ stand for $r$, and columns 1, \ldots, 6 give the values of $x_{p}^{(n)}$, $\ldots$, $x_{6}^{(n)}$. NIT denotes $n$, MRANF is a parameter determining the choice of the initial vectors, and $X$ is the initial point.
The results are reported in Table II, which is a reproduction of the original computer's output. In the Table, NIT denotes \( \sigma = 5 \), MRANF is a parameter which determines the choice of the initial vectors, \( N \) stands for \( r \) and runs from 20 to \( 5 \times 10^5 \) and the columns 1, 2, ..., 6 give the values of \( x_1^{(r)}, \ldots, x_6^{(r)} \).

First of all one observes that the quantities \( x_p^{(r)} \) appear to approach rather well defined limits with increasing \( r \). This property is better visualized if the results are plotted in a graph in a log-log scale; such graphs will be shown below in connection with the dependence on the initial data.

From the Table one observes furthermore that property 3 is well satisfied. More precisely, in Table III we report, for increasing values of \( r \), the corresponding values of \( x_1^{(r)} + x_6^{(r)}, x_2^{(r)} + x_5^{(r)}, x_3^{(r)} + x_4^{(r)} \) as computed from Table II; one thus observes that they appear to tend to zero, with increasing \( r \), with remarkable regularity (as \( 1/r \)).

For what concerns the independence of \( x_p^{(r)} \) from \( s \), we have checked it in several cases; for example, by changing \( s \) from 5 to 1 we got a Table analogous to Table II, with exactly all the same numbers.

Finally, for what concerns the independence from the choice of the initial vectors, we report in the Figure 2 as an example the results corresponding to a computation with initial data as in Table II, apart from the choice of the initial vectors. This is shown in Table IV which gives, for increasing values of \( r \), the corresponding values of \( \Delta x_p^{(r)} \), namely the difference between \( x_p^{(r)} \) in the first and in the second computation.

From this Table one observes that actually \( \Delta x_p^{(r)} \) appears to go to zero, with increasing \( r \), with a remarkable regularity (again as \( 1/r \)). The example of Sec. 6.2 also provides a good check of the independence form the choice of the initial vectors.

### Table II

<table>
<thead>
<tr>
<th>( r )</th>
<th>( x_1^{(r)} + x_6^{(r)} )</th>
<th>( x_2^{(r)} + x_5^{(r)} )</th>
<th>( x_3^{(r)} + x_4^{(r)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5 \times 10 )</td>
<td>( 1.1 \times 10^{-3} )</td>
<td>( 3.0 \times 10^{-2} )</td>
<td>( 2.9 \times 10^{-2} )</td>
</tr>
<tr>
<td>( 5 \times 10^2 )</td>
<td>( 1.1 \times 10^{-4} )</td>
<td>( 2.9 \times 10^{-3} )</td>
<td>( 2.8 \times 10^{-3} )</td>
</tr>
<tr>
<td>( 5 \times 10^3 )</td>
<td>( 1.1 \times 10^{-5} )</td>
<td>( 2.9 \times 10^{-4} )</td>
<td>( 2.8 \times 10^{-4} )</td>
</tr>
<tr>
<td>( 5 \times 10^4 )</td>
<td>( 1.1 \times 10^{-6} )</td>
<td>( 2.9 \times 10^{-5} )</td>
<td>( 2.8 \times 10^{-5} )</td>
</tr>
<tr>
<td>( 5 \times 10^5 )</td>
<td>( \approx 1 \times 10^{-7} )</td>
<td>( 2.9 \times 10^{-6} )</td>
<td>( 2.8 \times 10^{-6} )</td>
</tr>
</tbody>
</table>

### Table IV

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \Delta x_1^{(r)} )</th>
<th>( \Delta x_2^{(r)} )</th>
<th>( \Delta x_3^{(r)} )</th>
<th>( \Delta x_4^{(r)} )</th>
<th>( \Delta x_5^{(r)} )</th>
<th>( \Delta x_6^{(r)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5 \times 10 )</td>
<td>( 8.3 \times 10^{-3} )</td>
<td>( 9.9 \times 10^{-2} )</td>
<td>( -2.9 \times 10^{-2} )</td>
<td>( -5.3 \times 10^{-2} )</td>
<td>( 2.5 \times 10^{-2} )</td>
<td>( -2.4 \times 10^{-2} )</td>
</tr>
<tr>
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<td>( 8.4 \times 10^{-4} )</td>
<td>( 1.2 \times 10^{-2} )</td>
<td>( -8.0 \times 10^{-3} )</td>
<td>( -5.2 \times 10^{-3} )</td>
<td>( 2.4 \times 10^{-3} )</td>
<td>( -2.5 \times 10^{-3} )</td>
</tr>
<tr>
<td>( 5 \times 10^3 )</td>
<td>( 8.4 \times 10^{-5} )</td>
<td>( 1.2 \times 10^{-3} )</td>
<td>( -8.0 \times 10^{-4} )</td>
<td>( -5.2 \times 10^{-4} )</td>
<td>( 2.4 \times 10^{-4} )</td>
<td>( -2.5 \times 10^{-4} )</td>
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<td>( 5 \times 10^4 )</td>
<td>( 8.4 \times 10^{-6} )</td>
<td>( 1.2 \times 10^{-5} )</td>
<td>( -8.0 \times 10^{-5} )</td>
<td>( -5.2 \times 10^{-5} )</td>
<td>( 2.4 \times 10^{-5} )</td>
<td>( -2.5 \times 10^{-5} )</td>
</tr>
<tr>
<td>( 5 \times 10^5 )</td>
<td>( \approx 8 \times 10^{-7} )</td>
<td>( 1.2 \times 10^{-5} )</td>
<td>( -7.9 \times 10^{-6} )</td>
<td>( -5.2 \times 10^{-6} )</td>
<td>( 2.4 \times 10^{-6} )</td>
<td>( -2.5 \times 10^{-6} )</td>
</tr>
</tbody>
</table>

Figure 2. \( x_p^{(r)} \) as a function of \( r \) (\( p = 1, 2, 3 \)) for the symplectic diffeomorphism \( T^2_3 \) of the torus \( T^6 \), with \( a = -1.3 \), \( b = 0.1 \) and three different sets of initial data: dots, \((2.02, 0.0, 2.1, 0.0, 2.0, 0.02)\); squares, \((2.02, 0.01, 2.1, 0.0, 2.0, 0.02)\); circles, \((1.4, 0.9, 3.1, 2.6, 0.1, 0.25)\).

### 7.3

In general, a point of great interest in the study of a model is how different are the LCEs from each other and how they depend on the initial data. Although this is not our primary concern here, we report however some results as an indication of the possibilities offered by our technique.

Thus in Fig. 2 we give a graph for \( x_p^{(r)}(p = 1, 2, 3) \) as a function of \( r \) in log-log scale for the same mapping considered above and three different sets of initial data, indicated in the figure caption (one set is just that of Table II). As anticipated, one sees that the indication of convergence is quite good; the separation between the limit values is evident, as is the independence form the sets of initial data. The independence from the initial data suggests that the three initial points on the torus belong to the same 'stochastic component'. An anal-
Denote by \( x = (q, p) \in \mathbb{R}^6 \) a point in the phase space of our system, and let \( \{ T_t \} \) denote as usual the Hamiltonian flow induced by the Hamiltonian \( H \) in \( \mathbb{R}^6 \). In this case the tangent space \( T_x \mathbb{R}^6 \) is canonically isomorphic to \( \mathbb{R}^6 \). Then, given a point \( x \in \mathbb{R}^6 \) and an initial vector \( v \in \mathbb{R}^6 \), one has to compute the evolved points \( T_t x \) and the evolved vectors \( dT_t v \), whose coordinates satisfy the Hamiltonian equations with Hamiltonian \( H \) and the corresponding variational (i.e. linearized) equations respectively.

We are thus interested in the LCEs of the flow \( \{ T_t \} \), the restriction to \( \mathbb{R}^6 \) of the flow \( \{ T_t \} \), and in this connection the relevant framework is provided by Corollary I.9. Namely, we can make reference to the flow \( \{ T_t \} \) on \( \mathbb{R}^6 \) and compute its LCEs for a given \( x \). These are known a priori to be of the form

\[
X_1 (x) \geq X_2 (x) \geq X_3 (x) = 0 \geq -X_2 (x) \geq -X_1 (x).
\]

The Corollary I.9 insures that the LCEs of the flow \( \{ T_t \} \) at \( x \) are equal to

\[
X_1 (x) \geq X_2 (x) \geq X_3 (x) = 0 \geq -X_2 (x) \geq -X_1 (x).
\]

For what concerns the numerical method, the Hamiltonian as well as the corresponding variational equations were integrated by a central point method correct up to third order, with an integration step \( r \) in the interval \( 0.005 \leq r \leq 0.1 \). Our results were checked to be independent of the integration step in this range.

8.2. Having recalled these general facts, we pass now to describe the properties already established in [3] for this model through numerical computations of the maximal LCE \( X_1 \). The connected compact component \( \Gamma_{0.09} \) of the energy surface with energy \( H = 0.09 \) was found to be subdivided into three disjoint invariant regions:

1) A large stochastic region, with \( X_1 = 0.03 \),
2) an ‘ordered’ region, with \( X_1 = 0 \),
3) an ‘intermediate’ region.

The last region is of stochastic type, because on it one has \( X_1 > 0 \); on the other hand it has an intermediate character, namely is ‘less stochastic’ that the first one, because on it \( X_1 \) takes values significantly smaller than 0.03. Actually such region appears to be subdivided into several disjoint invariant components with different values of \( X_1 \); for example \( X_1 = 0.005 \) and \( X_1 = 0.002 \).

The regions of the types 1), 2), 3) are called, in a descriptive way which generalizes an established use, ‘big sea’, ‘islands’ and ‘small seas’ respectively. The coexistence of stochastic components (seas) and ordered regions (islands) was well known since the first numerical studies on the onset of stochasticity in dynamical systems [22], while the existence of intermediate regions (small seas) was reported only recently [23]. In the paper [3] the latter fact was confirmed and also an attempt was made to characterize such partially stochastic regions in terms of formal integrals of motion. It is then very natural to try to see how these intermediate regions behave with respect to the LCEs.

8.3. As said before, we computed the LCEs for the flow

\[
H(q, p) = \frac{1}{2} \sum_{i=1}^{3} \omega_i \left( q_i^2 + p_i^2 \right) + q_1^2 q_2 + q_2^2 q_3,
\]

where \( q = (q_1, q_2, q_3) \). This model, suggested by G. Contopoulos, is of some interest in astronomical problems, and has recently been investigated numerically for what concerns both its maximal LCE and its formal integrals of motion [3]. On this example we want to illustrate a possible significance of the additional information provided by knowledge of the LCEs different from the maximal one. For this model the energy surfaces \( H(q, p) = h \) have a unique compact component \( \Gamma_h \) for each \( h \) in an interval \( 0 < h < h \approx 0.097 \), and all the numerical computations were performed, as in [3], on the compact component \( \Gamma_h \) with \( h = 0.09 \).
Figure 4. $x_p^{(t)} (p = 1, 2, 3)$ as a function of $t$ for the Hamiltonian system of three degrees of freedom of Contopoulos. Two orbits are given with initial points in the «big sea». These are determined by the three harmonic energies: dots, $(0.01, 0.01, 0.07)$; circles, $(0.03, 0.03, 0.03)$.

Figure 5. Same as Fig. 4 with initial point in the «ordered region». Harmonic energies $(0.0005, 0.0895, 0.0)$.

Figure 6. Same as Fig. 4 with initial point in a «small sea». Harmonic energies $(0.0085, 0.00815, 0.0)$.

In Figs. 4, 5, 6 we report typical results obtained for the big sea, the ordered region and a small sea respectively. The graphs give $x_p^{(t)}$ as a function of $t$ in log-log scale $(p = 1, 2, 3)$, where $t = rt$, $r$ having the same meaning than in the previous section and $r$ being the integration step. The initial conditions were defined assigning arbitrary values to the 'harmonic energies' of the oscillators $E_1$, $E_2$, $E_3$, with vanishing values for the coordinates, i.e. $q_1 = q_2 = q_3 = 0$.

\[ p_i = \sqrt{\frac{2E_i}{\omega_i}} \quad (i = 1, 2, 3), \]

with given values for $E_1$, $E_2$, $E_3$, indicated in each figure caption.

In the figures, $x_3^{(t)}$ appears to tend to zero as expected. For what concerns $x_1$ and $x_2$, they appear to vanish in the ordered region (Fig. 5) and to be both positive (with $x_1 > x_2$ as expected) in the big sea (Fig. 4).

Coming finally to the small sea (Fig. 6), which is the point of main interest, $x_1$ appears to be positive there, with a value smaller than in the big sea, as already known, while the behaviour of $x_2^{(t)}$ thereby is not completely clear, although one might appear to be justified in guessing it is tending to zero. In any case, it is certain that the behaviour of $x_3^{(t)}$ is very different there than in the big sea; thus our results appear to confirm those already obtained in [3] through the study of formal integrals, and indicating that the small seas actually present an intermediate character between stochasticity and order. We hope to return to this intriguing problem in the future.

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REFERENCES.


Note added in proofs: in connection with the problem of Sec. 2.1 see also

