IMPROVED ESTIMATES ON THE EXISTENCE OF INVARIANT TORI FOR HAMILTONIAN SYSTEMS

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Abstract. The existence of invariant tori in nearly–integrable Hamiltonian systems is investigated. We focus our attention on a particular one–dimensional, time–dependent model, known as the *forced pendulum*. We present a KAM algorithm which allows us to derive explicit estimates on the perturbing parameter ensuring the existence of invariant tori. Moreover, we introduce some technical novelties in the proof of KAM theorem which allow us to provide results in good agreement with the experimental break–down threshold. In particular, we have been able to prove the existence of the golden torus with frequency $\frac{\sqrt{5}-1}{2}$ for values of the perturbing parameter equal to 92 % of the numerical threshold, thus significantly improving the previous calculations.

1. Introduction

We investigate the existence of invariant surfaces in nearly–integrable Hamiltonian systems. As is well known, the existence of such surfaces is stated by the so–called KAM theory ([25], [1] and [28]). However, the original versions of the theorems provide estimates on the size of the perturbing parameter which are quite far from *reality* (i.e., from the experimental or physical values). As an application, unrealistic estimates were obtained by M. Hénon[24] in the framework of the restricted three body problem, where the perturbing parameter represents the Jupiter–Sun mass–ratio and can be compared with the physical value. Recently, many efforts have been performed to obtain estimates close to the break–down threshold. In particular, *computer assisted* KAM proofs have been implemented in [7], [8] on a simple mathematical model, known as the *periodically forced pendulum*,

described by the Hamiltonian function

$$H(x,y,t) = y^2/2 + \varepsilon\,[\cos x + \cos(x-t)] \ , \tag{1}$$

where $y \in \mathbf{R}$, $(x,t) \in \mathbf{T}^2 \equiv (\mathbf{R}/2\pi\mathbf{Z})^2$ and ε is a positive (small) parameter. In [8] the existence of the torus with frequency equal to the golden ratio $\frac{\sqrt{5}-1}{2}$ has been proved for any $\varepsilon \leq 0.019$, whereas the critical break–down value (obtained applying, e.g., Greene's method, see [21] and [22]) amounts to

$$\varepsilon_c(\frac{\sqrt{5}-1}{2}) \simeq 0.0275856 \ . \tag{2}$$

Therefore the rigorous result of [8] is in agreement of the 70 % with the numerical expectation. Within discrete setting, the existence of the golden curve for the standard mapping was proved for values of the perturbing parameter equal to 86 % of the numerical guess provided by Greene's method (see [8]).

Applications to mathematical models of physical interest have been developed in [5], [6], [9]; in particular, the spin–orbit coupling in Celestial Mechanics and a restricted three–body model have been investigated, providing estimates in agreement with the astronomical observations in the former case and less than the actual mass-ratio in the latter case.

In the present paper we implement a computer–assisted algorithm improving significantly the rigorous estimates on the existence of invariant tori. The mathematical setting has been already presented in [26], [27] and is based on *classical* series expansions, while previous results (see, e.g., [7], [8]) used *superconvergent* techniques. As an application, we consider the periodically forced pendulum (1). The improvement with respect to the previous results is quite significant, being able to prove the existence of the golden torus for values of the perturbing parameter in agreement within 92 % of the numerical break–down threshold. The above result has been obtained using a computer–assisted technique in 79 hours of CPU–time on a modern computer (⋆). Nevertheless, we stress that our result depends very weakly on the progress of the computers: for instance, we could prove the existence of the golden torus for values of ε equal to 90 % of Greene's value, using about 5 hours of CPU–time. Moreover, we will show that we can guess in a short time the best rigorous result which can be achieved by means of the application of our method.

We briefly recall the main ideas of the algorithm implemented in this paper, referring to [26], [27] and [19] for further details. Consider the Thirring model[29], i.e. a system of weakly perturbed rotators described by the Hamiltonian function $H(p,q) = \nu \cdot p + J^{-1}p \cdot p + \varepsilon f(q)$, where $(p,q) \in \mathbf{R}^n \times \mathbf{T}^n$ are action–angle coordinates, J is the matrix of the moments of inertia and f is a trigonometric polynomial of degree K.

The strategy is based on the transformation in Kolmogorov normal form of the original Hamiltonian; this is obtained by means of a sequence of canonical transformations

(⋆) Hereafter, all informations about the CPU–time are referred to an AlphaServer 8200/440 EV5 with 4 Gb of RAM.

expressed in terms of Lie series. Iterating the algorithm, after $r-1$ steps one is led to consider an Hamiltonian

$$H^{(r)} = \omega \cdot p + \sum_{s=0}^{r-1} \varepsilon^s h^{(s)}(p,q) + \sum_{s \geq r} \varepsilon^s f^{(r,s)}(p,q) \ , \tag{3}$$

where $h^{(s)}$ and $f^{(r,s)}$ are trigonometric polynomials of degree sK, $h^{(s)}$ are quadratic functions of p and $f^{(r,s)}$ are at most quadratic. Let $\tilde{h}_r = \sum_{s=0}^{r-1} \varepsilon^s h^{(s)}$ and $\tilde{f}_r^{(j)} = \sum_{s=jr}^{(j+1)r-1} \varepsilon^{s-jr} f^{(r,s)}$; then we can rewrite (3) as

$$H^{(r)} = \omega \cdot p + \tilde{h}_r(p,q) + \sum_{j=1}^{\infty} (\varepsilon^r)^j \tilde{f}_r^{(j)}(p,q) \ , \tag{4}$$

where the functions $\tilde{h}_r$ and $\tilde{f}_r^{(j)}$ are trigonometric polynomials of degree rK and jrK, respectively. By applying KAM theorem in the style of [19], under the assumption that ν is a diophantine frequency, we get [(†)] a condition similar to the following one:

$$\varepsilon^r r^\beta < \varepsilon^* \ , \tag{5}$$

with positive β. This example shows that it is very useful to iterate the algorithm as long as possible, since the estimate on the perturbing parameter is a monotonically increasing function of r.

Our algorithm is based on the application of the following strategy:

(i) *Formal algorithm:* Let $H^{(r)}$ be the sequence of Hamiltonians obtained through canonical transformations: calculate explicitely (up to a given order R_I) the truncated expansions for $H^{(1)}$, $H^{(2)}$, ..., $H^{(R_I)}$ and for all the generating functions of the Lie series.

(ii) *Iteration of the estimates:* Provide recursive inequalities, estimating the functions involved in the expansion (3) corresponding to the Hamiltonian $H^{(R_{II})}$ for a (large) suitable order R_{II}.

(iii) *Application of KAM theorem:* Apply a KAM condition, ensuring the convergence of the algorithm for an infinite order of normalization.

It is worth stressing that point (ii) is actually performed up to $R_{II} \gg R_I$: in our computations we took $R_I = 108$ and $R_{II} = 65535$. We will discuss in detail the dependence of our algorithm on the parameters R_I and R_{II}.

We remark that our method provides a lower bound on the Lindstedt radius, which is actually different (usually smaller than) the break–down threshold. More precisely, an analytic torus with frequency ω is invariant for the Hamiltonian (1), if there exists an

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[(†)] Actually the Hamiltonian formula (4) is not of the same type of the Thirring model; however the proof presented in [19] can be easily adapted to this case.

analytic conjugating function $u : \mathbf{T}^2 \mapsto \mathbf{T}$ satisfying

$$D^2 u(\vartheta, t) - \varepsilon[\sin(u(\vartheta, t) + \vartheta) + \sin(u(\vartheta, t) + \vartheta + t)] = 0 \ , \tag{6}$$

where $D \equiv \omega\partial_\vartheta + \partial_t$. The break–down threshold, say $\varepsilon_c(\omega)$, is defined as the supremum over the values of ε such that (6) is satisfied for any analytic function u . Equation (6) can be studied perturbatively by expanding u in Fourier–Taylor series as

$$u(\vartheta, t; \varepsilon) = \sum_{n=1}^{\infty} u_n(\vartheta, t)\varepsilon^n = \sum_{n=1}^{\infty} \sum_{k \in \mathbf{Z}^2} \hat{u}_{n,k} \exp(ik \cdot (\vartheta, t))\varepsilon^n \ . \tag{7}$$

We refer to (7) as the *Lindstedt series*. The *Lindstedt radius* is defined as

$$\varrho(\omega) = \inf_{(\vartheta,t) \in \mathbf{T}^2} \left(\limsup_{n \to \infty} |u_n(\vartheta, t)|^{1/n} \right)^{-1} \ . \tag{8}$$

One immediately gets $\varrho(\omega) \le \varepsilon_c(\omega)$ (examples for which $\varrho(\omega) < \varepsilon_c(\omega)$ are provided in [3]). A proof of the convergence of the Lindstedt series for invariant KAM tori can be found in [12], [13], [14], [15] and references therein. The equivalence between the expansions of the Lindstedt series and that provided by the Lie series leading to the Kolmogorov normal form is shown in [19]. In (ii) we derive estimates on the Taylor coefficients of the series expansion in ε, namely we compute $\sup_{(\vartheta,t) \in \mathbf{T}^2} |u_n(\vartheta, t)|$ as in (8). Therefore our estimate on ε cannot be bigger than the Lindstedt radius $\varrho(\omega)$. However it is expected that for the forced pendulum $\varrho(\frac{\sqrt{5}-1}{2}) = \varepsilon_c(\frac{\sqrt{5}-1}{2})$ (see [4], where numerical evidence of the previous equality is provided for the standard map). Recent numerical results for the computation of the break–down threshold using renormalization techniques have been developed in [10], [11], [20].
We conclude this introduction by mentioning that in view of the results obtained in [8] for continuous and discrete systems, the application of our algorithm to the standard mapping would likely provide results which are probably extremely close to the numerical expectations. It would be also interesting to apply our algorithm to mathematical models of physical interest, like the restricted three body problem, for which the existing estimates ([9]) are still far from the astronomical observations. We defer this projects to later studies.

This paper is organized as follows. The description of the algorithm is presented in §2. The dependency on the parameters R_I, R_{II} is discussed in §3. Conclusions are drawn in §4. Technical details are collected in the appendixes.

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2. Description of the computer–assisted proof

In this section we present in detail how to perform steps (i)–(iii) discussed in the Introduction.

2.1 Formal algorithm

In order to use the formalism of the autonomous canonical transformations, let us extend the phase space by introducing the canonically conjugated variables (p_2, q_2), where $q_2 = t$. Replacing (y, x) with $(p_1 + \omega, q_1)$, Hamiltonian (1) becomes

$$H^{(0)}(p,q) = \omega p_1 + p_2 + p_1^2/2 + \varepsilon\,[\cos q_1 + \cos(q_1 - q_2)] \ , \tag{9}$$

where $(p_1, p_2) \in \mathbf{R}^2$, $(q_1, q_2) \in \mathbf{T}^2$, ω is a fixed frequency and we neglected an unessential additive constant. Following [25] (see also [2], [18] and [29]), we perform a sequence of canonical changes of variables transforming the original Hamiltonian in Kolmogorov normal form as

$$H^{(\infty)}(p,q) = \omega p_1 + p_2 + \mathcal{R}(p_1, q) \ , \tag{10}$$

where $\mathcal{R} = \mathcal{O}(p_1^2)$. Obviously the torus $\{p = 0\}$ is invariant for $H^{(\infty)}$ and the angular frequencies of the motion are constant: $\dot{q}_1 = \omega$ and $\dot{q}_2 = 1$.

Let us introduce some further notation in order to describe the iterative algorithm leading to the Kolmogorov normal form. For any $k \in \mathbf{Z}^2$, let $|k| = \max_j |k_j|$. We denote by $\mathcal{P}_{l,s}$ the set of functions which are monomials of degree l in p_1 and trigonometric polynomials of degree s. Then we can expand $g \in \mathcal{P}_{l,s}$ as

$$g(p_1, q) = \sum_{|k| \le s} c_k\, p_1^l\, \exp(ik \cdot q) \ . \tag{11}$$

We proceed by induction on the order r of normalization. Suppose that we have performed $r-1$ normalization steps and write the transformed Hamiltonian as

$$H^{(r-1)}(p,q) = \omega p_1 + p_2 + \sum_{s=0}^{r-1} \varepsilon^s h^{(s)}(p_1, q) + \sum_{s=r}^{\infty} \sum_{l=0}^{2} \varepsilon^s f_l^{(r-1,s)}(p_1, q) \ , \tag{12}$$

where $h^{(s)} \in \mathcal{P}_{2,s}$ and $f_l^{(r-1,s)} \in \mathcal{P}_{l,s}$. The fact that functions $h^{(s)}$ and $f_l^{(r-1,s)}$ are independent of p_2 is a consequence of the algorithm. Notice that for $r = 1$, the above Hamiltonian reduces to (9). The new Hamiltonian is obtained from the old one by the application of

two Lie series [‡]

$$H^{(r)} \equiv \exp\left(\varepsilon^r L_{\chi_2^{(r)}}\right)\left[\exp\left(\varepsilon^r L_{\chi_1^{(r)}}\right) H^{(r-1)}\right] , \tag{13}$$

where $\chi_2^{(r)}$ and $\chi_1^{(r)}$ are determined so that the terms $\mathcal{O}(\varepsilon^r)$ in $H^{(r)}$ are quadratic functions of p_1 . More precisely, we set $\chi_1^{(r)}(q) = X^{(r)}(q) + \xi^{(r)} q_1$, where $X^{(r)} \in \mathcal{P}_{0,r}$ and $\xi^{(r)} \in \mathbf{R}$ are solutions of

$$\begin{aligned} \left(\omega \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right) X^{(r)}(q) + f_0^{(r-1,r)}(q) = 0 \ , \\ \{\xi^{(r)} q_1, h^{(0)}(p_1)\} + \langle f_1^{(r-1,r)} \rangle = 0 \ ; \end{aligned} \tag{14}$$

here we denoted by $\langle \cdot \rangle$ the average over the angle variables. The generating function $\chi_2^{(r)} \in \mathcal{P}_{1,r}$ is determined as the solution of the equation

$$\left(\omega \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right) \chi_2^{(r)}(p_1, q) + L_{X^{(r)}} h^{(0)}(p_1) + f_1^{(r-1,r)}(p_1, q) - \langle f_1^{(r-1,r)} \rangle = 0 \ . \tag{15}$$

The expression of the Hamiltonian after the application of the first Lie series becomes

$$\exp\left(\varepsilon^r L_{\chi_1^{(r)}}\right) H^{(r-1)} = \omega p_1 + p_2 + \sum_{s=0}^{r-1} \varepsilon^s h^{(s)}(p_1, q) + \sum_{s=r}^{\infty} \sum_{l=0}^{2} \varepsilon^s \hat{f}_l^{(r,s)}(p_1, q) \ , \tag{16}$$

where the recursive equations providing $\hat{f}_l^{(r,s)} \in \mathcal{P}_{l,s}$ in terms of $\chi_1^{(r)}$, $h^{(s)}$, $f_l^{(r-1,s)}$ can be easily derived taking all terms having the same order in ε and the same degree in p_1 . We collect the recursive formulæ in appendix B. For simplicity of notation we do not distinguish anymore between old and new coordinates.
Analogously, after the application of the second Lie series

$$H^{(r)}(p, q) = \omega p_1 + p_2 + \sum_{s=0}^{r} \varepsilon^s h^{(s)}(p_1, q) + \sum_{s=r+1}^{\infty} \sum_{l=0}^{2} \varepsilon^s f_l^{(r,s)}(p_1, q) \ . \tag{17}$$

The explicit expressions for $h^{(r)} \in \mathcal{P}_{2,r}$ and $f_l^{(r,s)} \in \mathcal{P}_{l,s}$ in terms of $\chi_2^{(r)}$ and of the functions appearing in (16) are easily derived (see appendix B). The term $h^{(r)}$ in (17) is the new term of order $\mathcal{O}(\varepsilon^r)$ appearing in Kolmogorov normal form. The remark that Hamiltonian (17) can be reduced to (12) by substituting r with $r+1$ closes the recursive procedure.

[‡] Within the aims of this work, it is sufficient to recall the following elementary notions. The flow induced by a generic Hamiltonian χ is a near to identity canonical transformation. Consider two different coordinates systems, (p, q) and (p', q'), such that $(p, q) = \Phi^\varepsilon(p', q')$ where Φ^ε is the flow at time ε of the canonical vector field generated by χ. Thus, any function $g(p, q)$ is expressed in terms of the new coordinates (p', q') as $g'(p', q') = \exp(\varepsilon L_\chi) g(p, q) = \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} L_\chi^j g(p, q)$, where $L_\chi \cdot = \{\chi, \cdot\}$ and $\{\cdot, \cdot\}$ are the Poisson brackets. An exhaustive review on Lie series can be found, for example, in [23] and [16].

The coding of the formal algorithm on computer is the translation in a programming language of the formulæ appearing in this section and in appendix B. Therefore we calculated up to the order ε^{R_I} (for a fixed integer R_I) the truncated expansions of $H^{(1)}$, ..., $H^{(R_I)}$. More precisely, using interval arithmetics (see Appendix A) we computed the Fourier coefficients of $\chi_1^{(1)}$, $\chi_2^{(1)}$, ..., $\chi_1^{(R_I)}$, $\chi_2^{(R_I)}$, $h^{(1)}$, ..., $h^{(R_I)}$ and of $\hat{f}_l^{(r,s)}$, $f_l^{(r,s)}$ with $0 \le l \le 2$, $1 \le r \le R_I$ and $r \le s \le R_I$. In our computations we used $R_I = 108$.

We stress that the formal algorithm for the general case of a slightly perturbed analytic Hamiltonian $H(p,q) = H_0(p) + \varepsilon H_1(p,q)$ in n degrees of freedom is a trivial generalization of the previous one. Explicit formulæ can be found in [18].

2.2 Iteration of the estimates

We want to find recursive estimates for the functions $h^{(s)}$, $f_l^{(r,s)}$ appearing in (17). Having fixed a positive (hopefully big) integer R_{II}, our aim is to find three finite sequences $\{E_r\}_{r=1}^{R_{II}}$, $\{\eta_r\}_{r=1}^{R_{II}}$ and $\{a_r\}_{r=1}^{R_{II}}$ of positive real numbers satisfying the following inequalities:

$$\left\{ \begin{array}{ll} \left\| h^{(s)} \right\| \le E_r \eta_r^2 \, a_r^s & \forall \; 0 \le s \le r \\ \left\| f_l^{(r,s)} \right\| \le E_r \eta_r^l \, a_r^s & \forall \; s \ge r+1 \text{ e } \forall \; 0 \le l \le 2 \; . \end{array} \right. \tag{18}$$

The iteration of the estimates is performed as follows:

i) Estimate of the truncations up to the order $\varepsilon^{R_{II}}$ of the generating functions $\chi_1^{(1)}$, $\chi_2^{(1)}$, ..., $\chi_1^{(R_{II})}$, $\chi_2^{(R_{II})}$ and of $h^{(s)}$, $f_l^{(r,s)}$, for $1 \le s \le R_{II}$.

ii) Derivation of the upper bounds (18) on the infinite sequence of terms appearing in the expansion (17), for $1 \le r \le R_{II}$.

Before entering details, let us introduce some notations. For any $g \in \mathcal{P}_{l,s}$ as in (11), we introduce the following [(∅)] norm:

$$\|g\| = \sum_{|k| \le s} |c_k| \; . \tag{19}$$

Moreover, let $\gamma > 0$ and $\{\alpha_r\}_{r=1}^{\infty}$ be a sequence of positive numbers such that

$$\min_{0<|k|\le r} |k_1 \omega + k_2| \ge \gamma \alpha_r \; . \tag{20}$$

2.2.1 Estimates on the truncated Hamiltonians

Step i) above can be performed as follows. The idea is to replace the equations (14), (15) (as well as (33) and (34) of Appendix B) with recursive inequalities on the norms. To this end we substitute the second equation of (14) with

$$\left\{ \xi^{(r)} q_1 \; , \; \sum_{s=0}^{r-1} \varepsilon^s \langle h^{(s)} \rangle \right\} + \langle f_1^{(r-1,r)} \rangle = 0 \; . \tag{21}$$

[(∅)] Remark that we avoid considering norms on complex domains. This fact allows us to get better estimates on the KAM threshold.

This represents a minor modification of the technical estimates given, e.g., in [18]; however the improvement of the estimates turns out to be significant.

The first equation (14) and equations (15) and (21) (as well as (33) and (34) of Appendix B) can be easily reformulated as recursive inequalities by successive applications of the following lemmas.

Lemma 1: *Assume that $g \in \mathcal{P}_{l,s}$; then the following inequalities hold:*

$$\begin{aligned} \|L_{X^{(r)}}\, g\| \le l\, r \left\|X^{(r)}\right\| \|g\| \ , \qquad \left\|\frac{1}{j!} L^j_{\chi_1^{(r)}}\, g\right\| \le \binom{l}{j} \left(r \left\|X^{(r)}\right\| + \left|\xi^{(r)}\right|\right)^j \|g\| \ , \\ \left\|\frac{1}{j!} L^j_{\chi_2^{(r)}}\, g\right\| \le \frac{\prod_{i=1}^{j}[(l+i-1)r+s]}{j!} \left\|\chi_2^{(r)}\right\|^j \|g\| \ . \end{aligned} \tag{22}$$

Lemma 2: *Let $\omega \in \mathbf{R} \setminus \mathbf{Q}$, $\gamma > 0$ constant and $\{\alpha_r\}_{r=1}^{\infty}$ be a sequence of positive numbers such that condition (20) is satisfied. Moreover, writing $\sum_{s=0}^{r-1} \varepsilon^s \langle h^{(s)} \rangle = \mathcal{C}^{(r)} p_1^2/2$ with constant $\mathcal{C}^{(r)}$, assume that there exists a positive constant m such that $m \le \left|\mathcal{C}^{(r)}\right|$. Then the following inequalities apply:*

$$\begin{aligned} \left\|X^{(r)}\right\| \le \frac{1}{\gamma \alpha_r} \left\|f_0^{(r-1,r)}\right\| \ , \qquad \left|\xi^{(r)}\right| \le \frac{1}{m} \left\|\langle f_1^{(r-1,r)} \rangle\right\| \ , \\ \left\|\chi_2^{(r)}\right\| \le \frac{1}{\gamma \alpha_r} \left(\left\|f_1^{(r-1,r)}\right\| + 2r \left\|X^{(r)}\right\| \left\|h^{(0)}\right\|\right) \ . \end{aligned} \tag{23}$$

We omit the (simple) proofs of the Lemmas.
The formal algorithm can be translated into a set of recursive inequalities. This is just matter of rewriting the recursive formulæ of Appendix Creplacing everywhere the norms of the functions and the estimates of Lemmas 1 and 2. The resulting recursive estimates are reported in Appendix C.

2.2.2 Estimates on the infinite power series expansions

We now determine the finite sequences E_r , η_r and a_r appearing in (18). To this purpose we assume that we have calculated some constants that bound all the functions $h^{(s)}$, $f_l^{(r,s)}$, $X^{(r)}$, $\xi^{(r)}$ and $\chi_2(r)$ for r , s $\le R_{II}$. These constants are to be calculated making use of the recursive formulæ of Lemma 2 and Appendix C by just replacing the sign $\le$ with $=$. This is an explicit calculation that may be easily implemented on a computer. Then we reduce all these constants to three sequences $\{E_r\}_{r=1}^{R_{II}}$, $\{\eta_r\}_{r=1}^{R_{II}}$ and $\{a_r\}_{r=1}^{R_{II}}$ by applying the following

Lemma 3: *Suppose that inequalities (18) are verified by E_{r-1} , η_{r-1} and a_{r-1} ; then the same inequalities hold to the step r with*

$$E_r = \hat{E}_r \ , \quad \eta_r = \hat{\eta}_r \left(1 + r \left\|\chi_2^{(r)}\right\| \Big/ \hat{a}_r^{\,r}\right) \quad a_r = \hat{a}_r \left(1 + r \left\|\chi_2^{(r)}\right\| \Big/ \hat{a}_r^{\,r}\right)^{1/r} \ ,$$

where

$$\hat{E}_r = E_{r-1} \cdot \left[1 + \frac{2\left(r \left\|X^{(r)}\right\| + \left|\xi^{(r)}\right|\right) \eta_{r-1}}{a_{r-1}^{\ r}} \right]^2$$

$$\hat{\eta}_r = \frac{\eta_{r-1}}{\left[1 + \frac{2\left(r\left\|X^{(r)}\right\| + \left|\xi^{(r)}\right|\right)\eta_{r-1}}{a_{r-1}^{\ r}} \right]}$$

$$\hat{a}_r = a_{r-1} \ .$$

Here the quantities $\|X^{(r)}\|$, $|\xi^{(r)}|$ and $\|\chi_2(r)\|$ are to be replaced by their estimates computed as explained at beginning of this section. The proof of Lemma 3 is obtained by a direct application of Lemma 1 to the inequalities reported in Appendix C.

We complete the algorithm by determining E_1 , η_1 and a_1 . Computing explicitely the first step of the formal algorithm and using the inequalities of Appendix C, one obtains

$$\begin{aligned} &\left\|h^{(1)}\right\| \le \left\|\chi_2^{(1)}\right\| \ , && \left\|f_0^{(1,s)}\right\| \le \frac{s-1}{2} \left\|\chi_2^{(1)}\right\|^{s-2} \left\|X^{(1)}\right\|^2 \ , \\ &\left\|f_1^{(1,s)}\right\| \le (s-1) \left\|\chi_2^{(1)}\right\|^{s-1} \left\|X^{(1)}\right\| \ , && \left\|f_2^{(1,s)}\right\| \le \frac{s+1}{2} \left\|\chi_2^{(1)}\right\|^{s} \ . \end{aligned} \tag{24}$$

Since $R_{II} \gg 1$, a good choice for the parameters E_1 , η_1 and a_1 is provided by

$$E_1 = \left(\left\|X^{(1)}\right\| / \left\|\chi_2^{(1)}\right\|\right)^2 \frac{R_{II}}{e \log R_{II}} \ , \quad \eta_1 = \left\|\chi_2^{(1)}\right\| / \left\|X^{(1)}\right\| \ , \quad a_1 = R_{II}^{1/R_{II}} \left\|\chi_2^{(1)}\right\| \ , \tag{25}$$

where $\|X^{(1)}\|$ and $\|\chi_2^{(1)}\|$ are calculated using the explicit solutions of equations (14) and (15). From (24) and (25), one can easily check that (18) is satisfied for $r = 1$.

The whole calculation described above may be easily implemented on a computer up to some order R_{II} , by first calculating a table of quantities bounding all the estimated functions and then determining the three sequences $\{E_r\}_{r=1}^{R_{II}}$, $\{\eta_r\}_{r=1}^{R_{II}}$ and $\{a_r\}_{r=1}^{R_{II}}$. The computed values $E_{R_{II}}$, $\eta_{R_{II}}$ and $a_{R_{II}}$ will be used in the application of the theorem below.

2.3 Statement of KAM theorem

In order to derive an estimate of the perturbing parameter ε on the existence of invariant tori, we apply the following version of KAM theorem (we refer to [27] for the proof).

Theorem 1: *Let $H^{(R-1)}$ be the Hamiltonian obtained by $H^{(0)}$ after $R-1$ normalization steps. Assume the following hypotheses on the series expansion (17) with $r = R-1$:*

a) $h^{(s)} \in \mathcal{P}_{2,s}$ and $f_l^{(R-1,s)} \in \mathcal{P}_{l,s}$;

b) the inequalities (18) are satisfied with $r = R-1$ for given parameters E_{R-1} , η_{R-1} and a_{R-1} ;

c) ω is a diophantine number, i.e. condition (20) is satisfied with $\alpha_r \ge \frac{1}{r^\tau}$, where $\tau \ge 1$;

d) $\mathcal{C}^{(R)}$, defined by the relation $\mathcal{C}^{(R)} p_1^2/2 = \sum_{s=0}^{R-1} \varepsilon^s \langle h^{(s)} \rangle$, satisfies the following inequalities

$$0 < m' < m \le |\mathcal{C}^{(R)}| \ ,$$

where m' and m are positive numbers;

e) ε satisfies the conditions

$$\varepsilon \le \frac{1}{a'} \qquad \text{and} \qquad \frac{(\varepsilon a')^R}{1-\varepsilon a'} \le \frac{m-m'}{2E_{R-1}\bar\eta^{\,2}} \ , \tag{26}$$

where $\bar\eta = \eta_{R-1}e^{6/R}$ and a' is given by

$$a' = a_{R-1}\,(Z_1\,Z_2)^{2/R}\,e^{3/R^2}2^{(2\tau+12)b_{\bar l}} \ , \tag{27}$$

with[&] *$\bar l = \lfloor \log_2 R \rfloor$ and where the quantities Z_1 , Z_2 and $b_{\bar l}$ are defined by*

$$Z_1 = \max\left\{1, \frac{E_{R-1}\bar\eta}{\gamma}\right\} \ , \quad Z_2 = \max\left\{1, \frac{E_{R-1}\bar\eta^{\,2}}{m'}\right\} \ , \quad b_{\bar l} = \frac{\bar l+2}{2^{\bar l-1}} \ . \tag{28}$$

Then, there exists a canonical analytic transformation ψ transforming $H^{(R-1)}$ in the Kolmogorov normal form (10), namely $H^{(R-1)} \circ \psi = H^{(\infty)}$.

The latter statement is just a reformulation of the KAM theorem, which exploits the fact that the Hamiltonian is in normal form up to terms of order $\mathcal{O}(\varepsilon^R)$ and explicit estimates are available for $H^{(R-1)}$. This is useful in our situation because we actually know the required estimates, that have been determined by either explicit construction of the normal form or by recursive evaluation according to Lemma 3. The final achievement is the determination of the threshold for ε via condition (26).

2.4 Application to the problem of the forced pendulum

Now we describe the application of the theorem to the model (1) for the golden torus. First we performed the explicit construction (as described in section 2.1) up to $R_I = 108$; then we iterated the estimates (as described in section 2.2) up to $R_{II} = 65535$. In order to apply recursively Lemma 2, we computed the values of $\gamma\alpha_r$ for $109 < r < 65535$ (see definition (20)). Concerning the parameter m , we started with $m = 1$ and we checked step by step that condition $|\mathcal{C}^{(r)}| \ge m = 1$ of Lemma 2, is satisfied for all $\varepsilon \in [0, a']$ with a' as in (27). Hence we may fix $m = 1$ and $m' = 1/2$. Taking the values of E_1 , η_1 and a_1 as in (25), the recursive application of Lemma 3 provides $E_{65535} = \exp(45.558)$, $\eta_{65535} = \exp(23.398)$ and $a_{65535} = 38.9447$. Finally, we applied Theorem 1 with $R = 65536$ and the values E_{R-1} , η_{R-1} and a_{R-1} above. Moreover, in view of well known inequalities of diophantine theory, we set $\gamma = \frac{3-\sqrt5}{2}$, $\tau = 1$. A straightforward calculation of the expressions appearing in (26)–(28) provides the following

Corollary 2: *Consider Hamiltonian (1); if $\varepsilon < \varepsilon^* = 0.025375$, then there exists an invariant torus on which a quasiperiodic motion with frequency $\omega = \frac{\sqrt5-1}{2}$ takes place.*

We emphasize, once again, that all the calculations are made with interval arithmetics. Hence, our result is rigorous in the sense of computer–assisted proofs.

[&] *Given a real nonnegative number x , we denote by $\lfloor x \rfloor$ the integer part of x .*

3. The dependence of the algorithm on R_I, R_{II}

The calculation described in section 2 is rigorous, but time demanding. In this section we explore the possibility of foreseeing the final result using the information hidden in the calculation at finite values of R_I and R_{II}. To this purpose, we first try to describe the asymptotic behaviour of the threshold $\varepsilon^*_{R_I,R_{II}}$ with respect to R_I and R_{II}, and then try to find a method of extrapolation. The discussion in this section is clearly not rigorous: since we are interested to the qualitative behaviour, the results contained in this section are based on numerical experiments (without interval arithmetics).

3.1 On the dependency on R_{II}

As remarked before, a larger value of R_{II} implies a better estimate on ε^*. Nevertheless, iterating the estimates is time demanding: the CPU–time can be roughly evaluated as proportional to R_{II}^2. Therefore, we are strongly interested to predict in a short time the value of the limit threshold $\varepsilon^*_{R_I,\infty} \equiv \lim_{R_{II}\to\infty} \varepsilon^*_{R_I,R_{II}}$ that can be obtained applying the algorithm described in the previous sections.

To this end, consider the sequence $\{\|h^{(s)}\|\}_s$, where for $0 \le s \le R_I$ we calculate the norm using the explicit Fourier expansion of $h^{(s)}$ and for $R_I < s \le R_{II}$ we estimate the norms by iterating the inequalities described in section 2.2.1 and Appendix C. The iteration of the estimates turns out to have a very smoothing effect: a slight change of the slope of the curve is observed for $R_I = 108$ (Figure 1b), while for $s \ge R_I$ the growth of $\|h^{(s)}\|$ seems to be strictly geometric (Figure 1a and 1b). This seems to be typical. For instance, a similar behaviour is observed in [26], using a model describing the secular part of the Sun–Jupiter–Saturn system. Moreover, applying the root criterion to the estimates of section 2.2 we guess $\varepsilon^*_{R_I,\infty} = \lim_{s\to\infty}(\|h^{(0)}\| / \|h^{(s)}\|)^{1/s}$. Therefore, due to the regularity of the growth of $\{\|h^{(s)}\|\}_s$, we obtain that $\varepsilon^*_{R_I,\infty}$ can be approximated, e.g., as

$$\varepsilon^*_{R_I,\infty} \approx \left(\left\|h^{(\lfloor R_{II}/2\rfloor)}\right\| \Big/ \left\|h^{(R_{II})}\right\|\right)^{1/(R_{II}-\lfloor R_{II}/2\rfloor)} \qquad \text{for } R_{II} \gg R_I \ , \tag{29}$$

providing a practical criterion to evaluate $\varepsilon^*_{R_I,\infty}$.

The latter criterion may be better justified by looking at the qualitative behaviour of $\varepsilon^*_{R_I,R_{II}}$ as a function of R_{II}, deduced by Theorem 1. The key remark is that the sequence a_r calculated via Lemma 3 tends to the inverse of the convergence radius in ε of the Hamiltonian in Kolmogorov's normal form. This suggests that $\varepsilon^*_{R_I,\infty}$ should be evaluated as $\varepsilon^*_{R_I,\infty} = \lim_{R_{II}\to\infty} a^{-1}_{R_{II}}$. The qualitative behaviour of $\varepsilon^*_{R_I,R_{II}}$ as a function of R_{II} can be guessed by looking at Theorem 1. Replacing in (27) the values of Z_1, Z_2 and $b_{\bar l}$ given by (28), one finds that the dominant term in the ratio between a' and a_{R-1} is of order $\mathcal{O}(R^{1/R})$. Replacing this information in (26) one guesses

$$\varepsilon^*_{R_I,R_{II}} \sim a^{-1}_{R_{II}}[C(R_{II}+1)^\beta]^{-1/(R_{II}+1)} \ , \tag{30}$$

where C and β are some positive constants.

Figures 1c and 1d show that $\varepsilon^*_{R_I,R_{II}}$, as rigorously given by the theorem, $a^{-1}_{R_{II}}$ and the estimate of $\varepsilon^*_{R_I,\infty}$, given by the root criterion (29) seem to converge to the same limit as $R_{II} \to \infty$. In order to have a rough idea of the advantages provided by these numerical methods in terms of computational time, let us compare the root criterion

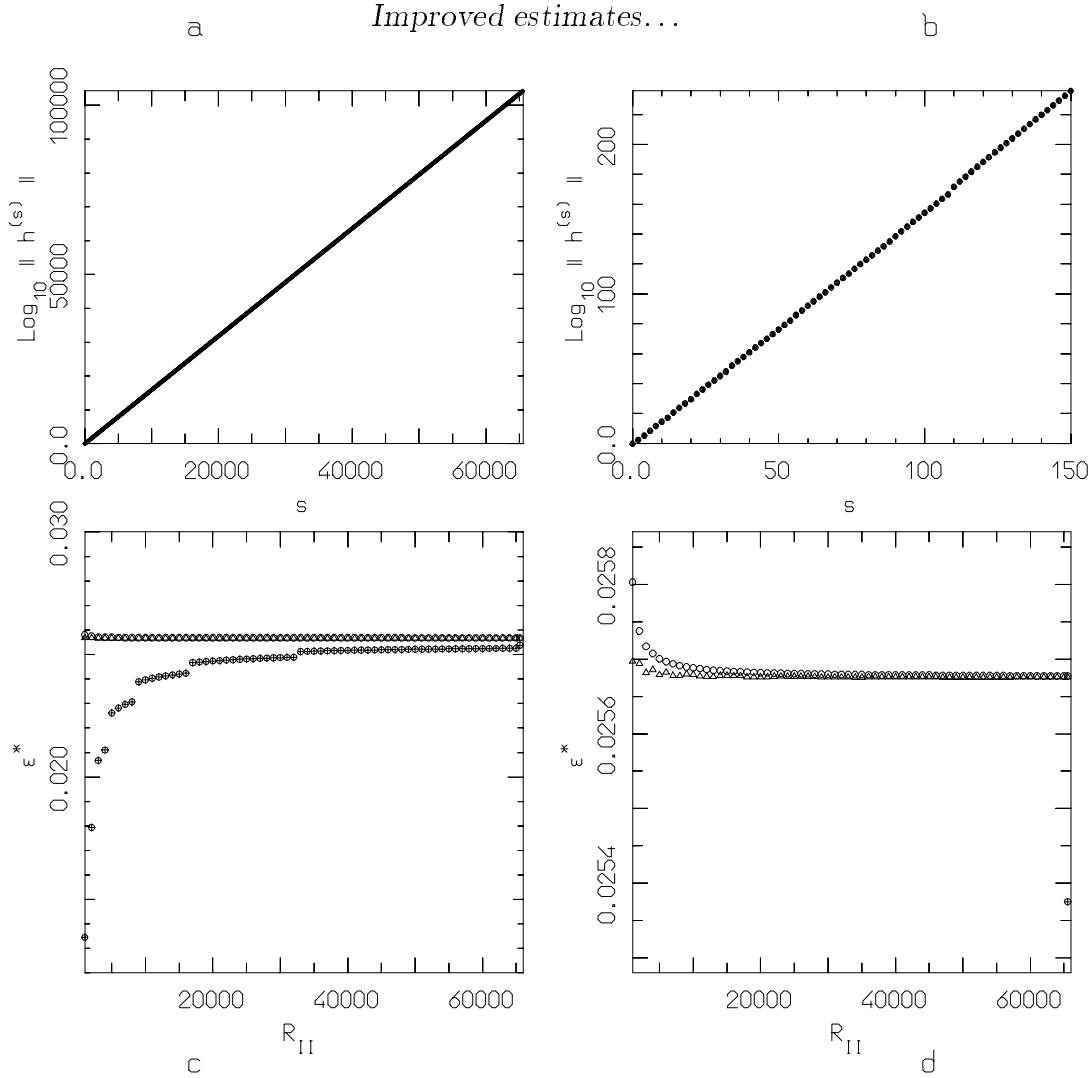

Figure 1. The growth of the sequence $\{\|h^{(s)}\|\}_s$ is reported in Figure 1a. Recall that $\|h^{(s)}\|$ is calculated directly from the Fourier expansion of $h^{(s)}$ for $0 \le s \le R_I \equiv 108$, while the norm is estimated as described in section 2.2.1 for $s > R_I$. In Figure 1c, the dependency of $\varepsilon^*_{R_I,R_{II}}$ on R_{II} is studied (where $R_I = 108$). Symbol $\triangle$ denotes the estimate of $\varepsilon^*_{108,R_{II}}$ by the root criterion (see formula (29)); $\bigcirc$ indicates $1/a_{R_{II}}$, where $a_{R_{II}}$ is obtained by iteration of Lemma 3 up to R_{II}; $\oplus$ corresponds to the value $\varepsilon^*_{108,R_{II}}$ provided by Theorem 1. Figures 1b and 1d are enlargements of Figures 1a and 1c, respectively.

with the rigorous final threshold value given by our algorithm. The latter one needs about 79 hours to calculate $\varepsilon^*_{R_I,65535}$. The computed value agrees within 1.2 % with $\bar{\varepsilon} \equiv (\|h^{(\lfloor R_{II}/2 \rfloor)}\| \,/\, \|h^{(R_{II})}\|)^{1/(R_{II}-\lfloor R_{II}/2 \rfloor)}$, with $R_{II} = 65535$; on the other hand, just 1000 iterations (i.e. 8 sec. of CPU–time) are enough for the root criterion to provide a value in agreement with $\bar{\varepsilon}$ within 0.1 %.

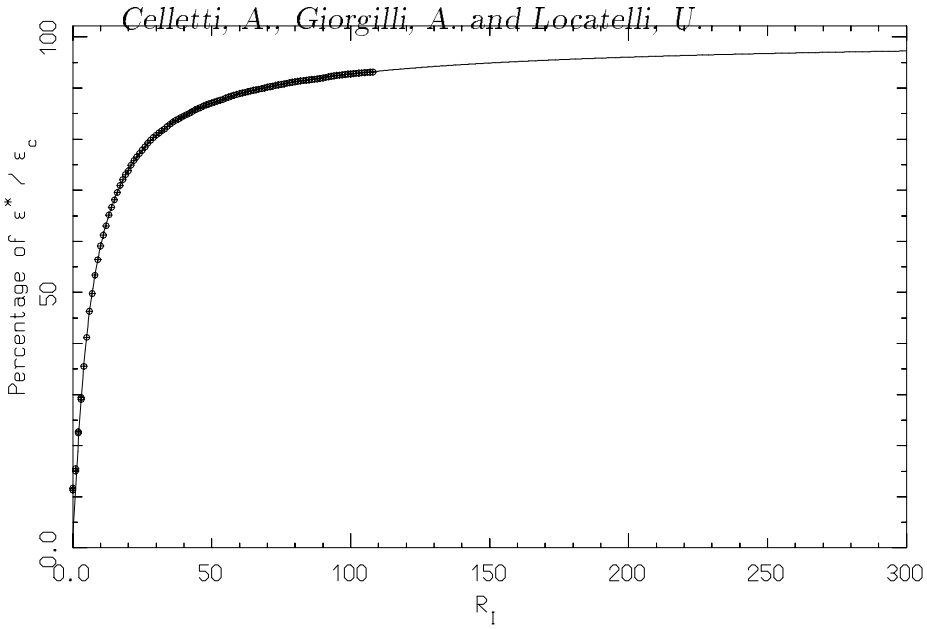

Figure 2. Plot of the ratio $\varepsilon^*_{R_I,\infty}/\varepsilon_c(\frac{\sqrt{5}-1}{2})$ as a function of R_I . $\varepsilon^*_{R_I,\infty}$ is estimated by means of the root criterion (see formula (29), with $R_{II} = 1000$).

3.2 On the dependency on R_I

The expansion in ε of the normalized Hamiltonian can be calculated explicitely up to an order R_I , usually not very large. Therefore, our method is in some sense more sensitive to R_I than to R_{II} . The study of the dependency on R_I may be useful in practical applications. Let us see if we can predict up to which order we need to calculate the expansion of the normalized Hamiltonian in order to get good results on the KAM threshold.

Figure 2 shows a regular plot of $\varepsilon^*_{R_I,\infty}$ provided by the root criterion (29) versus R_I [(\$)]. We interpolate these values using the same asymptotic behaviour of (30), namely (since R_I here is the independent variable)

$$\varepsilon^*_{R_I,\infty} \sim \bar{A}^{\,-1}[\bar{C}\,(R_I+1)^{\bar{\beta}}]^{-1/(R_I+1)} \ , \tag{31}$$

where $\bar{A}$, $\bar{C}$ and $\bar{\beta}$ are positive constants. In Figure 2, the solid curve represents the latter function, where the parameters $\bar{A}$, $\bar{C}$ and $\bar{\beta}$ are determined by a least squares fit. Denoting by σ_N the relative standard deviation of the best fit and by N the number of points, the relative error on $\bar{A}$ is evaluated as $\sigma_N\sqrt{N-2}$. This allows us to get an estimate[(△)] of the limit threshold $\varepsilon^*_{\infty,\infty}$, if the formal algorithm could be performed for an arbitrarily large

[(\$)] The behaviour of the function $\varepsilon^*_{R_I,\infty}$ is certainly dependent on the regularity of the continued fraction expansion of the frequency ω. For this reason the method developed in the present section cannot be extended to any frequency ω . However, one can expect good predictions at least when ω is of the type $\omega = [a_0, \ldots, a_{n-1}, (a_n)^\infty]$ $(a_0, ..., a_n \in \mathbf{Z})$ and R_I is large enough.

[(△)] In the calculation of the best fit, we neglected the first three points, considered too noisy.

number of steps; namely

$$\varepsilon^*_{\infty,\infty} \approx \bar{A}^{\,-1} \approx 0.0276 \ \pm 4 \times 10^{-4} \ . \tag{32}$$

Since from our algorithm one gets a threshold $\varepsilon^*_{R_I,R_{II}} < \varrho(\frac{\sqrt{5}-1}{2})$ (recall the definition (8) of the Lindstedt radius and the related discussion in the Introduction), the agreement between the asymptotic limit (32) and the numerical breakdown threshold (2) allows us to add a few remarks. First, the estimate (32) confirms the expectation that the Lindstedt radius for the golden mean and for the periodically forced pendulum should coincide with the breakdown threshold. Moreover, the extrapolation method outlined in this section could be used to give a value of the Lindstedt radius for rotation frequencies having a continued fraction expansion regular enough. Finally, the agreement between (32) and (2) leads us to conclude that spurious contributions due to technicalities should not have been introduced by our algorithm.

4. Conclusions

We have presented a constructive KAM algorithm based on the composition of Lie series. Since the proof is computer–assisted, the interval arithmetic technique is implemented to get rid of the computer rounding errors. We have been able to prove the existence of the golden torus in the case of the forced pendulum for values of the perturbing parameter ε in agreement within 92 % with the numerical expectation. This result represents a significant improvement with respect to previous KAM estimates. In order to discuss this point we make reference to [7] and [8], where a similar calculation gave an estimate of the breakdown threshold corresponding to the 70 % of the numerical one.

The improvement is mainly due to two technical elements, namely *a*) a more efficient algorithm for the expansions of the series and *b*) a better scheme of the estimates.

As discussed in sect. 2.1 our algorithm is based on the construction of the Kolmogorov's normal form for the Hamiltonian. The algorithm of [7] and [8] is based on an expansion of the equation (6) for the invariant torus in Lindstedt series. The latter approach seems to be more direct, because it avoids all the machinery required by canonical transformations, generating functions and so on. However, the use of the Lie series method for canonical transformations turns out to be definitely more effective if expansions at high order are required. The reason is that Lindstedt's algorithm actually needs an inversion of series, which in turn requires $\mathcal{O}(r^3)$ products of functions in order to perform the expansion up to order ε^r. Conversely, the Lie series method does not require inversion and can be implemented with $\mathcal{O}(r^2 \log r)$ products of functions. Taking into account that the product is the most time–demanding operation, the better performance of the Lie series algorithm is evident. For a detailed discussion of this point see [17].

Concerning the estimates, it is not easy to identify a particular point where our scheme is better. However, two remarks may be relevant. The first one concernes the qualitative behaviour of the estimate as described in (30). The dependence of the estimate on R_{II}, and in particular the exponent $1/R_{II}$, make the use of iterative estimates particularly advantageous. The second remark is concerned with the use of the norm (19). Contrary to the usual schemes we do not need to consider the extension of the functions to a complex

domain. Such an extension usually introduces a further parameter σ – the width of the domain in the complex – that has a quite strong impact on the final estimates. On the other hand, this parameter σ is completely arbitrary and it is not evident at all how to optimize it. Since we have removed it, we do not need any optimization.

We claim that our method could provide remarkable results also in more complicated applications of physical interest; in fact, though the algorithm has been developed for (1), it can be easily extended to a general case.

A. The interval arithmetics

A computer can represent exactly a finite set of numbers, which we denote as the *representable* set $\mathcal{R}$. The content of interval arithmetics is to substitute any "real" number r, with the smallest interval containing r and whose end–points are in $\mathcal{R}$. Therefore, operations among numbers are replaced by operations among intervals.

Let additions, subtractions, multiplications and divisions be denoted as *elementary operations*. The result of an elementary operation does not belong to $\mathcal{R}$, since the computer will, in general, round such result. Let us denote by $Up(a)$, $Down(a)$ some upper and lower bounds on a, such that $a \in (Down(a), Up(a))$. Elementary operations are therefore replaced by operations between intervals as follows: if $a \in (a_-, a_+)$ and $b \in (b_-, b_+)$ where $a, b \in \mathbf{R}$ and $a_\pm, b_\pm \in \mathcal{R}$, then

$$a + b \in (\mathrm{Down}(a_- + b_-), \mathrm{Up}(a_+ + b_+)) \equiv (a_-, a_+) + (b_-, b_+) \ ,$$

which defines the addition between intervals. The other elementary operations are treated in the same way. Other functions, like logarithms, exponentials, trigonometric functions etc., will be approximated by a finite sequence of elementary operations, using a truncation of the Taylor series expansion and simple inequalities to estimate the remainder.

In order to compute the upper and lower bounds on a representable number we proceed as follows. Double precision arithmetics guarantees that the results of elementary operations are exact up to the 16th decimal digit. To obtain $Up(a)$ and $Down(a)$ it suffices to multiply the number a by $1 \pm \eta$, where η is of the order of 10^{-16} and is computed according to the approximation rules adopted by the machine.

B. Recursive relations for constructing the Kolmogorov normal form

Recall that the Lie series of a generic function g is given by $\exp(\varepsilon^r L_{\chi_1^{(r)}}) g = \sum_{j=0}^{\infty} \frac{\varepsilon^{jr}}{j!} L^j_{\chi_1^{(r)}} g$, where $L_{\chi_1^{(r)}} \cdot = \{\chi_1^{(r)}, \cdot\}$. In order to derive the explicit expressions of the functions $\hat{f}_l^{(r,s)}$ appearing in (16), we collect all terms of order $\mathcal{O}(\varepsilon^s)$ and degree l in

p_1 .

$$
\begin{aligned}
\hat{f}_0^{(r,r)} &= L_{\chi_1^{(r)}}(\omega p_1 + p_2) + f_0^{(r-1,r)} = 0 && , \\
\hat{f}_0^{(r,r+m)} &= f_0^{(r-1,r+m)} && \forall\ 0 < m < r\ , \\
\hat{f}_0^{(r,2r+m)} &= \frac{1}{2} L^2_{\chi_1^{(r)}} h^{(m)} + L_{\chi_1^{(r)}} f_1^{(r-1,r+m)} + f_0^{(r-1,2r+m)} && \forall\ 0 \le m < r\ , \\
\hat{f}_0^{(r,kr+m)} &= \frac{1}{2} L^2_{\chi_1^{(r)}} f_2^{(r-1,(k-2)r+m)} + L_{\chi_1^{(r)}} f_1^{(r-1,(k-1)r+m)} + f_0^{(r-1,kr+m)} \\
& && \forall\ 0 \le m < r\ ,\ k \ge 3\ , \\
\hat{f}_1^{(r,r+m)} &= L_{\chi_1^{(r)}} h^{(m)} + f_1^{(r-1,r+m)} && \forall\ 0 \le m < r\ , \\
\hat{f}_1^{(r,kr+m)} &= L_{\chi_1^{(r)}} f_2^{(r-1,(k-1)r+m)} + f_1^{(r-1,kr+m)} && \forall\ 0 \le m < r\ ,\ k \ge 2\ , \\
\hat{f}_2^{(r,s)} &= f_2^{(r-1,s)} && \forall\ s \ge r\ .
\end{aligned}
\tag{33}
$$

Use (14) to get the generating function $\chi_1^{(r)}$. In a similar way, one can derive the following expressions for the functions $f_l^{(r,s)}$ appearing in (17):

$$
\begin{aligned}
f_0^{(r,kr+m)} &= \sum_{j=0}^{k-1} \frac{1}{j!} L^j_{\chi_2^{(r)}} \hat{f}_0^{(r,(k-j)r+m)} && \forall\ 0 \le m < r\ ,\ k \ge 1\ , \\
f_1^{(r,kr+m)} &= \sum_{j=0}^{k-1} \frac{1}{j!} L^j_{\chi_2^{(r)}} \hat{f}_1^{(r,(k-j)r+m)} && \forall\ 0 < m < r\ ,\ k \ge 1\ , \\
f_1^{(r,kr)} &= \frac{k-1}{k!} L^{k-1}_{\chi_2^{(r)}} \hat{f}_1^{(r,r)} + \sum_{j=0}^{k-2} L^j_{\chi_2^{(r)}} \hat{f}_1^{(r,(k-j)r)} && \forall\ k \ge 1\ , \\
f_2^{(r,kr+m)} &= \frac{1}{k!} L^k_{\chi_2^{(r)}} h^{(m)} + \sum_{j=0}^{k-1} \frac{1}{j!} L^j_{\chi_2^{(r)}} \hat{f}_2^{(r,(k-j)r+m)} && \forall\ 0 \le m < r\ ,\ k \ge 1\ .
\end{aligned}
\tag{34}
$$

Use (15) to compute the generating function $\chi_2^{(r)}$ and set $h^{(r)} = f_2^{(r,r)}$.

C. Iterative estimates on the norms of the truncated Hamiltonians

Applying repeatedly Lemma 1 and using equations (33), we get following relations:

$$
\begin{aligned}
\left\| \hat{f}_0^{(r,r)} \right\| &= 0 \qquad , \\
\left\| \hat{f}_0^{(r,r+m)} \right\| &= \left\| f_0^{(r-1,r+m)} \right\| \qquad \forall\ 0 < m < r\ , \\
\left\| \hat{f}_0^{(r,2r+m)} \right\| &\leq \left(r \left\| X^{(r)} \right\| + \left| \xi^{(r)} \right| \right)^2 \left\| h^{(m)} \right\| + \left(r \left\| X^{(r)} \right\| + \left| \xi^{(r)} \right| \right) \left\| f_1^{(r-1,r+m)} \right\| \\
&\quad + \left\| f_0^{(r-1,2r+m)} \right\| \qquad \forall\ 0 \leq m < r\ , \\
\left\| \hat{f}_0^{(r,kr+m)} \right\| &\leq \left(r \left\| X^{(r)} \right\| + \left| \xi^{(r)} \right| \right)^2 \left\| f_2^{(r-1,(k-2)r+m)} \right\| \\
&\quad + \left(r \left\| X^{(r)} \right\| + \left| \xi^{(r)} \right| \right) \left\| f_1^{(r-1,(k-1)r+m)} \right\| + \left\| f_0^{(r-1,kr+m)} \right\| \\
&\qquad \forall\ 0 \leq m < r\ ,\ k \geq 3\ ,
\end{aligned}
$$

$$
\begin{aligned}
\left\| \hat{f}_1^{(r,r+m)} \right\| &\leq 2 \left(r \left\| X^{(r)} \right\| + \left| \xi^{(r)} \right| \right) \left\| h^{(m)} \right\| + \left\| f_1^{(r-1,r+m)} \right\| \qquad \forall\ 0 \leq m < r\ , \\
\left\| \hat{f}_1^{(r,kr+m)} \right\| &\leq 2 \left(r \left\| X^{(r)} \right\| + \left| \xi^{(r)} \right| \right) \left\| f_2^{(r-1,(k-1)r+m)} \right\| + \left\| f_1^{(r-1,kr+m)} \right\| \\
&\qquad \forall\ 0 \leq m < r\ ,\ k \geq 2\ ,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\left\| \langle \hat{f}_1^{(r,r)} \rangle \right\| &= 0 \qquad , \\
\left\| \langle \hat{f}_1^{(r,r+m)} \rangle \right\| &\leq 2r \left\| X^{(r)} \right\| \left\| h^{(m)} \right\| + \left\| \langle f_1^{(r-1,r+m)} \rangle \right\| \qquad \forall\ 0 \leq m < r\ ,\ r > R_I\ , \\
\left\| \langle \hat{f}_1^{(r,r+m)} \rangle \right\| &\leq 2r \left\| X^{(r)} \right\| \left\| h^{(m)} \right\| + 2 \left| \xi^{(r)} \right| \left\| \langle h^{(m)} \rangle \right\| + \left\| \langle f_1^{(r-1,r+m)} \rangle \right\| \\
&\qquad \forall\ R_I/2 < r \leq R_I\ ,\ R_I - r < m < r\ , \\
\left\| \langle \hat{f}_1^{(r,s)} \rangle \right\| &\leq \left\| \hat{f}_1^{(r,s)} \right\| \qquad \forall\ s \geq 2r\ , \\
\left\| \hat{f}_2^{(r,s)} \right\| &= \left\| f_2^{(r-1,s)} \right\| \qquad \forall\ s \geq r\ .
\end{aligned}
$$

Analogously, using again Lemma 1 and the equations (34), we get:

$$\left\| f_0^{(r,kr+m)} \right\| \leq \sum_{j=0}^{k-1} \frac{\prod_{i=0}^{j-1}[(k-i-1)r+m]}{j!} \left\| \chi_2^{(r)} \right\|^j \left\| \hat{f}_0^{(r,(k-j)r+m)} \right\| \qquad \forall\ 0 \leq m < r\ ,\ k \geq 1\ ,$$

$$\left\| f_1^{(r,r)} \right\| = 0 \qquad ,$$

$$\left\| f_1^{(r,kr+m)} \right\| \leq \sum_{j=0}^{k-1} \frac{\prod_{i=0}^{j-1}[(k-i)r+m]}{j!} \left\| \chi_2^{(r)} \right\|^j \left\| \hat{f}_1^{(r,(k-j)r+m)} \right\| \qquad \forall\ 0 \leq m < r\ ,\ k \geq 1\ ,\ kr+m > r\ ,$$

(36)

$$\left\| \langle f_1^{(r,s)} \rangle \right\| \leq \left\| \langle \hat{f}_1^{(r,s)} \rangle \right\| \qquad \forall\ r \leq s < 2r\ ,$$

$$\left\| \langle f_1^{(r,s)} \rangle \right\| \leq \left\| f_1^{(r,s)} \right\| \qquad \forall\ s \geq 2r\ ,$$

$$\left\| f_2^{(r,kr+m)} \right\| \leq \frac{\prod_{i=0}^{k-1}[(k-i+1)r+m]}{k!} \left\| \chi_2^{(r)} \right\|^k \left\| h^{(m)} \right\| + \sum_{j=0}^{k-1} \frac{\prod_{i=0}^{j-1}[(k-i+1)r+m]}{j!} \left\| \chi_2^{(r)} \right\|^j \left\| \hat{f}_2^{(r,(k-j)r+m)} \right\| \qquad \forall\ 0 \leq m < r\ ,\ k \geq 1\ .$$

Finally, we need also the inequality $\|h^{(r)}\| \leq \|f_2^{(r,r)}\|$ to complete the inductive argument.

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