DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **7**, Number **2**, March **2007**

Website: http://AIMsciences.org
pp. **377–398**

INVARIANT TORI IN THE SUN–JUPITER–SATURN SYSTEM

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(Communicated by Àngel Jorba)

Abstract. We discuss the applicability of Kolmogorov's theorem on existence of invariant tori to the real Sun–Jupiter–Saturn system. Using computer algebra, we construct a Kolmogorov's normal form defined in a neighborhood of the actual orbit in the phase space, giving a sharp evidence of the convergence of the algorithm. If not a rigorous proof, we consider our calculation as a strong indication that Kolmogorov's theorem applies to the motion of the two biggest planets of our solar system.

1. **Introduction.** We reconsider the problem of the applicability of KAM theory (see [13], [22] and [1]) to physical systems and, in particular, to planetary systems. As it is well known, the announcement of Kolmogorov's theorem has been immediately considered by the scientific community as a major step towards the solution of the classical problem of the stability of planetary motions. On the other hand, the results of several numerical calculations of the orbits seem to indicate that the applicability of the theorem to the whole solar system is an hopeless task. Indeed, Kolmogorov's theorem states that for most initial conditions the orbits of the system should be quasi-periodic with fixed frequencies, while the numerical calculations strongly indicate that the actual orbits exhibit a limited chaotic behavior which is superimposed to the quasi-periodic motion (see, e.g., [28], [15] and [23]). As a matter of fact, a rigorous check of the applicability of Kolmogorov's theorem at least to the main part of our solar system is still lacking.

In this paper, we make a further step in this research field by considering the Sun–Jupiter–Saturn (hereafter SJS) system in the framework of the general problem of three bodies. For this particular case, *we produce a strong evidence of the existence of KAM tori, which are close to the actual orbits of the two biggest planets of our solar system.*

Our work is inspired to the method of proof known as "computer assisted" applied to nearly integrable Hamiltonian systems, as suggested by Gallavotti and

2000 *Mathematics Subject Classification.* Primary: 70F07; Secondary: 70F10, 37J40, 37N05, 70–08, 70H08.

Key words and phrases. Three–body problem, KAM theory, perturbation methods, Hamiltonian systems, Celestial Mechanics.

firstly applied in [6] (see [4] for a recent application of this kind of approach to a problem in Celestial Mechanics). The goal is typically to improve the threshold of the applicability of some rigorous results on perturbation theory, in particular Kolmogorov's theorem, so that the statement applies to physically interesting systems. The idea is to use the classical perturbation methods, in our case the normal form method, in order to reduce the size of the perturbation, and then apply analytical estimates. Roughly speaking, using action–angle variables $(\underline{p},\underline{q}) \in \mathcal{G} \times \mathbf{T}^n$, with $\mathcal{G}$ an open subset of $\mathbf{R}^n$, consider a Hamiltonian of the general form

$$H(\underline{p},\underline{q}) = h(\underline{p}) + \varepsilon f(\underline{p},\underline{q}) \ , \tag{1}$$

i.e., with a perturbation of order ε . Kolmogorov's theorem states that such a Hamiltonian possesses many invariant tori provided $\varepsilon < \varepsilon^*$, with some positive ε^*, the estimates of which turns out to be usually ridiculously small if applied to actual physical systems. Apply now the normal form method, in order to change the Hamiltonian to

$$H(\underline{p}',\underline{q}') = h(\underline{p}') + \varepsilon Z(\underline{p}',\underline{q}') + \varepsilon^R f'(\underline{p}',\underline{q}') \ , \tag{2}$$

with some normalization step $R \geq 1$, so that Z is in normal form and the size of the perturbation is reduced to ε^R . Then, the hypothesis to be fulfilled can be roughly written as $\varepsilon^R < \varepsilon^*$. Thus, the range of ε for which the theorem applies may be considerably enlarged, possibly ending up with realistic values. The method is computer assisted in the sense that the calculation of the normal form is performed by using an algebraic manipulation program. A fully rigorous proof can be achieved by using interval arithmetics.

In our case, the normal form we are looking for is that of Kolmogorov. Precisely, after some preparation, the starting Hamiltonian is given the form

$$H(\underline{p},\underline{q}) = \underline{\omega}^* \cdot \underline{p} + \varepsilon\big(A(\underline{q}) + \underline{B}(\underline{q}) \cdot \underline{p}\big) + \mathcal{O}(\|\underline{p}\|^2) \ , \tag{3}$$

with known functions $A(\underline{q})$, $\underline{B}(\underline{q})$. Here, $\underline{\omega}^*$ is the vector of the unperturbed frequencies. The Hamiltonian is said to be in Kolmogorov's normal form in case $A(\underline{q}) = 0$ and $\underline{B}(\underline{q}) = 0$. In the latter case the torus $\underline{p} = 0$ is clearly invariant, the flow on it being quasi–periodic.

The transformation to Kolmogorov's normal form consists in killing the unwanted terms $A(\underline{q})+\underline{B}(\underline{q})\cdot\underline{p}$. To this aim, we use a reformulation of Kolmogorov's algorithm in a form similar to the classical expansions in the perturbation parameter ε . After R steps the Hamiltonian takes the partial normal form

$$H^{(R)}(\underline{p},\underline{q}) = \underline{\omega}^* \cdot \underline{p} + \varepsilon^R\big(A^{(R)}(\underline{q}) + \underline{B}^{(R)}(\underline{q}) \cdot \underline{p}\big) + \mathcal{O}(\|\underline{p}\|^2) \ , \tag{4}$$

where the relevant fact is that the size of the unwanted terms is reduced to $\mathcal{O}(\varepsilon^R)$. The convergence of our procedure for $R \to \infty$ and ε small enough is proved in [11] and [12].

As a matter of fact, a complete application of the computer assisted method sketched above to the SJS system is still beyond our limits, because the amount of algebraic calculations required definitely exceeds the computer power available to us. Thus, we take the pragmatical attitude of choosing a value of the normalization step R that we can reach and checking that the size of the generating functions of the canonical transformations to normal form decreases geometrically as predicted by the theory. Although this cannot be taken as a rigorous proof, we consider it as a strong indication of the applicability of Kolmogorov's theorem to the SJS system in a neighborhood of the actual orbits of the planets.

The main difficulties in working out our program are due to *(a)* the degeneration of the Keplerian motion, and *(b)* the strong effect of the closeness of the frequencies of Jupiter and Saturn to the 5 : 2 resonance, i.e., the so called great inequality. The first problem has been widely discussed in [21], where a result similar to the present one was achieved by reducing the planetary masses by a factor 10 and by slightly changing the semi-major axis of the external planet. The second problem is overcome in the present paper, as explained in detail in Section 3 below.

The present work is somehow related to previous results that are essentially based on the work by Arnold ([2]), on the existence of invariant tori in planetary systems. The proof given by Arnold requires that *(a)* all the motions are coplanar, *(b)* the distances between the planets are extremely large, *(c)* the masses and the eccentricities of the planets are small enough. More recently, using a numerical assisted method, Robutel ([27]) has shown that the conditions *(a)* and *(b)* can be removed, thus considering the spatial case and the actual values of the semi–major axes. A detailed analytical proof of Arnold's theorem subject only to the condition *(c)* above has been given by Fejoz ([7]). However, in all these works no attention is paid to get a realistic value for the threshold of applicability on the masses and on the eccentricities.

The paper is organized as follows. In Section 2 we briefly recall the Hamiltonian of the three bodies and the use of Poincaré variables. In Section 3 we work out all the necessary canonical transformations in order to well approximate the motion of the secular variables by using just the secular part of the Hamiltonian. This part contains the major changes with respect to the previous paper [21], and it is essential in order to successfully carry out the whole algorithm. In Section 4, we give the Hamiltonian a suitable form for starting the Kolmogorov's normalization algorithm, by introducing appropriate action–angle variables for the secular coordinates. The construction of the invariant torus is described in Section 5. Finally, Section 6 contains the explicit application to the SJS system and some tests which validate our results. We tried to make the paper self–consistent by describing all the relevant steps of the procedure. The general scheme for applying the Kolmogorov's theorem to a planetary system in the framework of the problem of n bodies is illustrated in great detail in our previous paper [21].

2. **The Hamiltonian of the problem of three bodies.**

We consider the classical planetary problem of three bodies P_0, P_1, P_2, with masses m_0, m_1, m_2, interacting according to Newton's gravitational law. In this section we recall how the classical Poincaré variables are introduced, and perform a first expansion of the Hamiltonian around circular and horizontal orbits, i.e., orbits having zero eccentricity and inclination.

We follow here the formalism introduced by Poincaré (see [24] and [25]). A modern exposition can be found in [14] and [18]. We remove the motion of the center of mass by using heliocentric coordinates $\underline{r}_j = \overrightarrow{P_0P_j}$, with $j = 1,\,2$. Denoting by $\tilde{\underline{r}}_j$ the momenta conjugated to $\underline{r}_j$, the Hamiltonian of the system can be written as

$$H(\tilde{\underline{r}},\underline{r}) = T^{(0)}(\tilde{\underline{r}}) + U^{(0)}(\underline{r}) + T^{(1)}(\tilde{\underline{r}}) + U^{(1)}(\underline{r}) \ , \tag{5}$$

where

$$T^{(0)}(\underline{\tilde{r}}) = \tfrac{1}{2}\textstyle\sum_{j=1}^{2}\|\underline{\tilde{r}}_j\|^2\big(\frac{1}{m_j}+\frac{1}{m_0}\big)\ , \qquad T^{(1)}(\underline{\tilde{r}}) = \underline{\tilde{r}}_1\cdot\underline{\tilde{r}}_2\,/\,m_0\ ,$$

$$U^{(0)}(\underline{r}) = -\mathcal{G}\textstyle\sum_{j=1}^{2}\frac{m_0\,m_j}{\|\underline{r}_j\|}\ , \qquad U^{(1)}(\underline{r}) = -\mathcal{G}\frac{m_1\,m_2}{\|\underline{r}_1-\underline{r}_2\|}\ .$$

We now proceed to the *reduction of the angular momentum.* A simple geometrical argument (see, e.g., sect. 5.1 of [14]) shows that the longitudes of the nodes Ω_1, Ω_2 as defined with respect to the *invariant plane* (i.e. the plane orthogonal to the angular momentum $\underline{C}$) are always opposite. In other words, the relation $\Omega_1=\Omega_2+\pi$ holds true along the motion of the three bodies. This allows us to remove the inclinations and the longitudes of the nodes from the Hamiltonian, so that the actual number of degrees of freedom is reduced to 4. The system is conveniently described by the reduced set of Poincaré's canonical coordinates

$$\Lambda_j = \frac{m_0\,m_j}{m_0+m_j}\sqrt{\mathcal{G}(m_0+m_j)a_j}\ , \qquad \lambda_j = M_j+\omega_j\ ,$$
$$\xi_j = \sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\,\cos\omega_j\ , \qquad \eta_j = -\sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\,\sin\omega_j\ , \tag{6}$$

for $j=1\,,\,2\,$, where $a_j\,,\,e_j\,,\,M_j$ and ω_j are the semi-major axis, the eccentricity, the mean anomaly and the perihelion argument of the j–th body, respectively. One can remark that both ξ_j and η_j are of the same order of magnitude as the eccentricity e_j .

After having reduced both the center of mass and the angular momentum, the Hamiltonian can be given the form

$$H^{(R)}(\underline{\Lambda},\underline{\lambda},\underline{\xi},\underline{\eta}) = F^{(0)}(\underline{\Lambda})+\mu F^{(1)}(\underline{\Lambda},\underline{\lambda},\underline{\xi},\underline{\eta})\ , \tag{7}$$

where the dependency on the norm of the angular momentum $\|\underline{C}\|$ is omitted, because it is replaced by its actual numerical value. In the latter formula, $F^{(0)}=T^{(0)}+U^{(0)}$, $\mu F^{(1)}=T^{(1)}+U^{(1)}$, and we have inserted the small dimensionless parameter $\mu=\max\{m_1\,/\,m_0\,,\,m_2\,/\,m_0\,\}$, in order to highlight the different size of the terms appearing in the Hamiltonian above. Let us remark that the time derivative of each coordinate is $\mathcal{O}(\mu)$ but in the case of the angles $\underline{\lambda}$; in the following, we will refer to $\underline{\lambda}$ and (with a minor abuse) to their conjugate actions $\underline{\Lambda}$ as the *fast variables*, while $(\underline{\xi},\underline{\eta})$ will be called *secular variables.*

Writing the Hamiltonian in the form (7) is a quite standard matter. We now begin the process of constructing a first approximation of the Kolmogorov's normal form. To this end we choose a fixed frequency vector $\underline{n}^*$ for the fast angles and introduce the new actions

$$L_j = \Lambda_j-\Lambda_j^*\ , \qquad \forall\, j=1\,,\,2\,, \tag{8}$$

where the translation vector $\underline{\Lambda}^*$, is determined in such a way that

$$\left.\frac{\partial\,\langle H^{(R)}\rangle_{\underline{\lambda}}}{\partial\Lambda_j}\right|_{\substack{\underline{\Lambda}=\underline{\Lambda}^*\\ \underline{\xi}\,,\,\underline{\eta}=\underline{0}}} = n_j^*\ , \qquad \forall\, j=1\,,\,2\,. \tag{9}$$

In the latter formula, we denoted with $\langle f\rangle_{\underline{\alpha}}$ the average of the generic function f with respect to the angles $\alpha_1\,,\,\ldots\,,\,\alpha_j\,$, i.e., $\langle f\rangle_{\underline{\alpha}}=(2\pi)^{-j}\int_{\mathbf{T}^j}f\,{\rm d}\alpha_1\,\ldots\,{\rm d}\alpha_j\,$. Equation (9) can be solved in view of the fact that in the planetary case the matrix related to the linear part of the l.h.s. of (9) is non–degenerate provided the parameter μ is small enough. This is a consequence of the non–degeneracy of the main

Keplerian part $F^{(0)}(\underline{\Lambda})$ of the Hamiltonian $H^{(R)}$. We will denote by $\mathcal{T}_F$ the canonical transformation which translates the actions, leaving all the other coordinates $\underline{\lambda}$, $\underline{\xi}$ and $\underline{\eta}$ unchanged.

We now expand the Hamiltonian (7) in power series of $\underline{L}$, $\underline{\xi}$, $\underline{\eta}$ around the origin. Thus, by forgetting an unessential constant, we write the Hamiltonian as

$$H^{(T_F)}(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta}) = \underline{n}^* \cdot \underline{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(\mathrm{Kep})}(\underline{L}) + \mu \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(T_F)}(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta}) \ , \qquad (10)$$

where the functions $h_{j_1,j_2}^{(T_F)}$ are homogeneous polynomials of degree j_1 in the actions $\underline{L}$ and of degree j_2 in the secular variables $(\underline{\xi},\underline{\eta})$; moreover, the coefficients of such homogeneous polynomials do depend analytically and periodically on the angles $\underline{\lambda}$. Analogously, the terms $h_{j_1,0}^{(\mathrm{Kep})}$ of the Keplerian part are homogeneous polynomials of degree j_1 in the actions $\underline{L}$, the explicit expression of which can be determined in a straightforward manner. By applying some traditional methods of Celestial Mechanics, one is able to calculate the coefficients of the power series of $h_{j_1,j_2}^{(T_F)}$ as Fourier series with respect to $\underline{\lambda}$. We actually worked out these expansions for the case of the Sun–Jupiter–Saturn system (by using computer algebra) in the way described in sect. 2.1 of [20], where we followed the scheme sketched in sect. 3.3 of [27].

3. **Some preliminary transformations.** We now introduce some canonical transformations that are not quite standard, but are necessary in order to take into account the effects due to the quasi-resonance $5:2$, that is peculiar to the SJS system.

(a)Extended point transformation changing the angles.

The transformation introduced here is by no means essential in general, but turns out to be very useful in order to deal with the SJS case, since it helps to decrease the use of computer memory and improve the effectiveness of our algebraic calculations. In principle it could be ignored if one has enough memory and computer power available. The reason is the following. A generic term in the μ–dependent part of (10) has the form

$$L_1^{j_1} L_2^{j_2} \xi_1^{r_1} \xi_2^{r_2} \eta_1^{s_1} \eta_2^{s_2} \, {\sin \atop \cos}(k_1\lambda_1 + k_2\lambda_2)$$

with a numerical coefficient in front to be determined, where $j_1, j_2, r_1, r_2, s_1, s_2$ are non–negative integers and k_1, k_2 are generic integers. According to our setting in (10), this produces an infinite trigonometric series in $\underline{\lambda}$, that for the purpose of calculation we truncate by keeping only terms with $|\underline{k}| \le N_T$ for some suitably chosen truncation number N_T. Of course, we must keep N_T as small as possible in order to limit the number of terms in our expansions, since it has a major impact on the use of memory and computing time. On the other hand, choosing a too small truncation results in removing also significant terms connected with the $5:2$ resonance of the SJS system. Our remark is that the canonical transformation $(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta}) = \mathcal{U}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta})$, where

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{pmatrix} \ , \qquad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{pmatrix} \ . \qquad (11)$$

causes a reordering of the harmonics which lowers the trigonometric degree of the most critical terms. Let us remark that the latter matrix is unimodular (i.e., it induces a bijective correspondence on $\mathbf{Z}^2$), so that the Hamiltonian $\tilde{H}^{(T_F)} = H^{(T_F)} \circ$

$\mathcal{U}$ shall still be 2π–periodic with respect to the new angles $\underline{\tilde{\lambda}}$. Our choice of the matrix $\mathcal{U}$ takes into account the following facts.

1. Due to the D'Alembert rules[1], the effect of every term belonging to the expansion of $H^{(T_F)}$ in (10) and containing a Fourier harmonic of the type $\underline{k}\cdot\underline{\lambda}$ decreases exponentially with $|k_1+k_2|$; therefore, we should keep in our expansions most terms with low $|k_1+k_2|$.
2. On the other hand, the harmonics $\underline{k}\cdot\underline{\lambda}$ such that $k_1/k_2=-2/5$ have a strong influence on the dynamics due to the quasi resonance, although their impact is apparently reduced by the fact that they belong to terms of order at least three with respect to the eccentricities and the inclinations.

The usefulness of the unimodular transformation (11) can be understood by focusing on the actual values of the frequencies $\tilde{n}_1^*$ and $\tilde{n}_2^*$ which are related to the angular motions of $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, respectively, and are listed in Table 2. The smaller divisors $\tilde{k}_1\tilde{n}_1^*+\tilde{k}_2\tilde{n}_2^*$ arise when $\tilde{k}_1\,\tilde{k}_2<0$ and $|\tilde{k}_2|\ge 2|\tilde{k}_1|$. Therefore, one can easily remark that for this kind of terms the trigonometric degree $|k_1|+|k_2|$ is decreased by the transformation. For, $|k_1|+|k_2|=|\tilde{k}_1+\tilde{k}_2|+|\tilde{k}_1+2\tilde{k}_2|\ge 2|\tilde{k}_2|>|\tilde{k}_1|+|\tilde{k}_2|$. In particular, the harmonic $2\lambda_1-5\lambda_2$ is changed to $\tilde{\lambda}_1-3\tilde{\lambda}_2$, thus decreasing the trigonometric degree from 7 to 4. This nearly doubles the number of terms that are not truncated by the rule $|k_1|+|k_2|\le N_T$. For instance, our choice $N_T=20$ allows us to keep 5 terms related to the 5 : 2 resonance, while without changing the angles we should set $N_T=35$ in order to keep the same number of critical terms.

(b)The averaged Hamiltonian approximating the secular dynamics at order two in the masses

The further transformation that we perform here is crucial in order to deal with the SJS system with the true values of the masses and the semi–major axes of the planets. This is the main difference with the application discussed in our previous paper [21].

Let us focus on the Hamiltonian $\tilde{H}^{(T_F)}$, the expansion of which after the transformation (11) can be written as follows:

$$\tilde{H}^{(T_F)}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta})=\underline{\tilde{n}}^*\cdot\underline{\tilde{L}}+\sum_{j_1=2}^{\infty}\tilde{h}_{j_1,0}^{(\mathrm{Kep})}(\underline{\tilde{L}})+\mu\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\tilde{h}_{j_1,j_2}^{(T_F)}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta})\ , \tag{12}$$

where the functional dependence of $\tilde{h}_{j_1,0}^{(\mathrm{Kep})}$ and $\tilde{h}_{j_1,j_2}^{(T_F)}$ on their arguments is the same as that of the corresponding terms appearing in (10), i.e. $h_{j_1,0}^{(\mathrm{Kep})}$ and $h_{j_1,j_2}^{(T_F)}$, respectively.

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[1] Consider the summands $\underline{\tilde{r}}_1\cdot\underline{\tilde{r}}_2\,/\,m_0$ and $-\mathcal{G}m_1\,m_2\,/\,\|\underline{r}_1-\underline{r}_2\|$, entering the definition of the initial Hamiltonian (5), and imagine to expand them in Fourier series with respect to $\underline{\lambda}$ and in power series of $\underline{\Lambda}$, $\underline{z}$, $\underline{\bar{z}}$, $\underline{\zeta}$, $\underline{\bar{\zeta}}$, where $z_j=e_j\exp[\mathrm{i}(\omega_j+\Omega_j)]$ and $\zeta_j=\sin(i_j/2)\exp(\mathrm{i}\Omega_j)$, being i_j the inclination of the j–th planet and $j=1,2$. Let us introduce the following symbols: $\Xi_{\underline{\lambda}}=k_1+k_2$, where k_1 and k_2 are the Fourier harmonics related to λ_1 and λ_2, respectively, and $\deg_{\underline{z},\underline{\zeta}}$ and $\deg_{\underline{\bar{z}},\underline{\bar{\zeta}}}$ which are meant to be the total polynomial degree of the variables z, ζ and $\bar{z}$, $\bar{\zeta}$, respectively. Then, the so called D'Alembert rules claim that only terms with $\Xi_{\underline{\lambda}}=\deg_{\underline{\bar{z}},\underline{\bar{\zeta}}}-\deg_{\underline{z},\underline{\zeta}}$ are allowed to appear in the expansions of the initial Hamiltonian (see, e.g., sect. 5.2 of [14]). However, let us recall that the coordinates $(\underline{z},\,\underline{\zeta},\,\underline{\bar{z}},\,\underline{\bar{\zeta}})$ are not suitable for our purposes because they are not canonical. As a matter of fact, the D'Alembert rules imply that also the expansions (10) of $H^{(T_F)}$ as a function of the canonical coordinates $(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta})$ obey the restrictions that *every term appearing in such expansions is of order at least* $|\Xi_{\underline{\lambda}}|$ *with respect to the eccentricities and the inclinations.* Moreover, $\Xi_{\underline{\lambda}}$ *has the same parity as* $\deg_{\underline{\xi},\underline{\eta}}$.

We perform now a "Kolmogorov's like" normalization step by averaging on the fast angles $\tilde{\underline{\lambda}}$, so that we approximate the secular dynamics at order two in the masses. This allows us to obtain a secular Hamiltonian the flow of which gives a good representation of the motion of the coordinates $(\underline{\xi},\underline{\eta})$, as discussed in [20]. From the technical point of view, this normalization step is performed through a pair of canonical transformations which are written by means of the usual Lie series algorithm (see, e.g., [9]).

Let us introduce the new Hamiltonian $\hat{H}^{(\mathcal{O}2)} = \exp \mathcal{L}_{\chi_1^{(\mathcal{O}2)}} \tilde{H}^{(T_F)}$, with the generating function $\chi_1^{(\mathcal{O}2)}(\tilde{\underline{\lambda}},\underline{\xi},\underline{\eta})$ determined by the solution of the equation

$$\sum_{j=1}^{2} \tilde{n}_j^* \frac{\partial\,\chi_1^{(\mathcal{O}2)}}{\partial\tilde{\lambda}_j} + \mu \sum_{j_2=0}^{5} \left\lceil \tilde{h}_{0,j_2}^{(T_F)} \right\rceil_{\tilde{\underline{\lambda}};K_F} (\tilde{\underline{\lambda}},\underline{\xi},\underline{\eta}) = 0 \ , \tag{13}$$

where, for any function f, with the symbol $\lceil f \rceil_{\tilde{\underline{\lambda}};K_F}$ we mean the part of the Fourier expansion of f such that its harmonics $\underline{k}\cdot\tilde{\underline{\lambda}}$ satisfy the restriction $0<|\underline{k}|\le K_F$. The integer parameter K_F must be conveniently fixed, in order to include the most important perturbing terms in this first normalization step; we found that in the case of the SJS system the choice $K_F=5$ is suitable enough. Similarly, the upper limit 5 in the second sum is chosen so that the transformation takes into account the effect of the main terms related to the $5:2$ resonance. We notice that the equation (13) can be solved provided the frequencies $(\tilde{n}_1^*,\tilde{n}_2^*)$ are not resonant up to order K_F.

The Hamiltonian $\hat{H}^{(\mathcal{O}2)}$ can be written in the same form as in (12), by replacing the functions $\tilde{h}_{j_1,j_2}^{(T_F)}$ with $\hat{h}_{j_1,j_2}^{(\mathcal{O}2)}$, which can be determined by calculating the expansion of the Lie series $\exp \mathcal{L}_{\chi_1^{(\mathcal{O}2)}} \tilde{H}^{(T_F)}$ and by gathering all the terms having the same degree both in the fast actions and in the secular variables.

The construction of the Hamiltonian approximating the secular dynamics at order two in the masses is completed, by calculating $H^{(\mathcal{O}2)} = \exp \mathcal{L}_{\chi_2^{(\mathcal{O}2)}} \hat{H}^{(\mathcal{O}2)}$, where the generating function $\chi_2^{(\mathcal{O}2)}(\tilde{\underline{L}},\tilde{\underline{\lambda}},\underline{\xi},\underline{\eta})$ (which is linear with respect to $\tilde{\underline{L}}$) is determined by the solution of the equation

$$\sum_{j=1}^{2} \tilde{n}_j^* \frac{\partial\,\chi_2^{(\mathcal{O}2)}}{\partial\tilde{\lambda}_j} + \mu \sum_{j_2=0}^{5} \left\lceil \hat{h}_{1,j_2}^{(\mathcal{O}2)} \right\rceil_{\tilde{\underline{\lambda}};K_F} (\tilde{\underline{L}},\tilde{\underline{\lambda}},\underline{\xi},\underline{\eta}) = 0 \ . \tag{14}$$

The Hamiltonian $H^{(\mathcal{O}2)}$ can be written in a form similar to (12), namely

$$H^{(\mathcal{O}2)}(\tilde{\underline{L}},\tilde{\underline{\lambda}},\underline{\xi},\underline{\eta}) = \tilde{\underline{n}}^*\cdot\tilde{\underline{L}} + \sum_{j_1=2}^{\infty} \tilde{h}_{j_1,0}^{(\mathrm{Kep})}(\tilde{\underline{L}}) + \mu \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{O}2)}(\tilde{\underline{L}},\tilde{\underline{\lambda}},\underline{\xi},\underline{\eta};\mu) \ , \tag{15}$$

where the functions $h_{j_1,j_2}^{(\mathcal{O}2)}$ are calculated as previously explained for $\hat{h}_{j_1,j_2}^{(\mathcal{O}2)}$; moreover, they still have the same functional dependence on their arguments as that of the corresponding terms $h_{j_1,j_2}^{(T_F)}$ appearing in (10). Although it is not immediately evident, the two normalization steps just performed make the secular torus $\tilde{\underline{L}}=\underline{0}$ and $\underline{\xi}=\underline{\eta}=\underline{0}$ invariant up to terms $\mathcal{O}(\mu^2)$. In this sense the Hamiltonian (15) approximates the secular dynamics at order two in the masses. Indeed, the value of the parameter K_F is taken large enough, so that the perturbative terms appearing in the expansion (12) and not removed by these two normalization steps are about of the same size of the newly produced terms $\mathcal{O}(\mu^2)$. Let us make an example in order to clarify this point: the size of

$$\frac{1}{2}\mathcal{L}^2_{\chi_1^{(\mathcal{O}2)}}\tilde{h}^{(\mathrm{Kep})}_{2,0}(\underline{\tilde{L}})$$

(which is clearly $\mathcal{O}(\mu^2)$) compares with the size of the terms, which are at most linear in $\underline{\tilde{L}}$ but do not appear in the equations for the generating functions, i.e.

$$\mu\sum_{j_1=0}^{1}\sum_{j_2=0}^{\infty}\tilde{h}^{(T_F)}_{j_1,j_2}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta}) - \mu\sum_{j_1=0}^{1}\sum_{j_2=0}^{\infty}\left\lceil\tilde{h}^{(T_F)}_{j_1,j_2}\right\rceil_{\underline{\tilde{\lambda}};K_F}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta})\ .$$

Let us recall that the latter expression shrinks to zero when $K_F\to\infty$, due to the exponential decay of the Fourier harmonics in $\underline{\tilde{\lambda}}$.

In the following, we will denote with $\mathcal{K}_{\mathcal{O}2}$ the canonical transformation induced by the generating functions $\chi_1^{(\mathcal{O}2)}$ and $\chi_2^{(\mathcal{O}2)}$, i.e.

$$\mathcal{K}_{\mathcal{O}2}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta}) = \exp\mathcal{L}_{\chi_2^{(\mathcal{O}2)}}\Big(\exp\mathcal{L}_{\chi_1^{(\mathcal{O}2)}}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta})\Big)\ . \tag{16}$$

4. **Action–angle coordinates for the secular variables.** We now proceed by determining a first approximation of a torus carrying a quasi-periodic motion in the secular variables. Our procedure is strongly reminiscent of the one we followed in [21], with the necessary modifications due to the fact that we use the Hamiltonian (15) as the starting point, in place of (10) and, moreover, we change some technical parameters in order to improve the accuracy of the final numerical results.

We need to perform three steps, namely: (i) the diagonalization of the quadratic terms of the secular part of the Hamiltonian, (ii) the transformation to action–angle variables and a finite order Birkhoff's normalization in order to remove the degeneration of the unperturbed Hamiltonian, and (iii) the translation of the secular actions so that the origin of the actions coincides with an invariant torus with prescribed frequencies, in the integrable approximation of the Hamiltonian.

4.1. **Diagonalization of the quadratic terms of the secular part.** We first approximate the system (15) with an integrable one, the solutions of which are quasi–periodic with fast frequencies $\underline{\tilde{n}}^*$ and with slow frequencies for the secular motions. To this end we remove all terms $\tilde{h}^{(\mathrm{Kep})}_{j_1,0}$ which are at least quadratic in $\underline{\tilde{L}}$. Moreover, concerning the functions $\tilde{h}^{(T_F)}_{j_1,j_2}$, we first average over the fast angles $\underline{\tilde{\lambda}}$, then we put $\underline{L}=\underline{0}$, and finally we truncate the power expansions in $\underline{\xi}$, $\underline{\eta}$ keeping the quadratic part only. From the D'Alembert rules[2], one can deduce that the Hamiltonian $H^{(\mathcal{O}2)}$ averaged over $\underline{\tilde{\lambda}}$ contains only terms of even degree in $\underline{\xi}$, $\underline{\eta}$, so that we are actually considering the lowest degree part of the expansion in the secular variables, forgetting an unessential constant. As a matter of fact, we merely calculate $\langle h^{(\mathcal{O}2)}_{0,2}\rangle_{\underline{\tilde{\lambda}}}$, which is a quadratic form in $\underline{\xi}$, $\underline{\eta}$. We emphasize that this constitutes a better approximation of the secular dynamics with respect to

[2]As we discussed in the footnote n. 1, the terms appearing in the expansions of $H^{(T_F)}$ are such that $\Xi_{\underline{\lambda}}$ has the same parity as $\deg_{\underline{\xi},\underline{\eta}}$. One immediately realizes that the canonical transformation $\mathcal{K}_{\mathcal{O}2}$ would preserve the relation ensuring that every term has the same parity in $\Xi_{\underline{\lambda}}$ and in $\deg_{\underline{\xi},\underline{\eta}}$, if it were expressed as a function of $(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta})$. Therefore, $\langle H^{(\mathcal{O}2)}\circ\mathcal{U}^{-1}\rangle_{\underline{\lambda}}$ must contain only even terms in $(\underline{\xi},\underline{\eta})$. Since the average over the angles $\underline{\lambda}$ is equivalent to that with respect to $\underline{\tilde{\lambda}}$, one can conclude that also $\langle H^{(\mathcal{O}2)}\rangle_{\underline{\tilde{\lambda}}}$ contains just even terms in $(\underline{\xi},\underline{\eta})$.

the theory independently developed by Lagrange and Laplace, who considered only terms of order 1 in the masses. Thus, we focus on the so approximated Hamiltonian

$$\begin{aligned} H^{(Q2)}(\underline{\tilde{L}},\underline{\xi},\underline{\eta}) &= \underline{\tilde{n}}^* \cdot \underline{\tilde{L}} \\ &+\mu\left(\sum_{1\le i\le j\le 2} c_{i,j}\,\xi_i\xi_j + \sum_{\substack{1\le i\le 2\\ 3\le j\le 4}} c_{i,j}\,\xi_i\eta_{j-2} + \sum_{3\le i\le j\le 4} c_{i,j}\,\eta_{i-2}\eta_{j-2}\right)\ , \end{aligned} \tag{17}$$

where the quantities $c_{i,j}$ can be determined through an explicit calculation; the upper label $(Q2)$ recalls that this is the quadratic approximation in the slow variables at order 2 in the masses.

Consider now the matrix associated to the motion of the secular variables, i.e.,

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{1,2} & 2c_{2,2} & c_{2,3} & c_{2,4} \\ c_{1,3} & c_{2,3} & 2c_{3,3} & c_{3,4} \\ c_{1,4} & c_{2,4} & c_{3,4} & 2c_{4,4} \end{pmatrix}\ . \tag{18}$$

If the eigenvalues of this matrix are pure imaginary and distinct, which turns out to be the case for the SJS system, then $\underline{\xi} = \underline{\eta} = \underline{0}$ is an elliptic equilibrium point for the secular variables. Thus, there exists a linear canonical transformation $(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta}) = \mathcal{D}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi}',\underline{\eta}')$ (leaving unchanged the fast variables) such that in the new coordinates the Hamiltonian $H^{(Q2)}$ takes the diagonal form

$$H^{(Q2)}(\underline{\tilde{L}},\underline{\xi}',\underline{\eta}') = \underline{\tilde{n}}^* \cdot \underline{\tilde{L}} + \mu\sum_{j=1}^{2}\frac{\nu_j}{2}\left({\xi'_j}^2 + {\eta'_j}^2\right)\ .$$

4.2. **Birkhoff's normalization in action–angle variables.** We now come back to the Hamiltonian (15). We perform two canonical transformations, the first one being the change of coordinates $\mathcal{D}$ above, and a second one $(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi}',\underline{\eta}') = \mathcal{A}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{I},\underline{\varphi})$ introducing the usual action–angle variables defined by

$$\xi'_j = \sqrt{2I_j}\cos\varphi_j\ , \qquad \eta'_j = \sqrt{2I_j}\sin\varphi_j\ , \qquad j = 1,\ 2\ . \tag{19}$$

The complete Hamiltonian of the problem of three bodies now takes the form

$$\begin{aligned} H^{(A)}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{I},\underline{\varphi}) &= \underline{\tilde{n}}^* \cdot \underline{\tilde{L}} + \sum_{j_1=2}^{\infty}\tilde{h}^{(\mathrm{Kep})}_{j_1,0}(\underline{\tilde{L}}) + \mu\,\underline{\nu}\cdot\underline{I} + \mu\sum_{j_2=2}^{\infty}\big\langle h^{(A)}_{0,2j_2}\big\rangle_{\underline{\tilde{\lambda}}}(\underline{I},\underline{\varphi}) \\ &+\mu\sum_{j_2=0}^{\infty}\big\lfloor h^{(A)}_{0,j_2}\big\rfloor_{\underline{\tilde{\lambda}}}(\underline{\tilde{\lambda}},\underline{I},\underline{\varphi}) + \mu\sum_{j_1=1}^{\infty}\sum_{j_2=0}^{\infty} h^{(A)}_{j_1,j_2}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{I},\underline{\varphi})\ , \end{aligned} \tag{20}$$

where the terms $h^{(A)}_{j_1,j_2}$ are obtained from $h^{(\mathcal{O}2)}_{j_1,j_2}$ by performing the substitution $(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta}) = \mathcal{D}\circ\mathcal{A}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{I},\underline{\varphi})$. Hereafter, we use the following notation: for a generic function $f(\underline{\alpha})$ with $\underline{\alpha}\in\mathbf{T}^m$, we write $\lfloor f\rfloor_{\underline{\alpha}} = f - \langle f\rangle_{\underline{\alpha}}$, which is the part of f actually depending on the angles $\underline{\alpha}$. In formula (20): $\tilde{h}^{(\mathrm{Kep})}_{j_1,0}(\underline{\tilde{L}})$ is a homogeneous polynomial of degree j_1; $\big\langle h^{(A)}_{0,2j_2}\big\rangle_{\underline{\tilde{\lambda}}}(\underline{I},\underline{\varphi})$ is a homogeneous polynomial of degree $2j_2$ in $\sqrt{I_1}$, $\sqrt{I_2}$; $\big\lfloor h^{(A)}_{0,j_2}\big\rfloor_{\underline{\tilde{\lambda}}}(\underline{\tilde{\lambda}},\underline{I},\underline{\varphi})$ is a homogeneous polynomial $\mathcal{O}(\|\underline{I}\|^{j_2/2})$ and is an analytic and periodic function of $\underline{\tilde{\lambda}}$; finally, $h^{(A)}_{j_1,j_2}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{I},\underline{\varphi})$ is a homogeneous polynomial $\mathcal{O}(\|\underline{\tilde{L}}\|^{j_1})$ and $\mathcal{O}(\|\underline{I}\|^{j_2/2})$ and is again an analytic and periodic function of

$\underline{\tilde{\lambda}}$. Concerning the dependence on $\underline{\varphi}$, all functions above are trigonometric polynomials of the same degree as $\sqrt{I_1}$, $\sqrt{I_2}$, satisfying some further constraints due to the fact that they are originated by polynomials in $(\underline{\xi}', \underline{\eta}')$.

We now look for a nonlinear integrable approximation of the Hamiltonian (20) which is a suitable starting point for the Kolmogorov's normalization algorithm. To this end we use a Birkhoff normalization procedure which removes from the Hamiltonian all terms which are (a) at most linear in the fast actions $\underline{\tilde{L}}$, (b) independent of the fast angles $\underline{\tilde{\lambda}}$, (c) of degree at most r in the slow actions $\underline{I}$ and (d) truly dependent from the secular angles $\underline{\tilde{\varphi}}$. The Birkhoff normalization order r should be chosen high enough in order to have a good first approximation of the invariant torus and small enough so that the divergence of the Birkhoff series does not show up. We found that in the case of the SJS system the choice $r = 3$ is appropriate.

Our actual procedure consists in performing five different normalization steps with generating functions $B_{0,4}$, $B_{0,6}$, $B_{1,2}$, $B_{1,4}$ and $B_{1,6}$, where the transformation with $B_{s,2r}$ removes the unwanted terms of degree s in $\underline{\tilde{L}}$ and of degree r in $\underline{I}$. For instance, the first generating function $B_{0,4}$ is determined by solving the equation

$$\sum_{j=1}^{2} \nu_j \frac{\partial B_{0,4}}{\partial \varphi_j} + \big\langle h_{0,4}^{(A)} \big\rangle_{\underline{\lambda}}(\underline{I}, \underline{\varphi}) - \big\langle h_{0,4}^{(A)} \big\rangle_{(\underline{\lambda}\,,\,\underline{\varphi})}(\underline{I}) = 0\ . \tag{21}$$

The generating functions $B_{0,4}$, ..., $B_{1,6}$ can be determined if the frequencies $\underline{\nu}$ satisfy the nonresonance condition

$$\underline{k} \cdot \underline{\nu} \neq 0 \qquad \forall\ \underline{k} \in \mathbf{Z}^2 \setminus \{\underline{0}\}\ \text{ such that }\ |k_1| + |k_2| = 2\,,\, 4\,,\, 6\ . \tag{22}$$

Precisely, the canonical transformation of the coordinates writes

$$\mathcal{B}\big(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{I}, \underline{\varphi}\big) = \exp \mathcal{L}_{B_{1,6}} \circ \exp \mathcal{L}_{B_{1,4}} \circ \exp \mathcal{L}_{B_{1,2}} \circ \exp \mathcal{L}_{B_{0,6}} \circ \exp \mathcal{L}_{B_{0,4}} \big(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{I}, \underline{\varphi}\big)\ . \tag{23}$$

The transformed Hamiltonian is explicitely computed as

$$H^{(B)} = \exp \mathcal{L}_{B_{1,6}} \circ \exp \mathcal{L}_{B_{1,4}} \circ \exp \mathcal{L}_{B_{1,2}} \circ \exp \mathcal{L}_{B_{0,6}} \circ \exp \mathcal{L}_{B_{0,4}} H^{(A)}$$

and it has a form similar to $H^{(A)}$, namely

$$\begin{aligned} H^{(B)}(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{I}, \underline{\varphi}) = {} & \underline{\tilde{n}}^{*} \cdot \underline{\tilde{L}} + \sum_{j_1=2}^{\infty} \tilde{h}_{j_1,0}^{(\mathrm{Kep})}(\underline{\tilde{L}}) \\ & + \mu \Big[\underline{\nu} \cdot \underline{I} + \big\langle h_{0,4}^{(B)} \big\rangle_{\underline{\tilde{\lambda}}}(\underline{I}) + \big\langle h_{0,6}^{(B)} \big\rangle_{\underline{\tilde{\lambda}}}(\underline{I}) + \big\langle h_{1,0}^{(B)} \big\rangle_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}) \\ & \quad + \big\langle h_{1,2}^{(B)} \big\rangle_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}, \underline{I}) + \big\langle h_{1,4}^{(B)} \big\rangle_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}, \underline{I}) + \big\langle h_{1,6}^{(B)} \big\rangle_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}, \underline{I}) \Big] \\ & + \mu \sum_{j_1=0}^{1} \left[\sum_{j_2=4}^{\infty} \big\langle h_{j_1,2j_2}^{(B)} \big\rangle_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}, \underline{I}, \underline{\varphi}) + \sum_{j_2=0}^{\infty} \big\lfloor h_{j_1,j_2}^{(B)} \big\rfloor_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{I}, \underline{\varphi}) \right] \\ & + \mu \sum_{j_1=2}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(B)}(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{I}, \underline{\varphi})\ , \end{aligned} \tag{24}$$

where all functions are polynomial and trigonometric polynomials as the corresponding terms in $H^{(A)}$.

4.3. **First approximation of the unperturbed torus.** We look now for an unperturbed torus with prescribed frequencies $(n_1^*,\, n_2^*,\, \mu g_1^*,\, \mu g_2^*)$, where, in general $\underline{g}^*$ is close to but does not coincide with $\underline{\nu}$. To this end we determine a vector $\underline{I}^*$ by solving with respect to the unknown $\underline{I}^* \in \mathbf{R}^2$ the equation

$$\frac{\partial \left\langle h_{0,6}^{(B)} \right\rangle_{\underline{\lambda}}}{\partial I_j}(\underline{I}) + \frac{\partial \left\langle h_{0,4}^{(B)} \right\rangle_{\underline{\lambda}}}{\partial I_j}(\underline{I}) = g_j^* - \nu_j^* \ , \qquad j = 1,\, 2 \ . \tag{25}$$

We remark that this is a system of two equations that can be reduced to a single fourth order equation, e.g., in the unknown I_1 . This latter equation admits at least a solution close to the origin, provided $\|\underline{g}^* - \underline{\nu}^*\|$ is small enough and the non–degeneracy condition

$$\det\left(\frac{\partial^2 \left\langle h_{0,4}^{(B)} \right\rangle_{\underline{\lambda}}}{\partial I_i \partial I_j} \right)_{i\,,\,j=1\,,\,2} \neq 0 \tag{26}$$

is fulfilled. However, we need more, and precisely that the solutions I_1^*, I_2^* of the equation (25) are both positive and small, since we require that the canonical transformation $\mathcal{A}$ is real analytic in a suitably small neighborhood of $\underline{I}^*$ not too far from zero (recall the definition (19)).

We can now introduce the canonical transformation $(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{I}, \underline{\varphi}) = \mathcal{T}_S(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{J}, \underline{\psi})$ which is the identity in $\underline{\tilde{L}}, \underline{\tilde{\lambda}}$ and shifts the origin of the secular actions, in such a way that

$$I_j = J_j + I_j^* \ , \qquad \varphi_j = \psi_j \ , \qquad \forall\ j = 1,\, 2 \ . \tag{27}$$

This is the point where the transformation performed at point (b) of sect. 3, namely the average of the secular system at order 2 in the masses, plays a crucial role. Indeed, without such a preliminar transformation it turns out that I_1^*, I_2^* are not both positive if one considers the SJS system and replaces μg_1^*, μg_2^* with the observed secular frequencies. This is precisely due to the strong effect of the 5 : 2 resonance on the secular dynamics (as discussed in [26]), the main part of which is removed by averaging the first order on the masses, as we did.

After having performed the translation of the actions above, the Hamiltonian may be rearranged as

$$\begin{aligned} &H^{(\mathcal{T}_S)}(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{J}, \underline{\psi}) \\ &= \underline{\tilde{n}}^* \cdot \underline{\tilde{L}} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(\mathrm{Kep})}(\underline{\tilde{L}}) + \mu \underline{g}^* \cdot \underline{J} + \mu \sum_{l=2}^{\infty} \sum_{\substack{j_1 \geq 0\ j_2 \geq 0 \\ j_1 + j_2 = l}} h_{j_1,j_2}^{(\mathcal{T}_S)}(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{J}, \underline{\psi}) \\ &\quad + \mu \sum_{l=0}^{1} \sum_{\substack{j_1 \geq 0\ j_2 \geq 0 \\ j_1 + j_2 = l}} \left[\left\langle h_{j_1,j_2}^{(\mathcal{T}_S)} \right\rangle_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}, \underline{J}, \underline{\psi}) + \left\lfloor h_{j_1,j_2}^{(\mathcal{T}_S)} \right\rfloor_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{J}, \underline{\psi}) \right] \ , \end{aligned} \tag{28}$$

where the functions $h_{j_1,j_2}^{(\mathcal{T}_S)}$ are homogeneous polynomials of degree j_1 and j_2 with respect to the actions variables $\underline{\tilde{L}}$ and $\underline{J}$, respectively, while their dependence on the angles $\underline{\tilde{\lambda}}$ and $\underline{\psi}$ is analytic and periodic. By the way, the canonical transformation $\mathcal{T}_S$ has broken the relationships between the polynomial degrees of the secular actions and the Fourier harmonics of the secular angles which were satisfied in the expansions of the Hamiltonians $H^{(A)}$ and $H^{(B)}$, but this is harmless. The crucial point instead is that we recognize in the Hamiltonian (28) three parts, namely: (a) a term $\underline{\tilde{n}}^* \cdot \underline{\tilde{L}} + \mu \underline{g}^* \cdot \underline{J}$ which represents an isochronous system with the wanted frequencies $(n_1^*,\, n_2^*,\, \mu g_1^*,\, \mu g_2^*)$; (b) a part which is at least quadratic in the actions

$\underline{\tilde{L}}$, $\underline{J}$; and (c) a part which contains all terms which are independent of or linear in $\underline{\tilde{L}}$, $\underline{J}$. Let us call the part (c) the *critical part* of the Hamiltonian. This is isolated in the second line of formula (28) above.

We claim that the Hamiltonian (28) is in suitable form for starting the normalization in Kolmogorov's sense. However, this may be not immediately evident unless one understands that the critical part above, which does not belong to the Kolmogorov's normal form, is smaller than the rest of the Hamiltonian. Thus, let us clarify this point. The apparent problem is that the term $\mu\underline{g}^* \cdot \underline{J}$ looks at first sight to be of the same order of magnitude of the perturbation, being $\mathcal{O}(\mu)$. Actually, this is not true. Indeed, we can distinguish in the critical part two contributions, namely the terms $\lfloor h_{j_1,j_2}^{(T_S)} \rfloor_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}, \underline{\tilde{\lambda}}, \underline{J}, \underline{\psi})$ which do depend on the fast angles $\underline{\tilde{\lambda}}$ and the terms $\langle h_{j_1,j_2}^{(T_S)} \rangle_{\underline{\tilde{\lambda}}}(\underline{\tilde{L}}, \underline{J}, \underline{\psi})$ which are independent of them. Now, the terms which do depend on $\underline{\tilde{\lambda}}$ contain μ as a factor, so that they are smaller than the corresponding linear integrable part $\underline{\tilde{n}}^* \cdot \underline{\tilde{L}}$. On the other hand, the critical part of degree j_2 in $\underline{J}$ and independent of $\underline{\tilde{\lambda}}$ has been generated from terms which are at least cubic in $\|\underline{I}\|$, so that their coefficients are $\mathcal{O}(\|\underline{I}^*\|^{3-j_2})$. Since $\underline{I}^*$ is small, we may consider it as playing the role of a second perturbation parameter, say $\varepsilon \simeq \|\underline{I}^*\|$. From a more physical point of view, as a first approximation with respect to μ, one can show that this second small parameter $\varepsilon = \mathcal{O}\big((\sum_{j=1}^{2} e_j + i_j)^2\big)$, where e_j and i_j are the eccentricity and the inclination of the j–th planet, respectively.

The heuristic argument above shows that the critical part of the Hamiltonian can be considered *bona fide* as a small perturbation. This could be checked via analytic estimates leading to a rigorous proof. However, we take the pragmatical attitude of just checking by explicit calculations that the Kolmogorov's normalization procedure turns out to be effective.

5. **Construction of the invariant torus.** We come now to the construction of Kolmogorov's normal form in a neighbourhood related to the unperturbed torus $\underline{\tilde{L}} = 0$, $\underline{J} = 0$ of the Hamiltonian (28). The algorithm used here is essentially the same as in [21], where we considered the more general case of the planetary problem with $n+1$–bodies, and it is based on the procedure introduced in [11]. A detailed exposition including the adaptations that are convenient in the context of an explicit calculation using computer algebra may be found in [8]. We recall here the main steps for self–consistency purposes.

We denote by $\underline{p}$ all the actions $\underline{\tilde{L}}$ and $\underline{J}$ and by $\underline{q}$ all the angles $\underline{\tilde{\lambda}}$ and $\underline{\psi}$, so that $(\underline{p}, \underline{q}) \in \mathcal{G} \times \mathbf{T}^4$, where $\mathcal{G}$ is a neighbourhood of the origin of $\mathbf{R}^4$. Analogously, the frequency vector $\underline{\omega}^*$ is defined as $\underline{\omega}^* = (\tilde{n}_1^*, \tilde{n}_2^*, \mu g_1^*, \mu g_2^*)$. We also introduce the special class of functions $\mathcal{P}_{l,sK}$ as follows: for a fixed positive integer K and for integers $l \geq 0$ and $s \geq 0$, a function $f \in \mathcal{P}_{l,sK}$ is a homogeneous polynomial of degree l in the action variables and a trigonometric polynomial of degree at most sK with respect to the angular ones. Finally, we shall denote the l^1–norm of any vector $\underline{k} \in \mathbf{Z}^4$ by $|\underline{k}| = |k_1| + \ldots + |k_4|$.

The generic r–th step of the Kolmogorov's normalization algorithm is performed as follows. Assume that we have already performed $r-1$ steps and that the Hamiltonian is rearranged as

$$H^{(r-1)}(\underline{p}, \underline{q}) = \underline{\omega}^* \cdot \underline{p} + \sum_{s\geq 0} \sum_{l\geq 0} f_l^{(r-1,s)}(\underline{p}, \underline{q}) \ , \tag{29}$$

where $f_l^{(r-1,s)} \in \mathcal{P}_{l,sK}$ and none of the monomials in the expansion of $f_l^{(r-1,s+1)}$ belongs also to $\mathcal{P}_{l,sK}$, $\forall\ l \ge 0$ and $s \ge 0$. For $r = 1$, which is the starting point of the iteration, we just define $H^{(0)} = H^{(T_S)}$ and determine the functions $f_l^{(0,s)}$ by suitably rearranging the terms appearing in the expansion (28). In particular, we have

$$f_2^{(r-1,0)}(\underline{p}) = \frac{1}{2} C^{(r)} \underline{p} \cdot \underline{p} \ , \tag{30}$$

which defines the 4×4 numerical matrix $C^{(r)}$.

With the help of generating functions $\chi_1^{(r)}(\underline{q}) = X^{(r)}(\underline{q}) + \underline{\xi}^{(r)} \cdot \underline{q}$ (where $\underline{\xi}^{(r)} \in \mathbf{R}^4$) and $\chi_2^{(r)}(\underline{p}, \underline{q})$ we determine the transformed Hamiltonian

$$H^{(r)}(\underline{p}, \underline{q}) = \exp \mathcal{L}_{\chi_2^{(r)}} \circ \exp \mathcal{L}_{\chi_1^{(r)}} H^{(r-1)}(\underline{p}, \underline{q}) \ , \tag{31}$$

and rearrange the expansion so that it takes a form similar to (29), namely

$$H^{(r)}(\underline{p}, \underline{q}) = \underline{\omega}^* \cdot \underline{p} + \sum_{s \ge 0} \sum_{l \ge 0} f_l^{(r,s)}(\underline{p}, \underline{q}) \ , \tag{32}$$

where the functions $f_l^{(r,s)} \in \mathcal{P}_{l,sK}$ and none of the monomials in the expansion of $f_l^{(r,s+1)}$ belongs to $\mathcal{P}_{l,sK}$, $\forall\ l \ge 0$ and $s \ge 0$.

The generating functions are determined by solving with respect to $X^{(r)}(\underline{q})$, $\underline{\xi}^{(r)}$ and $\chi_2^{(r)}(\underline{p}, \underline{q})$ the equations

$$\sum_{j=1}^{4} \omega_j^* \frac{\partial X^{(r)}}{\partial q_j} + \sum_{s=1}^{r} f_0^{(r-1,s)}(\underline{q}) \quad = \quad 0 \ , \tag{33}$$

$$C^{(r)} \underline{\xi}^{(r)} \cdot \underline{p} + f_1^{(r-1,0)}(\underline{p}) \quad = \quad 0 \ , \tag{34}$$

$$\sum_{j=1}^{4} \omega_j^* \frac{\partial \chi_2^{(r)}}{\partial q_j} + \sum_{s=1}^{r} \hat{f}_1^{(r,s)}(\underline{p}, \underline{q}) = 0 \ . \tag{35}$$

Here, $\hat{f}_1^{(r,s)}(\underline{p}, \underline{q})$ comes from the expansion of the intermediate Hamiltonian

$$\hat{H}^{(r)}(\underline{p}, \underline{q}) = \exp \mathcal{L}_{\chi_1^{(r)}} H^{(r-1)}(\underline{p}, \underline{q}) = \underline{\omega}^* \cdot \underline{p} + \sum_{s \ge 0} \sum_{l \ge 0} \hat{f}_l^{(r,s)}(\underline{p}, \underline{q}) \ .$$

The equation for $\underline{\xi}^{(r)}$ admits a solution since the matrix $C^{(r)}$ is non-degenerate. Indeed, for $r = 1$ this is true in view of the non-degeneracy of the Keplerian part and of the condition (26), provided the parameters μ and ε are small enough. For $r > 1$, this must be checked and happens to be true for the SJS system. The equation for $X^{(r)}$ admits an unique solution satisfying $\langle X^{(r)} \rangle_{\underline{q}} = 0$, for any r, provided that the Diophantine condition[3]

[3]From a practical point of view, when one is interested in approximating an orbit of a *real* planetary system with a quasi-periodic motion on an invariant torus, the frequencies are obviously known up to a finite number of digits; thus, one can wonder how the diophantine non-resonant condition can be fulfilled. This is a minor concern. Indeed, in a concrete application, even the Kolmogorov's normalization algorithm can be performed up to a fixed finite number (let us say $r = R$). Thus, one has to check the non-resonance condition is satisfied up to the order RK , i.e. $\underline{k} \cdot \underline{\omega}^* \neq 0\ \forall\ \underline{k} \in \mathbf{Z}^4$ such that $0 < |\underline{k}| \le RK$; then, the unknown digits can be often determined in such a way $\underline{\omega}^*$ is diophantine (for instance, see sect. 2 of [5]). Because of the well known major role of the *small divisors* in the solution of the equations (33) and (35), in practice it is essential to just consider frequencies vector $\underline{\omega}^*$ staying safely far from the resonances with low order $|\underline{k}|$, in order that a very little "tuning" of the parameters is enough to make convergent the whole normalization procedure.

$$|\underline{k}\cdot\underline{\omega}^*| \geq \frac{\gamma}{|\underline{k}|^\tau} \qquad \forall\ \underline{k}\in\mathbf{Z}^4\setminus\{\underline{0}\}\ . \tag{36}$$

is satisfied for a pair of positive parameters γ and $\tau \geq 3$. The same holds true for the equation for $\chi_2^{(r)}$, with the further condition $\sum_{s=1}^{r}\langle\hat{f}_1^{(r,s)}\rangle_{\underline{q}} = 0$. The latter requirement is satisfied by a suitable rearrangement of terms in the intermediate Hamiltonian $\hat{H}^{(r)}$.

The Kolmogorov's normal form for the Hamiltonian is constructed by iterating the step above, thus constructing a sequence $H^{(1)}, H^{(2)},\ldots,H^{(r)}$ of Hamiltonians that, if the perturbation is small enough, converges to Kolmogorov's normal form. According to the theory of Lie series, the canonical transformation $(\underline{p},\underline{q}) = \mathcal{K}^{(r)}\big(\underline{p}^{(r)},\underline{q}^{(r)}\big)$ inducing the Kolmogorov's normalization up to the step r is given by

$$\mathcal{K}^{(r)}\big(\underline{p}^{(r)},\underline{q}^{(r)}\big) = \exp\mathcal{L}_{\chi_2^{(r)}}\circ\exp\mathcal{L}_{\chi_1^{(r)}}\circ\ldots\circ\exp\mathcal{L}_{\chi_2^{(1)}}\circ\exp\mathcal{L}_{\chi_1^{(1)}}\ \big(\underline{p}^{(r)},\underline{q}^{(r)}\big)\ , \tag{37}$$

where $\big(\underline{p}^{(r)},\underline{q}^{(r)}\big)$ are meant to be the new coordinates.

Let us end the description of the Kolmogorov's normalization algorithm with a few final remarks. Precisely, we discuss the problem of the convergence of the procedure and the choice of the integer parameter K.

Concerning the problem of the convergence, the theoretical estimates usually allow to prove that the size of the generating functions decreases geometrically with the normalization step r, i.e., $\|\chi_1^{(r)}\| \leq E_1\Theta^r$ and $\|\chi_2^{(r)}\| \leq E_2\Theta^r$, with a convenient functional norm $\|\cdot\|$ and suitable positive constants E_1, E_2 and $\Theta < 1$. This, of course, is not evident from the formal scheme above, since it requires an appropriate scheme of quantitative estimates. Here, we do not enter such a matter. We rather take the pragmatical attitude of computing the norms of the generating functions at every actually computed step and checking that they do indeed decrease geometrically. In particular, this implies that the critical unwanted terms in the Hamiltonian $H^{(r)}$ decrease geometrically too (for a proof see, e.g., [11]). Furthermore, this also implies that the non degeneracy condition is satisfied also for $r > 1$ provided that the perturbation is small enough, because the successive corrections to the matrix $C^{(1)}$ are proportional to the size of the generating functions.

Let us now come to the suitable choice of K. It might be surprising that in our case, dealing with a planetary system, it is enough to set $K = 2$. This is justified by the following considerations. The point is that we should exploit the exponential decay of the coefficients of the Fourier expansion of an analytic function, i.e. the coefficient of $\exp(\mathrm{i}\underline{k}\cdot\underline{q})$ has a size $\sim\exp(-|\underline{k}|\sigma)$ with some $\sigma > 0$. A natural choice appears to set $\exp(-K\sigma)\sim\mu$. However, a suitable theoretical scheme shows that a better choice is to set $K\sim 1/\sigma$ (see footnote n. 2 in Chap. 5 of [10]). On the other hand, for the purpose of an actual calculation we are interested to keep K as small as possible in order to decrease the need of memory and CPU–time. Now, in the case of a planetary system, for what concerns the fast angles, the parameter σ turns out to be an increasing function of the distance between the planets. Thus, if such a distance is large enough we can keep K quite small. Concerning the slow angles, the choice $K = 2$ is dictated by the fact that the original Hamiltonian has a polynomial form in the secular variables. Concerning the concrete application described below, we just checked that $K = 2$ is an appropriate value also for the fast angles, in order to observe a nice numerical convergence of the algorithm.

		Jupiter ($j = 1$)	Saturn ($j = 2$)
mass	m_j	$(2\pi)^2/1047.355$	$(2\pi)^2/3498.5$
semi-major axis	a_j	5.20092253448245	9.55716977296997
mean anomaly	M_j	6.14053316064644	5.37386251998842
eccentricity	e_j	0.04814707261917873	0.05381979488308911
perihelion argument	ω_j	1.18977636117073	5.65165124779163
inclination	i_j	0.006301433258242599	0.01552738031933247
longitude of the node	Ω_j	3.51164756250381	0.370054908914043

TABLE 1. Masses and heliocentric orbital elements (given with respect to the invariant plane) for Jupiter and Saturn. We adopt the UA as unit of length, the year as time unit and set the gravitational constant $\mathcal{G} = 1$. With these units, the solar mass is equal to $(2\pi)^2$. The data are taken by JPL at the Julian Day 2451220.5 .

6. **Application to the SJS system.** The actual application to the SJS system is nothing but a straightforward implementation of our algorithm with the help of a specially designed algebraic manipulator, so that all the prescribed canonical transformations are performed by a computer. This requires all the series to be truncated at a finite number of terms, of course.

6.1. **Initial data and parameters.** Our aim is to consider the *real* SJS system. Thus, we use the values of the masses and the initial data taken from astronomical observations. Such actual values are reported in Table 1. However, in order to give the Hamiltonian the form (10) we should preliminarily determine the frequencies of the orbital motion of Jupiter and Saturn. In first approximation they are given by the third Kepler law, but this is a too crude choice for our purposes. Thus, following Laskar (see, e.g., [16]), we integrate the motion of the system over a time interval of 2^{24} years and then we apply the frequency analysis method to the signals $\Lambda_j(t)\exp\big(\mathrm{i}\lambda_j(t)\big)$, with $j = 1$, 2. Such an integration has been performed by using the symplectic integrator $\mathcal{SBAB}_{C3}$ (see [19]) in quadruple precision with a time-step of 0.08 years. Moreover, the signals have been sampled with a time interval of 4 years and treated with the so-called Hanning filter. The values of the frequency vector $\underline{\omega}^*$ so obtained are reported in Table 2.

6.2. **Approximated construction of the invariant torus.** The construction of the invariant torus has been performed using the parameters above. Let us recall the main steps and the conditions to be checked.

(i) We calculate the initial expansion of the Hamiltonian $H^{(\mathcal{T}_F)}$ as in (10). Here, we had to check that equation (9) has a solution for values of $\tilde{n}_1^*$, $\tilde{n}_2^*$ close enough to their approximations given by the third Kepler law. This requires that the matrix

$$\left(\frac{\partial^2 \langle H^{(R)} \rangle_{\underline{\lambda}}}{\partial \Lambda_i \partial \Lambda_j} \bigg|_{\underline{\xi}\,,\,\underline{\eta}=\underline{0}} \right)_{i,j=1,2}$$

fast frequencies	$\tilde{n}_1^* = 0.31643597303390$	$\tilde{n}_2^* = 0.10298153012338$
secular frequencies	$\mu g_1^* = -0.00014577520419$	$\mu g_2^* = -0.00026201915143$

Table 2. Angular frequencies vector $\underline{\omega}^* = (\tilde{n}_1^*, \tilde{n}_2^*, \mu g_1^*, \mu g_2^*)$ related to the motion of the SJS system. Here, μg_1^* and μg_2^* are the average frequencies related to the perihelion argument of Jupiter and Saturn, respectively; while the frequencies $\tilde{n}_1^*$ and $\tilde{n}_2^*$ correspond to the angular motions of $\tilde{\lambda}_1 = \lambda_1 - \lambda_2$ and $\tilde{\lambda}_2 = \lambda_1 - 2\lambda_2$, where λ_1 and λ_2 are defined in (6). The components of $\underline{\omega}^*$ are numerically calculated by using the frequency analysis method and they are given in rad/year.

related to the linearization of (9) is non–degenerate, when Λ_1, Λ_2 are in a suitable neighborhood of the values corresponding to the initial conditions listed in Table 1.

(ii) We perform the transformations outlined in Sect. 3, checking that the frequencies $(\tilde{n}_1^*, \tilde{n}_2^*)$ are not resonant up to order $K_F = 5$.

(iii) We diagonalize the quadratic terms of the secular part and calculate the partial Birkhoff normal form for the secular variables up to terms of degree 6, as outlined in sects. 4.1 and 4.2. Here, we had to check *(a)* that the four eigenvalues of the matrix (18) are distinct and of the form $(\pm i\nu_1,\ \pm i\nu_2)$, and *(b)* that the computed values

$$\nu_1 = -0.00013730417614\ , \qquad \nu_2 = -0.00022241593589$$

fulfill the condition (22). This is true, because the smallest divisors of the form $|\underline{k}\cdot\underline{\nu}|$ such that $|k_1| + |k_2| = 2\,,\,4\,,\,6$ are $|\pm\nu_1 \mp \nu_2| = 0.000085\ldots > 0$, which is actually the difference between the frequencies.

(iv) We translate the origin to the unperturbed torus with the frequencies given in Table 2. This can be done because the hypothesis (26) is verified and the solutions I_1^*, I_2^* of the equation (25) are both positive and so small that the power series expansion of the Hamiltonian $H^{(B)}$ in (24) is uniformly convergent with respect to the angles $(\underline{\tilde{\lambda}}, \underline{\varphi})$ in a neighborhood of $\underline{\tilde{L}} = \underline{0}$, $\underline{I} = \underline{I}^*$.

(v) Finally, we performed 17 steps of the Kolmogorov's normalization algorithm; at every step r, we checked that the frequency vector $\underline{\omega}^* = (\tilde{n}_1^*, \tilde{n}_2^*, \mu g_1^*, \mu g_2^*)$ is non–resonant up to order rK, with $K = 2$, and that the matrix $C^{(r)}$ is non-degenerate.

All the expansions have been truncated at the orders specified in Table 3. As a matter of fact, such limitations have been selected, on one hand, by trying to get the best possible agreement between the approximated dynamics on the quasi–invariant torus and the true one (i.e. that computed by direct numerical integration), and, on the other hand, by the need to keep the number of coefficients within the limits allowed by the computer resources available to us. As a rough indication, we had to manipulate series containing up to about 10^7 coefficients and our calculations needed a total CPU–time of about one week. We used an AMD Athlon MP 2200 processor with CPU clock of 1.8 Ghz, equipped with 2 Gbytes of RAM.

6.3. **Validation of the results.** In order to validate our procedure, we made three tests, namely: *(a)* we check the geometrical decrease to zero of the norms of the

Hamiltonian	polynomial degrees	trigometric degrees
$\tilde{h}^{(\mathrm{Kep})}(\underline{\tilde{L}})$	$\underline{\tilde{L}}^{\underline{j}}$ with $\lvert\underline{j}\rvert \le 4$	
$\tilde{h}^{(T_F)}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta})$	$\underline{\tilde{L}}^{\underline{j}}$ with $\lvert\underline{j}\rvert \le 3$, $\underline{\xi}^{\underline{i}}\underline{\eta}^{\underline{j}}$ with $\lvert(\underline{i},\underline{j})\rvert \le 10$	$\underline{k}\cdot\underline{\tilde{\lambda}}$ with $\lvert\underline{k}\rvert \le 20$
$H^{(\mathcal{O}2)}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{\xi},\underline{\eta})$	$\underline{\tilde{L}}^{\underline{j}}$ with $\lvert\underline{j}\rvert \le 3$, $\underline{\xi}^{\underline{i}}\underline{\eta}^{\underline{j}}$ with $\lvert(\underline{i},\underline{j})\rvert \le 12$	$\underline{k}\cdot\underline{\tilde{\lambda}}$ with $\lvert\underline{k}\rvert \le 20$
$\tilde{H}^{(B)}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{I},\underline{\varphi})$	$\underline{\tilde{L}}^{\underline{j}}$ with $\lvert\underline{j}\rvert \le 3$, $\sqrt{\underline{I}}^{\underline{i}}$ with $\lvert\underline{i}\rvert \le 20$	$\underline{k}\cdot\underline{\tilde{\lambda}}$ with $\lvert\underline{k}\rvert \le 20$
$\tilde{H}^{(T_S)}(\underline{\tilde{L}},\underline{\tilde{\lambda}},\underline{J},\underline{\psi})$	$\underline{\tilde{L}}^{\underline{i}}\underline{J}^{\underline{j}}$ with $\lvert\underline{i}\rvert+\lvert\underline{j}\rvert \le 3$	$\underline{k}\cdot(\underline{\tilde{\lambda}},\underline{\psi})$ with $\lvert\underline{k}\rvert \le 34$
$\forall\ r\ge 1\,,\ H^{(r)}(\underline{p},\underline{q})$	$\underline{p}^{\underline{j}}$ with $\lvert\underline{j}\rvert \le 3$	$\underline{k}\cdot\underline{q}$ with $\lvert\underline{k}\rvert \le 34$

Table 3. Truncation rules on the expansions of the Hamiltonians, as we have explicitly calculated on a computer by applying the algorithm constructing the Kolmogorov's normal form to the SJS system. Such rules are shortly described by using the multi-index notation: for instance, $\underline{\tilde{L}}^{\underline{j}} = \tilde{L}_1^{j_1}\tilde{L}_2^{j_2}$ and $\lvert\underline{j}\rvert = \lvert j_1\rvert + \lvert j_2\rvert$. For what concerns $H^{(B)}$, the limitations over the expansions in the angles φ are consequent to those given for the actions $\underline{I}$ (see subsect. 4.2).

generating functions; *(b)* we evaluate the loss of accuracy due to the truncations in the expansions of the Hamiltonian related to the construction of the Kolmogorov's normal form; *(c)* we compare the dynamics on the quasi–invariant torus with the "true dynamics" calculated via numerical integration.

The results concerning the first test are reported in Figure 1, where the norms of the generating functions $\chi_1^{(r)}(\underline{q})$ and $\chi_2^{(r)}(\underline{p},\underline{q})$ are plotted as functions of the normalization step r . The norm is calculated by merely adding up the absolute value of the coefficients appearing in the expansions of the generating functions. It is quite evident that after a few initial steps the decrease of the norm becomes rather sharp. This corresponds to the (geometrical decreasing) behaviour expected from the theoretical estimates. Although this is far from being a formal proof, we consider it as a strong indication of the convergence of the algorithm. The second test goes as follows. Considering the Hamiltonian corresponding to a certain step of our procedure, we calculate the initial point of the orbit in the corresponding variables. Then, we compare the value of the energy given by such Hamiltonian with the corresponding value given by the original Hamiltonian in the original variables. The calculated differences are reported in Table 4. It is interesting to notice that the biggest error is introduced by the Birkhoff transformation.

As a third test we compare the motion calculated by numerical integration with the quasi–periodic motion on the approximated KAM torus. Let us refer to the

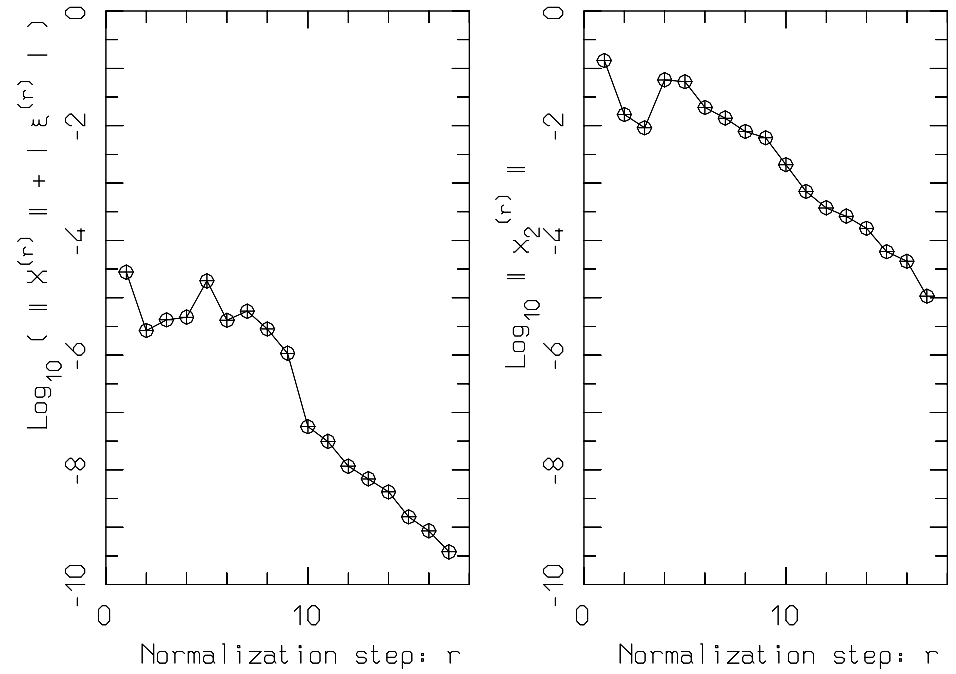

FIGURE 1. Numerical study of the convergence of the algorithm constructing the Kolmogorov's normal form for the SJS system. On the left, we consider the sum of the norms related to $X^{(r)}(\underline{q})$ and to $\underline{\xi}^{(r)}\cdot\underline{q}$ (i.e. the two terms giving the generating function $\chi_1^{(r)}(\underline{q})$). On the right, the norms of $\chi_2^{(r)}(\underline{p},\underline{q})$ are reported. In both cases, the plots are in a semi-log scale and the behaviours as a function of the normalization step r are studied.

calculation scheme

$$
\begin{array}{ccc}
\big(\underline{a}(0),\underline{\lambda}(0),\underline{e}(0),\underline{\omega}(0)\big) & \xrightarrow{\left(\mathcal{C}^{(\infty)}\right)^{-1}\circ\,\mathcal{E}^{-1}} & \big(\underline{p}(0)=\underline{0}\,,\,\underline{q}(0)\big) \\
 & & \Big\downarrow\,\Phi^t_{\underline{\omega}^*\cdot\underline{p}} \\
\big(\underline{a}(t),\underline{\lambda}(t),\underline{e}(t),\underline{\omega}(t)\big) & \xleftarrow{\mathcal{E}\circ\,\mathcal{C}^{(\infty)}} & \big(\underline{p}(t)=\underline{p}(0)\,,\,\underline{q}(t)=\underline{q}(0)+\underline{\omega}^*t\big)
\end{array}\ ,
\tag{38}
$$

where $(\underline{\Lambda},\underline{\lambda},\underline{\xi},\underline{\eta})=\mathcal{E}^{-1}(\underline{a},\underline{\lambda},\underline{e},\underline{\omega})$ is the (non-canonical) change of coordinates described in (6) and $\Phi^t_{\underline{\omega}^*\cdot\underline{p}}$ induces the quasi–periodic flow with frequency vector $\underline{\omega}^*$. We stress that the motion of the orbital elements $\underline{i}$ and $\underline{\Omega}$, which are missing in the calculation scheme above, can be obtained by using the conservation of the angular momentum $\underline{C}$, as described in sect. 5.1 of [14]. Let us explain the practical use of diagram (38) in some more detail.

In our concrete calculation we must replace the transformation $\mathcal{C}^{(\infty)}$ with a finite order transformation $\mathcal{C}^{(r)}$, meaning that we perform just r steps of the Kolmogorov's

relative error in energy	num. value
$\frac{1}{E}\Big[E - \tilde{H}^{(\mathcal{T}_F)} \circ \big(\mathcal{T}_F \circ \mathcal{U}\big)^{-1}(\underline{\Lambda}_0, \underline{\lambda}_0, \underline{\xi}_0, \underline{\eta}_0)\Big]$	0.549×10^{-9}
$\frac{1}{E}\Big[E - H^{(O2)} \circ \big(\mathcal{T}_F \circ \mathcal{U} \circ \mathcal{K}_{O2}\big)^{-1}(\underline{\Lambda}_0, \underline{\lambda}_0, \underline{\xi}_0, \underline{\eta}_0)\Big]$	1.162×10^{-9}
$\frac{1}{E}\Big[E - \tilde{H}^{(B)} \circ \big(\mathcal{T}_F \circ \mathcal{U} \circ \mathcal{K}_{O2} \circ \mathcal{D} \circ \mathcal{A} \circ \tilde{\mathcal{B}}\big)^{-1}(\underline{\Lambda}_0, \underline{\lambda}_0, \underline{\xi}_0, \underline{\eta}_0)\Big]$	6.188×10^{-8}
$\frac{1}{E}\Big[E - \tilde{H}^{(\mathcal{T}_S)} \circ \big(\mathcal{T}_F \circ \mathcal{U} \circ \mathcal{K}_{O2} \circ \mathcal{D} \circ \mathcal{A} \circ \tilde{\mathcal{B}} \circ \mathcal{T}_S\big)^{-1}(\underline{\Lambda}_0, \underline{\lambda}_0, \underline{\xi}_0, \underline{\eta}_0)\Big]$	6.189×10^{-8}
$\frac{1}{E}\Big[E - H^{(17)} \circ \big(\mathcal{C}^{(17)}\big)^{-1}(\underline{\Lambda}_0, \underline{\lambda}_0, \underline{\xi}_0, \underline{\eta}_0)\Big]$	6.188×10^{-8}

TABLE 4. Study of the loss of accuracy due to the truncation rules introduced on the expansions of the Hamiltonians, related to the construction of the Kolmogorov's normal form for the SJS system. The relative errors about the initial energy are reported. More precisely, the value of energy $E = H^{(R)}(\underline{\Lambda}_0, \underline{\lambda}_0, \underline{\xi}_0, \underline{\eta}_0)$, where $H^{(R)}$ is the Hamiltonian (7) and $(\underline{\Lambda}_0, \underline{\lambda}_0, \underline{\xi}_0, \underline{\eta}_0)$ are the initial conditions related to those in Table 1. The initial energy E is compared with the values assumed by the Hamiltonians $\tilde{H}^{(\mathcal{T}_F)}$, $H^{(O2)}$, $\tilde{H}^{(B)}$, $\tilde{H}^{(\mathcal{T}_S)}$ and $H^{(17)}$ (as defined in sects. 3–5) correspondingly to the canonical coordinates associated to $(\underline{\Lambda}_0, \underline{\lambda}_0, \underline{\xi}_0, \underline{\eta}_0)$. During such calculations, the same truncations rules reported in Table 3 have been applied on the expansions of the canonical transformations. The definition of $\mathcal{C}^{(17)}$ is given in (39).

normalization algorithm. Thus, the complete transformation from the orbital elements to the coordinates of the Kolmogorov's normal form at order r is

$$\mathcal{C}^{(r)} = \mathcal{T}_F \circ \mathcal{U} \circ \mathcal{K}_{\mathcal{O}2} \circ \mathcal{D} \circ \mathcal{A} \circ \mathcal{B} \circ \mathcal{T}_S \circ \mathcal{K}^{(r)} \ . \tag{39}$$

In our case, we set $r = 17$. Having fixed the initial values of the orbital elements, we perform in reverse order all the inverse transformations required by the algorithm, so that we determine the initial coordinates $(\underline{p}^{(r)}(0), \underline{q}^{(r)}(0))$ in the variables of the Hamiltonian $H^{(r)}$. As a matter of fact, we consider the initial data as lying on the "true" KAM torus $\underline{p}^{(\infty)} = \underline{0}$ with frequencies $\underline{\omega}^*$, which is just approximated by the manifold $\underline{p}^{(r)} = \underline{0}$.

Now, we calculate the evolution by replacing the flow of the Hamiltonian $H^{(r)}$ with the linear flow with frequencies $\underline{\omega}^*$. Finally, we perform all the transformations back to the orbital elements, thus getting $\big(\underline{a}(t), \underline{\lambda}(t), \underline{e}(t), \underline{\omega}(t)\big)$ calculated through the Kolmogorov's flow.

On the other hand, we calculate the orbital elements at time t via direct integration of the canonical equation for the Hamiltonian (5), thus getting, say, $\big(\underline{\tilde{a}}(t), \underline{\tilde{\lambda}}(t), \underline{\tilde{e}}(t), \underline{\tilde{\omega}}(t)\big)$. The differences $|\underline{a}(t) - \underline{\tilde{a}}(t)|$, etc., give an indication of the accuracy of the determination of the invariant torus.

The results for $r = 17$ are reported in Figure 2. We plotted the differences for the four elements above as a function of time for a time interval of 10^8 years. Let

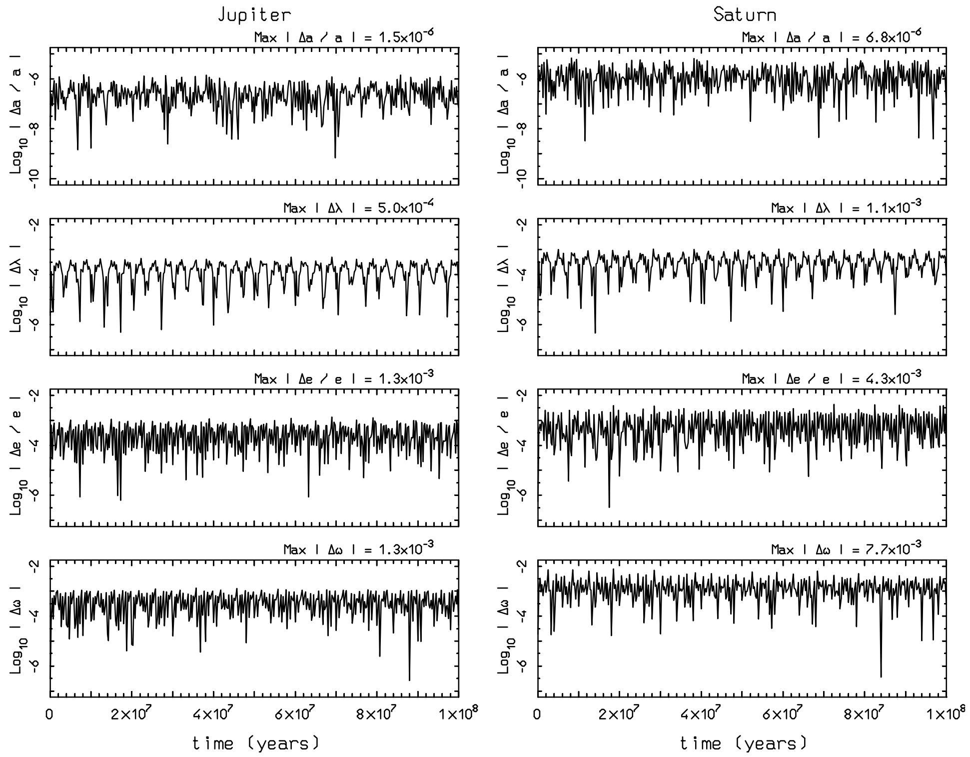

FIGURE 2. Test on the reliability of the construction of the Kolmogorov's normal form for the SJS system. All the boxes report (in a semi-log scale) the time dependency of the discrepancies about the orbital elements between a numerical integration and the semi-analytic one, based on an approximation of the scheme (38); see the text for more details. The comparisons related to Jupiter are on the left, while those about Saturn are on the right. Each drawing is based on 400 equally time-spaced points. The maximal value of the ordinate of the plotted points is reported near to the top–right corner of each box.

us stress that the time spanned by the comparison is a not completely negligible fraction of the age of the solar system (which is estimated about 5 billions of years).

In Figure 2 one sees that there is an excellent agreement for what concerns the semi-major axes $\underline{a}$, the discrepancies are somehow bigger for the mean longitudes $\underline{\lambda}$, while the worst case is that of the secular orbital elements $\underline{e}$ and $\underline{\omega}$. This should be expected, since the secular motion itself is due to the perturbing terms. Let us emphasize that each maximum related to the plots of $\Delta a(t)$, $\Delta\lambda(t)$, $\Delta e(t)$ and $\Delta\omega(t)$ (for both planets) does not significantly increase with respect to time. This on the one hand confirms that the motion of the SJS system is very close to be quasi-periodic (as results in the numerical investigation in [17]) and, on the other hand, that our determination of the frequencies is pretty good. Indeed, an error in the determination of the frequencies would be reflected in a steady drift of the difference in the angles $\underline{\lambda}$ and $\underline{\omega}$, and so the maximal error should sistematically increase in

	$\mathrm{Max}_t \left\{ \left| \frac{\Delta a(t)}{a(t)} \right| \right\}$	$\mathrm{Max}_t \left\{ \left| \Delta\lambda(t) \right| \right\}$	$\mathrm{Max}_t \left\{ \left| \frac{\Delta e(t)}{e(t)} \right| \right\}$	$\mathrm{Max}_t \left\{ \left| \Delta\omega(t) \right| \right\}$
Jupiter	1.0×10^{-6}	2.5×10^{-4}	9.5×10^{-4}	1.0×10^{-3}
Saturn	4.8×10^{-6}	7.1×10^{-4}	3.0×10^{-3}	4.5×10^{-3}

Table 5. Maximal discrepancies about the orbital elements of the *real* SJS system between a numerical integration and the semi-analytic one, based on an approximation of the scheme (38). The values reported here have been determined in exactly the same way as those reported in Figure 2. In the present case, the comparisons are made over a time interval ranging from 0.3 Myr in the past to 0.3 Myr in the future.

time. In fact, this effect has been observed in Figure 2 of [21], because in that case the significantly reduced size of the perturbation caused a worst determination of the secular frequencies. In the present case, the same effect should eventually be observed by performing a much longer integration.

It might be interesting to remark that a similar calculation, namely approximating the dynamics of the SJS system with a quasi–periodic motion, has been performed by Bretagnon and Simon ([3]). Their method was based on the technique introduced by Krylov and Bogoliubov (which is reminiscent of the Lindstedt's series method). Indeed, they calculated by formal expansions the solutions of the equations of motion corresponding to the initial data and the parameters of the real system. This makes their method much simpler than ours, from the viewpoint of the calculation of the planetary orbits. As a matter of fact, the comparison between the data in Table 15 of [3] and the corresponding Table 5 obtained by our method shows that the results are essentially the same within a factor 10. On the other hand, we emphasize that our goal was not to produce an effective method to compute the orbits of the planet, but rather to check the applicability of Kolmogorov's theory to a planetary system. Thus, we must take into account in our expansions also the actions of the system which makes the calculation definitely more difficult and time consuming.

Acknowledgements. The authors have been supported by the research program "Sistemi dinamici non lineari e applicazioni fisiche", prot. n. 2003015411, financed by MIUR.

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Received January 2006; revised July 2006; final version November 2006.

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