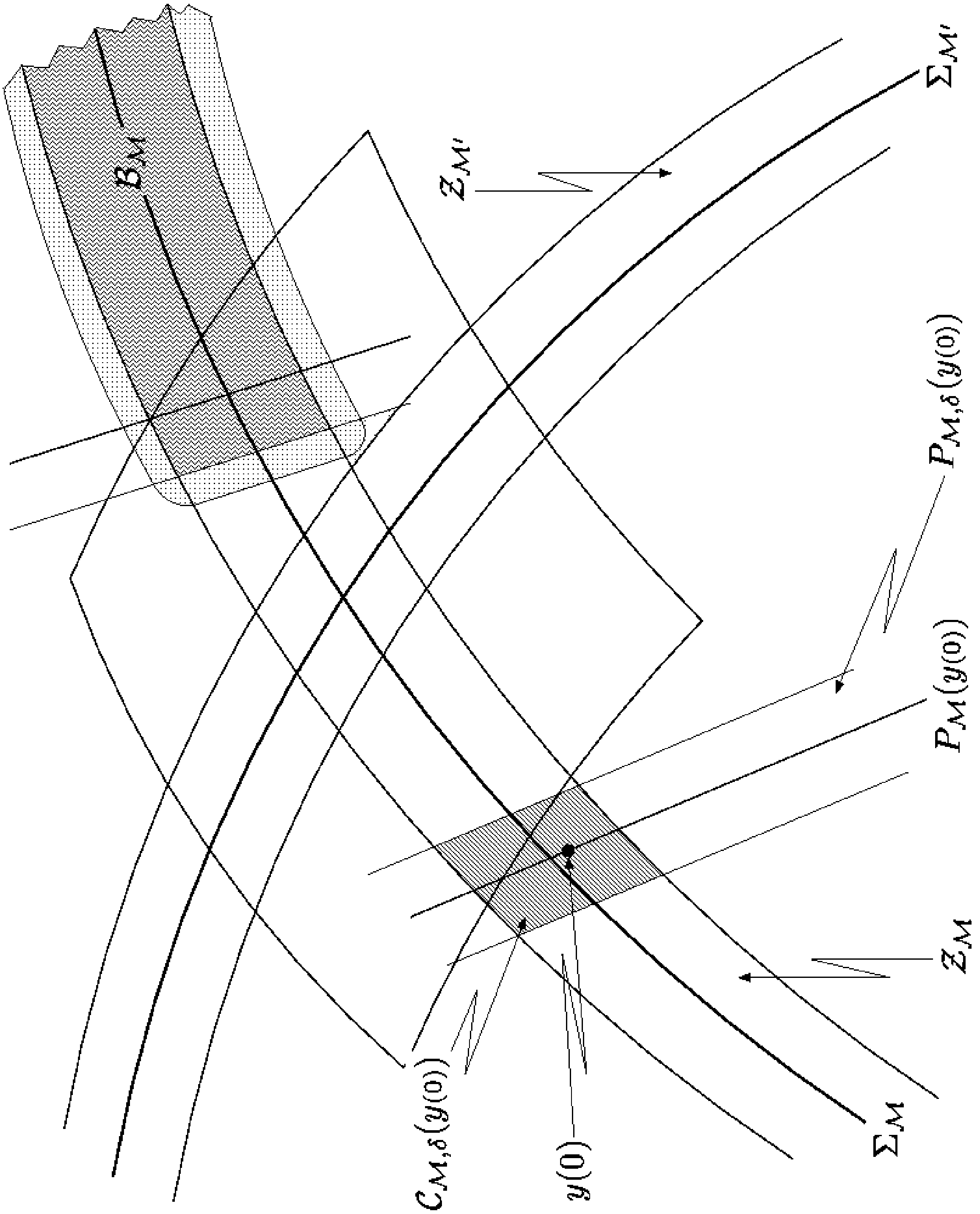
$\Sigma_{\mathcal{M}'}$
$\mathcal{Z}_{\mathcal{M}'}$
$B_{\mathcal{M}}$
$P_{\mathcal{M},\delta}(y(0))$
$P_{\mathcal{M}}(y(0))$
$\mathcal{Z}_{\mathcal{M}}$
$\Sigma_{\mathcal{M}}$
$y(0)$
$C_{\mathcal{M},\delta}(y(0))$

**Figure 3**

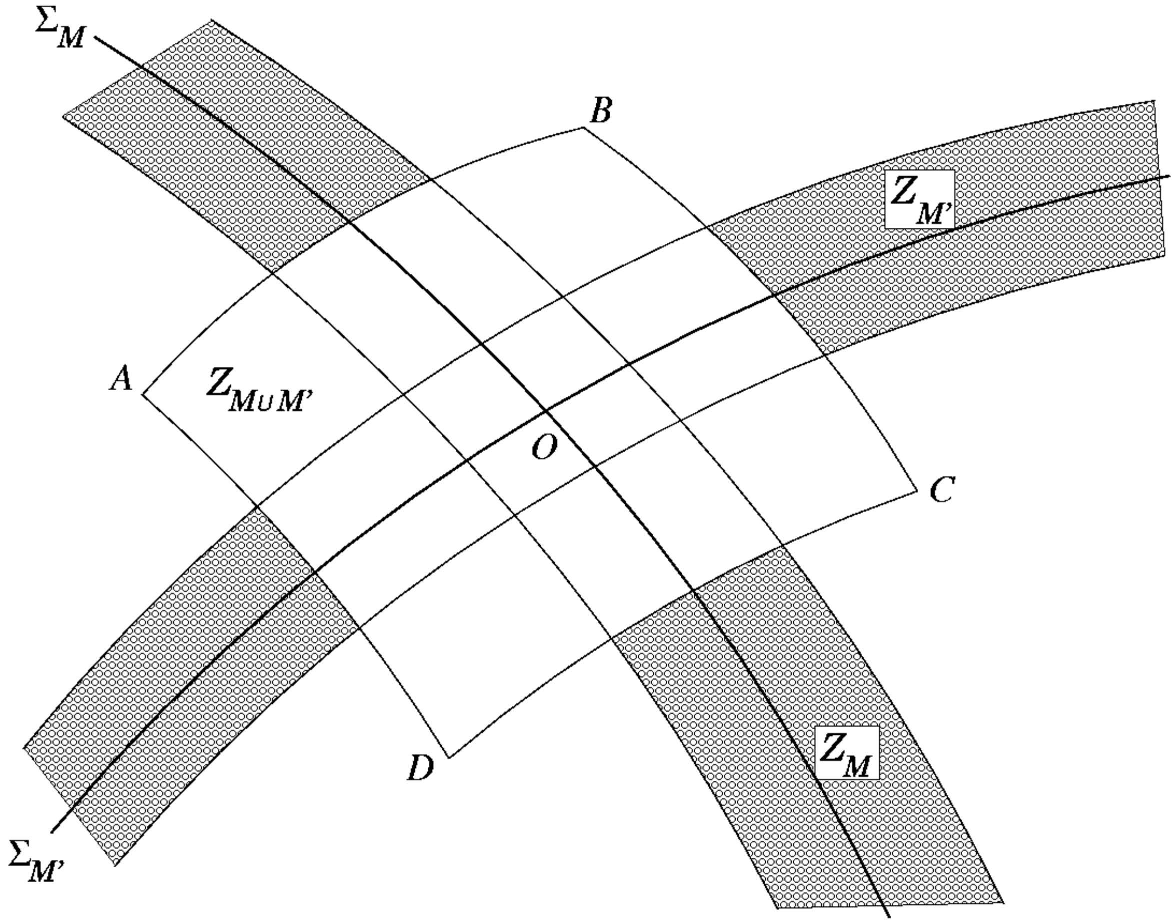

Figure 2

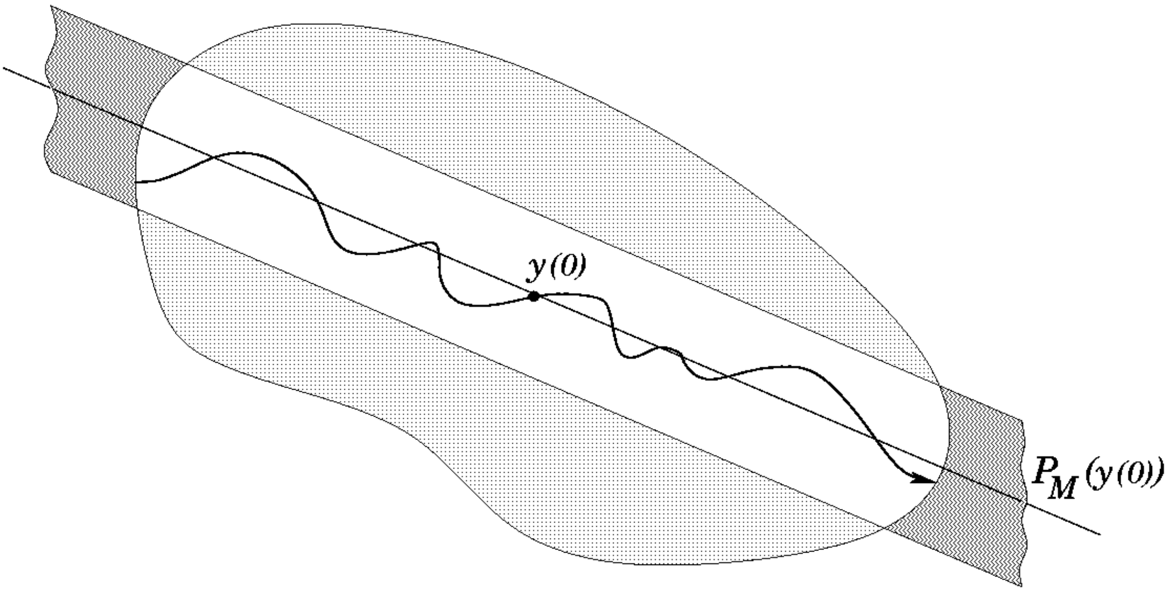

**Figure 1**

*while* $\{f, \xi\} = 0$.

We would like to thank Jürgen Pöschel for valuable comments.

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By the definition of the norm

$$\begin{aligned} \| \{f,g\} \|_{(1-d'-d)(\delta,\sigma)} < \sum_{k,k'} \Bigg[ \sum_{l=1}^{n} \Bigg( & |k_l| \left| \frac{\partial g_{k'}}{\partial y_l} \right|_{(1-d'-d)(\delta,\sigma)} |f_k|_{(\delta,\sigma)} \\ & + |k'_l| \left| \frac{\partial f_{k'}}{\partial y_l} \right|_{(1-d'-d)(\delta,\sigma)} |g_k|_{(1-d')(\delta,\sigma)} \Bigg) \Bigg] e^{(1-d'-d)|k+k'|\sigma} \ . \end{aligned} \tag{72}$$

Using now the Cauchy's inequalities

$$\left| \frac{\partial g_{k'}}{\partial y_l} \right|_{(1-d'-d)(\delta,\sigma)} \le \frac{1}{d\delta} |g'_k|_{(1-d')(\delta,\sigma)} \ , \qquad \left| \frac{\partial f_k}{\partial y_l} \right|_{(1-d'-d)(\delta,\sigma)} \le \frac{1}{(d'+d)\delta} |f_k|_{(\delta,\sigma)}$$

the right hand side of (72) is smaller than

$$\begin{aligned} & \frac{1}{d\delta} \sum_{k,k'} |g'_k|_{(1-d')(\delta,\sigma)} e^{(1-d')|k'|\sigma} |f_k|_{(\delta,\sigma)} e^{|k|\sigma} \sum_{l=1}^{n} |k_l| e^{-(d'+d)|k|\sigma} \\ & \qquad + \frac{1}{(d'+d)\delta} \sum_{k,k'} |g'_k|_{(1-d')(\delta,\sigma)} e^{(1-d')|k'|\sigma} |f_k|_{(\delta,\sigma)} e^{|k|\sigma} \sum_{l=1}^{n} |k'_l| e^{-d|k'|\sigma} \ . \end{aligned} \tag{73}$$

Remark now that the only terms depending on the index $l$ are $\sum_l |k_l| = |k|$ and $\sum_l |k'_l| = |k'|$; moreover, using the inequality

$$x e^{-\alpha x} \le \frac{1}{\alpha e} \quad \text{for } x > 0 \ , \ \alpha > 0 \ ,$$

one concludes

$$|k| e^{-(d'+d)|k|\sigma} \le \frac{1}{(d'+d)e\sigma} \ , \qquad |k'| e^{-d|k'|\sigma} \le \frac{1}{de\sigma} \ .$$

Reordering the terms, the expression in (73) is smaller than

$$\frac{2}{ed(d'+d)\delta\sigma} \sum_{k'} |g'_k|_{(1-d')(\delta,\sigma)} e^{(1-d')|k'|\sigma} \cdot \sum_{k} |f_k|_{(\delta,\sigma)} e^{|k|\sigma} \ ,$$

and the conclusion follows in view of the definition of the norms $\|f\|_{(\delta,\sigma)}$ and $\|g\|_{(1-d')(\delta,\sigma)}$ respectively. *Q.E.D.*

Similarly one proves the following estimates needed in order to relate our constants to those in ref. [9].

**Lemma 10:** *If* $f = f(x,y,\xi)$ *is analytic in* $\mathcal{G}_{(\delta,\sigma)}$ *then for* $0 < d < 1$ *and for* $1 \le j \le n$

$$\begin{aligned} \| \{f, y_j\} \|_{(1-d)(\delta,\sigma)} &\le \frac{1}{de\sigma} \|f\|_{(\delta,\sigma)} \\ \| \{f, x_j\} \|_{(1-d)(\delta,\sigma)} &\le \frac{1}{d\delta} \|f\|_{(\delta,\sigma)} \\ \| \{f, \eta\} \|_{(1-d)(\delta,\sigma)} &\le \frac{1}{d\sigma} \|f\|_{(\delta,\sigma)} \ , \end{aligned}$$

one finds, in view of lemma 8,

$$A = \frac{2e}{d^2\delta\sigma}\left[\left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^n |V|_{2\sigma} + \frac{e\delta}{2}\right] \ .$$

The value of $A$ in proposition 1 is obtained if we set $d = 1/(4\sqrt{n})$. Inserting $A$ and $h = e^{-K\sigma/2}$ in the estimate (69), the condition (54) guaranteeing the existence of the transformation into normal form becomes

$$\varepsilon = \frac{rA}{\alpha} + 2e^{-K\sigma/2} \leq \frac{1}{2} \ . \tag{70}$$

If (70) holds true one can prove that the remainder $\mathcal{R}^{(r)}$ in (55) as defined in (59) is estimated by

$$|\mathcal{R}^{(r)}|_{(1-2d)(\delta,\sigma)} < 2F\varepsilon^r \ , \tag{71}$$

with $F$ given in lemma 8 as

$$F = \left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^n |V|_{2\sigma} \ .$$

For a proof of the crucial estimate (71) we can again refer to [9], sect. 12. This finishes the proof of proposition 1.

We should add that the precise constants above follow from the reference [9] if one uses the following estimates for the Poisson bracket for our functions $f = f(x, y, \xi)$ defined by (40)

**Lemma 9:** *If $f$ and $g$ are analytic functions defined on $\mathcal{G}_{(\delta,\sigma)}$ and $\mathcal{G}_{(1-d')(\delta,\sigma)}$ respectively for some $0 \leq d' < 1$, then for every $0 < d < 1 - d'$*

$$\|\{f, g\}\|_{(1-d'-d)(\delta,\sigma)} \leq C\|f\|_{(\delta,\sigma)}\|g\|_{(1-d')(\delta,\sigma)} \ ,$$

*where*

$$C = \frac{2}{ed(d+d')\delta\sigma}$$

**Proof.** Using the Fourier expansions

$$f(y, x, \xi) = \sum_k f_k(y, \xi)e^{ik\cdot x} \ , \quad g(y, x, \xi) = \sum_k g_k(y, \xi)e^{ik\cdot x} \ ,$$

one computes

$$\{f, g\} = i\sum_{k,k'}\left[\sum_{l=1}^{n}\left(k_l\frac{\partial g_{k'}}{\partial y_l}f_k - k'_l\frac{\partial f_{k'}}{\partial y_l}g_k\right)\right] e^{i(k+k')\cdot x} \ .$$

**Lemma 8:** *Assume $V$ is analytic in the strip $|\mathrm{Im}\, x| < 2\sigma$ and $|\mathrm{Im}\, \xi| < 2\sigma$, and*

$$|V|_{2\sigma} < \infty .$$

*Then,*

$$\|H_j\|_\sigma = \|V_j\|_\sigma \le h^{j-1} F \ , \quad j \ge 1 \ ,$$

*with the constants $h$ and $F$ defined by*

$$h = e^{-K\sigma/2} \ , \quad F = \left( \frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}} \right)^n |V|_{2\sigma} \ .$$

**Proof.** The proof is an immediate consequence of the well known estimate

$$|v_k|_\sigma \le |V|_{2\sigma}\, e^{-2|k|\sigma}$$

for the Fourier coefficients $v_k(\xi)$ of the analytic function $V = V(x,\xi)$. Indeed, by definition of the norm,

$$\begin{aligned} \|V_j\|_\sigma &= \sum_{(j-1)K \le |k| < jK} |v_k|_\sigma e^{|k|\sigma} \\ &\le |V|_{2\sigma} \sum_{(j-1)K \le |k|} e^{-|k|\sigma} \ , \end{aligned}$$

and the claim follows in view of the following estimate. For any positive integer $m$,

$$\begin{aligned} \sum_{|k| \ge m} e^{-|k|\sigma} &\le e^{-m\sigma/2} \sum_{|k|\ge m} e^{-|k|\sigma/2} \\ &\le e^{-m\sigma/2} \left( \sum_{j \in \mathbf{Z}} e^{-|j|\sigma/2} \right)^n \\ &= e^{-m\sigma/2} \left( \frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}} \right)^n \ . \end{aligned}$$

*Q.E.D.*

We finally express the constants $G$ and $b$ in the assumptions of lemma 6 by their values found in lemma 7 and 8, and estimate

$$\begin{aligned} \frac{2Ge}{d^2\delta\sigma} + b &= \frac{2eF}{d^2\delta\sigma\alpha} + \frac{12(r-1)}{ed^2\delta\sigma\alpha} \left( F + \frac{e\delta}{2} \right) + 2h \\ &< \frac{2e^2 + 12(r-1)}{ed^2\delta\sigma\alpha} \left( F + \frac{e\delta}{2} \right) + 2h \qquad (69) \\ &\le \frac{2er}{d^2\delta\sigma\alpha} \left( F + \frac{e\delta}{2} \right) + 2h \end{aligned}$$

where we have used that $2e^2 + 12(r-1) \le 2e^2 r$ if $r \ge 1$. Defining

$$A = \frac{2e}{d^2\delta\sigma} \left( F + \frac{e\delta}{2} \right) \ ,$$

with $\Psi_s^{(2)}$ having all the Fourier coefficients in $\Psi_s$ belonging to $k \in \mathcal{M}$. We, therefore, set

$$Z_s = \Psi_s^{(2)} \in \mathcal{P}_s \cap \mathcal{N}_{\mathcal{M}} \ . \tag{62}$$

It remains to solve the partial differential equation

$$L_{H_0}\chi_s = \sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j}\chi_s = \Psi_s^{(1)} \ . \tag{63}$$

This is easily done by comparing the Fourier coefficients. Note that $y \in \mathcal{G}_\delta$ satisfies, by assumption, the estimate $|k \cdot y| \geq \alpha > 0$ if $k \notin \mathcal{M}$ and $|k| \leq N$. On the other hand, $\Psi_s^{(1)}$ contains only Fourier coefficients for $k \notin \mathcal{M}$ and $|k| \leq N$. Therefore, we find a solution $\chi_s$ which satisfies the estimate

$$\|\chi_s\|_{(1-d)(\delta,\sigma)} \leq \frac{1}{\alpha}\,\|\Psi_s\|_{(1-d)(\delta,\sigma)} \ . \tag{64}$$

In addition, $Z_s$ satisfies trivially

$$\|Z_s\|_{(1-d)(\delta,\sigma)} \leq \|\Psi_s\|_{(1-d)(\delta,\sigma)} \ . \tag{65}$$

Note that $\chi_s$ does not depend on the variable $\eta$, so that $T_\chi(\xi) = \xi$. Using the estimates (64) and (65) and the estimates for the Poisson brackets in lemmas 9 and 10 below, one concludes the following estimates for the generating sequence in terms of the given Hamiltonian:

**Lemma 7:** *Assume that $H = H_0 + \eta + H_1 + H_2 + \ldots$ with $H_s \in \mathcal{P}_s$ satisfies, on $\mathcal{G}_{(\delta,\sigma)}$, the estimates*

$$\|H_s\|_{(\delta,\sigma)} \leq h^{s-1}F \quad \text{for} \quad s \geq 1 \ , \tag{66}$$

*with two constants $h \geq 0$ and $F > 0$. Then, the generating sequence $\chi^{(r)} = \{\chi_1, \ldots, \chi_r\}$ defined above satisfies the estimates*

$$\|\chi_s\|_{(1-d)(\sigma,\delta)} \leq \frac{b^{s-1}}{s}G \quad \text{for} \quad 1 \leq s \leq r \ , \tag{67}$$

*with the constants $b$ and $G$ defined by*

$$b = \frac{12(r-1)}{ed^2\delta\sigma\alpha}\left(F + \frac{e\delta}{2}\right) + 2h \ , \quad G = \frac{F}{\alpha} \ , \tag{68}$$

*where $\alpha > 0$ is associated to our domain by (46), and where $rK = N$.*

The proof is carried out in [9], sect. 11. We next determine the constants $h$ and $F$ in the above lemma for our special Hamiltonian $H$ given by (45). Recall that $V = \sum_{j\geq 1} V_j$, and

$$H_j = V_j(x,\xi) = \sum_{(j-1)K \leq |k| < jK} v_k(\xi)e^{ik\cdot x}$$

does not depend on the variables $y$.

The proof of this lemma, with only minor modifications in the notations can be found in [9], sect. 10. We shall now use the lemma in order to construct a symplectic transformation $T = T_{\chi^{(r)}}$ with a finite generating sequence $\chi^{(r)} = \{\chi_1, \ldots, \chi_r\}$, i.e., $\chi_s = 0$ for $s > r$, transforming the Hamiltonian in the following normal form

$$T_{\chi^{(r)}} H = H_0 + \eta + Z^{(r)} + \mathcal{R}^{(r)} \ , \quad Z^{(r)} = Z_1 + \ldots + Z_r \ , \tag{55}$$

where $Z_j \in \mathcal{P}_j$ is in normal form, i.e., $Z_j \in \mathcal{N}_{\mathcal{M}}$.

In view of the expansion for $H$ given by (45) and in view of the definition (48) for $T$ the equation (55) becomes

$$\sum_{s\geq 1} E_s H_0 + \sum_{s\geq 1} E_s \eta + \sum_{s\geq 1} \sum_{l=1}^{s} E_{s-l} H_l = \sum_{s=1}^{r} Z_r + \mathcal{R}^{(r)} \ . \tag{56}$$

Introducing an artificial parameter $\varepsilon$ and writing (45) in the form

$$H = H_0(y) + \varepsilon\eta + \sum_{j\geq 1} \varepsilon^j H_j$$

we now compare the terms in (56) of the same order in $\varepsilon$ and find the following equations. Abbreviating

$$\begin{aligned} \Psi_1 &= H_1 \\ \Psi_s &= E_{s-1}\eta + \sum_{l=1}^{s-1} \frac{l}{s} L_{\chi_l} E_{s-l} H_0 + \sum_{l=1}^{s} E_{s-l} H_l \ , \quad 2 \leq s \leq r \ , \end{aligned} \tag{57}$$

for $2 \leq s \leq r$ one readily verifies by induction that $\Psi_s \in \mathcal{P}_s$. Moreover, $\Psi_s$ contains only $\chi_j$ for $1 \leq j \leq s-1$, so that it will be recursively known. With (57) the equation (56) becomes

$$L_{H_0} \chi_s + Z_s = \Psi_s \ , 1 \leq s \leq r \ , \tag{58}$$

and $\mathcal{R}^{(r)}$ is given by

$$\mathcal{R}^{(r)} = \sum_{s>r} H_s^{(r)} \ , \quad H_s^{(r)} \in \mathcal{P}_s \ , \tag{59}$$

where $H_s^{(r)}$ is defined as

$$H_s^{(r)} = E_s H_0 + E_{s-1}\eta + \sum_{l=1}^{s} E_{s-l} H_l \ , \quad s > r \ . \tag{60}$$

The problem now is first to solve the equations (58) for the two unknown functions $\chi_s \in \mathcal{P}_s$ and $Z_s \in \mathcal{P}_s \cap \mathcal{N}_{\mathcal{M}}$ with the recursively known right hand side $\Psi_s$. After this we shall have to estimate the remainder term $\mathcal{R}^{(r)}$. As to the first problem we observe that there is an unique splitting

$$\Psi_s = \Psi_s^{(1)} + \Psi_s^{(2)} \ , \tag{61}$$

It turns out that the formal inverse of $T_\chi$ exists and is represented by

$$T_\chi^{-1} = \sum_{s\geq 0} D_s \ , \tag{50}$$

with

$$D_0 = \mathrm{Id} \ , \quad D_s = -\sum_{j=1}^{s} \frac{j}{s} D_{s-j} L_{\chi_j} \ . \tag{51}$$

As a side remark we point out a familiar special case. Choosing $\chi$ to be the generating sequence with $\chi_1 = \varphi$ any function, and $\chi_s = 0$ for all $s > 1$, one finds $E_s = (L_\varphi)^s/s!$, and $T = T_\chi$ is given by

$$T = \exp(L_\varphi) \ ,$$

consequently, $T^{-1} = \exp(-L_\varphi) = \exp(L_{-\varphi})$.

It can be shown that $T = T_\chi$ defined by (48) preserves products and Poisson brackets, i.e., for two functions $f$ and $g$ one has $T(fg) = (Tf)\cdot(Tg)$ and $T\{f,g\} = \{Tf, Tg\}$. Consequently, if we let $T$ act on the canonical coordinate function $z$ we can define a symplectic transformation by setting

$$z' = T(z) \ . \tag{52}$$

It satisfies, moreover,

$$\big(Tf\big)(z) = f\big(T(z)\big) \ . \tag{53}$$

The proofs of all these statements are easy and can be found in [10].

After these algebraic considerations we formulate a quantitative existence statement used in the following. It gives conditions under which the formal expansions converge.

**Lemma 6:** *Let $\chi = \{\chi_s\}_{s\geq 1}$ be a generating sequence such that $\chi_s \in \mathcal{P}_s$ are analytic functions on $\mathcal{G}_{(\delta,\sigma)}$ satisfying the estimates $\|\chi_s\|_{(\delta,\sigma)} \leq b^{s-1}G/s$ for $s \geq 1$ with two constants $b \geq 0$ and $G > 0$. Then if $1/2 \leq d < 1$ and if*

$$\frac{2Ge}{d^2\delta\sigma} + b \leq \frac{1}{2} \tag{54}$$

*the transformation $T = T_\chi$ defined by (48) and (52) and $T^{-1}$ defined by (50) converge and define holomorphic and symplectic mappings satisfying*

$$\mathcal{G}_{(1-2d)(\delta,\sigma)} \subset T\mathcal{G}_{(1-d)(\delta,\sigma)} \subset \mathcal{G}_{(\delta,\sigma)} \ ,$$

*and the same for $T^{-1}$. Moreover, $T(\xi) = \xi$ and*

$$\begin{aligned} |T(y) - y|_{(1-d)(\delta,\sigma)} &< d\delta \ , \\ |T(\eta) - \eta|_{(1-d)(\delta,\sigma)} &< d\delta \ , \\ |T(x) - x|_{(1-d)(\delta,\sigma)} &< d\sigma \ . \end{aligned}$$

*The same estimates hold true if $T$ is replaced by $T^{-1}$.*

and belongs to $\mathcal{P}_j$. The Hamiltonian function

$$H(x,y,\xi,\eta)=\frac{1}{2}|y|^2+\eta+V(x,\xi)\ . \tag{44}$$

is, therefore, represented in the form

$$H=H_0+\eta+H_1+H_2+\dots \tag{45}$$

with $H_0=|y|^2/2$, and with

$$H_j=V_j\in\mathcal{P}_j\quad\text{for } j\geq 1\ .$$

We shall now fix a resonance module $\mathcal{M}$ and a positive integer $N=rK$, and assume that the domain $\mathcal{G}_\delta\subset\mathbf{C}^n$ satisfies the following condition: if $y\in\mathcal{G}_\delta$, then

$$|k\cdot y|\geq\alpha\quad\text{for } k\notin\mathcal{M} \text{ and } |k|\leq N \tag{46}$$

for some real parameter $\alpha>0$; recall that this was defined to be a nonresonance domain of type $(\mathcal{M},\alpha,\delta,N)$ in definition 2 above. We shall call a function $Z$ in normal form, and write $Z\in\mathcal{N}_{\mathcal{M}}$, if it is of the form

$$Z=\sum_{k\in\mathcal{M}}z_k(y,\xi)e^{ik\cdot x}\ , \tag{47}$$

i.e., if it contains only Fourier coefficients belonging to the distinguished resonance module $\mathcal{M}$.

The aim is to construct a local symplectic diffeomorphism which transforms the given Hamiltonian (45) on our chosen domain into normal form with respect to $\mathcal{M}$ up to finite order $r$. The standard procedure transforms successively into normal form of higher order by a finite composition of diffeomorphisms. In contrast, we shall proceed differently using a recursive technique designed for numerical purposes, and developed in [11].

We shall briefly outline this approach, proceeding at first algebraically on the level of formal expansions. Consider a sequence $\chi=\{\chi_s\}_{s\geq1}$ of functions $\chi_s\in\mathcal{P}_s$, in the following called a *generating sequence*; then we define a linear operator $T_\chi$ acting on functions as follows:

$$T_\chi=\sum_{s\geq0}E_s\ , \tag{48}$$

where $\{E_s\}_{s\geq0}$ is a sequence of linear operators recursively defined as

$$E_0=\mathrm{Id}\ ,\quad E_s=\sum_{j=1}^{s}\frac{j}{s}L_{\chi_j}E_{s-j}\ ,$$

where $L_{\chi_j}$ is the Lie derivative defined with the Poisson bracket by

$$L_{\chi_j}(f)=\{\chi_j,f\}\ . \tag{49}$$

## 6. Existence of the normal form

The aim is to prove proposition 1 which claims the existence of a normal form on a nonresonance domain $\mathcal{B}_{\mathcal{M}}$, where $\mathcal{M}$ is a fixed resonance $N$–module. We start with some notations.

First recall that for a real domain $\mathcal{G} \subset \mathbf{R}^n$ and for $\delta > 0$ we defined the complex neighbourhood $\mathcal{G}_\delta \in \mathbf{C}^n$ by

$$\mathcal{G}_\delta = \{y \in \mathbf{C}^n \mid |y - \mathcal{G}| < \delta\} \ ,$$

and, for $\sigma > 0$, the complex domain $\mathcal{G}_{(\delta,\sigma)}$ is defined by

$$\mathcal{G}_{(\delta,\sigma)} = \mathcal{G}_\delta \times \{\eta \in \mathbf{C}\} \times \{|\operatorname{Im} x| < \sigma\} \times \{|\operatorname{Im} \xi| < \sigma\} \ ,$$

the corresponding variables being $(y, \eta, x, \xi)$. The functions $f$ defined on $\mathcal{G}_{(\delta,\sigma)}$ and considered below will not depend on $\eta$. For these functions we introduce two norms, namely the supremum norm

$$|f|_{(\delta,\sigma)} := \sup_{\mathcal{G}_{(\delta,\sigma)}} |f| \tag{39}$$

and, with respect to the Fourier expansion on the torus given by

$$f(x, y, \xi) = \sum_{k \in \mathbf{Z}^n} f_k(y, \xi) e^{ik\cdot x} \ ,$$

we define another norm by

$$\|f\|_{(\delta,\sigma)} := \sum_{k \in \mathbf{Z}^n} |f_k|_{(\delta,\sigma)} e^{|k|\sigma} \ . \tag{40}$$

Clearly,

$$|f|_{(\delta,\sigma)} \le \|f\|_{(\delta,\sigma)} \ . \tag{41}$$

For a fixed integer $K \geq 1$ we shall denote by $\mathcal{P}_j$, for $j \geq 1$, the distinguished class of functions on $\mathcal{G}_{(\delta,\sigma)}$ defined as follows:

$$\mathcal{P}_j = \left\{ f \mid f(x, y, \xi) = \sum_{|k| < jK} f_k(y, \xi) e^{ik\cdot x} \right\} \ . \tag{42}$$

Accordingly, we shall write the given potential $V$ as a sum

$$V(x, \xi) = \sum_{j \geq 1} V_j(x, \xi) \ , \tag{43}$$

where $V_j$ is defined by

$$V_j(x, \xi) = \sum_{(j-1)K \le |k| < jK} v_k(\xi) e^{ik\cdot x}$$

as required in Proposition 4. We conclude that the time $T_0^*$ is indeed estimated exponentially by

$$T_0^* = T_* \varepsilon_0^{-r} \geq T_* \exp\left(\frac{N}{K_*}\right) = T_* \exp\left(\frac{\varrho}{\varrho_*}\right)^{1/a} \tag{38}$$

with $T_* = \frac{\delta\sigma}{8B}$ and

$$\varrho_* = K_*^a \varrho_0 \ .$$

Since $rK_* = N = (\varrho/\varrho_0)^{1/a}$ and $K_*$ and $r$ are positive integers we get the condition $\varrho \geq \varrho_*$. We turn now to the quantitative determination of the constants $\varrho_0$, $K_*$ and $r$. Concerning $\varrho_0$, we remark that, in view of (36) we can equivalently make a choice for $\delta_0$. Recalling the expression of $A$ in Proposition 1, we set

$$\delta_0 = \frac{2}{e}\left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^n |V|_{2\sigma} \ ,$$

so that

$$A = \frac{2^5 e^2 n}{\sigma} \ , \quad \varrho_0 = 2^{n+2}(n+1)!\,\delta_0 \ ;$$

$A$ depends on $\sigma$ and on the dimension $n$. Concerning $K_*$, in view of (37) and of the condition $\varepsilon_0 \leq 1/e$, we choose

$$K_* = \text{smallest integer } \geq \max\left(\frac{2eA}{\delta_0}, \frac{5}{\sigma}\right) \ ;$$

$K_*$ depends on $|V_{2\sigma}|$, $\sigma$ and $n$. Finally, we choose $r = r_*$ with

$$r_* = \text{integer part of } \frac{1}{K_*}\left(\frac{\varrho}{\varrho_0}\right)^{1/a} \ .$$

Due to the choice of $r_*$ and $K_*$, which are integers, we the get

$$\left(\frac{\varrho}{\varrho_0}\right)^{1/a} - 1 < N \leq \left(\frac{\varrho}{\varrho_0}\right)^{1/a}$$

instead of equality. However, the exponential estimate (38) still holds for $\varrho \geq \varrho_*$ with a minor change of the constant $T_*$, i.e.

$$T_* = \frac{\delta_0\sigma}{8eB} \ .$$

Inserting the expressions for the constants $\delta_0$ and $B$ theorem 2 follows in view of proposition 4.

Assume now $y(0) \in \mathcal{G} - 2\varrho$, and pick as time interval $[a,b] = [\tau^-, \tau^+]$, where $\tau^+$ and $\tau^-$ are the exit and entrance times of an orbit starting from $y(0)$ in the center of open ball of radius $\varrho$:

$$\begin{aligned} a &= \sup\{t > 0 \mid \mathrm{dist}(y(s), y(0)) < \varrho \text{ for } s \in [0,t]\} \\ b &= \sup\{t > 0 \mid \mathrm{dist}(y(s), y(0)) < \varrho \text{ for } s \in [-t,0]\} \ . \end{aligned}$$

From lemma 5 one concludes immediately

**Proposition 4:** *Let $r$, $K$, $N = rK$, $\varrho > 0$, $\sigma > 0$, $A$ and $B$ be as in lemma 5. Then for every solution with $y(0) \in \mathcal{G} - 2\varrho$*

$$\mathrm{dist}(y(t), y(0)) < \varrho \quad \text{if} \quad |t| \le T_0^* = \frac{\delta\sigma}{8B}\varepsilon_0^{-r} \ ,$$

*provided $\varepsilon_0 \le 1/2$.*

It remains to choose the parameters $N \ge 1$ and $r \ge 1$ for given $\varrho$ and $\sigma > 0$, so that $\varepsilon_0 \le 1/2$ and such that $T_0^*$ is as large as possible.

## 5. Choice of the parameters, Proof of theorem 2

In view of Proposition 4 it remains to choose the parameters in order to verify theorem 2. For given $\varrho > 0$ and $\sigma > 0$ the parameters to be chosen are $r$, $K$ and $N$, where $rK = N$. In order to first describe the idea of our choice recall the definition of the parameters $\beta_0$ and $\delta_0$ in definition 3, which depend on $\varrho$, and recall also the definition of the constants $A$ and $B$ in Proposition 1. We abbreviate for the following

$$a = \frac{n^2+n}{2} \ .$$

Then, for given $\varrho > 0$ we first choose

$$N = \left(\frac{\varrho}{\varrho_0}\right)^{1/a}$$

with an arbitrary dimensional constant $\varrho_0$; this gives

$$\delta_0 = \frac{\beta_0}{2N} = \frac{\varrho_0}{2^{n+2}(n+1)!} \ . \tag{36}$$

Consequently, $A$ and $B$ are also determined. In order to choose $K$ recall that $rK = N$, and compute

$$\varepsilon_0 = \frac{2rA}{\beta_0} + 2e^{-K\sigma/2} = \frac{C}{K} + 2e^{-K\sigma/2} \tag{37}$$

with $C = A\,2^{n+2}(n+1)!$. We finally choose $K = K_*$ so large that

$$\varepsilon_0 \le \frac{1}{e}$$

*in definition 3, define*

$$\varepsilon_0 = \frac{2rA}{\beta_0} + 2e^{-K\sigma/2}$$
$$T_0^* = \frac{\delta\sigma}{8B}\varepsilon_0^{-r}$$

*and assume* $\varepsilon_0 \le 1/2$. *If a solution satisfies* $y(t) \in \mathcal{G} - \varrho$ *for* $t \in [a,b]$, *then*

$$\mathrm{dist}(y(t), y(s)) < \varrho$$

*for all* $t, s \in [a,b]$, *provided* $b - a < T_0^*$.

**Proof.** We shall prove the existence of a resonance module of $\dim \mathcal{M} = s$ for some $s$ such that

$$y(t) \in \mathcal{C}_{\mathcal{M},\delta_s}(y^*) \quad \text{for} \quad t \in [a,b] \ . \tag{33}$$

The lemma then follows in view of $\operatorname{diam} \mathcal{C}_{\mathcal{M},\delta_s}(y^*) < \varrho$, which is the statement 3 of proposition 3. In order to prove (33) observe that due to the statement 1 in proposition 3 there exists for every $t \in [a,b]$ a $N$–module $\mathcal{M}_t$ such that $y(t) \in \mathcal{M}_t$, which has minimal dimension. Since the set of all $N$–moduli is finite there exists among these $N$–moduli $\mathcal{M}_t$, $t \in [a,b]$, a $N$–module $\mathcal{M}$ such that $\dim \mathcal{M} = s$ is minimal. Then $y(t_0) \equiv y^* \in \mathcal{B}_{\mathcal{M}}$ for some $t_0 \in [a,b]$. Let now $\tau^+$ be the exit time of the solution $y(t)$ starting at $y^*$ contained in the cylinder $\mathcal{C}_{\mathcal{M},\delta_s}(y^*)$, and assume that $\tau^+ \le t_0^*$. We claim that

$$y(t_0 + \tau^+) \in \partial \mathcal{Z}_{\mathcal{M}} \cap P_{\mathcal{M},\delta_s}(y^*) \ . \tag{34}$$

Indeed, by definition, $\beta_0 < \beta_s$ and $\delta_0 < \delta_s$. Moreover, $A(\delta_0) \ge A(\delta_s)$ and $B(\delta_0) = B(\delta_s)$, so that $\varepsilon_0 > \varepsilon_s$ and consequently $t_0^* < t_s^*$. The claim (34) follows therefore immediately from the previous lemma 4. Next we claim that

$$b < t_0 + \tau^+ \ . \tag{35}$$

Indeed, assume by contradiction that $b \ge t_0 + \tau^+$. Then we conclude that $y(t_0 + \tau^+) \in \mathcal{G} - \varrho$, and, by (34), that $y(t_0 + \tau^+) \notin \mathcal{Z}_{\mathcal{M}}$. Since, by proposition 3, statement 4, $\operatorname{Clos} \mathcal{B}_{\mathcal{M},\delta_s} \cap \mathcal{Z}_{\mathcal{M}'} = \emptyset$ for every $\mathcal{M}' \ne \mathcal{M}$ satisfying $\dim \mathcal{M} = \dim \mathcal{M}' = s$ we find

$$y(t_0 + \tau^+) \notin \bigcup_{\dim \mathcal{M} = s} \mathcal{Z}_{\mathcal{M}} = \mathcal{Z}_s^* \ .$$

Therefore, $y(t_0 + \tau^+) \in (\mathcal{G} - \varrho) \setminus \mathcal{Z}_s^*$. But, by proposition 3, statement 2,

$$(\mathcal{G} - \varrho) \setminus \mathcal{Z}_s^* \subset \bigcup_{\dim \mathcal{M} = s-1} \mathcal{Z}_{\mathcal{M}} \ .$$

Consequently, $y(t_0 + \tau^*) \in \mathcal{B}_{\mathcal{M}^*}$ with $\dim \mathcal{M}^* \le s - 1$. Since $t_0 + \tau^+ \in [a,b]$ this contradicts the definition of $\mathcal{M}$ which is assumed to have minimal dimension. Therefore, the claim $b < t_0 + \tau^+$ is proved. The same arguments show that $a > \tau^-$, and we conclude that $y(t) \in \mathcal{C}_{\mathcal{M},\delta_s}(y^*)$ for all $t \in [a,b]$, as claimed in (33). This proves the lemma. *Q.E.D.*

## 4. Global estimates depending on parameters

The previously described geography of resonances provides us with domains in $\mathbf{R}^n$ which cover $\mathbf{R}^n$ and which meet the assumptions of the local analytic statement in proposition 1. In order to reformulate proposition 2 we shall introduce the

**Definition 4:** *Assume* $\big(x(t), y(t), \xi(t), \eta(t)\big)$ *is a solution starting in* $y(0) \in \operatorname{Int} \mathcal{G}$ *for some set* $\mathcal{G} \subset \mathbf{R}^n$*. The exit resp. entrance time of this solution in* $\mathcal{G}$ *is defined as*

$$\tau^+ = \sup \{t > 0 \mid y(s) \in \operatorname{Int} \mathcal{G} \ \text{for} \quad 0 \le s \le t\}$$

*resp.*

$$\tau^- = \inf \{t < 0 \mid y(s) \in \operatorname{Int} \mathcal{G} \ \text{for} \quad t \le s \le 0\} \ .$$

The parameters $\beta_s$ and $\delta_s$ occurring in the following lemma are defined in definition 3. It should be recalled that they do depend on the parameters $N$ and $\varrho$.

**Lemma 4:** *Let* $r$ *and* $K$ *be positive integers,* $N = rK$ *and let* $\varrho > 0$ *and* $\sigma > 0$*. Consider the* $N$*–module* $\mathcal{M}$ *of* $\dim \mathcal{M} = s$ *with the associate parameters* $\beta_s$ *and* $\delta_s$*. With the constants* $A$ *and* $B$ *depending on* $|V|_\sigma$*,* $\sigma$*,* $n$ *and* $\delta = \delta_s$ *as in proposition 1 we set*

$$\varepsilon_s = \frac{2rA}{\beta_s} + 2e^{-K\sigma/2}$$

$$t_s^* = \frac{\delta\sigma}{8B}\varepsilon_s^{-r} \ .$$

*Assume* $\varepsilon_s \le 1/2$*. Then for a solution with* $y(0) \in \mathcal{B}_\mathcal{M}$ *and with exit (resp. entrance) time* $\tau^+$ *(resp.* $\tau^-$*) in the cylinder* $\mathcal{C}_{\mathcal{M},\delta_s}\big(y(0)\big)$ *the following holds true: if* $\tau^+ \le t_s^*$ *(resp. if* $\tau^- \ge -t_s^*$*) then*

$$y(\tau^+) \quad (\text{resp. } y(\tau^-)\ ) \ \in \big(\partial \mathcal{Z}_\mathcal{M} \cap P_{\mathcal{M},\delta_s}(y(0))\big) \ .$$

In other words a solution with $y(0) \in \mathcal{B}_\mathcal{M} \cap \mathcal{C}_{\mathcal{M},\delta_s}\big(y(0)\big)$ can leave the cylinder $\mathcal{C}_{\mathcal{M},\delta_s}\big(y(0)\big)$ within the time interval $|t| \le t_s^*$ only through its base (see fig. 3).

**Proof.** By assumption $y(0) \in \mathcal{B}_\mathcal{M} \cap \mathcal{C}_{\mathcal{M},\delta_s}\big(y(0)\big)$. By definition $\mathcal{B}_\mathcal{M} \subset \mathcal{B}_{\mathcal{M},\delta_s}$. In view of proposition 3, statement 6, the set $\mathcal{B}_{\mathcal{M},\delta_s}$ satisfies the assumptions of $\mathcal{G}$ in proposition 2, with $\delta$, however, replaced by $\delta_s$ and $\alpha$ replaced by $\beta_s/2$. Since $\mathcal{C}_{\mathcal{M},\delta_s}\big(y(0)\big) \subset \mathcal{B}_{\mathcal{M},\delta_s}$ we, therefore, conclude from proposition 2 that if $\tau^+ \le t_s^*$, then $y(\tau^+) \in \operatorname{Int} P_{\mathcal{M},\delta_s}$. Moreover, by definition of the exit time, $y(\tau^+) \in \partial \mathcal{C}_{\mathcal{M},\delta_s}\big(y(0)\big)$, so that $y(\tau^+) \in \partial \mathcal{C}_{\mathcal{M},\delta_s}\big(y(0)\big) \cap \operatorname{Int} P_{\mathcal{M},\delta_s}\big(y(0)\big)$, as claimed. Similarly for $\tau^-$ and $y(\tau^-)$ if $\tau^- \ge t_s^*$. From the definition of the corresponding sets one concludes readily that $\big(\partial \mathcal{C}_{\mathcal{M},\delta_s}\big(y(0)\big) \cap \operatorname{Int} P_{\mathcal{M},\delta_s}(y(0))\big) \subset \big(\partial \mathcal{Z}_\mathcal{M} \cap P_{\mathcal{M},\delta_s}(y(0))\big)$, and this finishes the proof. *Q.E.D.*

The crucial step in proving theorem 1 is the following lemma, which combines the analytical and the geometrical considerations.

**Lemma 5:** *With the parameters* $r$*,* $K$*,* $N = rK$*,* $\varrho > 0$ *and* $\sigma > 0$ *given, with the constants* $A$ *and* $B$ *depending on* $|V|_\sigma$*,* $\sigma$*,* $n$ *and* $\delta = \delta_0$*, and with* $\delta_0$ *and* $\beta_0$ *as given*

Pick $\overline{y} \in P_{\mathcal{M}}(y_0) \cap \Sigma_{\mathcal{M}}$; then $k \cdot \overline{y} = 0$ for $k \in \mathcal{M}$, and we conclude, since $y - y_* \perp P_{\mathcal{M}}(y_0)$, that

$$\begin{aligned} k \cdot y &= k \cdot (y - \overline{y}) \\ &= k \cdot (y - y_*) + k \cdot (y_* - \overline{y}) \\ &= k \cdot (y_* - \overline{y}) \ . \end{aligned}$$

Since $y \in \mathcal{Z}_{\mathcal{M}}$ there are $s$ independent vectors $k \in \mathcal{M}_N$ such that $|k \cdot y| < \beta_s$. Consequently,

$$|k \cdot (y_* - \overline{y})| < \beta_s$$

for these vectors, and since $y_* - \overline{y} \in P_{\mathcal{M}}$ we conclude by lemma 2,

$$\operatorname{dist}(y_*, \overline{y}) \leq sN^{s-1}\beta_s \ . \tag{31}$$

Using the triangle inequality we find by (30) and (31) that $\operatorname{dist}(y, \overline{y}) \leq sN^{s-1}\beta_s + \delta_s$. Therefore, if $y_1$ and $y_2 \in \mathcal{C}_{\mathcal{M},\delta_s}(y_0)$ we can estimate, again by the triangle inequality, $\operatorname{dist}(y_1, y_2) \leq 2sN^{s-1} + 2\delta_s$. The proof of the lemma is finished. *Q.E.D.*

In view of the definition of the sequences $\beta_s$ and $\delta_s$ we conclude in particular from lemma 3 that $\operatorname{diam} \mathcal{C}_{\mathcal{M},\delta_s}(y) < \varrho$ for $y \in \mathcal{B}_{\mathcal{M}}$, as claimed in statement 3.

Next, we claim that if $y \in \mathcal{B}_{\mathcal{M},\delta_s}$ then

$$|k \cdot y| \geq \beta_s \quad \text{for} \quad k \notin \mathcal{M} \quad \text{and} \quad |k| \leq N \ . \tag{32}$$

Indeed, in view of the definition of $\mathcal{B}_{\mathcal{M},\delta_s}$, there exists a $y' \in \mathcal{B}_{\mathcal{M}}$ such that $y \in \mathcal{C}_{\mathcal{M},\delta_s}(y')$, and, by lemma 1, we have $|k \cdot y'| \geq \beta_{s+1}$. Moreover, by the previous lemma 3,

$$|k \cdot (y' - y)| \leq |k| \operatorname{dist}(y', y) \leq N \operatorname{diam} \mathcal{C}_{\mathcal{M},\delta_s}(y') < N(2sN^{s-1}\beta_s + 2\delta_s) \ .$$

Consequently,

$$|k \cdot y| \geq |k \cdot y'| - |k \cdot (y' - y)| \geq \beta_{s+1} - N(2sN^{s-1}\beta_s + 2\delta_s) \ ,$$

which is $\geq \beta_s$ by our choice of the sequences in definition 3. This proves the claim (32).

In order to prove statement 6 let $y \in (\mathcal{B}_{\mathcal{M},\delta_s})_{\delta_s}$. By definition of the complex extension there exists an $y' \in \mathcal{B}_{\mathcal{M},\delta_s}$ such that $|y'_j - y_j| \leq \delta_s$ for $1 \leq j \leq n$. Consequently, if $|k| \leq N$,

$$|k \cdot (y' - y)| \leq \delta_s \sum |k_j| \leq N\delta_s \leq \frac{1}{2}\beta_s \ ,$$

by our choice of the sequences. Therefore, in view of (32), if $k \notin \mathcal{M}$ then

$$|k \cdot y| \geq |k \cdot y'| - |k \cdot (y' - y)| \geq \frac{1}{2}\beta_s \ ,$$

as claimed in statement 6.

In order to prove statement 4 assume, by contradiction, that $y \in \mathcal{B}_{\mathcal{M},\delta_s} \cap \mathcal{Z}_{\mathcal{M}'} \neq \emptyset$. Since $y \in \mathcal{Z}_{\mathcal{M}'}$ and $\mathcal{M} \neq \mathcal{M}'$, there is a $k \notin \mathcal{M}$, $|k| \leq N$, such that $|k \cdot y| < \beta_s$, in contradiction to estimate (32). Finally, the statement 5 is readily verified using the definition of the sequence $\delta_s$. The proof of proposition 3 is complete. *Q.E.D.*

*6) If $\mathcal{M}$ is a $N$–module of* $\dim\mathcal{M} = s$*, then for every $y$ belonging to the complex domain $(\mathcal{B}_{\mathcal{M},\delta_s})_{\delta_s}$ the following nonresonance condition holds true*

$$|k\cdot y| \geq \frac{\beta_s}{2} \quad \text{if } k \notin \mathcal{M} \text{ and } |k| \leq N\ .$$

**Proof of proposition 3.** The statements 1 and 2 follow readily from the definition. Indeed, observe that

$$\bigcup_{\dim\mathcal{M}=s} \mathcal{B}_{\mathcal{M}} = \bigcup_{\dim\mathcal{M}=s} \left(\mathcal{Z}_{\mathcal{M}} \setminus \mathcal{Z}^*_{s+1}\right) = \bigcup_{\dim\mathcal{M}=s} \mathcal{Z}_{\mathcal{M}} \setminus \mathcal{Z}^*_{s+1} = \mathcal{Z}^*_s \setminus \mathcal{Z}^*_{s+1}\ .$$

For $s = 0$, we have $\mathcal{Z}^*_0 = \mathcal{G} - \varrho$, which proves statement 2 for $s = 0$. For $s > 0$ we proceed by induction assuming statement 2 to be true up to $s-1$; then

$$\begin{aligned}(\mathcal{G}-\varrho)\setminus\mathcal{Z}^*_{s+1} &= [(\mathcal{G}-\varrho)\setminus\mathcal{Z}^*_s]\setminus\mathcal{Z}^*_{s+1} \cup \left(\mathcal{Z}^*_s\setminus\mathcal{Z}^*_{s+1}\right)\\ &\subset \Big(\bigcup_{\dim\mathcal{M}<s}\mathcal{B}_{\mathcal{M}}\Big)\setminus\mathcal{Z}^*_{s+1} \cup \Big(\bigcup_{\dim\mathcal{M}=s}\mathcal{B}_{\mathcal{M}}\Big)\\ &\subset \bigcup_{\dim\mathcal{M}\leq s}\mathcal{B}_{\mathcal{M}}\ .\end{aligned}$$

Setting $s = n$ the statement 1 follows, since $\mathcal{Z}^*_{n+1} = \emptyset$. In order to prove statement 3 we need the following geometric lemma which is of independent interest and which is proved in [6], appendix B:

**Lemma 2:** *Let $1 \leq s \leq n$, and let $(u^{(1)},\ldots,u^{(s)})$ be linearly independent vectors in $\mathbf{R}^n$ satisfying $|u^{(j)}| \leq N$ for some positive $N$ and for $1 \leq j \leq s$, and denote by* $\mathrm{Vol}(u^{(1)},\ldots,u^{(s)})$ *the $s$–dimensional volume of the parallelepiped with sides $u^{(1)},\ldots,u^{(s)}$; let, moreover, $w \in$* $\mathrm{span}(u^{(1)},\ldots,u^{(s)})$*, and $\alpha$ be a positive constant. Then, the following statement holds true: if $|w\cdot u^{(j)}| \leq \alpha$ for $1 \leq j \leq s$, then the Euclidean norm of $w$ is bounded by*

$$\|w\| < \frac{sN^{s-1}\alpha}{\mathrm{Vol}(u^{(1)},\ldots,u^{(s)})}\ .$$

We shall conclude from lemma 2

**Lemma 3:** *Let $\mathcal{M}$ be an $N$–module of* $\dim\mathcal{M} = s$*. Then for $y \in \mathcal{B}_{\mathcal{M}}$*

$$\mathrm{diam}\,\mathcal{C}_{\mathcal{M},\delta_s}(y) < 2sN^{s-1}\beta_s + 2\delta_s\ .$$

**Proof.** Changing the notation we recall that

$$\mathcal{C}_{\mathcal{M},\delta_s}(y_0) = \mathcal{Z}_{\mathcal{M}} \cap P_{\mathcal{M},\delta_s}(y_0) \quad \text{for} \quad y_0 \in \mathcal{B}_{\mathcal{M}}\ .$$

Let now $y \in \mathcal{C}_{\mathcal{M},\delta_s}(y_0)$ and pick $y_* \in P_{\mathcal{M}}(y_0)$ such that $y - y_* \perp P_{\mathcal{M}}$. In view of the definition of $P_{\mathcal{M},\delta_s}(y_0)$ we conclude that

$$\mathrm{dist}(y, y_*) \leq \delta_s\ . \tag{30}$$

**Lemma 1:** *Let* $\mathcal{M}$ *be an* $N$*–module of* $\dim\mathcal{M} = s$*. Then for every* $y \in \mathcal{B}_{\mathcal{M}}$ *the following nonresonance condition holds true:*

$$|k \cdot y| \geq \beta_{s+1} \quad \text{if} \quad k \notin \mathcal{M} \quad \text{and} \quad |k| \leq N \ .$$

**Proof.** Assume, by contradiction, that there is a $y \in \mathcal{B}_{\mathcal{M}}$ and a $k \notin \mathcal{M}$ with $|k| \leq N$ satisfying $|k \cdot y| < \beta_{s+1}$. Since $\mathcal{B}_{\mathcal{M}} \subset \mathcal{Z}_{\mathcal{M}}$, and since, by assumption, $\beta_s < \beta_{s+1}$ we conclude from the definition of $\mathcal{Z}_{\mathcal{M}}$ that there are $s+1$ independent vectors $k \in \mathcal{M}_N$ satisfying $|k \cdot y| < \beta_{s+1}$. Consequently, $y \in \mathcal{Z}_{\mathcal{M}'}$ for a resonant $N$–module $\mathcal{M}'$ with $\dim\mathcal{M}' = s+1$. Therefore, $y \in \mathcal{Z}_{\mathcal{M}'} \subset \mathcal{Z}^*_{s+1}$, and so $y \notin \mathcal{B}_{\mathcal{M}} = \mathcal{Z}_{\mathcal{M}} \setminus \mathcal{Z}^*_{s+1}$, contradicting $y \in \mathcal{B}_{\mathcal{M}}$. *Q.E.D.*

We next define the parameters in terms of $N$, $\varrho$ and $n$.

**Definition 3:** *Let* $N$ *be a positive integer and let* $\varrho > 0$*. Then we set*

$$\beta_0 = \left[2^{n+1}(n+1)!\, N^{(n^2+n-2)/2}\right]^{-1} \varrho \ ,$$
$$\beta_s = 2^s s!\, N^{[s(s-1)]/2}\beta_0 \quad \text{for } 1 \leq s \leq n \ ,$$

*and*

$$\delta_s = \frac{\beta_s}{2N} \quad \text{for } 0 \leq s \leq n \ .$$

*It then follows that*

$$\begin{aligned} &\beta_0 < \beta_1 \ldots < \beta_n \\ &\delta_0 < \delta_1 \ldots < \delta_n < \varrho \ . \end{aligned} \tag{29}$$

The main result of this section is the following

**Proposition 3:** *Let* $N$ *be a positive integer, let* $\varrho > 0$ *and consider the domain* $\mathcal{G} - \varrho \subset \mathbf{R}^n$*. Then*

*1) Covering:*

$$\mathcal{G} - \varrho \subset \bigcup_{\mathcal{M}} \mathcal{B}_{\mathcal{M}} \ .$$

*2) For* $0 \leq s \leq n$

$$(\mathcal{G} - \varrho) \setminus \mathcal{Z}^*_{s+1} \subset \bigcup_{0 \leq \dim\mathcal{M} \leq s} \mathcal{B}_{\mathcal{M}} \ .$$

*3) If* $\mathcal{M}$ *is a* $N$*–module of* $\dim\mathcal{M} = s$*, then for every* $y \in \mathcal{B}_{\mathcal{M}}$

$$\operatorname{diam} \mathcal{C}_{\mathcal{M},\delta_s}(y) < \varrho \ .$$

*4) If* $\mathcal{M}$ *and* $\mathcal{M}'$ *are* $N$*–moduli of* $\dim\mathcal{M} = \dim\mathcal{M}' = s$*, then*

$$\operatorname{Clos} \mathcal{B}_{\mathcal{M},\delta_s} \cap \mathcal{Z}_{\mathcal{M}'} = \emptyset \quad \text{if} \quad \mathcal{M} \neq \mathcal{M}' \ .$$

*5) For an* $N$*–module* $\mathcal{M}$ *of* $\dim\mathcal{M} = s$

$$(\mathcal{B}_{\mathcal{M},\delta_s})_{\delta_s} \cap \mathbf{R}^n \subset \mathcal{G} \ .$$

**Figure 3.** Construction of the cylinders and extended blocks for a resonant block of multiplicity 1. Starting with the point $y(0) \in \mathcal{B}_{\mathcal{M}}$, draw the plane $P_{\mathcal{M}}\big(y(0)\big)$, and extend it to $P_{\mathcal{M},\delta}\big(y(0)\big)$. The intersection with the resonant zone $\mathcal{Z}_{\mathcal{M}}$ defines the cylinder $\mathcal{C}_{\mathcal{M},\delta}\big(y(0)\big)$, represented by the dashed region. The condition of nonoverlapping of the resonances requires that the cylinder does not intersect with the resonant zone $\mathcal{Z}_{\mathcal{M}'}$; this is guaranteed on one hand by the convexity of $H_0$, which implies that the resonant manifold $\Sigma_{\mathcal{M}}$ and the resonant plane $P_{\mathcal{M}}\big(y(0)\big)$ intersect transversally, and on the other hand by $\beta_1 < \beta_2$, which forces the resonant region of multiplicity 2 (the big square–shaped area) to extend beyond the intersection of the zone $\mathcal{Z}_{\mathcal{M}}$ with $\mathcal{Z}_{\mathcal{M}'}$ both having multiplicity 1 (small square–shaped area). Repeating the construction for every point in $\mathcal{B}_{\mathcal{M}}$ and taking the union, an extended resonant block $\mathcal{B}_{\mathcal{M},\delta}$ is constructed, represented by the wave–dashed area. A further extension of $\delta$ in the complex, represented by the dotted area, defines the nonresonance domain suitable for the execution of the normal form.

**Figure 2.** The geography of the resonances in the plane case. Multiplicity 2: the resonant manifold is the point $O$; the resonant zone, region and block are the square–shaped area $ABCD$. Multiplicity 1: the resonant manifolds are the curves $\Sigma_{\mathcal{M}}$ and $\Sigma_{\mathcal{M}'}$, which intersect in the point $O$; the resonant zones are the strips around these curves; the resonant region is the cross-shaped area formed by the union of the strips, the resonant blocks are the dashed parts of the strips. Multiplicity 0: the resonant manifold, zone and region are the whole domain; the resonant blocks are the complement of the cross-shaped area which is the resonant region of multiplicity 1.

*9. Extended resonant block.* The extended resonant block is defined to be the set

$$\mathcal{B}_{\mathcal{M},\delta} = \bigcup_{y\in\mathcal{B}_{\mathcal{M}}} \mathcal{C}_{\mathcal{M},\delta}(y) \ . \tag{28}$$

The crucial sets in the following are the resonant blocks $\mathcal{B}_{\mathcal{M}}$. In view of the next lemma they are the likely candidates of the distinguished sets in $\mathbf{R}^n$ which meet the assumptions in proposition 1.

where $B(y, \varrho)$ is the open ball of radius $\varrho$ and center $y$. We shall first recall some concepts from [5] and [6]. We point out that we shall need only $\mathcal{G} = \mathbf{R}^n$ for our purpose.

*1. N–moduli and resonance parameters.* If $\mathcal{M} \subset \mathbf{Z}^n$ is a resonance module and $N$ a positive integer we introduce the abbreviation

$$\mathcal{M}_N = \{k \in \mathcal{M} \mid |k| \le N\} \tag{20}$$

and call a module $\mathcal{M}$ of $\dim \mathcal{M} = s$ an $N$–module if $\mathcal{M}_N$ contains $s$ independent vectors $k \in \mathbf{Z}^n$. To the $N$–moduli of $\dim \mathcal{M} = s$ we associate a positive parameter $\beta_s$ such that $\beta_0 < \beta_1 < \ldots < \beta_n$. They will be determined below.

*2. Resonant manifold.* With an $N$–module $\mathcal{M}$ we associate the resonant manifold $\Sigma_{\mathcal{M}}$ defined by

$$\Sigma_{\mathcal{M}} = \{y \in \mathbf{R}^n \mid k \cdot y = 0 \text{ for all } k \in \mathcal{M}\} \ . \tag{21}$$

*3. Resonant zone.* With an $N$–module $\mathcal{M}$ of $\dim \mathcal{M} = s$ we associate the resonant zone $\mathcal{Z}_{\mathcal{M}}$ defined by

$$\begin{aligned} \mathcal{Z}_{\mathcal{M}} = \{y \in \mathcal{G} - \varrho \mid & \text{ there are } s \text{ independent} \\ & \qquad k \in \mathcal{M}_N \text{ such that } |k \cdot y| < \beta_s\} \end{aligned} \tag{22}$$

*4. Resonant region of order s.* To the family of all $N$–moduli $\mathcal{M}$ of fixed $\dim \mathcal{M} = s$ we associate the subset

$$\mathcal{Z}^*_s = \bigcup_{\dim \mathcal{M} = s} \mathcal{Z}_{\mathcal{M}} \ . \tag{23}$$

By definition, $\mathcal{Z}^*_0 = \mathcal{G} - \varrho$; we also set $\mathcal{Z}^*_{n+1} = \emptyset$.

*5. Resonant block.* For an $N$–module $\mathcal{M}$ of $\dim \mathcal{M} = s$ we define the associated resonant block $\mathcal{B}_{\mathcal{M}} \subset \mathbf{R}^n$ to be the subset of $\mathcal{G} - \varrho$:

$$\mathcal{B}_{\mathcal{M}} = \mathcal{Z}_{\mathcal{M}} \setminus \mathcal{Z}^*_{s+1} \ . \tag{24}$$

*6. Resonant plane.* To $y \in \mathcal{B}_{\mathcal{M}}$ we associate the resonant plane

$$P_{\mathcal{M}}(y) = \{y + w \mid w \in P_{\mathcal{M}}\} \ . \tag{25}$$

*7. Extended plane.* For $\delta > 0$ and $y \in \mathcal{B}_{\mathcal{M}}$, the extended plane is defined to be the set

$$P_{\mathcal{M},\delta}(y) = \bigl\{z \in \mathbf{R}^n \mid \operatorname{dist}\bigl(z, P_{\mathcal{M}}(y)\bigr) \le \delta\bigr\} \ . \tag{26}$$

*8. Cylinder and its basis.* For $y \in \mathcal{B}_{\mathcal{M}}$ and $\delta > 0$ the cylinder is defined to be the set

$$\mathcal{C}_{\mathcal{M},\delta}(y) = P_{\mathcal{M},\delta}(y) \cap \mathcal{Z}_{\mathcal{M}} \ ; \tag{27}$$

its basis is defined as $\partial \mathcal{Z}_{\mathcal{M}} \cap P_{\mathcal{M},\delta}(y)$.

**Figure 1.** Confinement of the orbit in the strip around the plane $P_{\mathcal{M}}(y(0))$, unless it leaves the domain $\mathcal{G}$.

**Proof.** Let $\varphi$ be the symplectic diffeomorphism of proposition 1, and, in abuse of notation, let $y = \varphi(y')$; then

$$|\mu \cdot \big(y(t) - y(0)\big)| \leq |\mu \cdot \big(y(t) - y'(t)\big)| + |\mu \cdot \big(y'(t) - y'(0)\big)| + |\mu \cdot \big(y'(0) - y(0)\big)| \ .$$

Taking the supremum over $\mu \in \mathbf{R}^n$ with $|\mu| = 1$ and $\mu \perp \mathcal{M}$, the first and the third term on the right hand side are estimated by $\delta/4$ in view of the properties (12) of the diffeomorphism $\varphi$, while the second term is estimated, in view of (18), by

$$|y'(t) - y'(0)| \leq |t| \cdot \frac{2}{\sigma} \, |\mathcal{R}| \ ,$$

which is smaller than $\delta/4$, if $|t| \leq t^*$, in view of the estimate for $|\mathcal{R}|$ in proposition 1. This finishes the proof of proposition 2. *Q.E.D.*

If a solution leaves the local domain $\mathcal{G}$ considered above in a time shorter than $t^*$ we, of course, loose control (see fig. 1) and are forced to study the geometric pattern of all possible nonresonance domains. This will be done in the next section, where the special form of the integrable part which is given by $H_0 = |y|^2/2$ allows a simplification of earlier presentations.

## 3. Geography of resonances

If $\mathcal{G} \subset \mathbf{R}^n$ is an open set and $\varrho > 0$ we denote by $\mathcal{G} - \varrho$ the subset

$$\mathcal{G} - \varrho = \{ y \in \mathcal{G} \mid B(y, \varrho) \subset \mathcal{G} \} \ ,$$

The proposition allows us to gain already some crucial insight into the flow of (7) at least locally on $\mathcal{G}$. Namely, we conclude immediately that the functions $I_\mu$ defined, in the new variables on $\mathcal{G}_{(\delta,\sigma)\frac{1}{2}}$, by

$$I_\mu = \mu \cdot y \ , \quad \mu \in \mathbf{R}^n \quad \text{and} \quad \mu \perp \mathcal{M} \ , \tag{17}$$

are approximate integrals. Indeed, computing the time derivative along a solution of $H \circ \varphi$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\mu = \mu \cdot \frac{\mathrm{d}}{\mathrm{d}t} y = \mu \cdot \frac{\partial}{\partial x}(H \circ \varphi) = i \sum_{k \in \mathcal{M}} (\mu \cdot k) f_k(y,\xi) e^{ik\cdot x} + \mu \cdot \frac{\partial}{\partial x} \mathcal{R} \ .$$

Since $\mu \cdot k = 0$ for $k \in \mathcal{M}$, we find the estimate, on $\mathcal{G}_{\delta/2} \times \mathbf{T}^n \times \{|\operatorname{Im}\xi| < \sigma/2\}$,

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} I_\mu \right| \le \frac{2|\mu|}{\sigma} B \varepsilon^r \ , \tag{18}$$

where we have used (15) and the Cauchy estimate in order to estimate the derivative of a holomorphic function in terms of the supremum of the function. This consequence of the proposition 1 leads to the proposition 2 below in the original nonresonance domain $\mathcal{G} \subset \mathbf{R}^n$. In order to formulate it we need a definition.

If $\mathcal{M}$ is a resonance module and $y^* \in \mathbf{R}^n$ we define the plane $P_{\mathcal{M}}(y^*)$ containing $y^*$ by

$$P_{\mathcal{M}}(y^*) = \{ y \in \mathbf{R}^n \mid y = y^* + P_{\mathcal{M}} \} \ , \tag{19}$$

where $P_{\mathcal{M}}$ is introduced in definition 1.

**Proposition 2:** *Assume the potential $V(x,t)$ is real analytic and bounded on the complex strip $|\operatorname{Im} x| < 2\sigma$ and $|\operatorname{Im} t| < 2\sigma$ for some $\sigma > 0$. Let $r$ and $K$ be positive integers, $N = rK$. Assume $\mathcal{G} \subset \mathbf{R}^n$ is a nonresonance domain of type $(\mathcal{M}, \alpha, \delta, N)$. Then there are positive constants $A$ and $C$ depending on $|V|_{2\sigma}$, $\sigma$, $\delta$ and $n$ such that if*

$$\varepsilon := \left( \frac{rA}{\alpha} + 2e^{-K\sigma/2} \right) \le \frac{1}{2}$$

*then, the following holds true: if $\big(x(t), y(t), \xi(t), \eta(t)\big)$ is a solution of the Hamiltonian system (6) satisfying $y(0) \in \mathcal{G}$ and $y(t) \in \mathcal{G}$ for $\tau^- < t < \tau^+$, then*

$$\operatorname{dist}\big(y(t), P_{\mathcal{M}}(y(0))\big) \le \frac{3}{4}\delta$$

*for $t \in [\tau^-, \tau^+] \cap [-t^*, t^*]$ , where*

$$t^* = C\varepsilon^{-r} \ .$$

*The constant $A$ is defined in proposition 1, and $C$ is defined by*

$$C = \frac{\delta\sigma}{8B} \ ,$$

*with $B$ as in proposition 1.*

**Definition 2:** *Nonresonance domain. A set $\mathcal{G} \subset \mathbf{R}^n$ is called a nonresonance domain of type $(\mathcal{M}, \alpha, \delta, N)$ if*

$$|k \cdot y| > \alpha \quad \text{for all} \quad y \in \mathcal{G}_\delta \ , \quad k \in \mathbf{Z}^n \setminus \mathcal{M} \quad \text{and} \quad |k| \leq N \ .$$

*Here, $|k| = |k_1| + \ldots + |k_n|$, $\alpha$ and $\delta$ are real positive parameters, $N$ a positive integer and $\mathcal{M}$ a resonance module.*

After these preliminaries we are ready to formulate the analytical ingredient of the proof of the theorem.

**Proposition 1:** *Normal forms on nonresonance domains. Let $r$ and $K$ be positive integers, and let $N = rK$. Moreover, let $\mathcal{G} \subset \mathbf{R}^n$ be a nonresonance domain of type $(\mathcal{M}, \alpha, \delta, N)$. Then, there are positive constants $A$ and $B$ depending on $|V|_\sigma$, $\sigma$, $\delta$ and $n$ such that if*

$$\varepsilon := \left( \frac{rA}{\alpha} + 2e^{-K\sigma/2} \right) \leq \frac{1}{2} \tag{11}$$

*there exists a real analytic symplectic diffeomorphism $\varphi$ belonging to the subgroup $S$, such that $\varphi$ and $\varphi^{-1}$ are defined on the complex domain $\mathcal{G}_{(\delta,\sigma)\frac{1}{2}}$ and satisfy*

$$\operatorname{dist}(y, \varphi(y)) \leq \frac{1}{4}\delta \ , \quad \operatorname{dist}(y, \varphi^{-1}(y)) \leq \frac{1}{4}\delta \ . \tag{12}$$

*The map $\varphi$ transforms the Hamiltonian $H$ into the following normal form on $\mathcal{G}_{(\delta,\sigma)\frac{1}{2}}$:*

$$H \circ \varphi = \frac{1}{2}|y|^2 + \eta + \mathcal{N}_\mathcal{M} + \mathcal{R} \ , \tag{13}$$

*where*

$$\mathcal{N}_\mathcal{M} = \sum_{\substack{k \in \mathcal{M} \\ |k| \leq N}} f_k(y, \xi) e^{ik \cdot x} \ . \tag{14}$$

*The remainder $\mathcal{R}$ satisfies, on $\mathcal{G}_{(\delta,\sigma)\frac{1}{2}}$, the estimate*

$$|\mathcal{R}| \leq B\varepsilon^r \ . \tag{15}$$

*Moreover:*

$$\mathcal{G}_{(\delta,\sigma)\frac{1}{4}} \subset \varphi\big(\mathcal{G}_{(\delta,\sigma)\frac{1}{2}}\big) \subset \mathcal{G}_{(\delta,\sigma)\frac{3}{4}} \ . \tag{16}$$

*In addition, the same statement holds true for $\varphi^{-1}$ instead of $\varphi$. The constants are given by*

$$A = \frac{2^5 en}{\delta\sigma} \left[ \left( \frac{1 + e^{-\sigma/2}}{1 - e^{-\sigma/2}} \right)^n |V|_{2\sigma} + \frac{e\delta}{2} \right]$$

$$B = 2 \left( \frac{1 + e^{-\sigma/2}}{1 - e^{-\sigma/2}} \right)^n |V|_{2\sigma} \ .$$

The proof of the statement, based on a recursion procedure, is postponed to sect. 6 below. The ultimate aim is, of course, to choose the parameters such that $\varepsilon^r$ is small.

improved approach to Nekhoroshev's estimates due to J. Pöschel do not apply directly to our problem.

## 2. Local normal forms

It is a convenient trick to remove the time dependence by simply extending the phase space $\mathbf{R}^n \times \mathbf{T}^n$ by two additional variables $(\xi, \eta) \in \mathbf{R}^2$, which are canonically conjugate, and to consider the Hamiltonian function

$$H(x, y, \xi, \eta) = \frac{1}{2}|y|^2 + \eta + V(x, \xi) \ . \tag{6}$$

The following part of the Hamiltonian equations

$$\begin{aligned} \dot{x} &= \frac{\partial}{\partial y} H(x, y, \xi, \eta) = y \\ \dot{y} &= -\frac{\partial}{\partial x} H(x, y, \xi, \eta) = -V_x(x, \xi) \\ \dot{\xi} &= \frac{\partial}{\partial \eta} H(x, y, \xi, \eta) = 1 \end{aligned} \tag{7}$$

is then independent of $\eta$ and obviously equivalent to (3).

As usual, instead of solving the equations we shall make use of the Hamiltonian transformation theory in order to find local normal forms for the Hamiltonian functions. Since the $\xi$ variable is distinguished, the symplectic diffeomorphisms $\varphi$ considered later on in the transformation theory will belong to the following subgroup $S$ of symplectic diffeomorphisms:

$$S = \{\varphi = \mathrm{id} + \psi \mid \psi = \psi(x, y, \xi) \text{ and } \xi \text{ remains fixed}\} \ .$$

Examples of such diffeomorphisms are the maps belonging to the flow of a Hamiltonian vector field whose Hamiltonian function does not depend on the variable $\eta$.

In order to describe those subsets of the phase space on which the Hamiltonian will be transformed into normal form we need some definitions.

**Definition 1:** *Resonance module. If $M \subset \mathbf{Z}^n$ is a subgroup we denote by $P_M \subset \mathbf{R}^n$ the real subspace generated by $M$, and call the subgroup of $\mathbf{Z}^n$, defined by*

$$\mathcal{M} = P_M \cap \mathbf{Z}^n \ , \tag{8}$$

*a resonance module. The integer $\dim P_{\mathcal{M}}$ will be called the dimension or multiplicity of the resonance module.*

To a real domain $\mathcal{G} \subset \mathbf{R}^n$ and $\delta > 0$ we shall associate the complex neighbourhood

$$\mathcal{G}_\delta = \{y \in \mathbf{C}^n \mid |y - \mathcal{G}| < \delta\} \tag{9}$$

and denote by $\mathcal{G}_{(\delta,\sigma)}$ the complex domain

$$\mathcal{G}_{(\delta,\sigma)} = \mathcal{G}_\delta \times \{\eta \in \mathbf{C}\} \times \{|\operatorname{Im} x| < \sigma\} \times \{|\operatorname{Im} \xi| < \sigma\} \ . \tag{10}$$

*and for all* $\varrho \geq \varrho_*$. *The constants* $\varrho_* = R_*$ *and* $T_*$ *are defined in theorem 1.*

As a sideremark we should mention that the exponent $a$ is equal to 1 in the special case $n = 1$. Hence one concludes for all solutions of forced pendulum like equations defined by Hamiltonians of the form

$$H(x,y,t) = \frac{1}{2}y^2 + V(x,t)$$

with $V$ periodic in $x$, the estimate

$$|y(t) - y(0)| < \varrho$$

for all $t$ in

$$|t| \leq T_* \exp\left(\frac{\varrho}{\varrho_*}\right)$$

and for all $\varrho \geq \varrho_*$.

It should be recalled that the so called steepness of the "integrable part", which in our case is $H_0(y) = |y|^2/2$, is crucial for estimates over a time interval which is exponentially large. This is illustrated by the following example of an even time independent system due to N.N. Nekhoroshev[5]:

$$H(x,y) = \frac{1}{2}(y_1^2 - y_2^2) + \sin(x_1 - x_2)\ .$$

The special solution

$$\big(y_1^*(t), y_2^*(t), x_1^*(t), x_2^*(t)\big) = \big(-t, t, -\frac{1}{2}t^2, -\frac{1}{2}t^2\big)$$

satisfies

$$\operatorname{dist}\big(y^*(t), y^*(0)\big) = \sqrt{2}\,|t|$$

for all $t \in \mathbf{R}$, in sharp contrast to the statement in theorem 2.

We observe that rescaling the action and the time by setting

$$y = \frac{1}{\varepsilon}\eta \quad \text{and } t = \varepsilon\tau$$

leads to the Hamiltonian system described by the function

$$H(x,\eta,\tau) = \frac{1}{2}\,|\eta|^2 + \varepsilon^2 V(x,\varepsilon\tau)\ ,$$

which is equivalent to (1) and instead of studying solutions of (1) for large $y$ we therefore can as well study solutions for bounded $|y|$ choosing $\varepsilon$ small. This is an adiabatically perturbed system to which the ideas of Nekhoroshev [5] apply.

Indeed, the proof of theorem 2 is based on the underlying ideas of Nekhoroshev[5] and of Benettin, Galgani and Giorgilli[6]. Actually, it turns out that the proof requires only minor modifications of the latter work. However, some extra work is needed because the time dependence is not assumed to be periodic or quasiperiodic. For this reason also the approach by Benettin and Gallavotti[7], the simplified version of Nekhoroshev's theory due to P. Lochak[8] and the recent elegant and quantitatively

In order to formulate the result we assume $V(x,t)$ to be real analytic and, moreover, to have a holomorphic extension to a complex strip $|\operatorname{Im} x| < 2\sigma$ and $|\operatorname{Im} t| < 2\sigma$ for some $\sigma > 0$, such that

$$|V|_{2\sigma} = \sup_{\substack{|\operatorname{Im} x|<2\sigma \\ |\operatorname{Im} t|<2\sigma}} |V(x,t)| < \infty \ . \tag{5}$$

Denote by $B(R)$ the open ball in $\mathbf{R}^n$ centered at 0 with radius $R > 0$. Then the following statement of exponential stability replaces (4) in higher dimensions.

**Theorem 1:** *Let $\varphi^t\big(x(0),y(0)\big) = \big(x(t),y(t)\big)$ be the flow of the Hamiltonian vector field (3) on $\mathbf{T}^n \times \mathbf{R}^n$. Assume the potential $V(x,t)$ is real analytic on $\mathbf{T}^n \times \mathbf{R}$ and has an analytic extension to a complex strip $2\sigma > 0$ satisfying (5). Then there are two positive constants $R_*$ and $T_*$ depending on $|V|_{2\sigma}$, $\sigma$ and the dimension $n$, such that for $R \geq R_*$ the following statement holds true: if $y(0) \in B(R)$ then $y(t) \in B(2R)$ for all $t$ in*

$$|t| \leq T_* \exp\left(\frac{R}{R_*}\right)^{1/a} \ , \quad \text{with} \quad a = \frac{n^2+n}{2} \ .$$

*The constants are given by*

$$T_* = \frac{\sigma}{8e^2}$$

$$R_* = K_*^a \, \frac{2^{n+3}(n+1)!}{e} \left(\frac{1+e^{-\sigma/2}}{1-e^{-\sigma/2}}\right)^n |V|_{2\sigma}$$

*where $K_*$ is the smallest positive integer satisfying*

$$K_* \geq \frac{2^5 e^4 n}{\sigma} \left(\frac{1-e^{-\sigma/2}}{1+e^{-\sigma/2}}\right)^n \frac{1}{|V|_{2\sigma}} \quad \text{and} \quad K_* \geq \frac{5}{\sigma} \ .$$

Observe that there is no smallness requirement on the potential $V$; however, $V$ is assumed to be real analytic and to have a bounded holomorphic extension in a complex strip. The statement applies in particular to a potential $V$ which depends quasiperiodically on time $t$, $V = V(x,\omega t)$ with $V(x,\xi)$ being real analytic and periodic in all the variables $(x,\xi) \in \mathbf{T}^n \times \mathbf{T}^N$; the frequency vector $\omega \in \mathbf{R}^N$ is not required to meet diophantine conditions.

The above theorem follows immediately from the following stronger statement.

**Theorem 2:** *Let $\varphi^t\big(x(0),y(0)\big) = \big(x(t),y(t)\big)$ be the flow of the Hamiltonian vector field (3) on $\mathbf{T}^n \times \mathbf{R}^n$. Assume the potential $V(x,t)$ satisfies the assumptions of theorem 1. Then*

$$\operatorname{dist}\big(y(t),y(0)\big) < \varrho$$

*for all $t$ in*

$$|t| \leq T_* \exp\left(\frac{\varrho}{\varrho_*}\right)^{1/a} \quad \text{with} \quad a = \frac{n^2+n}{2} \ ,$$

## 1. Introduction and results

On the phase space $\mathbf{T}^n \times \mathbf{R}^n$ we consider a time–dependent Hamiltonian system given by the Hamiltonian function

$$H(x,y,t) = \frac{1}{2}|y|^2 + V(x,t) \ , \tag{1}$$

where $(x,y) \in \mathbf{R}^{2n}$ and $t \in \mathbf{R}$, and where $V$ depends periodically on $x$:

$$V(x+j,t) = V(x,t) \ , \quad j \in \mathbf{Z}^n \ . \tag{2}$$

It is, of course, well known that, in general, the flow of the corresponding Hamiltonian system

$$\begin{aligned} \dot{x} &= \frac{\partial}{\partial y} H(x,y,t) \\ \dot{y} &= -\frac{\partial}{\partial x} H(x,y,t) \end{aligned} \tag{3}$$

is extremely complicated. If the potential $V$, however, is bounded, the system can be viewed as a system close to an integrable one in the region of the phase space where $|y|$ is large; the integrable system is given by $H(y) = \frac{1}{2}|y|^2$. Therefore, one might be tempted to ask whether for the flow $\varphi^t\big(x(0),y(0)\big) = \big(x(t),y(t)\big)$ of (3) one has

$$\sup_{t\in\mathbf{R}} |y(t)| < \infty \ , \tag{4}$$

i.e., whether all the solutions are bounded for all times.

In the special case, $n=1$, this can indeed be proven for potentials which depend, in addition, periodically on time $t$ and which are sufficiently smooth[1][2][3]. More generally, if the potential is quasiperiodic in time, with frequencies $(\omega_1,\dots,\omega_N)$ satisfying diophantine conditions, it can be shown that all the solutions are bounded on the whole time interval, provided $V$ as a function on $\mathbf{T}^1 \times \mathbf{T}^N$ is sufficiently smooth. The bounds for the solutions are deduced from the existence of quasiperiodic solutions[4][1].

The situation is quite different in the case $n > 1$. If the potential is quasiperiodic in time, $V = V(x,\omega t)$, then there still exists an abundance of quasiperiodic solutions, hence of solutions which are bounded for all times, provided $V$ is sufficiently smooth[1]. However, in sharp contrast to the case $n=1$, the quasiperiodic solutions do not give bounds for all solutions, since the invariant surfaces covered by the quasiperiodic solutions do not bound open sets in the phase space, and boundedness of all solutions cannot be expected.

Recall now that for systems of the form $H(x,y) = H_0(y) + \varepsilon H_1(x,y)$ on $\mathbf{T}^n \times \mathbf{R}^n$ and $\varepsilon$ small N. N. Nekhoroshev[5] discovered estimates for all solutions, not over the whole time interval, but over an exponentially long interval of time, assuming the Hamiltonian to be not only smooth, but real analytic, and assuming, moreover, the integrable system $H_0$ to meet certain convexity type conditions. In view of these results one might guess that such exponential estimates hold true also for our global problem. This is indeed the case, as we shall prove.

# EXPONENTIAL STABILITY FOR TIME DEPENDENT POTENTIALS

A. GIORGILLI
*Dipartimento di Matematica dell'Università,*
*Via Saldini 50, 20133 MILANO, Italy*

E. ZEHNDER
Forschunginstitut für Mathematik ETH, CH–8093 ZÜRICH, Switzerland

**Abstract.** For a classical Hamiltonian system on a torus defined by a time dependent, bounded and analytic potential we establish global and quantitative bounds for the solutions over an exponentially long interval of time by using techniques which go back to Nekhoroshev.

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