2

THE POINCARÉ–SIEGEL CENTER PROBLEM


- *The non linear case*: resonances and small divisors.
2.1 The Poincaré center problem

Consider the holomorphic system
\[ \dot{z} = V(z) , \quad V(z) = \Lambda z + V_1(z) + V_2(z) + \ldots , \quad z \in \mathbb{C}^n , \]
with \( V_s(z) \) a homogeneous polynomial of degree \( s + 1 \).

- The series is assumed to be convergent in some neighbourhood of the origin.

**Problem** (Poincaré, 1879)

Find a near the identity transformation of the form (2.3) which conjugates the system (2.1) to its linear part, i.e., such that the transformed system reads
\[ \dot{w} = \Lambda w . \]

- Near the identity transformation: a power series
\[ w_j = z_j + \varphi_{1,j}(z) + \varphi_{2,j}(z) + \ldots , \quad j = 1, \ldots, n , \]
where \( \varphi_{s,j}(z) \) is a homogeneous polynomial of degree \( s + 1 \).

- Formal expansions (in rough terms): calculate the coefficients of a power series disregarding convergence.

- The problem of convergence of the series comes later.
2.1.1 Formal scheme of solution

Look for \( n \) power series

\[
w_j = z_j + \varphi_{1,j}(z) + \varphi_{2,j}(z) + \ldots , \quad j = 1, \ldots, n,
\]

where \( \varphi_{s,j}(z) \) is a homogeneous polynomial of degree \( s + 1 \), which are solutions of the equation

\[
(2.4) \quad L_V w_j = \lambda_j w_j .
\]

- Suppose a solution exists. Recall that \( \dot{w}_j = L_V w_j \). Then:
  - The functions \( w_1(z), \ldots, w_n(z) \) are formally independent;
  - the transformation (2.3) conjugates the system to its linear part, i.e.,
    \[
    \dot{w} = \Lambda w .
    \]

- Try to solve eq. (2.4) by comparison of coefficients. Recall (linearity):
  \[
  L_V = L_{\Lambda z} + L_{V_1} + L_{V_2} + \ldots
  \]
  Set
  \[
  \partial_\lambda = L_{\Lambda z} = \sum_{j=1}^{n} \lambda_j z_j \frac{\partial}{\partial z_j} .
  \]

- Split the system \( L_V w_j = \lambda_j w_j \) as
  \[
  (\partial_\lambda - \lambda_j) z_j = 0 ,
  \]
  \[
  (\partial_\lambda - \lambda_j) \varphi_{s,j} = \psi_{s,j} , \quad \psi_{s,j} = - \sum_{l=1}^{s} L_{V_l} \varphi_{s-l,j} , \quad s \geq 1 .
  \]

This is a recurrent system.
Problem: Solve for $\varphi$ the equation

$$(\partial \lambda - \lambda_j) \varphi = \psi$$

with $\psi$ a known homogeneous polynomial of degree $s \geq 2$, with a given complex vector $\lambda$.

Lemma 2.1: The linear operator $\partial \lambda$ is diagonal over the basis of monomials, since

$$\partial \lambda z^k = \langle k, \lambda \rangle z^k .$$

Corollary 2.2: For every $j \in \{1, \ldots, n\}$ the linear operator $(\partial \lambda - \lambda_j)$ is diagonal on the basis of monomials, since

$$(\partial \lambda - \lambda_j) z^k = (\langle k, \lambda \rangle - \lambda_j) z^k .$$

Solution of the equation $(\partial \lambda - \lambda_j) \varphi = \psi$.

- Write

$$\psi = \sum_{k \in \mathbb{N}_0^n} \psi_k z^k$$

with known (complex) coefficients $\psi_k$.

- The formal solution is $\varphi = \sum_k \varphi_k z^k$ with

$$\varphi_k = \frac{\psi_k}{\langle k, \lambda \rangle - \lambda_j} ,$$

... BUT ...

- Cave canem: some divisors might be zero!
2.1.2 Resonances: the case of Poincaré and the case of Siegel

A vector \( \lambda \in \mathbb{C}^n \) is said to be resonant in case one has \( \langle k, \lambda \rangle - \lambda_j = 0 \) for some \( k \in \mathbb{N}_0^n \) with \( |k| > 1 \) and some \( j \in \{1, \ldots, n\} \). The integer \( |k| = k_1 + \ldots + k_n \) is called the order of the resonance.

The plane \( \langle k, \lambda \rangle - \lambda_j = 0 \) is called a resonant plane.

Two different cases:

- **Poincaré domain:** the convex hull of the set \( \{\lambda_1, \ldots, \lambda_n\} \) does not contain the origin of \( \mathbb{C} \) (left).

- **Siegel domain:** The complement in \( \mathbb{C} \) of the Poincaré domain (right).

- Every \( k \) with \( |k| = 1 \) is resonant (obvius), but it is harmless because we must consider only \( |k| > 1 \).

**Example 2.3:** resonant and non resonant vectors. Let \( n = 2 \).
- \( \lambda_1 = 2\lambda_2 \) is a resonance of order 2. Set \( j = 1 \) and \( k = (0, 2) \).
- \( \lambda_1 = m\lambda_2 \) with \( m > 1 \) is a resonance of order \( m \). Set \( j = 1 \) and \( k = (0, m) \).
- \( \lambda_1 = -\lambda_2 \) satisfies infinitely many resonance relations or order \( 2s + 1 \) for \( s \geq 1 \). Set either \( j = 1 \) and \( k = (s + 1, s) \), or \( j = 1 \) and \( k = (s, s + 1) \).
- \( \lambda_1 = \lambda_2 \) is not a resonance, in view of \( |k| > 1 \).
- \( r\lambda_1 = s\lambda_2 \) with \( r, s > 1 \) is not a resonance relation.
Example 2.4: Resonant planes in the real plane. All resonant planes for:

- $2 \leq |k| \leq 3$ (upper left);
- $2 \leq |k| \leq 5$ (upper right);
- $2 \leq |k| \leq 9$ (lower left);
- $2 \leq |k| \leq 17$ (lower right).

- Poincaré domain: the first and third quadrant.
- Siegel domain: the second and fourth quadrant, including the axes.
2.1.3 The Poincaré domain

- The original condition of Poincaré:
  
  There is a straight line $\sigma$ through the origin that leaves all eigenvalues $\lambda_1, \ldots, \lambda_n$ on the same side.

  (Equivalent to the condition on the convex hull).

**Proposition 2.5:** Let $\lambda \in \mathbb{C}^n$ be in the Poincaré domain. Then the following statements hold true:

  (i) There are positive constants $K$ and $C$ such that

  \[ \frac{|\langle k, \lambda \rangle - \lambda_j|}{|k|} > C, \quad \text{for } |k| \geq K. \]

  (ii) The eigenvalues $\lambda$ satisfy at most a finite number of resonance relations.

  (iii) If $\lambda$ is nonresonant then there exists a positive constant $\gamma$ such that

  \[ \frac{|\langle k, \lambda \rangle - \lambda_j|}{|k|} > \gamma, \quad \text{for all } k \in \mathbb{N}_0^n, |k| > 1. \]

  (iv) If $\lambda$ is resonant then the claim (iii) holds true for $k$ non resonant (i.e., for all $k$ such that $\langle k, \lambda \rangle \neq \lambda_j$).

  (v) If $\lambda$ is nonresonant then there is a neighbourhood of $\lambda$ which is nonresonant (i.e., the resonant planes form a discrete set in the Poincaré domain).

**Proof.** Main remark: $\langle k, \lambda \rangle/|k|$ is the barycenter of $n$ masses $k_1, \ldots, k_n$ placed at the points $\lambda_1, \ldots, \lambda_n$ of the plane. Hence it lies inside the convex hull of $\lambda_1, \ldots, \lambda_n$. 

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*The Poincaré–Siegel center problem*
Claim (i).

- Define
  \[ d = \min_l \{ \text{dist}(\lambda_l, \sigma) \}, \quad D = \max_l \{ \text{dist}(\lambda_l, \sigma) \} \]
  and set \( K = 2[D/d] \).
- Hence
  \[ D \geq \frac{\text{dist}(\langle k, \lambda \rangle, \sigma)}{|k|} \geq d \quad \text{for } |k| \geq 1 \]
  (true for every point of the convex hull of \( \lambda_1, \ldots, \lambda_n \)).
- For \( |k| \geq K \) and \( j = 1, \ldots, n \) we have
  \[ \frac{\text{dist}(\lambda_j, \sigma)}{|k|} \leq \frac{D}{K} \leq \frac{d}{2} \]
  by definition of \( K \).
- Conclude
  \[ \frac{|\langle k, \lambda \rangle - \lambda_j|}{|k|} \geq \frac{\text{dist}(\langle k, \lambda \rangle - \lambda_j, \sigma)}{|k|} \geq \frac{\text{dist}(\langle k, \lambda \rangle, \sigma)}{|k|} - \frac{\text{dist}(\lambda_j, \sigma)}{|k|} \geq \frac{d}{2} . \]
  Hence (i) holds true with \( C = d/2 \).

Claim (ii).

- By (i), \( \langle k, \lambda \rangle - \lambda_j = 0 \) may occur only if \( |k| < K \), which represents a finite set.
Claim (iii).
- Let

$$\left(2.5\right) \quad C' = \min_{|k| < K} \min_{1 \leq j \leq n} \frac{|\langle k, \lambda \rangle - \lambda_j|}{|k|}. $$

- By non-resonance $C' > 0$ (the minimum over a finite set).
- Hence may set $\gamma = \min(C, C')$, with $C$ as in the proof of (i).

Claim (iv).
- Set $C'$ as in (2.5), but taking the minimum over the non resonant $k$'s.
- Repeat the proof of (iii).

Claim (v).
- For $\mu \in \mathbb{C}^n$ with $|\mu| < \gamma/2$ and for $|k| > 1$

$$\frac{|\langle k, \mu \rangle - \mu_j|}{|k|} \leq |\mu| + \frac{|\mu_j|}{|k|} < \frac{3\gamma}{4}. $$

- Then

$$\frac{|\langle k, \lambda + \mu \rangle - \lambda_j - \mu_j|}{|k|} \geq \frac{|\langle k, \lambda \rangle - \lambda_j|}{|k|} - \frac{|\langle k, \mu \rangle - \mu_j|}{|k|} > \frac{\gamma}{4},$$

and the claim follows since $\gamma > 0$.

Q.E.D.
2.1.4 The Siegel domain

Proposition 2.6: Let $\lambda$ be in the Siegel domain. Then the following statements hold true:

(i) the quantity $|\langle k, \lambda \rangle - \lambda_j|$ may assume an arbitrarily small value, i.e.,

$$\inf_{|k| > 1} \min_{1 \leq j \leq n} |\langle k, \lambda \rangle - \lambda_j| = 0 ;$$

(ii) there is a resonant plane arbitrarily close to every $\lambda$, i.e., the resonant planes are dense in the Siegel domain.

Example 2.7: small divisors. Let

$$\beta_0 = 1, \quad \beta_s = \min_{|k|=s+1} \min_{1 \leq j \leq n} |\langle k, \lambda \rangle - \lambda_j| .$$

Let also

$$\alpha_0 = \beta_0, \quad \alpha_s = \min(\alpha_{s-1}, \beta_s) .$$

• The sequences $\{\beta_s\}_{s \geq 1}$ (dots) and $\{\alpha_s\}_{s \geq 1}$ (crosses) for the complex vector

$$\lambda = (1 + i, -i\phi, -\sqrt{2}) ,$$

$$\phi = \frac{\sqrt{5} - 1}{2} .$$
Proof of proposition 2.6.

Claim (i)

- Enough to consider the case \( n = 3 \).
- Let \( \mathcal{A} \) be the sector determined by the half-lines \( O\lambda_1 \) and \( O\lambda_2 \).
- Say that \( \mu, \mu' \in \mathcal{A} \) are equivalent in case \( \mu - \mu' = s\lambda_1 + r\lambda_2 \) for some integers \( s \) and \( r \).
- Consider the infinite sequence \( -\lambda_3, -2\lambda_3, -3\lambda_3, \ldots \), as in figure. Let \( \mu_1, \mu_2, \mu_3, \ldots \) be the image of the sequence inside the parallelogram of sides \( \lambda_1, \lambda_2 \).
- If two of the \( \mu \)'s coincide, then \( \lambda \) is resonant, and belongs to a resonant plane: nothing else.
- If they are all distinct, then there is an accumulation point, i.e., there are \( |\mu_{s+r} - \mu_s| < \varepsilon \) with \( \varepsilon \) arbitray.
- Thus, \( -r\lambda_3 = j_1\lambda_1 + j_2\lambda_2 + \mu_{s+r} - \mu_s \) for some \( j_1, j_2 \), and so (claim (i))
  \[
  |j_1\lambda_1 + j_2\lambda_2 + (r + 1)\lambda_3 - \lambda_3| = |\mu_{s+r} - \mu_s| < \varepsilon .
  \]

Claim (ii).

- Let \( \lambda'_3 = \lambda_3 - \frac{\mu_{r+s} - \mu_r}{r} \). The vector \( \lambda' = (\lambda_1, \lambda_2, \lambda'_3) \) is resonant and satisfies (claim (ii))
  \[
  |\lambda' - \lambda| = \left| \frac{\mu_{r+s} - \mu_r}{r} \right| < \frac{\varepsilon}{r} .
  \]

Q.E.D.
2.1.5 Diophantine inequalities

**Definition 2.8:** A vector \( \lambda \in \mathbb{C}^n \) is said to be diophantine of class \((\gamma, \nu)\) with \(\gamma > 0\) and \(\nu \in \mathbb{R}\) in case
\[
|\langle k, \lambda \rangle - \lambda_j | \geq \frac{\gamma}{|k|^\nu}
\]
for all \(k \in \mathbb{N}_0^n, |k| \geq 2\) and for \(1 \leq j \leq n\).

Denote by \(\Gamma_{\gamma,\nu}\) the set of diophantine vectors of class \((\gamma, \nu)\).

- Poincaré domain: if \(\lambda\) is non resonant then it is diophantine of class \((\gamma, -1)\).
- Siegel domain: the diophantine inequality introduces a condition of strong non resonance.

**Proposition 2.9:** Let \(\nu > (n - 2)/2\). Then for every ball \(B_R \subset \mathbb{C}^n\) we have
\[
\text{Vol}(B_R \setminus \Gamma_{\gamma,\nu}) \leq c\gamma^2,
\]
where \(c\) is a positive constant depending on the radius \(R\) of the ball, on the dimension \(n\) and on \(\nu\).

- Hint for the proof:
  - Recall the proof that rational numbers on the interval \([0, 1]\) have measure zero.
  - Adapt it to the \(n\)-dimensional case (see next page).
Proof.

• Pick \( k \in \mathbb{N}_0^n \) (\( \mathbb{N}_0 \) the set of non negative integers). Let \( \alpha = \gamma/k^n \).

• The set

\[
G_{k,j} = \{ \lambda \in \mathbb{C}^n : |\langle k, \lambda \rangle - \lambda_j| < \alpha \}
\]

is a strip of width less that \( \alpha/\|k\| < \sqrt{n} \alpha/|k| \) around the \((k,j)\)-resonant plane.

• Counting the number of different \( k \)'s:

\[
\#\{k \in \mathbb{N}_0^n : |k| = s\} = \binom{s + n - 1}{n - 1} < s^{n-1}
\]

(see next page).

• Add up the contributions of all \( k \in \mathbb{N}_0^n, |k| \geq 2 \) and of \( 1 \leq j \leq n \), and get

\[
\text{Vol}\left( \bigcup_{|k| \geq 2} G_{k,j} \right) \leq C_1 n \gamma^2 \sum_{|k| \geq 2} \frac{1}{|k|^{2(\nu+1)}}
\]

\[
= C_1 n \gamma^2 \sum_{s \geq 2} \sum_{|k|=s} \frac{1}{s^{2(\nu+1)}}
\]

\[
= C_1 n \gamma^2 \sum_{s \geq 2} \left( \frac{s + n - 1}{n - 1} \right) \frac{1}{s^{2(\nu+1)}}
\]

\[
< C_1 n \gamma^2 \sum_{s \geq 2} \frac{1}{s^{2(\nu+1)-(n-1)}}.
\]

• The sum is finite if \( 2(\nu + 1) - (n - 1) > 1 \), i.e., if \( \nu > (n - 2)/2 \). Hence a constant \( c \) is readily found.

\[ Q.E.D. \]
Counting integer vectors: the number of \( k \in \mathbb{N}_0^n \) with \(|k| = s\).

I.e., counting the number of independent monomials of degree \( s \) in \( n \) variables.

**Lemma 2.10:** We have

\[
\# \{ k \in \mathbb{N}_0^n : |k| = s \} = \binom{s + n - 1}{n - 1}.
\]

**Proof.**

- Take \( s \) white balls and \( n - 1 \) black balls. Align them as, e.g.,

\[
\begin{array}{c}
\circ \ldots \circ \circ \ldots \circ \circ \ldots \circ \\
\hline
k_1 \quad k_2 \quad \ldots \quad k_n
\end{array}
\]

- Every vector \( k \) generates a single alignment.
- Every alignment corresponds to a single vector.
- There is a one-to-one correspondence between the set of \( k \)'s and the alignments of balls.
- From combinatorial analysis: the number of distinct alignments of \( s \) white balls and \( n - 1 \) black balls is

\[
\frac{(s + n - 1)!}{s!(n - 1)!} = \binom{s + n - 1}{n - 1}.
\]

Q.E.D.

**Exercise 2.11:** Prove that

\[
\# \{ k \in \mathbb{N}_0^n : |k| \leq s \} = \binom{s + n}{n}.
\]
2.2 Formal statements

Back to the initial problem:

Transform the non linear system

\[ \dot{z} = \Lambda z + V_1(z) + V_2(z) + \ldots \]

into a simpler (possibly linear) form.

2.2.1 The homological equation

- Want to solve \((\partial_\lambda - \lambda_j) \varphi = \psi\) with \(\psi(z)\) a known homogeneous polynomial of degree \(s \geq 2\).

- Introducing suitable linear spaces:
  - \(s^s\): the space of homogeneous polynomials of degree \(s\) in the \(n\) variables \(z_1, \ldots, z_n\).
  - \(N^s_j = (\partial_\lambda - \lambda_j)^{-1}(\{0\})\): the kernel of the linear operator \((\partial_\lambda - \lambda_j)\).
  - \(R^s_j = (\partial_\lambda - \lambda_j)(s^s)\): the range of the linear operator \((\partial_\lambda - \lambda_j)\).
  - Remark: both \(N^s_j\) and \(R^s_j\) are subspaces of \(s^s\).

**Lemma 2.12:** Let \(j \in \{1, \ldots, n\}\) be arbitrary but fixed. For any \(s > 2\) the following statements are true:

(i) The subspaces \(N^s_j\) and \(R^s_j\) satisfy

\[ N^s_j \cap R^s_j = \{0\}, \quad N^s_j \cup R^s_j = s^s. \]

(ii) If \(\lambda_1, \ldots, \lambda_n\) are non resonant then \(N^s = \{0\}\) for any \(s > 1\).

(iii) The operator \((\partial_\lambda - \lambda_j)\) restricted to its range \(R^s\) is uniquely inverted. I.e., for any \(\psi \in R^s_j\) there is a unique \(\varphi \in R^s\) such that \((\partial_\lambda - \lambda_j)\varphi = \psi\).

An obvious statement, since \((\partial_\lambda - \lambda_j)\) is a diagonal operator.

- The projection operators

\[ \Pi_{s^s} = (\partial_\lambda - \lambda_j)^{-1} \circ (\partial_\lambda - \lambda_j), \quad \Pi_{s^s} = 1 - \Pi_{s^s}, \]

satisfy \(\Pi_{s^s} \circ \Pi_{s^s} = \Pi_{s^s} \circ \Pi_{s^s} = 0\) and \(\Pi_{s^s} + \Pi_{s^s} = 1\).
The non resonant case of Poincaré and Siegel

Denote $V = \Lambda z + V_1 + V_2 + \ldots$ the vector field in eq. (2.6).

**Lemma 2.13:** Let $\lambda_1, \ldots, \lambda_n$ be non resonant. Then for every $j = 1, \ldots, n$ eq. $L_V \varphi = \lambda_j \varphi$ possesses a unique formal solution $\varphi = \varphi_0 + \varphi_1 + \ldots$ with $\varphi_0 = z_j$. The $n$ solutions so found are independent.

**Proof.** Recall that $L_V \varphi = \lambda_j \varphi$ splits into the recurrent system

\[
(\partial_\lambda - \lambda_j) \varphi_0 = 0 ,
\]

\[
(\partial_\lambda - \lambda_j) \varphi_s = \psi_s , \quad \psi_s = - \sum_{l=1}^s L_{V_l} \varphi_{s-l} , \quad s \geq 1 .
\]

- The first equation has the obvious solution $\varphi_0 = z_j$, which is unique.
- Let $s > 0$ and proceed by induction.
  - If $\varphi_0, \ldots, \varphi_{s-1}$ are known then so is $\psi_s$.
  - $\psi_s \in \mathbb{R}^s$, i.e., $\Pi_\lambda \psi_s = 0$, in view of non resonance (lemma 2.1).
  - Hence $\varphi_s = (\partial_\lambda - \lambda_j) \psi_s$ is uniquely determined, thus completing the induction.
- The independence follows from the independence of the linear parts $z_1, \ldots, z_n$.

Q.E.D.

**Proposition 2.14:** Let $\lambda_1, \ldots, \lambda_n$ be non resonant. Then there exists a formal near the identity transformation $w_j = z_j + w_{1,j}(z) + \ldots$ that conjugates the system to its linear part, i.e., $\dot{w}_j = \lambda_j w_j$. The transformation is unique.

**Proof.** The $n$ functions $w_j(z)$ are the solutions found in lemma 2.13.

Q.E.D.
2.2.3 The resonant case of Dulac

• Problem: the property $\Pi_{\mathcal{N}} \psi_s = 0$ fails to be true because there are resonant monomials.

• Way out (to be explored): change the homological equation to

$$ (\partial_\lambda - \lambda_j) \varphi_s + \eta_s = \psi_s , $$

with unknown $\varphi_s$ and $\eta_s$.

  o Split $\psi_s = \Pi_{\mathcal{N}} \psi_s + \Pi_{\mathcal{R}} \psi_s$.
  o Solve the homological equation by setting $\eta_s = \Pi_{\mathcal{N}} \psi_s$, $\varphi_s = (\partial_\lambda - \lambda_j)^{-1} \psi_s$.

  o The solution is not unique; it is made unique by asking $\varphi_s \in \mathcal{R}^s$;
  o an arbitrary term $\overline{\psi}_s \in \mathcal{N}^s$ may be added, if there are good reasons.

• Reformulate the initial problem as:

  Find a near the identity transformation which changes the system (2.1) into the non-linear normal form

$$ (2.9) \quad \dot{w}_j = \lambda_j w_j + \eta^{(j)}(w) , \quad j = 1, \ldots, n $$

with $(\partial_\lambda - \lambda_j) \eta^{(j)}(w) = 0$.

• Problem: show that the equation $L_V \varphi = \lambda_j \varphi$ can be consistently solved.

  o Reformulate in proper manner the recursive equations;
  o find a formal solution.
Assume that $\lambda_1, \ldots, \lambda_n$ are resonant, but belong to the domain of Poincaré.

- Recall: there is a straight line $\sigma$ through the origin that leaves $\lambda_1, \ldots, \lambda_n$ on the same side.

- Reorder the eigenvalues so that

$$\text{dist}(\lambda_1, \sigma) \leq \text{dist}(\lambda_j, \sigma) \leq \ldots \leq \text{dist}(\lambda_n, \sigma).$$

(the resonance relations are not affected).

- Remark: the resonance relation $\langle k, \lambda \rangle = \lambda_j$ may occur only if $k_j = \ldots = k_n = 0$.

**Proposition 2.15:** *(Dulac, 1912)* Let the eigenvalues $\lambda_1, \ldots, \lambda_n$ belong to the Poincaré domain, and be ordered as in (2.10). Then there exists a formal near the identity transformation that gives the system

$$\dot{z} = \Lambda z + V_1(z) + \ldots$$

the normal form

$$\dot{w}_1 = \lambda_1 w_1,$$

$$\dot{w}_j = \lambda_j w_j + \eta^{(j)}(w_1, \ldots, w_{j-1}), \quad j = 2, \ldots, n,$$

where $\eta^{(j)}(w_1, \ldots, w_{j-1})$ is a (non homogeneous) polynomial of finite degree belonging to the kernel of the operator $(\partial_{\lambda} - \lambda_j)$. The normal form for $j = 2, \ldots, n$ needs not be unique.
Proof. Proceed step by step.

- Let $j = 1$.
  - None of the divisors $\langle k, \lambda \rangle - \lambda_1$ can be zero.
  - Solve the recurrent system as in the non resonant case. Get
    \[
    w_1 = z_1 + \varphi_{1,1}(z) + \varphi_{2,1}(z) + \ldots
    \]
  - Invert (standard procedure for analytic functions)
    \[
    z_1 = w_1 + \bar{\varphi}_{1,1}(w_1, z_2, \ldots, z_n) + \bar{\varphi}_{2,1}(w_1, z_2, \ldots, z_n) + \ldots
    \]
  - Replace in the initial system of equations, and get the new system
    \[
    (2.12) \quad \dot{w}_1 = \lambda_1 w_1; \quad \dot{z}_j = \lambda_j z_j + V'_{1,j}(w_1, z_2, \ldots, z_n) + V'_{2,j}(w_1, z_2, \ldots, z_n) + \ldots, \quad j = 2, \ldots, n,
    \]
    with $n - 1$ equations to be solved.
- Let $j = 2$, assume that $k_1 \lambda_1 = \lambda_2$ for some $k_1 \geq 2$. (If not, there are no resonances: nothing new.)
  - There is only one resonance relation (obvious).
  - Split the system (2.12) as
    \[
    (\partial_\lambda - \lambda_2)\varphi_{s,2} = \psi_s \text{ for } s \neq k_1; \quad (\partial_\lambda - \lambda_2)\varphi_{k_1,2} + \eta^{(2)} = \psi_{k_1},
    \]
    with $\eta^{(2)}$ an unknown polynomial of degree $k_1 + 1$, and
    \[
    \psi_1 = -V'_{1,2}, \quad \psi_s = -V'_{s,2} - \sum_{l=1}^{s-1} L V'_{l,2} \varphi_{s-l,2}.
    \]
  - For $s \neq k_1$ there are no resonances; nothing new.
  - For $s = k_1$ the monomial $w_1^{k_1}$, and only that, is resonant: move it into $\eta^{(2)}$ (depends only on $w_1$).
  - Get the solution
    \[
    w_2 = z_2 + \varphi_{1,2}(w_1, z_2, \ldots, z_n) + \varphi_{2,2}(w_1, z_2, \ldots, z_n) + \ldots \quad \text{so that} \quad \dot{w}_2 = \lambda_1 w_2 + \eta^{(2)}(w_1).
    \]
• By induction, let $1 < j \leq n$, and assume we know

\[ w_l = z_l + \varphi_{1,2}(w_1, \ldots, w_{l-1}, z_l, \ldots, z_n) + \varphi_{2,2}(w_1, \ldots, w_{l-1}, z_l, \ldots, z_n) + \ldots \]

that change the first $j - 1$ equations to

\[ (2.13) \quad \dot{w}_1 = \lambda_1, \ldots, \dot{w}_{j-1} = \lambda_{j-1} w_{j-1} + \eta^{(j-1)}(w_1, \ldots, w_{j-1}) , \]

- Invert (standard procedure)
  \[ z_l = w_l + \tilde{\varphi}_{1,2}(w_1, \ldots, w_{l-1}, z_l, \ldots, z_n) + \tilde{\varphi}_{2,2}(w_1, \ldots, w_{l-1}, z_l, \ldots, z_n) + \ldots, \quad l = 1, \ldots, j - 1. \]
- Replace in the initial system and get the new system (2.13) plus the $n - j + 1$ equations
  \[ \dot{z}_l = \lambda_l z_l + V''_{1,l}(w_1, \ldots, w_{j-1}, z_j, \ldots, z_n) + V''_{2,l}(w_1, \ldots, w_{j-1}, z_j, \ldots, z_n) + \ldots, \quad l = j, \ldots, n \]
- Split the equation for $z_j$ as
  \[ (\partial_{\lambda} - \lambda_j) \varphi_{s,j} + \eta_s^{(j)} = \psi_s, \quad s \geq 2, \]

- There is a finite number of resonant terms, hence only a finite number of $\eta_s^{(j)}$ may be non zero.
- Resonant monomials: $w_1^{k_1} \cdots w_{j-1}^{k_{j-1}}$ with $k = (k_1, \ldots, k_{j-1}, 0, \ldots, 0)$ resonant. I.e., they depend only on $w_1, \ldots, w_{j-1}$.
- Hence the equation for $w_j$ may be solved as
  \[ w_j = z_j + \varphi_{1,2}(w_1, \ldots, w_{j-1}, z_j, \ldots, z_n) + \varphi_{2,2}(w_1, \ldots, w_{j-1}, z_j, \ldots, z_n) + \ldots \]

• Iterate the procedure for $j = 2, \ldots, n$.

Q.E.D.
2.3 Quantitative methods

The basic tools for investigating the convergence of the formal expansions.

2.3.1 Polynomial norm

- For a homogeneous polynomial \( f(z) = \sum_{|k| = r} f_k z^k \) of any degree \( r \) define the polynomial norm as
  \[
  \|f\| = \sum_{|k| = r} |f_k|.
  \]

- For a homogeneous polynomial vector field \( X(z) = (X_1(z), \ldots, X_n(z)) \) define the norm as
  \[
  \|X\| = \max_j \|X_j\|,
  \]

- Consider a (closed) polydisk \( \Delta_\varrho \) of radius \( \varrho \) around the origin of \( \mathbb{C}^n \), i.e.,
  \[
  \Delta_\varrho = \{ z \in \mathbb{C}^n : |z_j| \leq \varrho, j = 1, \ldots, n \}
  \]
  - For a homogeneous polynomial \( f_r(z) \) of degree \( r \) we have
    \[
    |f_r(z)| \leq \|f_r\|\varrho^r \quad \text{for all} \quad z \in \Delta_\varrho.
    \]
  - Let \( f = \sum_{r \geq 0} f_r(z) \) be a power series. If \( \|f_r\| \leq C^{r-1}F \) for some \( C, F > 0 \) then the series is absolutely and uniformly convergent in any polydisk of radius \( \varrho < 1/C \).
2.3.2 Technical estimates

Estimate of the Lie derivative.

Lemma 2.16: Let $X_r$ be a homogeneous polynomial vector field of degree $r + 1$ and $f_m$ be a homogeneous polynomial of degree $m + 1$. Then $L_{X_r} f_m$ is homogeneous of degree $s + m + 1$, and

$$\|L_{X_r} f\| \leq (m + 1) \|X\| \|f\|.$$ 

Proof. Write $X_{r,l} = \sum_{|k'|=r+1} c_{l,k'} z^{k'}$ and $f_m = \sum_{|k|=m+1} b_k z^k$.

- Get

$$L_{X_r} f_m = \sum_{l=1}^{n} \sum_{k,k'} c_{l,k'} k_l b_k z^{k+k'},$$

- By definition of norm

$$\|L_{X_r} f_m\| = \sum_{l=1}^{n} \sum_{k} |k_l| |b_k| \sum_{k'} |c_{l,k'}| \leq \sum_{k} \sum_{l=1}^{n} |k_l| |b_k| \|X_{r,l}\|.$$ 

- Use

$$\sum_{k'} |c_{l,k'}| = \|X_{r,l}\| \leq \|X_r\|, \quad \sum_{l} |k_l| = |k| = m + 1, \quad \text{and} \quad \sum_{k} |b_k| = \|f_m\|,$$

which gives the claim. Q.E.D.
Solution of the homological equation.

- Define the sequence \( \{\beta_s\}_{s \geq 0} \) as

\[
(2.14) \quad \beta_0 = 1, \quad \beta_s = \min_{|k| = s+1} \min_{1 \leq j \leq n} |\langle k, \lambda \rangle - \lambda_j|, \quad s \geq 1.
\]

- Poincaré domain, \( \lambda \) non resonant:

\[
\beta_s \geq \gamma(s+1), \quad \gamma > 0.
\]

- Siegel domain, \( \lambda \) non resonant and diophantine:

\[
\beta_s \geq \frac{\gamma}{(s+1)^\nu}, \quad \gamma > 0, \quad \nu > \frac{n-2}{2}.
\]

**Lemma 2.17:** Let \( \lambda \) be non resonant, and let \( \psi_s \) be a homogeneous polynomial of degree \( s + 1 \). Then we have

\[
\left\| (\partial_\lambda - \lambda_j)^{-1} \psi_s \right\| \leq \frac{\|\psi_s\|}{\beta_s}
\]

with the sequence \( \{\beta_s\}_{s \geq 1} \) defined by (2.14).

**Proof.** Straightforward in view of the diagonal form of the operator \( (\partial_\lambda - \lambda_j) \) stated by corollary 2.2.

- Recall

\[
(\partial_\lambda - \lambda_j)z^k = (\langle k, \lambda \rangle - \lambda_j)z^k.
\]

- Just divide the coefficient of every monomial \( z^k \) by the corresponding eigenvalue \( \langle k, \lambda \rangle - \lambda_j \), which has lower bound \( \beta_s \).

Q.E.D.
2.3.3 The theorem of Poincaré

**Theorem 2.18:** (Poincaré, 1879) Let the system \( \dot{z} = \Lambda z + V_1 + \ldots \) be analytic in a neighbourhood of the origin of \( \mathbb{C}^n \), and let the eigenvalues of the matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) belong to the Poincaré domain and be non resonant. Then in a neighbourhood of the origin there exists an analytic diffeomorphism that conjugates the system to its linear normal form (2.2), i.e., \( \dot{w} = \Lambda w \).

**Proof.** By Proposition 2.14 there are \( n \) independent formal series \( w_j = z_j + \varphi_{1,j}(z) + \ldots \) which give the wanted normal form. Prove that the formal series are convergent in a polydisk.

- Recall the recurrent system

  \[
  (\partial_\lambda - \lambda_j) \varphi_0 = 0 , \\
  (\partial_\lambda - \lambda_j) \varphi_s = \psi_s , \quad \psi_s = -\sum_{l=1}^{s} LV_l \varphi_{s-l} , \quad s \geq 1 .
  \]

  to be solved for \( \varphi_0 = z_j \), hence \( \|\varphi_0\| = 1 \).

- By analyticity there are \( B \geq 0 \) and \( E > 0 \) such that

  \[
  \|V_s\| \leq B^{s-1}E , \quad s > 0
  \]

- By lemmas 2.16 and 2.17 and using \( \beta_s \geq \gamma s \) get

  \[
  \|\varphi_0\| = 1 ; \quad \|\varphi_s\| \leq \frac{1}{\gamma} \sum_{l=1}^{s} \frac{(s - l + 1)}{(s + 1)} \|V_l\| \|\varphi_{s-l}\| \leq \frac{E}{\gamma} \sum_{l=1}^{s} B^{l-1} \|\varphi_{s-l}\| .
  \]

- Hence \( \|\varphi_s\| \leq \mu_s \) with the arithmetic sequence

  \[
  \mu_0 = 1 , \quad \mu_s = \frac{E}{\gamma} \sum_{l=1}^{s} B^{l-1} \mu_{s-l} ; \quad \text{hence} \quad \mu_s \leq \frac{E}{\gamma} \left( B + \frac{E}{\gamma} \right)^{s-1} .
  \]

- Conclude: the series for \( \varphi \) is absolutely convergent in any polydisk \( \Delta_\rho \) with \( \rho < \left( B + \frac{E}{\gamma} \right)^{-1} \).

Q.E.D.
2.3.4 On the case of Siegel

Consider the simplified model

\[ \dot{z} = \Lambda z + V_1(z). \]

- The recurrent system simplifies to

\[ (\partial_{\lambda} - \lambda_j)z_j = 0, \quad (\partial_{\lambda} - \lambda_j)\varphi_{s,j} = -LV_1\varphi_{s-1,j}, \quad s \geq 1. \]

- The recurrent estimate gives

\[ \|\varphi_s\| \leq \frac{s!}{\beta_1 \cdots \beta_s}. \]

- Two sources of divergence:
  - the factorial at the numerator, due to estimates of derivatives;
  - the accumulation of divisors at the denominator estimated via the diophantine inequality.

- Accumulation of divisors controlled by a number theoretical lemma due to Siegel:

\[ \frac{1}{\beta_1 \cdots \beta_s} \leq K^s \]

for some \( K > 1 \).

- Need to change the algorithm in order to remove the factorial.

The obstacle will be removed using the normal form theory.