

Talk 2/5

Vector bundles, sheaves and divisors

- ① Holomorphic manifolds / projective varieties
- ② Vector bundles and sheaves
- ③ Line bundles and hypersurfaces
- ④ Short exact sequences (of sheaves)

- ① Problem: defining a geometric space X , with enough structure to do geometry / physics

tools from algebra / analysis

$$\left\{ \begin{array}{l} \text{continuously} \\ \text{flexibility} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{holomorphic} \\ \text{BIG DIFFERENCE} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{polynomials} \\ \text{rigidly} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{open sets, neighborhoods} \\ \text{continuous functions} \\ \text{(convergence of sequences)} \end{array} \right\}$$

To start with, X topological space

definition A holomorphic manifold on X is given

open cover $\{U_\alpha\}$ and maps $\varphi_\alpha: U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^n$

(φ_α are topological isomorphisms) s.t. $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic

on $\varphi_\beta(U_\alpha \cap U_\beta)$.

example $P_{\mathbb{C}}^m := \{0 \neq x \in \mathbb{C}^{m+1}\} / \sim$ $x \sim \lambda x \quad \lambda \in \mathbb{C}^*$

$P_{\mathbb{C}}^m \supset V_i := \{x_i \neq 0\} / \sim$ $\varphi_i: V_i \xrightarrow{1:1} \mathbb{C}^m$

$$(x_0, \dots, x_n) \mapsto \left(\frac{x_1}{x_i}, \dots, \frac{x_n}{x_i} \right)$$

$$\varphi_{\mathbb{P}^n}(\mathbb{C}^{\times}) : (z_0 \neq 0) \quad (z_1, \dots, z_n) \mapsto \left(\frac{z_1}{z_0}, \dots, \frac{z_{n-1}}{z_0}, \frac{1}{z_0}, \frac{z_n}{z_0} \right)$$

is holomorphic

$$\text{similarly, } (\mathbb{P}^n)^V = \{ \text{hypersurfaces in } \mathbb{C}^{n+1} \}$$

definition A holomorphic submanifold Y of X is (locally on open sets) the zero locus of f_1, \dots, f_k holomorphic functions on X

(*) $\text{rk } \left(\frac{\partial f_i}{\partial z_j} \right) = k$ ANALYSIS : \Rightarrow a submanifold (implicit function theorem) is in fact a manifold.

An analytic subvariety V of X : drop (*)

irreducible : set of smooth points is connected.

definition A projective manifold/variety is a holomorphic analytic subvariety of \mathbb{P}^n .

remark : They are closed in \mathbb{P}^n , hence compact.

Example,

homogeneous polynomial of degree d

$$f(x_0, \dots, x_n) = \sum_{|x_i|=d} \text{deg } d \text{ monomials in the } x_i$$

$\{ x \in \mathbb{P}^n \mid f_i(x) = 0 \}$ well defined when f_i are homogeneous

(A) $\{ x_m = 0 \} \subset \mathbb{P}^n$ hyperplane $\cong \mathbb{P}^{n-1}$

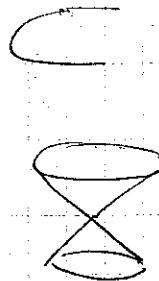


(B) $\{ x_1 x_2 = 0 \} \subset \mathbb{P}^2$

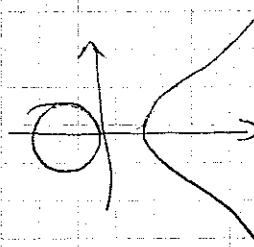


$$(C) \{x_0x_1 - x_2^2 = 0\} \subset \mathbb{P}^2$$

$$\cap \mathbb{P}^3$$



$$(D) \{x_0x_2^2 = x_1^3 + ax_1x_0 + bx_0^3\} \subset \mathbb{P}^2$$



$$(E) \begin{cases} x_1=0 \\ x_2=0 \end{cases} \subset \mathbb{P}^3$$

$$\begin{aligned} z &= \frac{x_3}{x_0} \\ r &= \frac{x_2}{x_0} \\ x &= \frac{x_1}{x_0} \end{aligned}$$

Name: hypersurface when $\mathcal{V}X$ given by 1 equation.

(A...D) hypersurfaces, E is not.

Theorem (Chow's lemma) A projective variety is always locally the zero locus of polynomials

- ② { holomorphic vector bundles on X } \cong { sheaves of \mathcal{O}_X -modules on X }
- motivation:
- 1) language of sheaves convenient for VB
 - 2) tangent to singular varieties is NOT VB
 - 3) there are other interesting sheaves
(push-forward)

definition let $\pi: E \rightarrow X$ holomorphic map of real manifolds

π is a vector bundle if there is $\{U_\alpha\}$ open cover of X , and iso

$$\psi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}^n$$

$$1) \pi = \text{pr}_1 \circ \psi_\alpha$$

$$2) U_\alpha \cap U_\beta \times \mathbb{C}^n \xleftarrow{\psi_\alpha} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\psi_\beta} U_\alpha \cap U_\beta \times \mathbb{C}^n$$

$$\psi_\beta \circ \psi_\alpha^{-1} = (\text{id}, g_{\alpha\beta})$$

$g_{\alpha\beta}$ LINEAR MAPS

construction of the "associated sheaf": let $\pi: E \rightarrow X$ be V.B.

E "associated sheaf" via rule, $\forall U \subset X$ OPEN

$$U \mapsto \mathcal{E}(U) := \left\{ s: U \rightarrow E \text{ local} \mid \pi \circ s = \text{id}_U \right\} \text{ sections}$$

Properties:

$$\begin{aligned} 1) \quad s_1 \in \mathcal{E}(U_1) & \text{ s.t. } s_1 = s_2 \Rightarrow \exists s \in \mathcal{E}(U_1 \cup U_2) \\ s_2 \in \mathcal{E}(U_2) & \text{ in } \mathcal{E}(U_1 \cup U_2) \quad s|_{U_1} = s_1 \\ & s|_{U_2} = s_2 \end{aligned}$$

2) $\mathcal{E}(U)$ is an abelian group

INTERMETTO: the trivial line bundle $\underline{\mathbb{C}}$ is $\mathbb{C} \times \mathbb{P}^1$

associated sheaf of $\underline{\mathbb{C}}$ is called \mathcal{O}_X

$$\mathcal{O}_X(U) = \{ f: U \rightarrow \mathbb{C} \text{ holomorphic} \}$$

3) can multiply sections with holomorphic functions

$$\mathcal{O}_X(U) \times \mathcal{E}(U) \rightarrow \mathcal{E}(U)$$

$$(f, s) \mapsto f \cdot s$$

4) " \mathcal{E} is locally free"; $\exists \{U_\alpha\}_{\alpha \in I}$ open cover

$$\mathcal{E}(U_\alpha) \cong \{ s: U_\alpha \rightarrow \mathbb{C}^n \} \cong \mathcal{O}_X(U_\alpha)^{\oplus n}$$

holomorphic

definition: A sheaf of abelian groups on X via "rule"

$$\begin{array}{ccc} \forall U \subset X & U \mapsto G(U) & (\text{THINK: } g(u) \text{ are} \\ \text{OPEN} & \text{OPEN} & \text{sections over } U) \\ & \text{AB. GROUP} & \end{array}$$

satisfying ① ②

A sheaf satisfying ③ is a sheaf of G -modules.

Proposition {VB on $X = \{$ locally free (4)
sheaves of \mathcal{O}_X -modules on $X\}$

example M_X : sheaf of meromorphic functions

$$M_X(U) = \{ f: U \rightarrow \mathbb{C} \text{ s.t. } \exists g, h \in \mathcal{O}_X(U) \text{ s.t. } f = g/h \}$$

• M_X is NOT locally free.

• constructions are done with sheaves of \mathcal{O}_X -modules

$$\oplus \quad E_1 \sim g_{\alpha\beta}^1 \quad E_1 \oplus E_2 \left(\begin{matrix} g_{\alpha\beta}^1 \\ g_{\alpha\beta}^2 \end{matrix} \right) \quad \mathcal{E}_1(U) \oplus \mathcal{E}_2(U)$$

$$E_2 \sim g_{\alpha\beta}^2$$

$$\otimes \quad E_1 \otimes E_2 \sim g_{\alpha\beta}^1 \cdot g_{\alpha\beta}^2 \quad \mathcal{E}_1(U) \otimes \mathcal{E}_2(U)$$

• dual

$$E^* \sim g_{\alpha\beta}^{-1}$$

$$\text{Sym}^k(E), \Lambda^k(E), \det(E) = \Lambda^{kn}(E)$$

$$f^*(F)(U) = F(f^{-1}(U)) \quad \text{push-forward}$$

$$f^*(F)(U) = "F(f(U))" \quad \text{pull-back}$$

Example

(A) tangent space

$$x \in \mathbb{C}^n \text{ given by } f_1 = 0, \dots, f_m = 0$$

$$T_p(x) = \{v \in \mathbb{C}^m \mid \left(\frac{\partial f_i}{\partial x_j}\right)(p)v_j = 0\}$$

x smooth \Rightarrow tangent BPL
otherwise \Rightarrow tangent sheaf

$$\bullet XY = 0 \quad Df = (Y, X) \quad \text{normally rank 1}$$

$$\bullet XY - Z^2 = 0 \quad Df = (Y, X, -2Z) \quad \text{in } (0, 0, 0) \text{ rank 2}$$

(C) "ideal sheaf" fix $Y \subseteq X$ subvariety

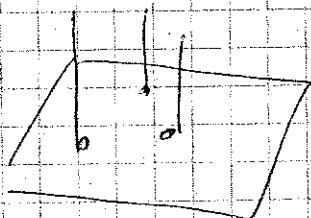
$$M_Y(U) := \{ f \in \mathcal{O}_X(U) \mid f(p) = 0 \forall p \in Y \}$$

not locally free, $M_Y \subset \mathcal{O}_X$.

(D) $Y \subseteq X$ (* \mathcal{O}_Y) sheaf supported on Y

closed subm.

$\mathcal{I} Y = \{p_1, \dots, p_n\}$ points \mathcal{O}_Y called skyscraper sheaf



(E) very important example $S^1_X = \mathbb{C}^* T^* X$ cotangent

$\wedge^{top} S^1_X =: k_X$ canonical line bundle

(3) Motivation: equivalence on X

$\{$ hypersurfaces $\} \xrightarrow{\sim} \{$ line bundles on $X \}$

operation on the set of \mathcal{O}_X -modules

$$u \mathcal{E}_1 + v \mathcal{E}_2 : \mathcal{E}_1 \otimes \mathcal{E}_2$$

Remark:

commutative, associative,

0 neutral no inverse

restrict this operation to the set of line bundles

definition $\{ \text{line bundles on } X \} /_{\text{iso}} =: \text{Pic}(X)$

$$n - \sum n_i = \sum v_i$$

definition A Weil divisor D on X is $D = \sum e_i V_i$ finite
 $e_i \in \mathbb{Z}$, V_i mod anal-hypersurface

$\text{Div}(X)$ group of divisors

D effective: $e_i \geq 0 \forall i$

On $\text{Div } X$ equivalence \sim

Fix $U \subseteq X$ open
 $V \subseteq U$ mod subreg

1) $g \in \mathcal{O}_X(U)$ $\text{ord}_V(g) :=$ order of vanishing of g at V

2) $f \in M_X(U)$

$f = g/h$ $\text{ord}_V(f) := \text{ord}_V(g) - \text{ord}_V(h)$

$D \sim D' \Leftrightarrow \exists f \in M_X(X)$
 global meromorphic

$$D' = D + \text{div}(f)$$

$$\text{div}(f) := \sum_{V \text{ hypersurf}} \text{ord}_V(f) \cdot V.$$

correspondence:

$$D \in \text{Div}(X) \leftrightarrow \{(U_\alpha, f_\alpha)\} \quad \{U_\alpha\} \text{ open covers}$$

$(D \text{ effective precisely when one can choose } f_\alpha \in \mathcal{O}_X(U_\alpha) \forall \alpha)$

$$f_\alpha \neq 0 \in M_X(U_\alpha)$$

$$D|_{U_\alpha} = \text{div}(f_\alpha)$$

$\chi: \text{Div}(X) \rightarrow \text{Pic}(X)$

$$D \mapsto \mathcal{O}(D)$$

$$\mathcal{O}(D)(U_\alpha) := \{g \cdot f_\alpha^{-1} : g \in \mathcal{O}_X(U_\alpha)\}$$

equivalently: meromorphic functions $g_{XB} = f_\alpha/g_\beta$ on $M_X(U_\alpha)$

proposition $(X \text{ holom, } \frac{\text{proj. smooth}}{\text{smooth}}) \xrightarrow[\sim]{\text{Div}(x)} \text{Pic}(X)$ is onto

D effective $\Leftrightarrow D = \sum a_i V_i$

$\mathcal{O}(D) = \{ \text{merom functions with pole at } V_i \leq a_i \}$

$\mathcal{O}(-D) = \{ \text{holom functions vanishing at } V_i \text{ of order } \geq a_i \}$

Ample divisors

let D be a divisor. FACT: $\mathcal{O}(D)(x)$ is finite

dimensional
vector space

$\varphi_D: X \dashrightarrow \mathbb{P}(\mathcal{O}(D)(x))$

$p \mapsto \{ s \in \mathcal{O}(D)(x) \mid s(p) = 0 \}$

def D is very ample if φ_D is well defined and
closed embedding

def D is ample if $\exists n \in \mathbb{N}$ nD is very ample

def X holom manifold

Fano: $-K_X$ ample

CY: $K_X \cong \mathcal{O}_X$ ($+ b_1(x) = 0$)

General type: K_X ample

Examples

1) $\text{Pic}(\mathbb{P}^n)$

(A) (line bundles)

$$\mathcal{O}(-1) \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$$

↑
↑

$$\{(x, v) \mid v \in \mathbb{C}^n\}$$

$$\mathcal{O}(n) := \begin{cases} \mathcal{O}(-1)^{\otimes (-n)} & n \leq 0 \\ (\mathcal{O}(-1))^\vee \otimes \mathbb{C}^n & n > 0 \end{cases}$$

(B) (divisors / n)

$$f_1 \text{ hypersurfaces deg of } V_1 \quad \text{div}(f_1/f_2) = V_1 - V_2$$

f_2

V_2

↓

$V_1 \cup V_2$

conclusion $\mathbb{Z} \cong \{\mathcal{O}(n)\}_{n \in \mathbb{Z}} = \text{Pic}(\mathbb{P}^n)$

Hyper surface of degree $n \quad \mathcal{O}(V) = \mathcal{O}(n)$

2) $\text{Pic}(\mathbb{P}^n \times \mathbb{P}^m) \cong \text{Pic}(\mathbb{P}^n) \oplus \text{Pic}(\mathbb{P}^m)$

(in general)

$$\text{Pic}(\mathbb{X}) + \text{Pic}(\mathbb{Y})$$

\oplus

Pic

Theorem (Grothendieck) E vector bundle on \mathbb{P}^1

$$\exists k_1, \dots, k_n \in \mathbb{Z} \quad E \cong \bigoplus_{i=1}^m \mathcal{O}(k_i)$$

(1)

(4) Motivation: very powerful language.

Help us calculate K_X .

def $0 \rightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \rightarrow 0$ short exact sequence

means: α, β morphisms of sheaves ($F(U) \xrightarrow{\alpha_U} G(U)$)

$V_P \cdot \exists V_P \ni p$
OPEN

A) $\alpha_{V_P}: \mathcal{F}(U_P) \rightarrow \mathcal{G}(U_P)$ INJECTIVE

B) $\beta_{V_P}: \mathcal{G}(U_P) \rightarrow \mathcal{H}(U_P)$ SURJECTIVE

C) $\left\{ \begin{array}{l} \text{s sections} \\ \text{of } \mathcal{G}(U_P) \end{array} \right\} = \left\{ \begin{array}{l} \text{s sections} \\ \text{of } \mathcal{H}(U_P) \end{array} \right\}$
 $s = \alpha(t)$ $\beta(s) = 0$

FACT (linear algebra) $\det(\mathcal{Y}) \cong \det(\mathcal{F}) \otimes \det(\mathcal{H})$.

Examples

1) $0 \rightarrow \mathbb{Z} \hookrightarrow \mathcal{O}_X \xrightarrow{\quad} \mathcal{O}_X^* \rightarrow 0$
 $f \mapsto f^{-1}$

\mathcal{O}_X^* = local functions of
 $f(P) \neq 0 \forall P$.

2) $D \subseteq X$ need hypersurface

$$0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O}_X \xrightarrow{\quad} \mathcal{O}_D \rightarrow 0$$

3) $Y \subseteq X$ closed submanifolds

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \xrightarrow{\quad} \mathcal{O}_Y \rightarrow 0$$

4) Euler sequence on P^n

$$0 \rightarrow \mathcal{O}_{P^n} \xrightarrow{\quad} \mathcal{O}(1) \xrightarrow{\quad} T_{P^n} \rightarrow 0$$

$$1 \mapsto (x_0, \dots, x_n)$$

$$(x_0, \dots, x_n) \mapsto \sum x_i \frac{\partial}{\partial x_i}$$

short exact

sequence : Use Euler theorem

on homogeneous functions

corollary (FACT, Euler) $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$

5) $c: X \rightarrow Y$ closed emb. of lsdm submanifold

$$0 \rightarrow T_X \rightarrow c^* T_Y \rightarrow N_{X/Y} \rightarrow 0$$

corollary 1 (FACT) $K_X \cong c^* K_Y \otimes \det(N_{X/Y})$

lemma $D \subset Y$ hypersurface $N_{D/Y} \cong \mathcal{O}(D)$

corollary 2 $X \subset \mathbb{P}^n$ hypersurface of degree d

$$K_X = c^*(\mathcal{O}(-n-1+d))$$

corollary 3 $X \subset \mathbb{P}^n$ hypersurface of deg d

$$X \cong \begin{cases} \text{Fano} & d < n+1 \\ \text{CY} & d = n+1 \\ \text{general type} & d > n+1 \end{cases}$$

remark If $n > 2$, X is also simply connected

(follows from Lefschetz hyperplane theorem + Veronese embedding)