Gravitational effects of the faraway matter on the rotation curves of spiral galaxies

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Abstract

It was recently shown that in cosmology the gravitational action of faraway matter has quite relevant effects, if retardation of the forces and discreteness of matter (with its spatial correlation) are taken into account. Indeed, far matter was found to exert, on a test particle, a force per unit mass of the order of $0.2cH_0$. It is shown here that such a force can account for the observed rotational velocity curves in spiral galaxies, if the force is assumed to be decorrelated beyond a sufficiently large distance, of the order of 1 kpc. In particular we fit the rotation curves of the galaxies NGC 3198, NGC 2403, UGC 2885 and NGC 4725 without any need of introducing dark matter at all. Two cases of galaxies presenting faster than keplerian decay are also considered.

Keywords: galaxies: general – galaxies: kinematics and dynamics – galaxies: fundamental parameter

1 Introduction

Perhaps, the actual problem of gravitation theory is to justify the large speeds observed in objects of galactic or larger sizes. This problem became increasingly puzzling, starting from the end of the years seventies. Indeed precise measurements (with a variety of techniques) of the speed of gases in spiral galaxies made apparent that the action of the galaxy on the gas, computed according to the standard theory of gravitation, cannot account for the measured velocities, if the gas is assumed to be a stable component of the galaxy. In the literature, while a lot of efforts are devoted to computing the gravitational action of a galaxy on the nearby gas, no attention is paid to the gravitational effects exerted by the other galaxies, particularly the far away ones. This attitude, of neglecting the faraway matter, would indeed be correct if the other galaxies were uniformly distributed. However, up to the present limit of observations (at least 300 Mpc, according to Gabrielli et al. [1999, 2004]; Sylos Labini et al. [1998]), it was suggested that the galaxies may instead be distributed according to a very complex fractal structure. It was shown by Carati et al. ([2008]) that the gravitational action of faraway matter computed according to the General Relativity has quite relevant effects, if both the retardation of the forces and the discreteness of matter (with its spatial correlation) are taken into account. Indeed, the far matter was found to exert, on a test particle, a

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force per unit mass of the order of $0.2 \, c H_0$, where $H_0$ is the Hubble constant, and $c$ the speed of light. It was also shown how the gravitational force due to the faraway matter may account for the speed of the galaxies in clusters.

The aim of the present paper is to estimate the effect that the faraway matter produces on the rotation curves in spiral galaxies. We show that such an effect becomes very relevant if one assumes that the gravitational field due to the faraway matter is decorrelated beyond a sufficiently large distance (of the order of 1 kpc). So such a force goes unnoticed for objects of the solar system’s size (or smaller), while starts becoming important from objects of galactic size on, being the predominant one for clusters of galaxies and larger structures. In particular, we show that the rotation curves of four galaxies (with very different masses and sizes) can be fitted taking into account such a force, leading to reasonable values of the luminosity–mass ratio (see Table I of Section 4). Two galaxies presenting a faster than keplerian decay are also considered.

The paper is organized as follows. In Section 2, we recall the model that was introduced by Carati et al. ([2008]) in order to deal with the gravitational effect of faraway matter. In Section 3 the effect on the rotation curves is evaluated, and in Section 4 the fit to rotational curves for the galaxies NGC 3198, NGC 2403, NGC 2885, NGC 4725 is reported. Two cases of rotation curves presenting a faster than keplerian decay are also briefly discussed. Two Appendices, in which some technical computations are reported, complete the paper.

2 Definition of the model

We briefly recall in this section the model that was introduced by Carati et al. ([2008]) in order to estimate the gravitational action of faraway matter. To this end, first of all one has to estimate the gravitational field generated by the distant galaxies (thought of as point masses) considered as sources in Einstein’s equation, in the approximation in which their positions and velocities are assigned, in the simplest way compatible with observational cosmology.

To this end, the positions $q_j$ of the galaxies are considered as random variables, whose statistical properties will be discussed later. The velocity field of the sources is instead taken according to Hubble’s law, the peculiar velocities being altogether neglected. Namely, taking a locally Minkowskian coordinate system centered about an arbitrary point, a particle with position vector $q$ is assumed to have a velocity

$$q = H_0 \, q,$$

where, for the sake of simplicity of the model, the Hubble constant $H_0$ is assumed to be time independent. It is easily established that the considered chart has a local Hubble horizon $R_0 = c/H_0$, where the galaxies’ speed equals that of light.

The energy–momentum tensor $T^{\mu\nu}$ is then given by

$$T^{\mu\nu} = \sum_{j=1}^{N} \frac{1}{\sqrt{g}} \, M_j \, \gamma_j \, \delta(x - q_j) q_j^\mu q_j^\nu,$$

where $N$ is the number of galaxies, $M_j$ and $\gamma_j$ are the mass and the Lorentz factor of the $j$–th one, $g$ is the determinant of the metric tensor (which is considered as an unknown of the problem), $\delta$ the Dirac delta function, and the dot denotes derivative with respect to proper time along the worldline of the source.
The study of the solutions of Einstein’s equations with the energy–momentum tensor (2) as a source is a formidable task, and so the study of Carati et al. ([2008]) was performed within the limits of a perturbation approach, considering the energy–momentum tensor $T^{\mu\nu}$ (2) as a perturbation of the vacuum. Following the standard procedure (see Einstein [1922] or Weinberg [1972]), one then has to determine a zero–th order solution (the vacuum solution), and solve the Einstein equations, linearized about it. The simplest consistent zero–th order solution is the flat metric because, coherently, the perturbation is then shown to be small (at least if the parameters of the model, such as the density, are chosen in agreement with the observations).

Thus the metric tensor $g_{\mu\nu}$ is written as a perturbation of the Minkowskian background $\eta_{\mu\nu}$, namely, as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and it is well known that in the linear approximation the perturbation $h_{\mu\nu}$ then has to satisfy essentially the wave equation with $T^{\mu\nu}$ as a source. More precisely, one gets

$$\Box \left[ h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \right] = -\frac{16\pi G}{c^4} T_{\mu\nu} \, ,$$

where $G$ is the gravitational constant, $h$ the trace of $h_{\mu\nu}$, and $\Box = (1/c^2) \partial^2_t - \Delta$. The solutions are the well known retarded potentials

$$h_{\mu\nu} = -\frac{2G}{c^4} \sum_{j=1}^N \frac{M_j}{\gamma_j} \frac{2q^{(j)}_\mu q^{(j)}_\nu - c^2 \eta_{\mu\nu}}{|x - q_j|} \bigg|_{t=t_{ret}} \, ,$$

(with $q^{(j)} \equiv q_j$).

In the present case in which we are concerned with the rotation curves of spiral galaxies, one has also to take into account the contribution of local matter, so that, in the spirit of the considered approximation, the total metric tensor is given by the sum of that due to the local matter, and of that due to the distant one. For the local metric $h^{loc}_{\mu\nu}$, one furthermore makes the Newtonian approximation, i.e., all components are assumed to vanish, apart from $h^{loc}_{00}$, which is set equal to the Newtonian potential $V^{loc}(x)/2c^2$ due to local masses. By local matter we mean the one contained in the galaxy, so that we forget, in the rest of the paper, the possible contribution of the dark matter halo. The total metric is thus written as

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2V^{loc}(x)}{c^2} \delta_{0\mu} \delta_{0\nu} + h_{\mu\nu} \, ,$$

with $h_{\mu\nu}$ given by (4).

So the equation of motion for a test particle contains, in addition to the Newton force produced by the mass inside the galaxy, coming from the term $h^{loc}_{\mu\nu}$, also some terms coming from the field due to the distant matter. How large is this additional term with respect to the local Newton force? The answer depends, on the assumptions made on the positions $q_j$ of the constituents of the far away matter which appear in formula (4).

Carati et al. ([2008]) assumed the positions $q_j$ to be random variables, having a probability distribution which is isotropic (rotationally invariant). A simple computation then shows that the mean force (per unit mass) acting on a test particle vanishes. Thus the order of magnitude of the force is given by its standard deviation.

The estimate of the standard deviation requires an additional hypothesis on the distribution of the quantities $q_j$. If they are assumed to be independent and identically distributed, then the force turns out to be negligible.
On the other hand, it is known that the positions of different galaxies are correlated, having a distribution with a fractal character, and it has been suggested that this may happen up a certain quite large distance (see Sylos Labini et al. [1998]; Gabrielli et al. [2004, 1999]). Carati et al. ([2008]), assuming that the distribution is fractal at all distance scales, show that the force per unit mass due to the distant galaxies is of the order of $0.2 c H_0$.

Now, as pointed out by Milgrom ([1983, 1987]), it is precisely when the acceleration of the stars is of the order of $c H_0$ that the rotation curves of the spiral galaxies depart from the behaviour expected from the Newtonian force due to local visible matter. Our purpose is then to compute the effect of the far field, precisely in such a region. It will be shown that such an the effect can be described as corresponding to an effective potential which acts on the stars by modifying the profile of the rotation curves as to fit the actual observed one.

3 The effect of far away matter on the rotation curves

In this Section, we estimate the effect of the field produced by the far away matter on the rotation curves. So, first of all, one has to estimate the influence of the far field on the motion of stars.

In general, as there is no reason to expect that the far field should be spherically symmetric, its first effect amounts to destroy the circular orbits, from which every discussion about the rotation curves usually starts. To tackle this problem, one can imagine that the effect of the far field is not too large, and that the orbits are still close to circular ones. In this case one can average the radial component of the equation of motion along a circle, thus obtaining an effective radial equation from which the angular velocity of the stars as a function of the radius may be estimated.

Using cylindrical coordinates $(r, \theta, z)$, and considering only motions which lie in the galactic plane $z = 0$ (so that $\dot{z} = \ddot{z} = 0$), the radial equation of motion turns out to be

$$2 \frac{d}{dr} \left( g_{rr} r + g_{r\theta} \dot{\theta} + g_{r0} c \right) = c^2 \partial_r g_{00} + \dot{\theta}^2 \partial_{\theta} g_{\theta\theta} + 2 c \dot{\theta} \partial_{\theta} g_{\theta0}$$

$$+ 2 r \dot{\theta} \partial_r g_{r0} + 2 c \dot{r} \partial_r g_{\theta r} + r^2 \partial_r g_{rr} .$$

(5)

In particular, in terms of the cartesian components $h_{\mu\nu}$ given by (4) and of the local potential $h_{\mu\nu}^{loc}$, the relevant components are given by the formulas

$$g_{rr} = r^2 \left( 1 + h_{11} \cos^2 \theta + h_{22} \sin^2 \theta + h_{12} \sin \theta \cos \theta \right)$$

$$g_{r\theta} = r \left( (h_{11} - h_{22}) \sin \theta \cos \theta + h_{12} \cos 2\theta \right) \overset{\text{def}}{=} r \mathcal{D}$$

$$g_{r0} = h_{01} \cos \theta + h_{02} \sin \theta$$

$$g_{\theta0} = r^2 \left( 1 + h_{11} \sin^2 \theta + h_{22} \cos^2 \theta + h_{12} \sin \theta \cos \theta \right) \overset{\text{def}}{=} r^2 (1 + \mathcal{A})$$

$$g_{00} = r \left( h_{02} \cos \theta - h_{01} \sin \theta \right) \overset{\text{def}}{=} r \mathcal{B}$$

$$g_{00} = 1 + h_{00} + \frac{2}{c^2} \psi^{loc}(r) .$$

(6)

The quantities $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$, implicitly defined in such formulas, will be used later. In our setting, equation (5) is a stochastic one, because of the stochastic character of the terms $h_{\mu\nu}$, and it is very hard to be dealt with.
To reduce our problem to a tractable one, we make essentially two hypotheses. First of all we suppose that it is meaningful to average the equation with respect to $\theta$, i.e., we suppose that $\theta$ is a fast variable with respect to $r$, and consequently the derivative $\dot{r}$ must be thought of as “small”. Thus, in the averaged equation, we will forget all terms proportional to $\dot{r}$.

Now, as the galaxies do not change their dimensions on the time scale of the revolution of a typical star, one can readily give an estimate of the term $d(g_{rr}\dot{r})/dt$ of (5), which turns out to essentially vanish. Obviously, also the terms which are the derivatives with respect to $\theta$ of components of the metric tensor will have a vanishing averages. One is therefore left with the following equation in the unknown $v \equiv r\dot{\theta}$:

$$0 = \partial_r V_{\text{loc}} + \left[1 + \frac{1}{2\pi r} \oint (\mathcal{A} + r\partial_r \mathcal{A}) \, dl \right] \frac{\nu^2}{r}$$

$$+ \left[\frac{c}{2\pi r} \oint (\mathcal{B} + r\partial_r \mathcal{B}) \, dl - \frac{1}{2\pi r} \oint \partial_t \mathcal{D} \, dl \right] \frac{\nu}{r}$$

$$- \frac{c}{2\pi r} \oint \partial_t g_{00} \, dl + \frac{c^2}{2\pi r} \oint \partial_t h_{00} \, dl ,$$

where use was made of expressions (6) for the components of the metric tensor. This equation has to be compared (see for example Bertin [2000]) with the familiar one

$$\frac{\nu^2}{r} = -\partial_r V_{\text{loc}},$$

which is obtained when the only acting force is assumed to be the Newtonian one due to local matter. So, at first sight, the structure of the equation seems to have changed just a little bit: there appears a term which is linear in $\nu$, but this modification turns out to be negligible. However, a more relevant contribution (in some hypotheses to be discussed in a moment) is given by the term

$$\frac{c}{2\pi r} \oint \partial_t g_{00} \, dl \overset{\text{def}}{=} -\partial_r V_{\text{eff}}.$$  

In this respect, the faraway field acts as to produce a modification of the local potential, giving rise to a sort of effective potential; this effect might also be mimicked by a local distribution of matter (of an unknown type).

The relevant difference between our formula (7) and the standard one, consists in the fact that in (7) the terms depending on the far fields have a stochastic character, so that, as already pointed out, we cannot predict their exact values, but can only estimate their order of magnitude by looking at their average and variance.

Notice that, as we are integrating along a circular path, the magnitude of the random terms will depend on the correlation of the field at different points of the path. Being interested only in the variance, it will suffice to just consider the two–point correlations.

In principle such correlations should be computed from the expression (4), but at the moment we are unable to do this in a consistent way. Therefore we make here our second assumption, i.e., it will be assumed that the correlations decay in an exponential way on a certain length scale $l$, namely, we assume

$$\langle \partial_\sigma g_{\mu\nu}(x) \partial_\sigma g_{\mu\nu}(y) \rangle = \exp \left( -\frac{|x - y|}{l} \right) \left( \langle \partial_\sigma g_{\mu\nu}(x) \rangle \right)^2 .$$

Here $l$ is considered as a free parameter, to be fitted by comparison with the actual observations of galaxy rotation curves. If one obtains consistent values of $l$ for different
so that their order of magnitude is given by their standard deviation. Remember that explained in the Introduction, we are dealing with random variables having zero mean, the motion of stars in a galaxy, and smooth instead, for example for what concerns the motion of planets around a star.

We recall that, via the Wiener–Khinchin formula (see Appendix B), one can obtain the correlation of a function, from the correlations of its derivatives. In our case, we can be considered as smooth, becoming more stochastic as it decreases. We will find for a value of 1 kpc, so that the far field can be considered as stochastic for what concerns the motion of stars in a galaxy, and smooth instead, for example for what concerns the motion of planets around a star.

Using (9) and (10), the averages and variances of all terms involved in equation (7) can be computed as shown in Appendix A. The results of such computations are the following. Denote

\[
\begin{align*}
\mathcal{A}_1 & \equiv \frac{1}{2\pi r} \int \mathcal{A} \, dl, \quad \mathcal{A}_2 \equiv \frac{1}{2\pi r} \int \partial_\gamma \mathcal{A} \, dl, \\
\mathcal{B}_1 & \equiv \frac{1}{2\pi r} \int \mathcal{B} \, dl, \quad \mathcal{B}_2 \equiv \frac{1}{2\pi r} \int \partial_\gamma \mathcal{B} \, dl, \\
C & \equiv \frac{c}{2\pi r} \int \partial_\gamma g_{0r} \, dl, \quad D_1 \equiv \frac{1}{2\pi r} \int \partial_\gamma d \, dl.
\end{align*}
\]

Then, for the averages and the variances one finds the following results:

\[
\begin{align*}
\langle \mathcal{A}_1 \rangle &= -\frac{1}{2}, \\
\langle \mathcal{A}_2 \rangle &= 0, \quad \sigma_\mathcal{A}_1^2 = \frac{1}{\pi r} \langle \partial_\gamma \mathcal{A} \rangle^2 = \frac{\gamma_A}{\pi} \frac{H_0}{c} \frac{1}{r}, \\
\langle \mathcal{B}_1 \rangle &= 0, \quad \sigma_\mathcal{B}_1^2 = \frac{1}{\pi r} \langle \partial_\gamma \mathcal{B} \rangle^2 = \frac{3 \gamma_B}{4\pi} \frac{H_0^2}{c^2} \frac{1}{r^2} \left( \frac{l}{r} \right)^{0.8}, \\
\langle \mathcal{B}_2 \rangle &= 0, \quad \sigma_\mathcal{B}_2^2 = \frac{1}{\pi r} \langle \partial_\gamma \mathcal{B} \rangle^2 = \frac{\gamma_B}{4\pi} \frac{H_0}{c} \frac{1}{r}, \\
\langle C \rangle &= 0, \quad \sigma_C^2 = \frac{c^2 l}{\pi r} \langle \partial_\gamma g_{0r} \rangle^2 = 0.05 H_0^2 c^2 \frac{1}{\pi r}, \\
\langle D_1 \rangle &= 0, \quad \sigma_D_1^2 = \frac{1}{r} \langle \partial_\gamma g_{0r} \rangle^2 = \frac{1}{4\pi} \gamma_B H_0 \frac{1}{r},
\end{align*}
\]

(11)

where the constants are given by \( \gamma_A \approx 0.05 \), \( \gamma_B \approx 0.026 \) and \( \gamma_0 \approx 0.08 \).

Notice that in the first line the variance is not computed because, being the average not vanishing, it is just the average which gives the order of magnitude of the terms in equation (7). Using relations (11), for \( r \) in the range of galactic distances, i.e., \( r \leq 30 \) kpc, one checks that all terms in (7) are negligible, except for three of them: the local Newtonian potential, the quadratic term in the tangential velocity \( v \), and the term containing \( \oint \partial_\gamma g_{0r} \, dl/2\pi r \). Thus (7) takes the simpler form

\[
\frac{3 \gamma^2}{2} \frac{l}{r} = -\partial_\gamma V^{\text{loc}}(r) - \partial_\gamma V^{\text{eff}}, \quad \text{with } \partial_\gamma V^{\text{eff}} \approx 0.2 H_0 c \sqrt{\frac{l}{r}}.
\]

(12)

Here, the value indicated after the sign \( \approx \) is just the standard deviation. Indeed, as explained in the Introduction, we are dealing with random variables having zero mean, so that their order of magnitude is given by their standard deviation. Remember that
Figure 1: Plot of the rotational velocity $v$ vs. distance $r$ from the galactic center, for the galaxy NGC 3198. Squares (□) are the velocities determined from HI observations (see Bergman [1989]), solid line is the theoretical curve (see text). Dashed line is the contribution due to the local matter.

the sign can be positive or negative: if the term is positive it acts as a pressure and so it helps keeping the galaxy stable, while the opposite occurs if it is negative. So one can conjecture that the positive case occurs more frequently in observations, but it would be possible in principle to observe also cases in which the term has a negative value.

We show in the next section that, if the term is positive, the magnitude of the correction due to the far field is able to flatten the rotation curve in the external region of the galaxy.

4 Fitting some observed rotation curves

To make a comparison between the predictions given by (12), and the observed rotation curves of galaxies, one should have available an expression for $V_{\text{loc}}(r)$, or, equivalently, one should assign the local distribution of matter. In the literature, there are several papers which illustrate different ways in which the distribution of matter can be estimated, starting for example from the measured distribution of luminosity of the galaxies (see Toomre [1963]; Freeman [1970]; Carignan&Freeman[1985]; Carignan et al. [1988]).

We take instead the simpler path which consists in assuming a functional form for
Figure 2: Plot of the rotational velocity $v$ versus distance $r$ from the galactic center, for the galaxy NGC 2403. Squares (□) are the velocities determined from HI observations (see Bergman [1987]), while triangles (△) are Hα observations (see Blais-Ouellette et al. [2004]). The solid line is the theoretical curve (see text), while the dashed line is the contribution due to the local matter.
Figure 3: Plot of the rotational velocity $v$ versus distance $r$ from the galactic center, for the galaxy NGC 4725. Squares ($\square$) are the velocities determined from HI observations (see Wevers et al. [1984]). The solid line is the theoretical curve (see text), while the dashed line is the contribution due to the local matter.
Figure 4: Plot of the rotational velocity $v$ versus distance $r$ from the galactic center, for the galaxy UGC 2885. Squares ($\Box$) are the velocities determined from HI observations (see Roelfsema & Allen [1985]), while triangles ($\triangle$) are H$\alpha$ observations (see Rubin et al. [1985]). The solid line is the theoretical curve (see text), while the dashed line is the contribution due to the local matter.
$V^{\text{loc}}(r)$, with a minimal number of parameters (essentially two, total mass and radius of the galaxy) and then trying to determine these parameters by a best fit with the observed rotation curves. In particular we concentrate on the rotation curves of the four galaxies NGC 3198, NGC 2403, UGC 2885 and NGC 4725.

For what concerns the local potential $V^{\text{loc}}(r)$, in the literature one often makes use of potentials of the form

$$V^{\text{loc}}(r, z) = \frac{GM}{\sqrt{r^2 + (a + \sqrt{z^2 + b^2})^2}},$$

first introduced by Miyamoto & Nagai ([1975]). Here $M$ is the total mass of the object, to be understood in the following sense. First one introduces the matter density $\rho$ from the Poisson equation $\Delta^2 V^{\text{loc}} = 4\pi G \rho$, then one easily checks that the integral of $\rho$, over the whole space, is equal to the parameter $M$.

For what concerns the other two parameters $a$ and $b$, one has first of all that the ratio $b/a$ determines the flatness of the mass distribution $\rho$: for $b/a > 1$, the distribution is essentially spherical, while it reduces to a singular disk if the ratio vanishes. The parameter $a$ (at fixed ratio $b/a$) is then the length scale of the distribution $\rho$.

So, for the galaxies NGC 3198 and NGC 4725 we take the potential $V^{\text{loc}}(r)$ in the Miyamoto–Nagai form, supposing that the only relevant contribution to the potential comes from the star disk. In the case of the other two galaxies, i.e., NGC 2403 and UGC 2885, it appears that there is also a non negligible contribution from the inner part of the galaxy (the so called bulge), so that we take the local potential as the sum of two Miyamoto–Nagay potentials, with different parameters, i.e., we take

$$V^{\text{loc}}(r, z) = \frac{GM_1}{\sqrt{r^2 + (a_1 + \sqrt{z^2 + b_1^2})^2}} + \frac{GM_2}{\sqrt{r^2 + (a_2 + \sqrt{z^2 + b_2^2})^2}},$$

the first term referring to the bulge, the second to the disk. In this case, the total mass $M$ of the galaxy will be given by $M = M_1 + M_2$.

Now, using the above expressions for the local potentials, we made a best fit to the actual observed curve in order to determine both the parameters of the potential and the correlation length $l$. The resulting rotation curves are reported as solid lines in Figures 1–4. The dashed lines are instead the rotation curves which would be obtained if the contribution of the distant matter were neglected. As one sees, the contribution of far matter is essential to keep flat the rotation curves at large distances. The values of the masses $M$ (in solar units $M_\odot$), the mass luminosity ratios (in solar mass over solar luminosity $L_\odot$ units), and the correlation lengths $l$ in kiloparsecs for the four galaxies are reported in Table 1.

Table 1: Values of total mass, mass–luminosity ratio and correlation length for the four galaxies considered.

<table>
<thead>
<tr>
<th>Galaxy</th>
<th>Mass</th>
<th>Mass/Luminosity</th>
<th>Correlation length</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC 3198</td>
<td>$4.0 \times 10^{10} M_\odot$</td>
<td>$4.6 M_\odot/L_\odot$</td>
<td>0.6 Kpc</td>
</tr>
<tr>
<td>NGC 2403</td>
<td>$3.5 \times 10^{10} M_\odot$</td>
<td>$4.4 M_\odot/L_\odot$</td>
<td>0.8 Kpc</td>
</tr>
<tr>
<td>NGC 2885</td>
<td>$1.0 \times 10^{12} M_\odot$</td>
<td>$2.1 M_\odot/L_\odot$</td>
<td>1.7 Kpc</td>
</tr>
<tr>
<td>NGC 4725</td>
<td>$1.1 \times 10^{11} M_\odot$</td>
<td>$2.1 M_\odot/L_\odot$</td>
<td>3.1 Kpc</td>
</tr>
</tbody>
</table>
Figure 5: Plot of the rotational velocity $v$ versus distance $r$ from the galactic center, for the galaxy NGC 864. Squares ( Sabha) are the velocities determined from HI observations (see Espada et al. [2005]), solid line is the theoretical curve (see text).
Figure 6: Plot of the rotational velocity $v$ versus distance $r$ from the galactic center, for the galaxy AGC 400848. Squares ($\square$) are the velocities determined from H$\alpha$ observations (see Catinella et al. [2005]), solid line is the theoretical curve (see text).
At variance with the effect due to dark matter, the far away matter could in principle also produce a negative pressure, because, as already explained, the term $\partial_r V^{(\text{eff})}$ in equation (12) can be positive or negative. This would imply that the rotation curve decays, beyond the galaxy’s edge, faster than keplerian. By a quick search, we found in literature at least two cases of this behavior: the rotation curve of the galaxy NGC 864, (see Espada et al. [2005]) and that of galaxy AGC 400848 (see Catinella et al. [2005]). We are able to fit also such curves, as shown in Figures 5 and 6, with again a value of the correlation length $l$ of the order of 1 kpc.

In any case, these curves are shown here only for the sake of illustration: in fact, first of all the contribution of the local matter was computed with an “ad hoc” potential without a deep confrontation with the data taken from luminosity to see if they are compatible. More importantly, we did not investigate whether there are some perturbations coming from known objects (such as nearby galaxies), which could perturb the velocity profile.

5 Conclusions

It was shown here that the faraway matter can exert a force of such a magnitude as to account for the observed rotation velocity curves in spiral galaxies, if the force is assumed to be decorrelated beyond a sufficiently large distance, of the order of 1 kpc. In particular we fit the rotation curves of the galaxies NGC 3198, NGC 2403, UGC 2885 and NGC 4725 without any need to introduce dark matter at all.

We also showed that in principle faraway matter could act as a negative pressure, which would steepen the fall of of the rotation curves (with respect to the keplerian decay), as apparently observed for some galaxies. We considered, in a preliminary way, two such cases, for which our approach seems to apply, giving a quite good fit. This subject is however left for a future work.

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A Computation of variances

In this appendix we compute the average and the variance of the terms which appear in the equation of motion (7). We recall that, from the fourth of (6) of Section 3 and from the expression of $h_{\mu\nu}$ given by 4 of Section 2, one has

$$\mathcal{A} = -\frac{4G}{c^2} \sum_j M_j \left( \frac{\dot{q}_k^j \cos \theta + \dot{q}_l^j \sin \theta}{|q^j - \mathbf{x}|} \right)^2 - \frac{2G}{c} \sum_j \frac{M_j}{|q^j - \mathbf{x}|},$$

(13)

where we are using cylindrical coordinates $(r, \theta, z)$, and the point $\mathbf{x}$ at which we are evaluating the field lies on the galactic plane $z = 0$.

We have now to consider $\mathcal{A}_1 = \frac{1}{2\pi r} \oint \mathcal{A} dl$, and in particular $\langle \mathcal{A}_1 \rangle$. However, one obviously has

$$\langle \mathcal{A}_1 \rangle = \frac{1}{2\pi r} \oint \langle \mathcal{A} \rangle dl = \langle \mathcal{A} \rangle,$$

14
because $\langle A \rangle$ is independent of the angle $\theta$, due to the isotropy of the $q^{(j)}$ distribution. So we can take $\theta = 0$, and we obtain

$$\langle A \rangle = -\frac{4GH_0^2}{c^4} \left( \sum_j M_j \frac{q^{(j)}_e \cos \theta + q^{(j)}_y \sin \theta}{|q^{(j)} - x|} \right) - \frac{2G}{c^2} \left( \sum_j M_j \right) |q^{(j)}|,$$

where $R_j = |q^{(j)}|$ is the distance of the source $q^{(j)}$ from the galaxy center, $\theta_j$ is the angle between the source $q^{(j)}$ and the galactic plane, and $\phi_j$ a third angle on the galactic plane. We have obviously omitted any term of order $|x|/R_j$. To compute the average, we can take the continuum limit (as was done in paper Carati et al. [2008]), setting $M_j = \rho_{\text{eff}} \, dV$, so that one gets

$$\langle A \rangle = -\frac{2G \rho_{\text{eff}}}{c^4} \left[ 2H_0^2 \int R^3 \sin^2 \phi \sin^3 \theta \, dR \, d\theta \, d\phi + c^2 \int R \sin \theta \, dR \, d\theta \, d\phi \right] = -\frac{2G \rho_{\text{eff}} H_0^2 R_0^3 (\pi/3 + \pi)}{c^4} = -\frac{1}{2},$$

where we have used the expression (8) of Carati et al. [2008]) for the effective density $\rho_{\text{eff}}$. This gives the first of (11).

We come now to the second of (11). In the first place one has to consider that, in computing the derivative which appears in the forces, one gets many terms, the largest of which comes from the fact the time at which we take the position of the sources is retarded, i.e. is a function of $x$. One has then

$$\partial_r A = -\frac{4G}{c^4} \sum_j M_j \frac{2 \left( q^{(j)}_e \cos \theta + q^{(j)}_y \sin \theta \right) \left( q^{(j)}_e \cos \theta + q^{(j)}_y \sin \theta \right)}{|q^{(j)} - x|} - \frac{2G}{c^2} \sum_j M_j \frac{\partial_r |q^{(j)} - x|}{|q^{(j)}|} + \text{smaller terms},$$

where “smaller terms” has to be understood in the sense that they give a smaller contribution to the mean and to the variance. Now one has

$$\partial_r |q^{(j)} - x| = \partial_r \left( q^{(j)} + \frac{2 \left( q^{(j)}_e \cos \theta + q^{(j)}_y \sin \theta \right) \left( q^{(j)}_e \cos \theta + q^{(j)}_y \sin \theta \right)}{|q^{(j)}|} + O(r/q^{(j)}) \right) = \frac{\left( q^{(j)}_e \cos \theta + q^{(j)}_y \sin \theta \right)}{|q^{(j)}|} + O(r/q^{(j)}),$$

where we have used the fact that $q^{(j)} \gg r$. We get then, using Hubble’s law, the following expression

$$\partial_r A = -\frac{8GH_0^3}{c^4} \sum_j M_j \frac{2 \left( q^{(j)}_e \cos \theta + q^{(j)}_y \sin \theta \right)^3}{|q^{(j)}|^2},$$

(14)

where, from now on, we consider only the most relevant terms in the expressions. Now, due to the rotational invariance of the distribution, one gets immediately

$$\langle \frac{1}{2\pi r} \oint \partial_r A \, dl \rangle = \frac{1}{2\pi r} \oint \langle \partial_r A \rangle \, dl = 0,$$
i.e. the first equality of the second line in (11). The estimate of the variance $\sigma_{\mathcal{A}_1}$ is obtained in the following way. By definition one has

$$
\sigma_{\mathcal{A}_1} = \left( \frac{1}{4\pi^2 r^2} \int d\lambda_1 \int d\lambda_2 \partial_r A(x) \partial_r A(y) \right),
$$

and using the hypothesis on the correlations one gets

$$
\sigma_{\mathcal{A}_1} = \frac{1}{4\pi^2 r^2} \int d\lambda_1 \int d\lambda_2 \langle (\partial_r A)^2 \rangle \exp \left( -\frac{|x-y|}{l} \right).
$$

Now, due to the rotational invariance of the distribution, the average inside the integral does not depend on the angle $\theta$ and can be taken out from the integral. So one gets

$$
\sigma_{\mathcal{A}_1} = \frac{\langle (\partial_r A)^2 \rangle}{4\pi^2 r^2} \int d\lambda_1 \int_{0}^{2\pi} d\theta \exp \left( -\frac{2r}{l} \sin \theta / 2 \right) = \gamma_1 \frac{l}{r} \langle (\partial_r A)^2 \rangle.
$$

Now, the factor $\gamma_1$, which is essentially the value of the inner integral, depends indeed on the ratio $r/l$, but in the range of interest is essentially equal to its asymptotic value $1/\pi$ (such a value can be computed for example by applying Laplace method to the inner integral). So we simply put

$$
\int_{0}^{2\pi} d\theta \exp \left( -\frac{2r}{l} \sin \theta / 2 \right) = \frac{1}{\pi r},
$$

having stipulated that $l/r$ is sufficiently large.

We are left with the computation of the variance $\langle (\partial_r A)^2 \rangle$, which can be performed as follows. From relation (14), taking the square and using the rotational invariance of the distribution to compute it for $\theta = 0$, one gets the following expression

$$
\langle (\partial_r A)^2 \rangle = \left( \frac{8GH^3}{c^5} \right)^2 M^2 \sum_{j,k} \frac{(q^{(j)})^3 (q^{(k)})^3}{|q^{(j)}|^2 |q^{(k)}|^2} = \gamma_A \left( \frac{8GH^3}{c^5} \right)^2 M^2 R_u^2,
$$

with $M_{\text{tot}} = NM$, where $N$ the total number of galaxies, while $\gamma_A$ is a numerical coefficient which depends on the distribution of the galaxies. If we choose the distribution as discussed in Section 2, one can estimate it by numerical computations, which give $\gamma_A \approx 0.05$. Now, using for $M_{\text{tot}}$ the expression

$$
M_{\text{tot}} = \frac{(4/3)\pi \rho_{\text{eff}} R_u^3},
$$

with $\rho_{\text{eff}}$ as given by (8) of Carati et al. ([2008]), one eventually finds

$$
\sigma_{\mathcal{A}_1} = \gamma_A \frac{l}{r} \left( \frac{H_0}{c} \right)^2,
$$

which is the second relation in the second line of (11).

Let us now come to the third and fourth lines of (11). The average of $\mathcal{B}_1$ vanishes by symmetry, as can be seen from its expression

$$
\mathcal{B}_1 = -\frac{4GH_0}{c^3} \frac{1}{2\pi r} \int \sum_j M_j \frac{q^{(j)}_c \cos \theta - q^{(j)}_s \sin \theta}{|q^{(j)}|} d\lambda.
$$

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where use was made of Hubble’s law. For what concerns the variance \( \sigma_{B_1} \) one has

\[
\sigma_{B_1} = \frac{1}{4\pi^2 r^2} \int dl_1 \int dl_2 \langle B(x) B(y) \rangle = \frac{4 \pi^2}{2 \pi} \int_0^{2\pi} d\theta \frac{l}{2r \sin \theta/2} \left( 1 - e^{-2r \sin \frac{\theta}{2}} \right) \left( 1 + \frac{r}{l} \sin \frac{\theta}{2} \right),
\]

where we used (19) of Appendix B to estimate the correlation of a function in terms of the correlations of its derivatives. Use was also made of the isotropy of the distribution, so that \( \langle \nabla B^2 \rangle \) turns out to be constant along the integration path. The integral in the second line can be computed numerically, and with a good accuracy (in the range of interest) one finds the value \( 2\pi l^2 (l/r)^{0.8} \) so that one has

\[
\sigma_{B_1} \simeq \langle (\nabla B)^2 \rangle l^2 \left( \frac{1}{r} \right)^{0.8}. \tag{16}
\]

Exactly in the same way one can check that the average of \( B_2 \) vanishes, while the variance \( \sigma_{B_2} \) is given by

\[
\sigma_{B_2} \simeq \langle (\partial_r B)^2 \rangle \frac{l}{\pi r}. \tag{17}
\]

The expression of \( \partial_r B \), retaining only the largest terms, is

\[
\partial_r B = -\frac{4G_c}{c^4} \sum_j M_j \frac{q_j^{(j)} y \cos \theta - q_j^{(j)} x \sin \theta}{|q_j^{(j)}|^2},
\]

so that, using Hubble’s law, one gets

\[
\langle (\partial_r B)^2 \rangle = \left( \frac{4GH_0^2 M}{c^4} \right)^2 \left( \sum_{jk} \frac{q_j^{(j)} q_k^{(k)} y^2 q_i^{(i)}}{|q_j^{(j)}|^2 |q_k^{(k)}|^2} \right) = \gamma_B \left( \frac{4GH_0^2 M_{tot}}{c^4} \right)^2.
\]

Here, as before, one has \( M_{tot} = NM \) where \( N \) is the number of galaxies, while the numerical coefficient \( \gamma_B \) depends on the distribution of the galaxies; for our fractal model a numerical estimate gives \( \gamma_B \approx 0.026 \). Again, inserting this expression into relation (17), and using for \( M_{tot} \) the value (15), one gets the fourth line of relation (11). The third line is obtained simply by observing that

\[
\langle (\nabla B)^2 \rangle = 3 \langle (\partial_r B)^2 \rangle.
\]

We come now to the fifth line of (11). One has

\[
c_{gr} = -\frac{4G_c}{c^2} \sum_j M_j \frac{q_j^{(j)} x \cos \theta + q_j^{(j)} y \sin \theta}{|q_j^{(j)}|^2},
\]

so that one gets (forgetting as usual all small terms) the relation

\[
c_{\partial_r g_r} = -\frac{4GH_0^2}{c^2} \sum_j M_j \frac{q_j^{(j)} x \cos \theta - q_j^{(j)} y \sin \theta}{|q_j^{(j)}|^2} = \frac{4GH_0^2}{c^2} \sum_j M_j \frac{q_j^{(j)} \cdot \hat{r}}{|q_j^{(j)}|^2}.
\]

\]
From this expression it follows immediately that \( \langle \partial_i g_{,0} \rangle = 0 \) so that one has also

\[
\langle C \rangle = \frac{1}{2\pi r} \int_0^{2\pi} \left( c \partial_i g_{,0} \right) dl = 0,
\]

which is the first relation of the last line in (11). The computation of the variance \( \sigma^2_C \) is made in exactly the same way as was done before for all other variances, obtaining

\[
\sigma^2_C = \frac{c^2}{4\pi r^2} \int_0^{2\pi} dl_1 \int_0^{2\pi} dl_2 \langle \partial_i g_{,0}(x) \partial_i g_{,0}(y) \rangle = \frac{c^2 \langle (\partial_i g_{,0})^2 \rangle}{2\pi}.
\]

We are reduced to the computation of \( \langle (\partial_i g_{,0})^2 \rangle \), and one gets

\[
\langle (\partial_i g_{,0})^2 \rangle = \gamma_C H_0^2,
\]

with the factor \( \gamma_C \approx 0.05 \) for our fractal model. This numerical estimate was already done by Carati et al. ([2008]), in providing the estimate (11) of page four. So also the fifth line of (11) is proven.

We finally come to the last relation (11). One has (again taking only the largest terms)

\[
\partial_i \mathcal{D} = \frac{4G}{c^4} \sum_j M_j \frac{q_s^{(j)} q_s^{(i)} - q_s^{(j)} q_i^{(i)}}{|q^{(j)} - x|} \sin 2\theta
\]

\[
\quad + \frac{4G}{c^4} \sum_j M_j \frac{q_s^{(j)} q_s^{(i)} + q_s^{(j)} q_i^{(i)}}{|q^{(j)} - x|} \cos 2\theta,
\]

\[
= \frac{4G H_0^3}{c^4} \sum_j M_j \left( \frac{q_s^{(j)}}{|q^{(j)} - x|} \right)^2 \sin 2\theta + M_j \frac{2q_s^{(j)} q_i^{(i)}}{|q^{(j)} - x|} \cos 2\theta,
\]

which shows, in a very simple way, that one has \( \langle \partial_i \mathcal{D} \rangle = 0 \). For what concerns the variance \( \sigma^2_{\mathcal{D}_i} \) of \( \mathcal{D}_i \) the usual correlation argument shows that

\[
\sigma^2_{\mathcal{D}_i} = \frac{1}{\pi r} \langle (\partial_i \mathcal{D})^2 \rangle.
\]

As before, in the expression of \( \langle (\partial_i \mathcal{D})^2 \rangle \) one can put, for example, \( \theta = 0 \), so that one gets

\[
\langle (\partial_i \mathcal{D})^2 \rangle = \left( \frac{4G H_0^3 M}{c^4} \right)^2 \sum_{jk} \frac{4q_s^{(j)} q_s^{(k)} q_s^{(j)} q_s^{(k)}}{|q^{(j)}||q^{(k)}|} = \gamma_D \left( \frac{16G H_0^3 M_{\text{tot}}}{c^4} \right) R_0^2,
\]

where \( M_{\text{tot}} = NM \) is the total mass of the galaxies, \( R_0 = c/H_0 \) is the radius of the universe, \( \gamma_D \) a numerical factor which depends on the distribution of the galaxies. For our fractal model, a numerical estimate provides \( \gamma_D \approx 0.08 \). Now, using for \( M_{\text{tot}} \) the value given by (15), one eventually finds

\[
\langle (\partial_i \mathcal{D})^2 \rangle = \frac{\gamma_D}{4} H_0^2,
\]

which, inserted in the expression for \( \sigma^2_{\mathcal{D}_i} \), gives the second of the last line of (11). This completes the proof.
\section*{B The Wiener–Khinchin theorem}

We deal here with the problem of how one computes the autocorrelation \( \langle f(x)f(0) \rangle \) if the autocorrelation \( \langle \partial_x f(x) \partial_x f(0) \rangle \) of the derivatives is known (see (9)). We will show that if one assumes

\[
\langle \partial_x f(x) \partial_x f(0) \rangle = \langle (\partial_x f(0))^2 \rangle \exp(-\frac{|x|}{l}),
\]

(18)

then one has

\[
\langle f(x)f(0) \rangle = l^3 \frac{1}{(2\pi)^3} \int \frac{p^3}{l} \left( 1 - e^{-|x|/(2l)} \right) \left( \langle (\partial_x f(0))^2 \rangle \right) \exp(-|x|/l) \, dp.
\]

(19)

This gives formula (10), if we take for \( f(x) \) the function \( g_{\mu\nu} \).

To show this, we make use of the Wiener–Khinchin formula (see Wiener [1930]; Khinchin [1934])

\[
\langle f(x)f(0) \rangle = \frac{1}{(2\pi)^3} \int \frac{p^3}{l} \left( 1 - e^{-|x|/(2l)} \right) \left( \langle (\partial_x f(0))^2 \rangle \right) \exp(-|x|/l) \, dp.
\]

(20)

where \( \hat{f}(k) \) is the Fourier transform of \( f(x) \). Now, the Fourier transform of \( \partial_x f(x) \) is simply given by \( k \hat{f}(k) \), so that, once \( \langle \partial_x f(x) \partial_x f(0) \rangle \) is known, the spectrum \( |\hat{f}(k)|^2 \) (and thus the correlation) is easily found.

Indeed, inverting the Wiener–Khinchin formula, the spectrum is given by

\[
k^2 |\hat{f}(k)|^2 = \frac{1}{(2\pi)^3} \int \frac{p^3}{l} \left( 1 - e^{-|x|/(2l)} \right) \left( \langle (\partial_x f(0))^2 \rangle \right) \exp(-|x|/l) \, dp.
\]

where, in the second line, we used polar spherical coordinates \( (r, \phi, \theta) \) in \( \mathbb{R}^3 \) with the \( z- \)axis parallel to \( k \). The integral is elementary, and one obtains

\[
k^2 |\hat{f}(k)|^2 = \frac{1}{(2\pi)^3} \frac{p^3}{l} \int \frac{\langle (\partial_x f(0))^2 \rangle}{1 + l^2 k^2} \, dp.
\]

which gives the spectrum

\[
|\hat{f}(k)|^2 = \frac{\langle (\partial_x f(0))^2 \rangle}{1 + l^2 k^2} \frac{p^3}{2\pi^2 k^2}.
\]

(21)

Now the correlation follows from the Wiener-Khinchin formula quite easily because, using spherical coordinates in the momentum space \( k \) with the \( z- \)axis parallel to \( x \), and the expression (21) just found for the spectrum, relation (20) gives

\[
\langle f(x)f(0) \rangle = \frac{\langle (\partial_x f(0))^2 \rangle}{2\pi^2} \frac{p^3}{1 + l^2 k^2} \exp(i|\mathbf{x}|k \cos \theta) \sin \theta \, dk \, d\theta \, d\phi.
\]

The integral in the second line can be computed for example by using the method of the residues, thus obtaining relation (19).
References


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