Nonlinearity 6 (1993) 905–914. Printed in the UK

Asymptotic character of the series of classical electrodynamics and an application to bremsstrahlung

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Received 15 February 1993
Recommended by P Goddard

Abstract. The Lorentz–Dirac equation, which describes the self-interaction of a classical charged particle with the electromagnetic field, is studied, for the case of scattering and in the non-relativistic approximation, in the framework of the theory of singular perturbation problems. We prove that the series expansions, which are usually given for the solutions in terms of the electric charge, in general are divergent and have asymptotic character. A closer inspection of such series leads to recognition of two types of particle motions, namely those qualitatively similar to purely mechanical ones (corresponding to vanishing charge), and those qualitatively dissimilar. For an attractive Coulomb potential, the distinction turns out to depend on the value of the initial angular momentum, the threshold being of the order of magnitude of e^2/c. Finally, we discuss the implications for the radiated spectrum, showing that the threshold in angular momentum should correspond to a frequency cutoff of the order of magnitude of the de Broglie frequency.

PAC numbers: 0350, 0365, 0230.

1. Introduction

It is very well known that the series expansions occurring in quantum electrodynamics are in general expected to be divergent and (at most) of asymptotic type (see [1], [2] page 84, and [3] chapter 37), and it thus seems quite natural to ask whether the same characteristics are shared by the series expansions of classical electrodynamics, typically those obtained from the Lorentz–Dirac equation (see for example [4], section 17.6). However, to our surprise, we couldn't find any clear statement on the subject, and even met with statements suggesting a possible analytic character. Indeed, in the concluding section of a review article by a well respected scientist (see [5] page 362, and also [6]) one finds the following sentence: 'The existence of solutions of the Lorentz–Dirac equation was proven. But their uniqueness has not been proven. *Nor is the dependence of these solutions on the charge e known. Are*

0951-7715/93/060905 + 10$07.50

they analytic in e? Under what conditions do perturbation expansions converge?' (for related problems, see also [7, 8] and [9, 10]).

Here we show that in general such series are divergent and have asymptotic character. In fact this should be considered today as almost obvious, due to the singular character of the Abraham–Lorentz equation. And indeed, in the present paper we study such an equation in the general framework of singular perturbation theory (see, for example, [11–13]), which is familiar in boundary layer theory and in many other fields of applied mathematics, but apparently not so much in classical electrodynamics. Furthermore, we point out that in connection with such series there exists, for problems of scattering by a Coulomb potential, a kind of threshold in angular momentum, the physical interpretation of which might be of some interest in the framework of the general relations between classical mechanics and quantum mechanics.

2. The Lorentz–Dirac equation

The equation under discussion is that of Lorentz–Dirac for a point electron, which we consider for simplicity's sake in the non-relativistic approximation, namely

$$\varepsilon \dddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}} - \frac{1}{m}\boldsymbol{F}(x) \qquad (1)$$

where $\boldsymbol{x}$ is the position vector of the electron and m its (renormalized) mass, $\boldsymbol{F}(x)$ is an external force field, and

$$\varepsilon = \frac{2}{3}\frac{e^2}{mc^3} \qquad (2)$$

is the 'small parameter', e and c being the electron charge and the velocity of light. We recall that the quantity ε has the dimensions of time, and has for the electron a value of the order of 10^{-23} s; it will be shown below how a pure number related to ε will appear naturally for any solution of the equation. We do not enter here into a discussion on the justification of the Lorentz–Dirac equation; for example, in our opinion it is not clear whether it is justified for motions not satisfying the ingoing condition $\ddot{x}(t) \to 0$ for $t \to -\infty$ (for a potential vanishing at infinity), required typically for scattering problems. Equation (1) is an ordinary differential equation in the 'enlarged phase space' $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$, $\ddot{\boldsymbol{x}}$, which for $\varepsilon = 0$ reduces to the 'mechanical equation' $m\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ in the 'ordinary or mechanical phase space' $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$, thus losing one order of differentiation. Problems related to equations with such a property, named singular perturbation problems, are well known to lead in general to asymptotic expansions about $\varepsilon = 0$; in any case, we give here a direct discussion for the Lorentz–Dirac equation.

Preliminarly, we discuss the problem of the subsidiary conditions that have to be added to (1). Conditions of a Cauchy type, which assign position, velocity and acceleration at one time, give as usual existence, uniqueness, and continuity with respect to initial data and to the parameter ε, for $\varepsilon \neq 0$. However, as first apparently pointed out by Dirac [14, 15], it occurs that for generic initial data the solutions have runaway character, i.e. have the property that the acceleration diverges† exponentially for $t \to +\infty$, even in the absence of an external force.

Now, the generic appearence of runaways does not exclude the possibility of exceptional solutions with bounded acceleration, and the proposal of Dirac was just to add such a requirement on the solutions. In particular, for scattering problems non-runaway solutions of (1) are selected by imposing subsidiary conditions of mixed type, namely conditions of ordinary Cauchy type for position and velocity (so defining a unique solution for the corresponding 'reduced' or 'mechanical' problem $m\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$), and an asymptotic outgoing condition for the acceleration, precisely $\ddot{\boldsymbol{x}}(t) \to 0$ for $t \to +\infty$. This requirement eliminates the runaways by definition, but we stress that it is only sufficient, and not necessary, for a solution to be non-runaway; on this point we will come back below, and for the moment we restrict ourselves to solutions satisfying the Dirac outgoing condition. From the mathematical point of view, such a condition, together with conditions of a Cauchy type on the initial position and velocity, leads to a problem of the Sturm–Liouville type, which in general might admit no solution. However, with ordinary potentials vanishing at infinity the reduced mechanical problem does admit scattering states for suitable initial data, and so the same might be expected to hold also for the Lorentz–Dirac equation (1) with subsidiary conditions of Dirac type, at least for suitable initial data. A mathematical discussion of the existence of solutions with the Dirac outgoing asymptotic condition was given in 1961 by Hale and Stokes [16].

In the present paper we assume the existence of solutions of the Lorentz–Dirac equation satisfying the Dirac outgoing condition, and discuss the character of the corresponding series expansions in ε. The force field $\boldsymbol{F}(\boldsymbol{x})$ is assumed to be analytic, which implies that the solutions $\boldsymbol{x}(t, \varepsilon)$ are analytic in t, and also in ε for $\varepsilon \neq 0$; the problem at hand is then just the analyticity in ε at $\varepsilon = 0$. Furthermore, we will consider solutions $\boldsymbol{x}(t, \varepsilon)$ having at least a singularity in the complex t plane, which is clearly the generic situation for a nonlinear force field $\boldsymbol{F}(\boldsymbol{x})$. Actually such singularities will be shown below to play a relevant role.

3. The series expansion: its divergence and asymptotic character

For a given force field $\boldsymbol{F}(\boldsymbol{x})$, consider a solution $\boldsymbol{x}(t, \varepsilon)$ of the Lorentz–Dirac equation satisfying the outgoing Dirac condition

$$\lim_{t \to +\infty} \ddot{\boldsymbol{x}}(t, \varepsilon) = 0. \tag{3}$$

The series expansion in ε for the solution $\boldsymbol{x}(t, \varepsilon)$ is usually defined through an

† This can be seen in a particularly perspicuous way if one takes the point of view of the qualitative theory of dynamical systems. Considering for simplicity the case of one degree of freedom, equation (1) can be written in the form

$$\dot{x} = v \qquad \dot{v} = a \qquad \dot{a} = \frac{1}{\varepsilon}(a - F(x)/m).$$

So, in the enlarged phase space x, v, a, outside a 'small' layer about the 'slow manifold' defined by $a - F(x)/m = 0$ the vector field defining the differential equation is 'practically infinite' and essentially parallel to the a-axis, being directed away from the slow manifold. Thus, with a by now standard terminology, the existence of runaways is described by simply saying that for the Lorentz–Dirac equation the 'fast foliation' is parallel to the acceleration and away from the slow manifold.

expansion for the corresponding acceleration $\ddot{x}$. This is given in the form

$$m\ddot{x}(t,\varepsilon)=\sum_{n=0}^{N}c_n(t,\varepsilon)\varepsilon^n+R_N(t,\varepsilon) \tag{4}$$

with coefficients

$$c_n(t,\varepsilon)=D_t^n F(x(t,\varepsilon)) \tag{5}$$

where we have denoted $D_t^n=\mathrm{d}^n/\mathrm{d}t^n$, and remainder

$$R_N(t,\varepsilon)=\varepsilon^{N+1}\int_0^{+\infty}\mathrm{e}^{-u}D_{t+\varepsilon u}^{N+1}F(x(t+\varepsilon u,\varepsilon))\,\mathrm{d}u. \tag{6}$$

Writing out explicitly the first two terms, the series reads

$$m\ddot{x}=F(x)+\varepsilon\sum_{k=1}^{3}\frac{\partial F}{\partial x_k}(x)\cdot\dot{x}_k+O(\varepsilon^2). \tag{7}$$

A simple deduction of the expansion is the following: by the variation of constants formula, the Lorentz–Dirac equation (1) with the outgoing condition (3) is rewritten in the integro-differential form

$$m\ddot{x}(t,\varepsilon)=\int_t^{+\infty}\frac{\mathrm{e}^{(t-s)/\varepsilon}}{\varepsilon}F(x(s,\varepsilon))\,\mathrm{d}s \tag{8}$$

and then N repeated integrations by parts are performed. This gives for the remainder the expression

$$R_N(t,\varepsilon)=\varepsilon^N\int_t^{+\infty}\mathrm{e}^{(t-s)/\varepsilon}D_s^{N+1}F(x(s,\varepsilon))\,\mathrm{d}s$$

which reduces to (6) by a trivial change of the integration variable†. Series expansions of the type (4), with coefficients depending on the expansion parameter ε itself, are sometimes called 'of generalized type', as contrasted to series 'of regular or Poincaré type', whose coefficients are independent of ε.

The fact that the expansion (4), (5) diverges for any $\varepsilon\neq 0$, as $N\to\infty$, is an immediate consequence of the assumed existence of a singularity in t for the solution $x(t,\varepsilon)$. It suffices to consider the Taylor expansion of $F(x(t,\varepsilon))$ about t,

$$F(x(t+s,\varepsilon))=\sum_{n=0}^{\infty}\frac{s^n}{n!}D_t^n F(x(t,\varepsilon))$$

and use the obvious inequality $|s|^n/n!<|\varepsilon|^n$, which holds for any $\varepsilon\neq 0$ and for any s, for large enough n. This shows that the convergence of the series expansion (4), (5) in ε would imply $F(x(t+s,\varepsilon))$ to be an entire function of s, against the assumption that $x(t,\varepsilon)$ has a singularity in the complex t plane.

The discussion about the asymptotic character of the expansion (4), (5) is a little more delicate. We recall that a series is said to be asymptotic if, for all fixed N, one has $|R_N|/\varepsilon^N\to 0$ as $\varepsilon\to 0$. So, preliminarly, since the series is associated to any given solution $x(t,\varepsilon)$, one has to define which family of solutions depending on ε is

† By the way, the expansion (4) can be extended to the case $N=-1$, when it reduces to (8) provided one makes the convention that the sum $\sum_{n=0}^{-1}$ vanishes.

considered: the natural choice is just to keep fixed the initial data $\boldsymbol{x}_0$, $\dot{\boldsymbol{x}}_0$ in the mechanical phase space.

From the expression (6) of the remainder it is clear that, in order to prove the asymptotic character of the series, one has to give a bound for all derivatives $D_t^n \boldsymbol{x}(t, \varepsilon)$, for $n \geqslant 1$ and all t. A useful control is provided by the assumed singularity of the solution $\boldsymbol{x}(t, \varepsilon)$ in the variable t, making use of the Cauchy estimate

$$|D_t^n \boldsymbol{F}(\boldsymbol{x}(t, \varepsilon))| \leqslant \mathscr{F}(t, \varepsilon) \frac{n!}{d(t, \varepsilon)^n}.$$

Here $d(t, \varepsilon)$ is for example half the distance of t from the nearest singularity of $\boldsymbol{x}(t, \varepsilon)$, and $\mathscr{F}(t, \varepsilon) = \sup |\boldsymbol{F}(\boldsymbol{x}(z, \varepsilon)|$ for $|z - t| < d(t, \varepsilon)$. This gives immediately the uniform bound

$$\frac{|\boldsymbol{R}_N(t, \varepsilon)|}{\varepsilon^N} < \varepsilon \widetilde{\mathscr{F}}(\varepsilon) \frac{(N+1)!}{\tau(\varepsilon)^{N+1}} \tag{9}$$

where $\tau(\varepsilon)$ is half the amplitude of the analyticity strip of the solution in the complex t plane, and $\widetilde{\mathscr{F}}(\varepsilon) = \sup \mathscr{F}(t, \varepsilon)$ for all real t. So we conclude that the divergent series expansion for the given family of solutions is asymptotic if $\bar{\tau} > 0$, where $\bar{\tau} = \inf_{\varepsilon > 0} \tau(\varepsilon)$.

4. Motions of mechanical type and of non-mechanical type

It is well known that, for any fixed ε, the divergence of a series $\sum_n c_n \varepsilon^n$ representing asymptotically a given function does not imply at all that the series be useless. Rather, the expansion can provide a good estimate of the function by truncation to an optimal order N_{opt}, defined by the property that the modulus of the remainder has a minimum; moreover, the remainder usually turns out to be exponentially small with $1/\varepsilon$, if $N_{\text{opt}} > 0$.

In our case, with the estimate for the remainder given above, one immediately finds $N_{\text{opt}}(\varepsilon) = [\tau(\varepsilon)/\varepsilon]$, where $[\cdot]$ denotes integer part. Thus the situation strongly depends on the value of $\tau(\varepsilon)/\varepsilon$. Indeed, if such a ratio is larger than 1 one has $N_{\text{opt}} > 0$, and correspondingly the remainder turns out to have an exponentially small estimate; precisely one easily finds

$$|\boldsymbol{R}_{N_{\text{opt}}}(\varepsilon)| \leqslant \tilde{C}[\tau(\varepsilon)/\varepsilon]^{-1/2} \widetilde{\mathscr{F}}(\varepsilon) \mathrm{e}^{-[\tau(\varepsilon)/\varepsilon]}$$

with a suitable constant $\tilde{C}$. Instead, if the ratio is smaller than 1 one has $N_{\text{opt}} = 0$, and the estimate for the remainder just reduces to that given by (9) evaluated for $N = 0$, namely

$$|\boldsymbol{R}_0| \leqslant \frac{\varepsilon}{\tau(\varepsilon)} \widetilde{\mathscr{F}}(\varepsilon)$$

which is not 'small', because $\varepsilon/\tau(\varepsilon)$ is then larger than 1.

So we see that the usefulness of the series depends on the value of $\tau(\varepsilon)/\varepsilon$, which is characteristic for any given solution $\boldsymbol{x}(t, \varepsilon)$ at any given value of ε. If $\tau(\varepsilon)/\varepsilon > 1$, from the asymptotic expansion of $\boldsymbol{x}(t, \varepsilon)$ one can extract a partial sum which is a small perturbation of the solution of the corresponding mechanical problem (i.e. that obtained for $\varepsilon = 0$). This means that the solution $\boldsymbol{x}(t, \varepsilon)$ is qualitatively similar to the corresponding mechanical one, or is, as one can say, of 'mechanical type'. But a radically different situation occurs if $\tau(\varepsilon)/\varepsilon < 1$, because the expansion becomes then useless, the remainder being no longer small. However, this doesn't mean anything special for the solution $x(t, \varepsilon)$ of the complete equation, apart from the fact that it is no more a perturbation of the solution of the purely mechanical equation; in such a case we say that the solution is of 'non-mechanical type'. In this sense we say that the Lorentz–Dirac equation, which takes into account the interaction of a charged particle with 'its own' field, should admit solutions of two kinds: those qualitatively similar to purely mechanical ones, and those qualitatively dissimilar. The difference depends on the value of the ratio $\tau(\varepsilon)/\varepsilon$ being greater or smaller than 1, where $\tau(\varepsilon)$ is the amplitude of the analyticity strip of the solution in the complex t plane. As far as we know, this remark was never made before.

5. *Angular momentum threshold distinguishing between the two types of motions* for a Coulomb attractive potential

But how can one conceive of such non-mechanical motions, i.e. of solutions of the Lorentz–Dirac equation which are qualitatively dissimilar from solutions of the purely mechanical equation with the same initial data for position and velocity? The easiest and most natural way is to think of solutions of the Lorentz–Dirac equation still corresponding to situations of scattering experiments, namely satisfying the ingoing asymptotic conditions $\ddot{x}(t) \rightarrow 0$ for $t \rightarrow -\infty$, but not satisfying the Dirac outgoing asymptotic condition $\ddot{x}(t) \rightarrow 0$ for $t \rightarrow +\infty$; indeed, this would just describe a situation with a charged particle being captured in a scattering experiment. Now, in the purely mechanical model such a situation does not occur, because one has just two possibilities, either bound states or scattering states. But things are different within the much wider mathematical model of classical electrodynamics, where the particle, by losing enough energy, might be captured, in which case the Dirac outgoing condition would be no longer satisfied. Naturally there remains open the problem whether the Lorentz–Dirac equation (which should be considered as an *approximation giving a closed equation for the particle, starting from the coupled* equation describing particle and field) does indeed admit non-runaway solutions corresponding to capture. Here we admit that this occurs, and content ourselves with heuristic considerations aimed at characterizing the domain of the mechanical phase space which is expected to lead to solutions of non-mechanical type, presumably corresponding to particle capture. The conclusion will be that, in the case of a Coulomb attractive potential $V(r) = K/r$ ($r = |x|$, $K = -Ze^2$, with Z the *atomic number), solutions of non-mechanical type are expected if, in a scattering* experiment, the modulus l_0 of the initial angular momentum is smaller than a certain threshold value, a first-order estimate of the latter being

$$\bar{l} = 6Z^{2/3}\frac{e^2}{c}. \tag{10}$$

The difficulty of the problem is due to the fact that one is trying to estimate the domain of initial data in the mechanical phase space leading to non-mechanical solutions of the Lorentz–Dirac equation, while these were characterized above in terms of the singularities of the solutions themselves of the complete equation, which we are unable to control a priori. But a way out, which is expected to lead at least to a first-order estimate, consists in estimating the complementary domain, namely the domain of initial data leading to solutions of mechanical type; indeed by definition such solutions are well approximated by solutions of the purely mechanical equation, which are known.

The solutions of mechanical type of the complete equation are characterized by the condition that the optimal truncation order in the expansion (4) be larger than one, or equivalently $|\boldsymbol{c}_1 \varepsilon| < |\boldsymbol{c}_0|$, i.e. (see (7))

$$\varepsilon \left| \sum_{k=1}^{3} \frac{\partial \boldsymbol{F}}{\partial x_k}(\boldsymbol{x}) \cdot \dot{x}_k \right| < |\boldsymbol{F}(\boldsymbol{x})| \tag{11}$$

for all times. For a Coulomb attractive potential, in the case of a one-dimensional motion, such a condition becomes $2\varepsilon v/r < 1$ (with $v = |\dot{\boldsymbol{x}}|$), while it is easily seen to become

$$\frac{v}{r} < \frac{1}{5\varepsilon} \tag{12}$$

for a three-dimensional motion. One has now to work out the first-order approximation, namely to find conditions on the initial data ensuring (12) for all times, for the solution of the corresponding mechanical problem; as shown in the appendix, a necessary and sufficient condition is $l_0 > l^*$ where l^* is an angular momentum slightly larger than the value $\bar{l}$ given by (10). So, for a Coulomb attractive potential one has a first-order estimate of the domain of initial data of phase space leading to solutions of the complete equation which are expected to be qualitatively different from the corresponding solutions of the mechanical problem: non-mechanical motions should occur for small enough values of the initial angular momentum l_0, an approximate value for the threshold being $\bar{l}$ given by (10). Taking for e the electron charge, such an estimated threshold of angular momentum is of the order of e^2/c, which, as a first approximation in the spirit of perturbation theory, might be considered to be not very dissimilar to that of Planck's constant $\hbar = 137e^2/c$. It would be very interesting to understand whether a more accurate analytical estimate could lead to an action nearer to $\hbar$ and also possibly independent of the atomic number Z.

6. Possible physical interpretation of the threshold as giving a frequency cutoff in the radiated spectrum: analogues of the Duane–Hunt law and of the de Broglie relation

We add now some considerations of a completely heuristic character, which, in our opinion, might be of interest for the general problem of the connections between classical electrodynamics and quantum mechanics. Let us consider a typical situation occurring in scattering experiments, with a beam of charged particles (say electrons) impinging on a target: the particles can be assumed to come all with the same velocity v_0 and all possible values of the impact parameter b, each with a corresponding value $l_0 = mv_0 b$ of angular momentum in the range $0 \leqslant l_0 < \infty$. Thus, if l_0 is larger than the threshold $\bar{l}$, the motion is expected to be of mechanical type, namely near to a hyperbola, producing a certain emission of radiation in the continuous spectrum. Instead, if l_0 is below threshold, i.e. b is sufficiently small, then the motion is expected to be of a qualitatively different type, possibly leading to capture, with emission of radiation of a possibly different nature. So it seems reasonable to assume that, in order to study the emission of continuous spectrum in bremsstrahlung, *to a first approximation one should take into account only the* impact parameters b corresponding to l_0 above threshold.

On the other hand, it is very well known that for a given impact parameter b the emitted spectrum has a cutoff frequency $\bar{\omega} = \bar{\omega}(b)$, above which the spectrum essentially vanishes: to a first approximation (motions near to uniform ones) one has indeed (see [17–20], or [21] page 88)

$$\bar{\omega} = v_0/b \tag{13}$$

or equivalently $(1/2)l_0\bar{\omega} = (1/2)mv_0^2$. Thus, because of the limitation $l_0 > \bar{l}$, one obtains that a beam with initial velocity v_0 is expected to emit a continuous spectrum up to a maximal frequency $\bar{\omega}$, which is estimated by

$$3Z^{2/3}\frac{e^2}{c}\bar{\omega} = \tfrac{1}{2}mv_0^2. \tag{14}$$

This is somehow analogous to the well known Duane–Hunt law (or inverse photoelectric effect, or Einstein relation; see [22])

$$\hbar\bar{\omega} = \tfrac{1}{2}mv_0^2. \tag{15}$$

Now, after the works of Weizsäcker [23] and Williams [24] following the work of Fermi [17, 18] (see also [25], appendix 6), it is very well known that the quantum mechanical computations for bremsstrahlung are to a surprisingly high accuracy approximated by purely classical computations, if one adds the external constraint that the impact parameter should be limited by the corresponding condition on angular momentum $l_0 > \hbar$. The motivation is that for lower values of l_0 one violates the uncertainty principle, so that classical mechanics should be abandoned†. In this connection, the considerations presented here might be considered to be of some interest, inasmuch as they give indications that prescriptions somehow analogous to quantum ones might be afforded by classical mechanics (or rather classical electrodynamics) itself, as conditions of internal consistency.

We add now another comment concerning a rather interesting conception of Fermi, which is not well known. In fact, the work of Fermi which stimulated the works of Weizsäcker and Williams quoted above was based essentially on the idea that, from the point of view of the effects produced on matter, a charged particle should be considered as equivalent to the classical electromagnetic field that accompanies it‡. On the other hand, such a field has a spectrum extending up to infinity, and Fermi just added the quantum prescription that the true or effective field should be the classical one, only truncated at a frequency $\bar{\omega}$ defined by the Einstein relation (15) (see [18], formula 7). This is a conception very near to that of de Broglie, the main difference being that in Fermi's conception one is referring to a realistic wave (precisely an electromagnetic wave) rather than just, in de Broglie's words, to an undefined 'periodic phenomenon' or 'nonmaterial wave'. In the

† Such a procedure is very similar to that of Bohr in dealing with bound states, where one considers classical motions and adds from outside a 'quantum condition' on the angular momentum.

‡ '... there naturally occurs the hypothesis that the electric field of the particle produce on the atom the same excitation or ionization effects that would be produced by the equivalent light' (see [18], page 143).

present paper we have pointed out that classical electrodynamics alone (in the form of the Lorentz–Dirac equation) seems to contain an internal criterion which, in a scattering experiment, assigns to a charged particle a characteristic cutoff frequency in the sense of Fermi (namely referring to the electromagnetic field accompanying the particle). Such a frequency $\bar{\omega}$ is estimated by (14), namely has the same structure as the de Broglie frequency defined by the Einstein relation (15), but in place of Planck's constant $\hbar$ has an action proportional to e^2/c with a factor which in a first approximation is estimated by $3Z^{2/3}$. At the moment we are unable to say anything more precise, and just content ourselves with having exhibited such an analogy.

7. Final remarks

In conclusion, we have shown, in a rather straightforward way, that the series expansions of classical electrodynamics should in general be divergent and have asymptotic character. The fact that such a simple remark went apparently unnoticed till now supports, in our opinion, the hope that a renewed interest in classical electrodynamics might disclose new relevant aspects of it. Indeed we reported here some heuristic considerations indicating the possibility that new insights in the correspondence between classical and quantum mechanics be disclosed, in the spirit of [26] (see also [27–29]). But even independently of this, it seems to be plausible that interesting aspects of classical electrodynamics might be revealed by exploiting the known techniques of asymptotic series, especially the most recent ones [30, 31], such as resummation of the divergent tail of the remainder after optimal truncation, resurgence and so on. In particular, the appearance and disappearance of small exponentials across Stokes lines [32, 33], being usually related to physical effects such as tunnelling [34], might be of interest. These are open problems for future work, which should also include an effort at a closer investigation of the solutions which were here called 'of non-mechanical type'.

Appendix

We show here that, for solutions of the purely mechanical equation with an attractive Coulomb potential, condition (12) is guaranteed for all t, if $l_0 > \bar{l}$, with $\bar{l}$ given by (10). One remarks that $v(t)$ takes its maximal value $v_{\max}$ just when $r(t)$ takes its minimal value $r_{\min}$, and moreover one has $v_{\max} = l_0/mr_{\min}$. Thus, in order to satisfy (12) for all t it is necessary to have

$$\frac{l_0}{mr_{\min}^2} \leqslant \frac{1}{5\varepsilon}. \tag{16}$$

Now, $r_{\min}$ is the positive solution of the equation

$$-\frac{Ze^2}{r} + \frac{l_0^2}{2mr^2} = E$$

so that

$$\frac{1}{r_{min}} = \frac{mZe^2}{l_0^2}\left(1 + \left(1 + \frac{2El_0^2}{mZ^2e^4}\right)^{1/2}\right). \tag{17}$$

So (16) becomes, with the expression (2) for ε

$$l_0 \geqslant (10/3)^{1/3} Z^{2/3} \frac{e^2}{c} \left(1 + \left(1 + \frac{2El_0^2}{mZ^2e^4}\right)^{1/2}\right)^{2/3} \tag{18}$$

The RHS of (18) is a positive increasing function of l_0, with a derivative which is seen to be always less than 1; so there exists a value l^* (depending on v_0 and Z), such that the inequality is satisfied only if $l_0 > l^*$. It is obvious that l^* is larger than the value of the RHS of (18) evaluated at $l_0 = 0$, and it is also easy to check that indeed it is less than two times such a value, namely one has $l^* \leqslant 4(5/3)^{1/3} Z^{2/3} e^2/c$. Then, *a fortiori*, inequality (12) is satisfied for all $l_0 > \bar{l}$ with $\bar{l}$ given by (10), as claimed.

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