Inverse Scattering Theory and Transmission Eigenvalues

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Research supported by grants from AFOSR and NSF
We consider the propagation of sound waves of small amplitude in $\mathbb{R}^3$ viewed as a problem in fluid dynamics. Let $v(x, t), x \in \mathbb{R}^3$, be the velocity potential of a fluid particle in an inviscid fluid and

$$p(x, t) = \text{pressure}, \quad \rho(x, t) = \text{density}, \quad S(x, t) = \text{specific entropy}.$$ 

Then, if there are no external forces, we have

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \frac{1}{\rho} \nabla p = 0 \quad \text{(Euler’s equation)}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \quad \text{(equation of continuity)}$$

$$p = f(\rho, S) \quad \text{(equation of state)}$$

$$\frac{\partial S}{\partial t} + v \cdot \nabla S = 0 \quad \text{(adiabatic hypothesis)}$$

where $f$ is a function depending on the fluid.
Assuming \( v(x, t) \), \( p(x, t) \), \( \rho(x, t) \) and \( S(x, t) \) are small, we perturb around the static case \( v = 0, p = p_0 = \text{constant}, \rho = \rho_0(x), S = S_0(x) \) with \( p_0 = f(\rho_0, S_0) \):

\[
\begin{align*}
  v(x, t) &= \epsilon v_1(x, t) + O(\epsilon^2), \\
  p(x, t) &= p_0 + \epsilon p_1(x, t) + O(\epsilon^2), \\
  \rho(x, t) &= \rho_0(x) + \epsilon \rho_1(x, t) + O(\epsilon^2), \\
  S(x, t) &= S_0(x) + \epsilon S_1(x, t) + O(\epsilon^2),
\end{align*}
\]

where \( 0 < \epsilon << 1 \). Substituting (2) into (1) implies that

\[
\frac{\partial v_1}{\partial t} + \frac{1}{\rho_0} \nabla p_1 = 0, \quad \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 v_1) = 0
\]

\[
\frac{\partial \rho_1}{\partial t} + c^2(x) \left( \frac{\partial \rho_1}{\partial t} + v_1 \cdot \nabla \rho_0 \right)
\]

where the sound speed \( c \) is defined by

\[
c^2(x) = \frac{\partial}{\partial \rho} f(\rho_0(x), S_0(x)).
\]
We now have that

\[
\frac{\partial^2 p_1}{\partial t^2} = c^2(x) \rho_0(x) \nabla \cdot \left( \frac{1}{\rho_0(x)} \nabla p_1 \right).
\]

If \( p_1(x, t) = \text{Re} \left\{ u(x) e^{-i\omega t} \right\} \) we have that \( u \) satisfies

\[
\rho_0(x) \nabla \left( \frac{1}{\rho_0(x)} \nabla p_1 \right) + \frac{\omega^2}{c^2(x)} u = 0.
\]

Making the further assumption that \( \nabla \rho_0 \) can be ignored, we arrive at

\[
\Delta u + \frac{\omega^2}{c^2(x)} u = 0 \quad (3)
\]

We now assume that the slowly varying inhomogeneous medium is of compact support and is imbedded in \( \mathbb{R}^3 \) where the sound speed is \( c(x) = c_0 = \text{constant} \).
Scattering by an Inhomogeneous Medium

We further assume that the wave motion is caused by an incident field $u^i$ satisfying (3) with $c(x) = c_0$. We then arrive at the scattering problem of determining $u$ such that

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3$$

$$u = u^s + u^i$$

$$\lim_{r \to \infty} r \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

where $k = \omega / c_0 > 0$ is the wave number, the incident field $u^i$ is an entire solution of the Helmholtz equation $\Delta u + k^2 u = 0$ and $u^s$ is the scattered field.

- The function $n(x)$ is called the refractive index.
- More generally if medium is absorbing $n(x) = n_1(x) + \frac{n_2(x)}{k}$.
- $n(x) = 1$ outside the inhomogeneous medium $D$.
- $n(x)$ is piece-wise smooth, $\Re(n(x)) \geq n_0 > 0$, $\Im(n(x)) \geq 0$. 

The Helmholtz Equation

We look for solutions of the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad k > 0$$

in the form

$$u(x) = f(k|x|) Y_n^m(\hat{x})$$

where $x \in \mathbb{R}^3$, $\hat{x} = x/|x|$, $Y_n^m$ is a spherical harmonic

$$Y_n^m(\theta, \varphi) := \left( \frac{2n + 1}{4\pi} \frac{(n - |m|)!}{(n + |m|)!} \right)^{1/2} P_n^m(\cos \theta) e^{im\varphi}$$

$m = -n, \cdots, n$, $n = 0, 1, 2, \cdots$, $(\theta, \varphi)$ are the spherical angles of $\hat{x}$ and $P_n^m$ is an associated Legendre polynomial.

Note that $\{ Y_n^m \}$ is a complete orthonormal system in $L^2(S^2)$ where $Y_0^0 = \frac{1}{\sqrt{4\pi}}$ and

$$S^2 := \{ x : |x| = 1 \}.$$
The function $f$ is a solution of the spherical Bessel equation

$$z^2 f''(z) + 2zf'(z) + [z^2 - n(n + 1)] f(z) = 0$$

with two linearly independent solutions $j_n(z)$, $y_n(z)$ called, respectively, the spherical Bessel function and the spherical Neumann function. The functions

$$h_n^{(1)}(z) := j_n(z) + iy_n(z) \quad h_n^{(2)}(z) := j_n(z) - iy_n(z)$$

are called spherical Hankel functions of order $n$. In particular, $h_n^{(1)}(kr)$ satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} r \left( \frac{\partial u}{\partial r} - iku \right) = 0,$$

i.e. if $u(x) = h_n^{(1)}(kr) Y_m(\hat{x})$ then $u(x)e^{-i\omega t}$ (where $\omega$ is the frequency and $t$ is time) is an outgoing wave.
Solutions of the Helmholtz equation satisfying the Sommerfeld radiation condition uniformly in $\hat{x}$ are called radiating.

Now let $D$ be a bounded domain such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and assume that $\partial D$ is of class $C^2$ with unit normal $\nu$ directed into the exterior of $D$. Let

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y$$

be the radiating fundamental solution to the Helmholtz equation.

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|} \left\{ e^{-ik\hat{x} \cdot y} + O \left( \frac{1}{|x|} \right) \right\}$$

$$\frac{\partial}{\partial \nu_y} \frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|} \left\{ \frac{\partial}{\partial \nu_y} e^{-ik\hat{x} \cdot y} + O \left( \frac{1}{|x|} \right) \right\}$$

as $|x| \to \infty$, $\hat{x} = x/|x|$ uniformly for all $y \in \partial D$. 

The Helmholtz Equation

Using Green’s second identity

\[ \int_D (u \Delta v - v \Delta u) \, dx = \int_D \left( u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \, ds \]

we can deduce Green’s formula for functions \( u \in C^2(D) \cap C^1(\overline{D}) \):

\[
 u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu} \Phi(x, y) - u \frac{\partial}{\partial y} \Phi(x, y) \right\} \, ds \\
- \int_D \left\{ \Delta u + k^2 u \right\} \Phi(x, y) dy, \quad x \in D.
\]

Theorem

Let \( u \in C^2(D) \cap C^1(\overline{D}) \) be a solution to the Helmholtz equation in \( D \). Then \( u \) is real analytic in \( D \).
The Helmholtz Equation

Holmgren’s Theorem

Let \( u \in C^2(D) \cap C^1(\overline{D}) \) be a solution to the Helmholtz equation in \( D \) such that

\[
u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma
\]

for some open subset \( \Gamma \subset \partial D \). Then \( u \) is identically zero in \( D \).

For \( x \) in the exterior of \( D \) we have the following theorem:

Theorem

Let \( u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D) \) be a radiating solution to the Helmholtz equation

\[
\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}.
\]

Then we have Green’s formula

\[
u(x) = \int_{\partial D} \left\{ u(y) \frac{\partial}{\partial \nu_y} \Phi(x, y) - \frac{\partial u(y)}{\partial \nu_y} \Phi(x, y) \right\} ds, \quad x \in \mathbb{R}^3 \setminus \overline{D}.
\]
Corollary

An entire solution to the Helmholtz equation satisfying the radiation condition must vanish identically.

Corollary

Every radiating solution to the Helmholtz equation has the asymptotic behavior of an outgoing spherical wave

\[ u(x) = \frac{e^{ik|x|}}{|x|} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty \]

uniformly in all directions \( \hat{x} = x/|x| \).

The function \( u_\infty \) defined on the unit sphere \( S^2 \) is called the far field pattern of \( u \).
Rellich’s Lemma

Let $u \in C^2(\mathbb{R}^3 \setminus \overline{D})$ be a solution to the Helmholtz equation satisfying

$$\lim_{r \to \infty} \int_{|x|=r} |u(x)|^2 \, dx = 0.$$ 

Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

Corollary

Assume $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus \overline{D})$ is a radiating solution to the Helmholtz equation such that

$$\text{Im} \int_{\partial D} u \frac{\partial u}{\partial \nu} \, ds \geq 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. 
Let us set $m := 1 - n$. Hence

$$D := \{ x \in \mathbb{R}^3 : m(x) \neq 0 \} .$$

We again let

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x - y|}, \quad x \neq y.$$ 

**Theorem**

Given two bounded domains $D$ and $G$, the volume potential

$$(V\varphi)(x) := \int\limits_{D} \Phi(x, y)\varphi(y)dy, \quad x \in \mathbb{R}^3$$

defines a bounded operator $V : L^2(D) \to H^2(G)$. 
We now show that the scattering problem (4) – (6) is equivalent to solving the Lippmann-Schwinger integral equation

\[ u(x) = u^i(x) - k^2 \int_{\mathbb{R}^3} \Phi(x, y)m(y)u(y) \, dy, \quad x \in \mathbb{R}^3. \]  

**Theorem**

If \( u \in H^2_{\text{loc}}(\mathbb{R}^3) \) is a solution of (4) – (6) then \( u \) is a solution of (7). Conversely, if \( u \in C(\mathbb{R}^3) \) is a solution of (7) then \( u \in H^2_{\text{loc}}(\mathbb{R}^3) \) and \( u \) is a solution of (4) – (6).

**Theorem**

Suppose that \( m(x) = 0 \) for \( |x| \geq a \) and \( k^2 < 2/Ma^2 \) where \( M = \sup_{|x| \leq a} |m(x)| \). Then there exists a unique solution to the Lippmann-Schwinger integral equation
From (7) we see that

$$u^s(x) = -k^2 \int_{\mathbb{R}^3} \Phi(x, y) m(y) u(y) \, dy, \quad x \in \mathbb{R}^3$$

and hence

$$u^s(x) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty$$

where the far field pattern $u_\infty$ is given by

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} m(y) u(y) \, dy, \quad \hat{x} = \frac{x}{|x|}.$$  

Assuming $k$ is sufficiently small and replacing $u$ by the first term in solving (7) by iteration (the weak scattering assumption) gives the Born approximation

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} m(y) u^i(y) \, dy.$$
Unique Continuation Principle

Let $G$ be a domain in $\mathbb{R}^3$ and suppose $u \in H^2(G)$ is a solution of

$$\Delta u + k^2 n(x) u = 0$$

in $G$ such that $n$ is piecewise continuous in $G$ and $u$ vanishes in a neighborhood of some $x_0 \in G$. Then $u$ is identically zero in $G$.

Theorem

For each $k > 0$ there exists a unique solution $u \in H^2_{loc}(\mathbb{R}^3)$ to the scattering problem (4) – (6) and $u$ depends continuously with respect to the maximum norm on the incident field $u^i$. 
Now let $u^i(x) = e^{ikx \cdot d}$, $|d| = 1$, and consider the scattering problem

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3$$

$$u(x) = u^i(x) + u^s(x)$$

$$\lim_{r \to \infty} r \left( \frac{\partial u^s}{\partial r} -iku^s \right) = 0$$

with corresponding far field pattern $u_\infty(\hat{x}) = u_\infty(\hat{x}, d)$ defined by

$$u^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ u_\infty(\hat{x}, d) + O \left( \frac{1}{|x|} \right) \right\}.$$ 

Reciprocity Principle

Let $u_\infty(\hat{x}, d)$ be the far field pattern corresponding to (1) – (3). Then

$$u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x}).$$
Proof: Let $D \subset \{ x : |x| < a \}$ where $D := \{ x : m(x) \neq 0 \}$. We now use Green’s second identity to obtain

$$0 = \int_{|y|=a} \left\{ u^i(y, d) \frac{\partial}{\partial \nu} u^i(y, -\hat{x}) - u^i(y, -\hat{x}) \frac{\partial}{\partial \nu} u^i(y, d) \right\} \, ds(y)$$

$$0 = \int_{|y|=a} \left\{ u^s(y, d) \frac{\partial}{\partial \nu} u^s(y, -\hat{x}) - u^s(y, -\hat{x}) \frac{\partial}{\partial \nu} u^s(y, d) \right\} \, ds(y).$$

The second corollary to Green’s formula implies that

$$4\pi u_\infty(\hat{x}, d) = \int_{|y|=a} \left\{ u^s(y, d) \frac{\partial}{\partial \nu} u^i(y, -\hat{x}) - u^i(y, -\hat{x}) \frac{\partial}{\partial \nu} u^s(y, d) \right\} \, ds(y)$$

$$4\pi u_\infty(-d, -\hat{x}) = \int_{|y|=a} \left\{ u^s(y, -\hat{x}) \frac{\partial}{\partial \nu} u^i(y, d) - u^i(y, d) \frac{\partial}{\partial \nu} u^s(y, -\hat{x}) \right\} \, ds(y).$$
Subtracting the last of the above equations from the sum of the first three gives

\[ 4\pi [u_\infty(\hat{x}, d') - u_\infty(-d, -\hat{x})] \]

\[ = \int_{|y|=a} \left\{ u(y, d) \frac{\partial}{\partial \nu} u(y, -\hat{x}) - u(y, -\hat{x}) \frac{\partial}{\partial \nu} u(y, d) \right\} ds(y) \]

\[ = 0 \]

by Green’s second identity.
We now define the **far field operator** $F : L^2(S^2) \to L^2(S^2)$ by

$$ (Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d)g(d)ds(d). $$

Since $u_\infty(\hat{x}, d)$ is infinitely differentiable with respect to each of its variables, $F$ is clearly compact.

The corresponding **scattering operator** $S : L^2(S^2) \to L^2(S^2)$ is defined by

$$ S := I + \frac{ik}{4\pi} F. $$
The Far Field Operator

Lemma

For \( g, h \in L^2(S^2) \) define the Herglotz wave functions \( v^i \) and \( w^i \) with kernels \( g \) and \( h \) respectively by

\[
v^i(x) := \int_{S^2} e^{ikx \cdot d} g(d) ds(d) \quad w^i(x) := \int_{S^2} e^{ikx \cdot d} h(d) ds(d).
\]

Let \( v \) and \( w \) be the solutions of the scattering problem (1) – (3) corresponding to the incident fields \( v^i \) and \( w^i \) respectively. Then

\[
(ik)^2 \int_D \mathcal{G}(n)vw \, dx = 2\pi (Fg, h) - 2\pi (g, Fh) - ik (Fg, Fh).
\]

Theorem

If \( \mathcal{G}(n) = 0 \), then the far field operator \( f \) is normal, i.e. \( F^*F = FF^* \), and the scattering operator \( S \) is unitary, i.e. \( SS^* = S^*S = I \).
The Far Field Operator

We now introduce the transmission eigenvalue problem: Determine $k > 0$ and $v, w \in L^2(D)$, $v - w \in H^2_0(D)$ such that $v$ and $w$ are not identically zero and

\[
\begin{align*}
\Delta w + k^2 n w &= 0 \quad \text{in} \quad D \\
\Delta v + k^2 v &= 0 \quad \text{in} \quad D \\
w &= v \quad \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D
\end{align*}
\]

Such values of $k$ are called transmission eigenvalues.

Recall that $D := \{x : n(x) \neq 1\}$ and it is assumed that $D$ is bounded with $C^2$ boundary $\partial D$ such that $\mathbb{R}^3 \setminus \overline{D}$ is connected.

**Theorem**

Let $F$ be the far field operator corresponding to the scattering problem (1) – (3). Then $F$ is injective if $k$ is not a transmission eigenvalue.
Proof: Suppose $Fg = 0$. Then the far field pattern $w_\infty$ of the scattered field $w^s$ corresponding to the incident field

$$w^i(x) = \int_{S^2} e^{ikx \cdot d} g(d) ds(d)$$

vanishes. By Rellich’s lemma $w^s = w - w^i$ vanishes outside $D$. Then $w = w^s + w^i$ satisfies $\Delta w + k^2 nw = 0$ in $\mathbb{R}^3$ and $w - w^i = 0$ on $\partial D$, $\frac{\partial}{\partial \nu}(w - w^i) = 0$ on $\partial D$. If $k$ is not a transmission eigenvalue then $w^i = w = 0$ which implies $g = 0$, i.e. $F$ is injective.

Corollary

Let $F$ be the far field operator corresponding to the scattering problem (1) – (3). Then $F$ has dense range if $k$ is not a transmission eigenvalue.
The Far Field Operator

Proof: From \((Fg, h) = (g, F^* h)\) we have \(R(F)^\perp = N(F^*)\). Hence we must show that \(F^* h = 0 \implies h = 0\).

To this end, using reciprocity, we have that \(F^* h = 0\)

\[
\implies \int_{S^2} u_\infty(d, \hat{x})h(d)ds(d) = 0
\]

\[
\implies \int_{S^2} u_\infty(- \hat{x}, -d)h(d)ds(d) = 0
\]

\[
\implies \int_{S^2} u_\infty(\hat{x}, d)h(-d)ds(d) = 0
\]

\[
\implies h = 0 \text{ by the previous theorem.}
\]