Inverse Scattering Theory and Transmission Eigenvalues

Fioralba Cakoni

Department of Mathematical Sciences
University of Delaware
Newark, DE 19716, USA
email: cakoni@math.udel.edu

Research supported by grants from AFOSR and NSF
Alternative approach to weak scattering approximation or nonlinear optimization techniques are the qualitative methods. Such methods seek partial information such as the support $D$ and estimates on $n$

- Singular sources method (POTTHAST (2001)) . . .
- . . .

CAKONI-COLTON (2014), A Qualitative Approach to Inverse Scattering Theory, Springer.

The linear sampling method is based on solving the far field equation

\[(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k), \quad \text{for} \quad g \in L^2(S^2), \quad \hat{x} \in S^2, \quad z \in \mathbb{R}^m\]

for a fixed \(k\), where \(\Phi_\infty(\hat{x}, z, k) := e^{-ik\hat{x} \cdot z}\).

If \(k\) is not a transmission eigenvalue, then there is a \(g_\varepsilon := g_{z,k,\varepsilon}\) satisfying

\[\|Fg_\varepsilon - \Phi_\infty(\cdot, z, k)\|_{L^2(S^2)} < \varepsilon\]

such that

- for \(z \in D\) \(\lim_{\varepsilon \to 0} \|v_{g_\varepsilon}\|_{L^2(D)}\) exists
- for \(z \notin D\) \(\lim_{\varepsilon \to 0} \|v_{g_\varepsilon}\|_{L^2(D)} = \infty\)

where \(v_g(x) := \int_{S^2} e^{ikx \cdot d} g(d) ds(d)\)
The Linear Sampling Method

Two main ingredients:

- If $k$ is not a transmission eigenvalue than there exits a unique solution $v, w \in L^2(D)$, $v - w \in H^2(D)$, of the inhomogeneous interior transmission problem

  \begin{align*}
  \Delta w + k^2 nw &= 0 \quad \text{in} \quad D \quad (1) \\
  \Delta v + k^2 v &= 0 \quad \text{in} \quad D \quad (2) \\
  w - v &= \Phi(\cdot, z) \quad \text{on} \quad \partial D \quad (3) \\
  \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} &= \frac{\partial}{\partial \nu} \Phi(\cdot, z) \quad \text{on} \quad \partial D. \quad (4)
  \end{align*}

- The set $\{v_g : g \in L^2(S^2)\}$ is dense in

  $\{u \in L^2(D) : \Delta u + k^2 u = 0 \text{ in } D\}$. 

The linear sampling method applies to more general class of problems for which the above two ingredients are valid.

A problem with the linear sampling method is that, in general, there does not exist a solution to

\[ Fg = \Phi_\infty(\cdot, z) \]

for noise free data and hence it is not clear what solution is obtained by using Tikhonov regularization to solve this equation.

In particular, it is not clear whether Tikhonov regularization indeed leads to the approximations predicted by the above theorem.

The problem can be resolved by using the theory of the factorization method.
The Factorization Method

To fix the idea assume that $\mathcal{S}(n) = 0$ (i.e. the far field operator is normal) and $n(x) > 1$.

We re-write the scattering problem as

$$\Delta u^s + k^2 n u^s = k^2 m e^{ikx \cdot d} \quad \text{in } \mathbb{R}^3$$

and $u^s$ satisfies the Sommerfeld radiation condition

More generally, for $f \in L^2(D)$ we consider

$$\Delta w + k^2 n w = mf \quad \text{in } \mathbb{R}^3$$

and $w$ satisfies the Sommerfeld radiation condition, which has a unique solution $w \in H^2_{\text{loc}}(\mathbb{R}^3)$.

- Define the $G : L^2(D) \rightarrow L^2(S^2)$

$$f \rightarrow w_\infty$$
Factorization of the far field operator

Theorem

Let $F$ and $G$ be defined as above. Then

$$F = 4\pi k^2 GS^* G^*$$

where $S^*$ is the adjoint of $S : L^2(D) \to L^2(D)$ defined by

$$(S\psi)(x) := -\frac{\psi(x)}{m(x)} - k^2 \int_D \Phi(x, y)\psi(y)dy, \quad x \in D$$

and $G^* : L^2(S^2) \to L^2(D)$ is the adjoint of $G$. 
Proof: We have that \( u_\infty = k^2 G u^i \). We now define the Herglotz operator \( H : L^2(S^2) \to L^2(D) \) by

\[
(Hg)(x) := \int_{S^2} e^{ikx \cdot d} g(d) ds(d), \quad x \in D,
\]

and note that \( Fg \) is the far field pattern corresponding to the incident field \( Hg \), i.e.

\[
F = k^2 GH.
\]

Since

\[
(H^* \psi)(\hat{x}) := \int_D e^{-ik\hat{x} \cdot y} \psi(y) dy, \quad \hat{x} \in S^2
\]

we have that \( H^* \psi = 4\pi w_\infty \) where \( w_\infty \) is the far field pattern of

\[
w(x) := \int_D \Phi(x, y) \psi(y) dy, \quad x \in \mathbb{R}^3.
\]
But

\[ \Delta w + k^2 nw = -m \left( \frac{\psi}{m} + k^2 w \right) \]

and hence

\[ H^* \psi = 4\pi w_{\infty} = -4\pi G \left( \frac{\psi}{m} + k^2 w \right) = 4\pi GS \psi \]

i.e. \( H^* = 4\pi GS \). Thus \( H = 4\pi S^* G^* \) and since \( F = k^2 GH \) the theorem follows.

**Lemma**

For \( z \in \mathbb{R}^3 \) let

\[ \Phi_{\infty}(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x} \cdot z} \]

be the far field pattern of \( \Phi(x, z) \).

Then \( z \in D \) if and only if \( \Phi_{\infty}(\cdot, z) \) is in the range of \( G \).
Proof: For \( z \in D \) choose a cut-off function \( \rho \in C^\infty(\mathbb{R}^3) \) which vanishes near \( z \) and equals one in \( \mathbb{R}^3 \setminus D \). Then \( v(x) = \rho(x)\Phi(x, z) \) has \( \Phi_\infty(\hat{x}, z) \) as its far field pattern. Hence

\[
\Phi_\infty(\cdot, z) = Gf
\]

where

\[
f := \frac{1}{m} \left( \Delta v + k^2 n v \right) \in L^2(D).
\]

Now assume that \( z \notin D \) and \( \Phi_\infty(\cdot, z) = Gf \) for some \( f \in L^2(D) \). By Rellich’s lemma \( \Phi(\cdot, z) = u^s \) in the exterior of \( D \cup \{z\} \) and this is a contradiction since \( u^s \) is smooth near \( z \) but \( \Phi(\cdot, z) \) is singular at \( z \).
Theorem

Let $X$ and $H$ be Hilbert spaces and assume that $F : H \to H$, $B : X \to H$ and $T : X \to X$ are bounded linear operators that satisfy

$$F = BTB^* \quad \text{where } B^* \text{ is the adjoint of } B$$

$$\text{Im} \left( Tf, f \right) \neq 0 \quad \text{for all } f \in B^* (H) \text{ with } f \neq 0$$

and $T = T_0 + C$ where $C$ is compact such that

$$\left( T_0 f, f \right) \in \mathbb{R}, \quad \left( T_0 f, f \right) \geq c \| f \| ^2 \quad \text{for all } f \in B^* (H) \text{ and some } c > 0$$

In addition let the operator $F$ be compact, injective and assume that $I + i \gamma F$ is unitary for some $\gamma > 0$.

Then the ranges $B(X)$ and $(F^* F)^{1/4} (H)$ coincide.
The Factorization Method

Theorem

Let $S : L^2(D) \rightarrow L^2(D)$ be defined as above and let $S_0 : L^2(D) \rightarrow L^2(D)$ be given by

$$S_0 \psi := -\frac{\psi}{m}.$$ 

Then

1. $S_0$ is bounded, self-adjoint and satisfies

$$\langle S_0 \psi, \psi \rangle \geq \frac{1}{\|m\|_{\infty}} \|\psi\|^2, \quad \psi \in L^2(D).$$

2. $S - S_0 : L^2(D) \rightarrow L^2(D)$ is compact.

3. $S$ is an isomorphism from $L^2(D)$ onto $L^2(D)$.

4. $\text{Im} (S \psi, \psi) \leq 0$ for all $\psi \in L^2(D)$ with strict inequality holding for all $\psi \in G^* (L^2(D))$ with $\psi \neq 0$. 
Proof:

1. This follows since $m(x) > 0$ for $x \in \bar{D}$.

2. This follows from the fact that volume potentials map $L^2(D) \to H^2(D)$ and $H^2(D)$ is compactly embedded into $L^2(D)$.

3. From its definition, $S_0$ clearly has a bounded inverse. Hence by 2. and the Riesz-Fredholm theory it suffices to show that $S$ is injective. Suppose $S\psi = 0$. Then $\varphi = \frac{\psi}{m}$ satisfies the homogeneous Lippmann-Schwinger equation and hence $\varphi = 0$ and then $\psi = 0$.

4. Proof is technical - not given.
Theorem

Assume that $n(x) > 1$ for $x \in \overline{D}$ and $k > 0$ is not a transmission eigenvalue.

Then $z \in D$ if and only if $\Phi_{\infty}(\cdot, z)$ is in the range of $(F^* F)^{1/4}$.

Remark

The theorem is also true if we assume that $0 < n(x) < 1$ for $x \in \overline{D}$.

To construct $D$, let $\lambda_n$ and $\psi_n$ be the eigenvalues and eigenfunctions of $F$. Then $\left(\sqrt{|\lambda_n|}, \psi_n, \psi_n\right)$ is a singular system for $(F^* F)^{1/4}$ and by Picard’s theorem

$$z \in D \iff \sum_{n=1}^{\infty} \frac{|(\psi_n, \Phi_{\infty}(\cdot, z))|^2}{|\lambda_n|} < \infty.$$
The factorization method can be used to provide the following justification of the linear sampling method.

Assume $k$ is not a transmission eigenvalue. Consider

$$(F^* F)^{1/4} \varphi_z = \Phi_\infty(\cdot, z, k) \quad \text{and} \quad (\alpha I + F^* F)g_{z,k,\alpha} = F^* \Phi_\infty(\cdot, z)$$

Then $c \| \varphi_z \|^2 \leq \lim_{\alpha \to 0} |v_{g_{z,k,\alpha}}(z)| \leq C \| \varphi_z \|^2$, i.e.

- for $z \in D$ \quad $\lim_{\alpha \to 0} |v_{g_{z,k,\alpha}}(z)|$ \quad exists
- for $z \notin D$ \quad $\lim_{\alpha \to 0} |v_{g_{z,k,\alpha}}(z)| = \infty$

Hence to implement the linear sampling method one can plot $|v_{g_{z,k,\alpha}}(z)|$ or (as done in practice) $\| g_{z,k,\alpha} \|_{L^2(\Omega)}$