Abstract

In this paper we first introduce a non symmetric notion of commutativity between a relation $S$ and an equivalence relation $R$, which coincides with Smith commutativity in the case $S$ is an equivalence relation too. We then introduce a related notion of ns-centralizer for an equivalence relation $R$, defined as an equivalence relation which is the largest among relations that commute with $R$ in this non symmetric sense. After proving that any action accessible category in the sense of D. Bourn and G. Janelidze ([11]) has ns-centralizers for equivalence relations, we show that the existence of ns-centralizer for any equivalence relation actually characterizes action accessibility for exact protomodular categories.

Keywords: faithful split extension, centralizer, commutator, protomodular

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Introduction

The notion of action accessible category was recently introduced by D. Bourn and G. Janelidze in [11], and developed in a more general concept by D. Bourn in [6], in the context of pointed protomodular categories (categories where split short five lemma holds, see [1]). Via the equivalence between internal actions and split extensions in the category of groups as in any other semiabelian category (see [17]), it is possible to represent any action on a fixed object $X$ by a split extension with kernel $X$. The central notion of faithful split extension introduced in [11] is an interpretation in terms of split extensions of the classical notion of faithful action. A pointed protomodular category $\mathcal{C}$ is said to be action accessible if any split extension in $\mathcal{C}$ admits a morphism in a faithful one. This assignment is not unique, but we’ll show that it can be given in a canonical way (see Corollary 2.9), as it happens in groups, where this procedure is obtained by zeroing the elements acting trivially.

The relevance of action accessibility was pointed out in the very recent paper [12] by D. Bourn and A. Montoli, where action accessible categories which are
Barr-exact turn out to be the “right” context for an internal version of Schreier-Mac Lane theorem on obstructions to extensions. The reason relies on a very important property of action accessible categories: the existence of centers, and more generally of centralizers. This means that in an action accessible category, for any normal subobject \( X \) of \( A \), there exists a normal subobject \( Z(X, A) \) of \( A \), commuting with \( X \) in \( A \) in the sense of Huq (see Definition 2.1), which is larger than any other subobject of \( A \) with the same property. Moreover, D. Bourn and G. Janelidze showed that a similar property holds for any equivalence relation \( R \) on \( A \): namely, there exists a larger equivalence relation \( E_A(R) \) commuting with \( R \) in the sense of Smith (see Definition 3.2). In fact, they also showed that in a homological action accessible category two normal subobjects commute in the sense of Huq if and only if the corresponding equivalence relations commute in the sense of Smith.

Note that the centralizer of an equivalence relation \( R \) is the largest among equivalence relations that commute with \( R \), while the centralizer of a normal subobject \( X \) is the largest among (not necessarily normal) subobjects that commute with \( X \). This is a stronger property, which has no counterpart in terms of commutativity of equivalence relations.

In order to face this lack of analogy, in this paper we introduce a non symmetric notion of commutativity between a relation \( S \) and an equivalence relation \( R \), which coincides with Smith commutativity in the case \( S \) is an equivalence relation too (see Proposition 3.6). Consequently, we introduce a related notion of centralizer, which we call ns-centralizer for equivalence relations, defined as an equivalence relation which is the largest among relations that commute with \( R \) in the non symmetric sense. It turns out that any action accessible category admits also ns-centralizers for any equivalence relation (see Proposition 4.2), showing the analogy existing in this case with centralizers of subobjects. This stronger property suggested us a way to find a characterization of action accessible categories via the existence of ns-centralizers. We show that, in any homological category with ns-centralizers for equivalence relations, faithful split extensions are exactly those with trivial ns-centralizer, in the sense of Proposition 4.3. This property is typical of action accessible categories (as shown in [6] for the more general case of faithful groupoids). Actually, we prove that, for an exact category, the existence of ns-centralizers is equivalent to action accessibility.

1. Action accessible categories

Let \( C \) be a pointed protomodular category, essentially a category where split short five lemma holds (see [1] for the definition and several characterizations). Given an object \( X \in C \), a split extension with kernel \( X \) is a diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{x} & A \\
& & \xrightarrow{p} \downarrow \uparrow s \\
& & B
\end{array}
\]

such that \( ps = 1_B \) and \( x = \ker p \). We will denote such a split extension by \((B, A, p, s, x)\). Given another split extension \((D, C, q, t, k)\) with the same kernel
X, a morphism \((g, f) : (B, A, p, s, x) \rightarrow (D, C, q, t, k)\) is a pair \((g, f)\) of arrows such that \(k = fx, qf = gp\) and \(fs = tg\):

\[
\begin{array}{cccccc}
X & \xrightarrow{x} & A & \xrightarrow{p} & B \\
& & \downarrow{s} & & \\
& & f & & \\
X & \xrightarrow{k} & C & \xrightarrow{q} & D
\end{array}
\]

(1)

Split extensions with fixed kernel \(X\) and morphisms between them form a category, which we will denote by \(\text{SplExt}_\mathcal{C}(X)\), or simply by \(\text{SplExt}(X)\).

Note that, whenever it exists, the morphism \(f\) in diagram (1) is uniquely determined by \(g\) (and \(1_X\)), and this follows from the fact that the pair \((x, s)\) is jointly strongly epimorphic by protomodularity. Moreover, in every diagram of the form (1) the square \((*)\) is a pullback by protomodularity, since \(p\) and \(q\) have the same kernel \(X\).

It is well known that, in the category of groups, every split extension \(X \xrightarrow{x} A \xrightarrow{p} B \xleftarrow{s} C \xleftarrow{q} D\) gives rise to an action of the group \(B\) on \(X\), namely the action given by elements of \(B\) via conjugation in \(A\). This correspondence holds more in general in every pointed category \(\mathcal{C}\) with finite limits and finite coproducts, as explained in [3], where the notion of internal action was introduced. We recall here the definition: let \(\text{Pt}_B(\mathcal{C})\) the category of points over \(B\) in \(\mathcal{C}\), then we have an adjunction:

\[
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\perp} & \text{Pt}_B(\mathcal{C}) \\
\downarrow{F} & & \downarrow{G} \\
\mathcal{C} & \xrightarrow{B} & \mathcal{C}
\end{array}
\]

(2)

where, on objects:

\[
\begin{array}{ccc}
X & \xrightarrow{F} & B + X \\
\downarrow{h} & & \downarrow{[1,0]} \quad \text{and} \quad \downarrow{\iota} \\
B & & B
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{G} & \ker p \\
\downarrow{s} & & \downarrow{p} \\
B & & B
\end{array}
\]

And the corresponding monad \(GF(-)\) on \(\mathcal{C}\) is denoted by \(B\delta(-)\) (so, as an object, we will indicate \(B\delta X = \ker[1, 0]\)).

**Definition 1.1.** The algebras for the monad \(B\delta(-)\) induced by the adjunction (2) above are called internal \(B\)-action in \(\mathcal{C}\). We denote by \(\mathcal{C}^B\) the category of these algebras.

The comparison functor \(\text{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}^B\) associates with every point \((A, p, s)\) a \(B\)-action \(\xi\) as described in the following diagram (where \(X\) is the kernel of \(p\)
and $\xi$ is induced by the universal property of $X$):

$$
\begin{array}{c}
B\eta X \xrightarrow{\ker[1,0]} B + X \xrightarrow{[1,0]} B \\
\downarrow \downarrow \downarrow \downarrow \\
\xi \downarrow \downarrow \downarrow \\
\langle \ast \rangle \downarrow \downarrow \downarrow \\
X \xrightarrow{k} A \xrightarrow{p} B
\end{array}
$$

When $C$ is the category of groups, given a group action $\xi$ of $B$ over $K$, we can always associate with it a semidirect product $K \rtimes_{\xi} B$ and then a point $X \rtimes_{\xi} B \xrightarrow{\pi_B} B$. It turns out that the corresponding $B$-action is exactly the starting $\xi$. But it is not possible in general to associate a point to every action, unless, as in the case of semi-abelian categories, the comparison functor is an equivalence. In fact, this last condition is equivalent, when the category $C$ is pointed, to the categorical definition of semidirect product given by D. Bourn and G. Janelidze in [10]. However, we can always look at split extensions of kernel $X$ as actions on $X$, so that it makes sense, for example, to translate in terms of split extensions the classical notion of faithful action.

**Definition 1.2** ([11] Definition 1.2). An object in $\text{SplExt}(X)$ is said to be **faithful** if any object in $\text{SplExt}(X)$ admits at most one morphism into it.

The term **faithful** is justified by the fact that, if $C$ is the category of groups, the notion of faithful split extension corresponds, via the canonical equivalence between split extensions and actions given by the semidirect product construction, to the classical notion of faithful action.

**Definition 1.3** ([11], Definition 2.1). Let $C$ be a pointed protomodular category. An object in $\text{SplExt}_C(X)$ is said to be **accessible** if it admits a morphism into a faithful one. If, for any $X \in C$, every object in $\text{SplExt}_C(X)$ is accessible, we say that $C$ is an action accessible category.

In particular, if $C$ is action representative in the sense of [2], then it is action accessible. In fact, the property of $C$ to be action representative means that for any $X$ in $C$ the category $\text{SplExt}_C(X)$ has a terminal object, given by the split extension classifier of $X$. The converse is not true: indeed, for example, the category $\text{Rg}$ of rings is action accessible ([11], Proposition 2.2) but not action representative, as shown in [4].

Other examples of action accessible categories can be obtained from the following result:

**Proposition 1.4** ([11] Proposition 2.3). If $C$ is an action accessible homological category and $\mathcal{B}$ is a Birkhoff subcategory of $C$, then $\mathcal{B}$ is also action accessible.

Moreover A. Montoli proved in [18] that every category of interest in the sense of Orzech [19] is action accessible. This is the case of groups, rings, Lie
and Leibniz algebras, Poisson algebras and others, but not, for example, of Jordan algebras. Finally, D. Bourn proved in [6] that all topological models of action accessible varieties are action accessible.

2. Properties of action accessible categories: centralizers

First of all we need some elementary results linking commutators and morphisms of split extensions. The notion of commutator we refer to was introduced by S.A. Huq [16] and later developed by D. Bourn [5] in the context of unital categories. For a complete treatment also refer to [1].

**Definition 2.1.** Let $C$ be a pointed protomodular category. Two morphisms $f$ and $g$ with the same codomain cooperate if there exists a factorization $\varphi$ making the following diagram commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{(1,0)} & X \times Y & \xleftarrow{(0,1)} & Y \\
\downarrow{f} & & \downarrow{\varphi} & & \downarrow{g} \\
& & Z & & \\
\end{array}
$$

When it exists, the morphism $\varphi$ is unique and it is called the cooperator of $f$ and $g$.

**Definition 2.2.** Let $C$ be a pointed protomodular category, and consider two coterminal morphisms $f$ and $g$. We define the commutator quotient $Q = Q(f, g)$ as the colimit (when it exists) of the solid arrows in the diagram below:

$$
\begin{array}{ccc}
X & \xrightarrow{(1,0)} & X \times Y & \xleftarrow{(0,1)} & Y \\
\downarrow{f} & & \downarrow{\varphi} & & \downarrow{g} \\
& & Z & & \\
\end{array}
$$

The kernel of $q$ is denoted by $[X,Y]_Z$ and we will call it the Huq’s commutator of $X$ and $Y$ in $Z$.

Note that when $f$ and $g$ cooperate in the sense of Definition 2.1, then $Q(f, g)$ coincide with $Z$ and $[X,Y]_Z = 0$ and we’ll say that $X$ and $Y$ commute in $Z$.

**Lemma 2.3.** Given a split extension $(B, A, p, s, x)$ of $X$ in a pointed protomodular category and a subobject $J \rightarrow B$, then the following are equivalent:
1. \([X, J]_A = 0\), that is, there exists a cooperator \(X \times J \xrightarrow{\varphi} A\) for the maps \(x\) and \(sj\);

2. \(j\) gives rise to a morphism \((j, \varphi)\) of split extensions:

\[
\begin{array}{c}
X \xrightarrow{(1,0)} X \times J \xrightarrow{\pi_j} J \\
\downarrow \varphi \downarrow j \\
X \xrightarrow{x} A \xleftarrow{p} B
\end{array}
\]

**Proof.** Let \(j \xrightarrow{\cdot} B\) be a subobject of \(B\) such that \([X, J]_A = 0\), that is, there exists a cooperator \(X \times J \xrightarrow{\varphi} A\) for the maps \(x\) and \(sj\).

Consider the following diagram:

\[
\begin{array}{c}
X \xrightarrow{(1,0)} X \times J \xrightarrow{\pi_j} J \\
\downarrow \varphi \downarrow j \\
X \xrightarrow{x} A \xleftarrow{p} B
\end{array}
\]

First observe that \(\varphi(1, 0) = x\) and \(\varphi(0, 1) = sj\), since \(\varphi\) is the cooperator.

Moreover

\[
\begin{cases}
p\varphi(0, 1) = psj = j = j\pi_j(0, 1) \\
p\varphi(1, 0) = px = 0 = j\pi_j(1, 0)
\end{cases}
\]

so \(p\varphi = j\pi_j\), since \((1, 0)\) and \((0, 1)\) are jointly epic.

Vice versa, if \(j\) gives rise to a morphism \((j, \varphi)\) of split extensions, by the commutativity of the diagram of split monomorphisms, \(\varphi\) comes out as a cooperator for the maps \(x\) and \(sj\).

**Lemma 2.4.** For every morphism of split extensions in a pointed protomodular category:

\[
\begin{array}{c}
X \xrightarrow{x} A \xleftarrow{p} B \\
\downarrow f \downarrow g \\
X \xrightarrow{k} C \xleftarrow{q} D
\end{array}
\]

if \(I \xrightarrow{i} D\) is a subobject of \(D\) with \([X, I]_C = 0\), then the pullback \(j \xrightarrow{\cdot} B\) of \(i\) along \(g\) is such that \([X, J]_A = 0\).
Proof. In the diagram since $[X,I]_C = 0$, by Lemma 2.3, the bottom right hand square is a pullback, as well as the rear one. By the universal property of pullbacks, there exists a morphism $\varphi : X \times J \to A$ making the top right hand square a pullback. Again by Lemma 2.3, we can conclude that $[X,J]_A = 0$. □

Lemma 2.5. For every morphism of split extensions in a pointed protomodular category:

$\begin{array}{c}
X \\ \downarrow^x \\
\downarrow^i \\
X \\
\end{array} \rightarrow
\begin{array}{c}
A \\ \downarrow^p \\
\downarrow^f \\
C \\
\end{array} \rightarrow
\begin{array}{c}
B \\ \downarrow^g \\
\downarrow^s \\
D \\
\end{array}$

the kernel of $g$ is a normal subobject of $A$ that commutes with $X$ in $A$.

Proof. Let’s call $\begin{array}{c}
Z \\ \downarrow^z \\
\end{array} B$ the kernel of $g$: it is the pullback of 0 in $C$, and since trivially $[X,0]_C = 0$, by Lemma 2.4, $[X,Z]_A = 0$. Furthermore the map $sz$ is the kernel of $f$ and then $Z$ is normal in $A$. □

From now on in this section, assume that the category $\mathcal{C}$ is action accessible. The following proposition is inspired by the results in [11] and gives a definition of centralizer relative to a split extension.

Proposition 2.6. Given any morphism from a split extension in $\mathcal{C}$ to a faithful one:

$\begin{array}{c}
X \\ \downarrow^x \\
\downarrow^i \\
X \\
\end{array} \rightarrow
\begin{array}{c}
A \\ \downarrow^p \\
\downarrow^f \\
C \\
\end{array} \rightarrow
\begin{array}{c}
B \\ \downarrow^g \\
\downarrow^t \\
D \\
\end{array}$

the kernel of $g$ is the greatest subobject of $B$ commuting with $X$ in $A$. We’ll call this object $Z(X,B)$, the centralizer of $X$ in $B$.  

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Proof. Such a morphism always exists in action accessible categories by definition and, by Lemma 2.5, $Z(X, B)$ is a normal subobject of $A$ commuting with $X$. Conversely, let $J \to B$ be a subobject of $B$ such that $[X, J]_A = 0$, that is, there exists a cooperator $X \times J \to A$ for the maps $x$ and $sj$. Consider the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{(1,0)} & X \times J \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{x} & A \xleftarrow{s} B \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{k} & C \xleftarrow{t} D
\end{array}
\]

By Lemma 2.3, the upper rectangle is a morphism of split extensions, as well as the lower one, so $(f \varphi, gj)$ is a morphism of split extensions by composition. On the other hand, there is another morphism between the same two extensions:

\[
\begin{array}{ccc}
X & \xrightarrow{(1,0)} & X \times J \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{k \pi_x} & C \xleftarrow{t} D
\end{array}
\]

and since the lower split extension is faithful, $(f \varphi, gj) = (k \pi_x, 0)$ and in particular $gj = 0$, so $J$ is contained in the kernel of $g$.

As a corollary, we recover the result of Proposition 5.2 in [11], which gives a construction of the classical centralizer of a normal subobject:

**Corollary 2.7.** For any normal subobject $X \to A$ of an object $A$, let $R[p]$ be the equivalence relation on $A$ associated with $X$ (that is the kernel pair of the quotient $A \xrightarrow{p} A/X$), and consider the following morphism of split extension:

\[
\begin{array}{ccc}
X & \xrightarrow{(0,x)} & R[p] \xleftarrow{r_a} A \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{k} & C \xleftarrow{t} D
\end{array}
\]

where the lower split extension is faithful.

Then the kernel of $g$ is the centralizer $Z(X, A)$ of $X$ in $A$, that is, the largest subobject of $A$ commuting with $X$ in $A$.

Proof. We already know, by Proposition 2.6, that $\ker(g)$ is the largest subobject of $A$ commuting with $X$ in $R[p]$. But note that for any $Y \to A$, $[X, Y]_{R[p]} = 0$.
if and only if $[X,Y]_A = 0$, since if $\eta$ is the cooperator of $(0,x)$ and $s_0y$, then $r_1\eta$ is the cooperator of $x$ and $y$, vice versa if $\varphi$ is the cooperator of $x$ and $y$, then $(0,x)$ and $s_0y$ cooperate by means of the arrow $(y\pi Y, \varphi)$. \qed

**Remark 2.8.** Given any $Y \xrightarrow{y} A$, thanks to the following morphism:

\[
\begin{array}{c}
X \xrightarrow{x'} P \xrightarrow{p'} Y \\
\downarrow \downarrow \downarrow \downarrow \\
X \xrightarrow{(0,x)} R[p] \xrightarrow{s_0} A
\end{array}
\]

where $(*)$ is constructed as a pullback and $x' = \ker(p')$, we can construct the centralizer $Z(X,Y)$ as the intersection $Z(X,Y) = Z(X,A) \wedge Y$. As a special case we recover the centralizer of Proposition 2.6 as $Z(X,B) = Z(X,A) \wedge B$.

The result in Proposition 2.6 is independent from the chosen faithful extension and, if moreover the category $C$ is regular, it leads to the following construction.

**Corollary 2.9.** Every split extension in a homological action accessible category $C$ admits a morphism onto a canonical faithful extension:

\[
\begin{array}{c}
X \xrightarrow{x} A \xleftarrow{p} B \\
\downarrow \downarrow \downarrow \downarrow \\
X \xrightarrow{\tau} T_1 \xleftarrow{\tau_1} T_0
\end{array}
\]

with the property that any other morphism $(f,g)$ from $X \xrightarrow{x} A \xleftarrow{p} B$ to a faithful extension factors through $(\tau_1, \tau_0)$, which is its regular epi part.

**Proof.** Since $C$ is action accessible, the above extension always admits a morphism to a faithful one:

\[
\begin{array}{c}
X \xrightarrow{x} A \xleftarrow{p} B \\
\downarrow \downarrow \downarrow \downarrow \\
X \xrightarrow{f} C \xleftarrow{g} D
\end{array}
\]

by Proposition 2.6, the kernel $Z(X,B) \xrightarrow{z} B$ of $g$ is independent from the chosen faithful extension and $sz$ is a normal subobject of $A$. Let us take $(\tau_1, \tau_0) = (\coker(sz), \coker(z))$. Then, by the property of cokernels, a split
extension \((p, \pi)\) is induced:

\[
\begin{array}{c}
X \xrightarrow{x} A \xrightarrow{s} B \\
\downarrow \tau_1 \quad \downarrow (\ast) \quad \downarrow \tau_0 \\
X \xrightarrow{\pi} T_1 \xrightarrow{\pi} T_0
\end{array}
\]

since \(\ker(\tau_1) = \ker(\tau_0) = Z(X, B)\), by protomodularity \((\ast)\) is a pullback and \(\ker(p) = X\). Now, given any other morphism to another faithful extension:

\[
\begin{array}{c}
X \xrightarrow{x} A \xrightarrow{s} B \\
\downarrow f' \quad \downarrow g' \\
X \xrightarrow{k'} C' \xrightarrow{q'} D'
\end{array}
\]

since \(\ker(f') = \ker(g') = Z(X, B)\) and \((\tau_1, \tau_0) = (\coker(sz), \coker(z))\), then \((f', g')\) factor through \((\tau_1, \tau_0)\):

\[
\begin{array}{c}
X \xrightarrow{x} A \xrightarrow{s} B \\
\downarrow \tau_1 \quad \downarrow \tau_0 \\
X \xrightarrow{\pi} T_1 \xrightarrow{\pi} T_0 \\
\downarrow m_1 \quad \downarrow m_0 \\
X \xrightarrow{k'} C' \xrightarrow{q'} D'
\end{array}
\]

but, \(\tau_0\) and \(g'\) having the same kernel, the following diagram is a pullback by protomodularity:

\[
\begin{array}{c}
B \xrightarrow{\tau_0} B \\
\downarrow g' \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\]

then \(m_0\) is a monomorphism because pullbacks reflect monomorphisms in protomodular categories, and a similar argument holds for \(m_1\).

So \((m_1, m_0)\) is a monomorphism into a faithful extension and this implies that \(X \xrightarrow{\pi} T_1 \xrightarrow{\pi} T_0\) is faithful (see [11]).

The existence of centralizers and their normality yields a lot of useful properties.
Proposition 2.10. Let $C$ be a homological category with normal centralizers. Given a split extension $X \xrightarrow{x} A \xrightarrow{p} B$, and a normal subobject $K \xrightarrow{k} B$ with $[X,K]_A = 0$, then $K$ is a normal subobject of $A$.

Proof. By the property of the centralizer, $[X,K]_A = 0$ implies that $K$ is contained in $Z(X,B)$, so the map $\tau_0$ of Corollary 2.9 factors through the quotient $B/K$. Now, consider the following diagram:

$$
\begin{array}{c}
\begin{array}{c}
X \\
\downarrow x \hspace{1cm} \downarrow x' \hspace{1cm} \downarrow x' \hspace{1cm} \downarrow x \\
\end{array} \\
\begin{array}{c}
A \\
\downarrow a \hspace{1cm} \downarrow a' \\
\end{array} \\
\begin{array}{c}
A/K \\
\downarrow b \\
\end{array} \\
\begin{array}{c}
B \\
\downarrow p \\
\end{array} \\
\begin{array}{c}
P \\
\downarrow p' \\
\end{array} \\
\begin{array}{c}
B/K \\
\downarrow \tau_0 \\
\end{array} \\
\begin{array}{c}
T_1 \\
\downarrow \tau_1 \\
\end{array} \\
\begin{array}{c}
T_0 \\
\downarrow \tau_0 \\
\end{array}
\end{array}
$$

where $b'b = \tau_0$, (2) is constructed as a pullback of $p$ and $b'$ (so that $\ker(p') = X$) and $a$ is the unique arrow such that $a'a = \tau_1$. Since the rectangle (1)+(2) is a pullback, then (1) is a pullback as well, so $\ker(a) = \ker(b) = K$, $P = A/K$ and $K$ is normal in $A$.

Finally, on the other hand, the failure of the property of having normal centralizers provide us a criterion to prove that a given category is not action accessible. For example, the category of Jordan algebras is not action accessible, since G. Orzech proved in [19] that in this category set-theoretic centralizers are not even subobjects in general. Here follow other examples of categories which are very close to the action accessible category of rings, but which are not action accessible.

Example 2.11. Consider the category $C$ whose objects are abelian groups with an additional binary distributive (not necessarily associative) operation $\ast$, and morphisms are group homomorphisms preserving $\ast$. Here normal subobjects are ideals in the sense of Higgins (see [15]), i.e. normal subgroups closed under multiplication $\ast$ by every element. Let $A$ be the object in $C$ given by the group $A = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}t$, endowed with a distributive product whose multiplication table is the following:

$$
\begin{array}{c|ccc}
\ast & x & y & t \\
\hline
x & 0 & y & \\
y & 0 & x & \\
t & y & x & t \\
\end{array}
$$

The subobject $K = \mathbb{Z}x + \mathbb{Z}y$ generated by $x$ and $y$ is a normal subobject of $A$, whereas the centralizer $Z(K,A) = \mathbb{Z}y$ is not a normal subobject (since $y\ast t = x$). This implies that the category $C$ is not action accessible.

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Example 2.12. The second example is given by the category \( \mathcal{C} \) whose objects are groups with an additional binary associative and distributive operation \(*\), and morphisms are group homomorphisms preserving \(*\). Again, normal subobjects are ideals. Consider in \( \mathcal{C} \) the additive group:

\[ B = \langle x, y, t \rangle \quad \text{with} \quad x + y = y + x, \ y + t = t + y \]

endowed with associative and distributive multiplication:

\[
\begin{array}{ccc}
  x & y & t \\
  x & 0 & 0 \\
  y & 0 & y \\
  t & 0 & t \\
\end{array}
\]

If \( K \) is the ideal of \( B \) generated by \( x \), then the centralizer \( Z(K, B) = \langle y \rangle \) is not a normal subobject of \( B \), since \( y \ast t = t \notin Z(K, B) \). So \( \mathcal{C} \) is not action accessible.

3. A non symmetric version of commutativity of relations

Throughout this section let \( \mathcal{C} \) be a regular homological category. Since in this section we deal with relations, we state here the following lemma, which will be useful later:

Lemma 3.1. Every pair of parallel morphisms in \( \text{SplExt}(X) \):

\[
\begin{array}{c}
X \\ _k \downarrow \quad \downarrow f_0 \\
\quad C \\ _q \downarrow \quad \downarrow g_0 \\
D \\
\quad B \\
\end{array}
\]

factors through a couple of jointly monic pairs (i.e. relations on \( A \) and \( B \) respectively).

Proof. First of all recall that, by protomodularity, the right hand squares with parallel arrows of the same index are pullbacks (\( q \) and \( p \) having the same kernel \( X \)). Now take the (regular epi, mono) factorizations of the pair \( (\langle f_0, f_1 \rangle, \langle g_0, g_1 \rangle) \):
We are going to prove that the commutative square (\( \ast \)) above is a pullback. As already observed, the right hand squares in diagram 4 are pullbacks, so that \( f_0 \) and \( g_0 \) have the same kernel, and the same holds for \( f_1 \) and \( g_1 \). Moreover, for any two morphisms \( u \) and \( v \), \( \ker \langle u, v \rangle \cong \ker u \land \ker v \), thus \( \ker(f_0, f_1) \cong \ker f_0 \land \ker f_1 \cong \ker g_0 \land \ker g_1 \cong \ker(g_0, g_1) \). But by construction \( f = \coker(\ker(f_0, f_1)) \) and \( g = \coker(\ker(g_0, g_1)) \), so they are two regular epimorphisms with isomorphic kernels, and this implies, by protomodularity, that the square (\( \ast \)) is a pullback.

As a consequence, \( \ker q' = X \) and diagram 4 factorizes as follows:

\[
\begin{array}{ccc}
X & \overset{k}{\longrightarrow} & C \\
\downarrow & & \downarrow \\
X & \overset{k'}{\longrightarrow} & C' \\
\downarrow & & \downarrow \\
X & \overset{x}{\longrightarrow} & A
\end{array}
\]

\[
\begin{array}{ccc}
C & \overset{q}{\longrightarrow} & D \\
\downarrow & & \downarrow \\
C' & \overset{q'}{\longrightarrow} & D' \\
\downarrow & & \downarrow \\
A & \overset{m}{{\pi}_i m} & B
\end{array}
\]

where \( m_i = \pi_i m, n_i = \pi_i n \) (\( i = 0, 1 \)), and this completes the proof, since by construction \( (m_0, m_1, (n_0, n_1)) \) is a couple of jointly monic pairs.

We recall here the definition of commutativity of equivalence relations in the sense of Smith (see for example [7] or [13]):

**Definition 3.2.** Two equivalence relations \( R \) and \( S \) on \( A \) commute, and we’ll write \( [R, S]_A = 0 \), if and only if there exists a double centralizing relation between them, that is an equivalence relation \( C \) on both \( R \) and \( S \) such that, in the diagram below, the four squares where parallel arrows have the same index are pullbacks:

\[
\begin{array}{ccc}
C & \overset{p_0}{\longrightarrow} & S \\
\downarrow & & \downarrow \\
R & \overset{r_0}{\longrightarrow} & A
\end{array}
\]

\[
\begin{array}{ccc}
C & \overset{p_1}{\longrightarrow} & S \\
\downarrow & & \downarrow \\
R & \overset{r_1}{\longrightarrow} & A
\end{array}
\]

We can extend this definition to the non symmetric case, where only one of the relations is requested to be an equivalence.

**Definition 3.3.** Let \( S \) be a relation on \( A \) and \( R \) an equivalence relation on \( A \). We’ll say that \( S \) commutes with \( R \), and we’ll write \( [S, R]_A = 0 \), if and only if there exists \( (C, p_0, p_1, s_0) \) equivalence relation on \( S \) with \( (C, d_0, d_1) \) relation on \( R \) such that
1. In the diagram below, the four squares where parallel arrows have the same index are pullbacks:

```
\[ \begin{array}{c}
C \xrightarrow{p_0} S \\
\downarrow d_1 \quad \downarrow p_1 \\
R \xrightarrow{r_0} A
\end{array} \]
```

2. If \( k : X \to C \) is a kernel of \( p_0 \) (or \( p_1 \) equivalently), then \( d_0 k = d_1 k \).

Notice that the fact that \( C \) is a relation on \( R \) comes for free because \( d_0 \) and \( d_1 \) are jointly monic, since they are pullbacks of \( v_0 \) and \( v_1 \) respectively.

As easy consequences of Definition 3.3, we obtain:

**Lemma 3.4.** If \( \lvert S, R \rvert_A = 0 \), in the following diagram

```
\[ \begin{array}{c}
X \xrightarrow{k} C \xleftarrow{p_0} S \\
\downarrow d_1 \quad \downarrow p_1 \\
X \xrightarrow{x} R \xleftarrow{s_0} A
\end{array} \]
```

\( x = d_0 k = d_1 k \) is a kernel of \( r_0 \) and both \((d_0, v_0)\) and \((d_1, v_1)\) are morphisms of split extensions.

**Proof.** The left hand squares commute by definition. Moreover, in the diagram below:

```
\[ \begin{array}{c}
S \xrightarrow{t} C \xleftarrow{p_0} S \\
\downarrow v_0 \quad \downarrow v_1 \\
A \xrightarrow{s_0} R \xleftarrow{r_0} A
\end{array} \]
```

the whole rectangle and the right hand squares commute, so the left hand square also commute since \( r_0 \) and \( r_1 \) are jointly monic. The same argument holds by replacing \((d_0, v_0)\) with \((d_1, v_1)\).

**Lemma 3.5.** If \( \lvert S, R \rvert_A = 0 \) and \( h : Y \to C \) is a kernel of \( d_0 \), then \( p_0 h = p_1 h \).

**Proof.** \( y = p_0 h \) is a kernel of \( v_0 \). We claim that \( ty = h \). It is sufficient to check that the two maps are coequalized by \( d_0 \) and \( p_0 \), that are jointly monic. But, by definition, \( p_0 ty = y = p_0 h \), and \( d_0 ty = s_0 v_0 y = 0 = d_0 h \). Then \( p_0 h = p_0 ty = p_1 ty = p_1 h \).

The following proposition explains the link between the classical and this non symmetric version of commutativity in the case of two equivalence relations.
Proposition 3.6. Let $R$ and $S$ be equivalence relations on $A$, then:

$[S,R]_A = 0 \iff [S,R]_A = 0$

Proof. Suppose that $R$ and $S$ commute in the classical sense, then we have an equivalence relation $C$ on both $R$ and $S$:

$$
\begin{tikzcd}
C & S \\
\downarrow{d_1} & \downarrow{d_0} \\
R & A
\end{tikzcd}
$$

where the four commutative squares are pullbacks, and using the reflexivity of $C$ as equivalence relation on $R$, as in Lemma 3.5, the condition 2 in Definition 3.3 comes for free. So the implication $[S,R]_A = 0 \Rightarrow [S,R]_A = 0$ is proved.

Viceversa, suppose that $R$ and $S$ commute in the sense of Definition 3.3:

$$
\begin{tikzcd}
X & C & S \\
\downarrow{k} & \downarrow{c} \\
X & R & A
\end{tikzcd}
$$

To conclude the proof we have to show that $C$ is an equivalence relation on $R$, and in order to do this, since the category is Mal’cev, it suffices to exhibit a common section for $d_0$ and $d_1$.

Now consider the pullback given by the four arrows of index 0. Since $v_0ru_0 = r_0$, then there exists a unique arrow $c: R \to C$ such that:

$$
\begin{cases}
p_0c = ur_0 \\
d_0c = 1_R
\end{cases}
$$

Moreover

$$
\begin{cases}
p_0cs_0 = ur_0s_0 = u = p_0tu \\
d_0cs_0 = s_0 = s_0v_0u = d_0tu
\end{cases} \iff cs_0 = tu
$$

because $p_0$ and $d_0$ are jointly monic.

And finally

$$
\begin{cases}
p_0cx = u = 0 = p_0k \\
d_0cx = x = d_0k
\end{cases} \iff cx = k
$$

Now recall that the category is protomodular, so $x$ and $s$ are jointly epic and

$$
\begin{cases}
d_1cs_0 = d_1tu = s_0v_1u = s_0 \\
d_1cx = d_1k = d_0k = d_0cx = x
\end{cases} \Rightarrow d_1c = 1_R
$$

Therefore $c$ is a common section for $d_0$ and $d_1$, so that $C$ is an equivalence relation on $R$ and this proves the implication $[S,R]_A = 0 \Rightarrow [S,R]_A = 0$. \(\square\)
Proposition 3.7. If $[S, R]_A = 0$, the subobjects $r_1 x : X \to A$ and $v_1 y : Y \to A$ commute in $A$, that is $[X, Y]_A = 0$.

Proof. Consider the following commutative diagram:

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\pi_y} & Y \\
\Downarrow_{\psi} & \searrow_{h} \\
X & \to & C \\
\Downarrow_{t_0} & \searrow_{t_0} \\
X & \to & R \\
\Downarrow_{x} & \searrow_{v_0} \\
& \to & A
\end{array}
$$

By hypothesis the right lower square is a pullback, and since $v_0 y \pi_Y = 0 = r_0 x \pi_X$ there exists a unique $\psi : X \times Y \to C$ such that:

$$
\begin{cases}
  p_0 \psi = y \pi_Y = p_0 h \pi_Y \\
  t_0 \psi = x \pi_X = t_0 k \pi_X
\end{cases}
$$

Now observe that $\psi$ is the cooper of $h$ and $k$ in $C$, since $\psi(1, 0) = k$ (and similarly $\psi(0, 1) = h$) because:

$$
\begin{cases}
  p_0 \psi(1, 0) = p_0 h \pi_Y(1, 0) = 0 = p_0 k \\
  t_0 \psi(1, 0) = t_0 k \pi_X(1, 0) = t_0 k
\end{cases}
$$

and $p_0, t_0$ are jointly monic. Now define $\eta = r_1 t_1 = v_1 p_1$, then we have:

$$
\begin{cases}
  \eta \psi(1, 0) = r_1 t_1 \psi(1, 0) = r_1 t_1 k = r_1 x \\
  \eta \psi(0, 1) = v_1 p_1 \psi(0, 1) = v_1 p_1 h = v_1 y
\end{cases}
$$

that is $[X, Y]_A = 0$ and $\eta \psi$ is the needed cooper.

\[\square\]

4. A characterization of action accessibility

Now we are going to find a characterization of faithful split extensions by means of a more general kind of centralizers, based on the previous definition of commutativity. We need then the following definition:

Definition 4.1. A homological category $\mathcal{C}$ has ns-centralizers for equivalence relations ("ns" for "non-symmetric") if any equivalence relation $R$ on $A$ admits an equivalence relation $E_A(R)$ on $A$ such that:

1. $[E_A(R), R]_A = 0$
2. $E_A(R)$ contains any relation $S$ on $A$ with $[S, R]_A = 0$

Given a normal subobject $X$ of $A$, we denote by $Z_A(X)$ the normalization in $A$ of $E_A(R[p])$ (where $p : A \to A/X$). Obviously $Z_A(X)$ commutes with $X$ in $A$ (since $[E_A, R[p]]_A = 0$ implies that the corresponding kernels commute).
In [11] Bourn and Janelidze showed that in homological action accessible categories any equivalence relation $R$ admits a largest equivalence relation commuting with $R$, in the sense of Definition 3.2. Actually, a stronger version of this property holds:

**Proposition 4.2.** Let $C$ be a homological action accessible category. Then $C$ has ns-centralizers for equivalence relations (in the sense of 4.1).

**Proof.** Let $(R, r_0, r_1, s_0)$ be an equivalence relation on $A$ in $C$ and consider the associated split epimorphism $(R, r_0, s_0)$. Given any relation $S$ on $A$ with $]S, R[ A = 0$, by Lemma 3.4, we know that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{k} & C \\
\downarrow & & \downarrow \xrightarrow{p_0} \downarrow \xrightarrow{v_1} \\
X & \xrightarrow{z= R[p]} & A \\
\end{array}
\]

This means that $S$ must be contained in the kernel pair $R[ f_0 ]$ of $f_0$. But in [11] Bourn and Janelidze showed that this kernel pair commutes with $R$, that is $[R[f_0], R] A = 0$, so $R[ f_0 ]$ satisfies the two conditions in Definition 4.1 and is the required ns-centralizer: $R[ f_0 ] = E_A(R)$.

Observe that the normal subobject $Z_A(X)$ associated with $E_A(R)$ actually coincide with $Z(X, A)$, as proved by Bourn and Janelidze (see [11], Proposition 5.2).

Now we are ready to give our characterization of faithful split extensions:

**Proposition 4.3.** Let $C$ be a homological category with ns-centralizers for equivalence relations. Given a split extension $X \xrightarrow{x} A \xrightarrow{p} \xrightarrow{s} B$ , we can define $E_B$ as the pullback of $E_A(R[p])$ along $s \times s$ (and its normalization $Z_B$ as the pullback of $Z_A(X)$ along $s$):

\[
\begin{array}{ccc}
E_B & \xrightarrow{E_A} & E_A \\
\downarrow \xrightarrow{\langle w_0, w_1 \rangle} & & \downarrow \xrightarrow{\langle z_0, z_1 \rangle} \\
B \times B & \xrightarrow{s \times s} & A \times A \\
\end{array}
\]
The following are equivalent:

1. $X \xrightarrow{x} A \xleftarrow{p} B$ is faithful
2. $E_B = \Delta_B$
3. $Z_B = 0$

Proof. If $Z_B \neq 0$ (and then $E_B \neq \Delta_B$), since $Z_B \leq Z_A$, $[X, Z_B]_A = 0$. Then we have already seen by Lemma 2.3 that $Z_B$ gives rise to two (different) morphisms of split extension into $X \xrightarrow{x} A \xleftarrow{p} B$:

$$
\begin{array}{c}
\xymatrix{X \ar[r]^{(1,0)} \ar[d] & X \times Z_B \ar[r]^{\pi x_B} \ar[d]^\varphi & Z_B \ar[d]^j \ar[d] \\
X \ar[r]^{x} & A \ar[r]^{p} & B}
\end{array}
$$

This means that $X \xrightarrow{x} A \xleftarrow{p} B$ is not faithful.

Vice versa, let $E_B = \Delta_B$ and suppose there exist two morphisms of split extension into $X \xrightarrow{x} A \xleftarrow{p} B$:

$$
\begin{array}{c}
\xymatrix{X \ar[r]^{k} \ar[d] & C \ar[r]^{q} \ar[d] & D \\
X \ar[r]^{x} & A \ar[r]^{p} & B}
\end{array}
$$

By Lemma 3.1, we can take $(m_0, m_1), (n_0, n_1)$ jointly monic pairs. This means that $C$ is a relation on $A$. Since both the right hand squares are pullbacks, the corresponding morphisms between the kernel pairs $R[q]$ and $R[p]$ give rise to four pullbacks:

$$
\begin{array}{c}
\xymatrix{R[q] \ar[r]^{p_0} \ar[d]_{t_0} & C \ar[r]^{q} \ar[d]_{r_0} & D \\
R[p] \ar[r]^{p_1} \ar[d]_{t_1} & A \ar[r]^{p} & B}
\end{array}
$$

Since $R[q]$ is an equivalence relation on $C$, in order to show that $[C, R[p]]_A = 0$, we need to verify Condition 2 of Definition 3.3. Consider $(0, k) : X \to R[q]$ as a
kernel of \( p_0 \). \( t_0(0,k) = (0,x) \), because

\[
\begin{align*}
  r_0 t_0(0,k) &= m_0 p_0(0,k) = 0 = r_0(0,x) \\
  r_1 t_0(0,k) &= m_0 p_1(0,k) = m_0 k = x = r_1(0,x)
\end{align*}
\]

Since also \( m_1 k = x \), the same argument shows that \( t_1(0,k) = (0,x) \), so that also Condition 2 is fulfilled and we can conclude that \([C,R[p]]_A = 0 \) and then \( C \leq E_A(R[p]) \). So there exists a monomorphism \( i : C \to E_A \) such that \( z_0 i = m_0 \) and \( z_1 i = m_1 \) and this induces a monomorphism \( j : D \to E_B = \Delta_B \):

Now we are ready to state the main result of the present paper. Before to do this, we recall a useful characterization of commutativity of equivalence relations (Theorem 5.2 in [8], adapted to the case where normal subobjects coincide with kernels):

**Proposition 4.4.** In a pointed exact protomodular category \( C \), let \( R \) and \( S \) be two equivalence relations on an object \( A \) and \( y : Y \to A \) the normal subobject associated to \( S \). Then \([R,S] = 0 \) if and only if \( s_0 y : Y \to R \) is normal.

**Theorem 4.5.** Let \( C \) be a pointed exact protomodular category, so that kernel pairs coincide with equivalence relations. The following are equivalent:

1. \( C \) is action accessible;
2. \( C \) has ns-centralizers for equivalence relations;

**Proof.** We have already seen in Proposition 4.2 that homological action accessible categories have ns-centralizers. Now we are going to prove that if \( C \) has ns-centralizer, then it is action accessible.

Given any split extension \( X \xrightarrow{x} A \xleftarrow{s} B \), we want to find a morphism into a faithful one. If the given one is itself faithful, simply take the identity. If not, by Proposition 4.3, \( E_B \neq \Delta_B \), hence \( E_A \neq \Delta_A \). It suffices to find a faithful split extension for the canonical split extension on \( X \) generated by \( R[p] \), then by means of the following inclusion

\[
\begin{align*}
  X \xrightarrow{x} A &\xleftarrow{s} B \\
  X \xrightarrow{(0,x)} R[p] &\xleftarrow{r_0} A
\end{align*}
\]
our task is concluded.

By hypothesis, $E_A$ commutes with $R[p]$, so there exists a double centralizing equivalence relation $C$:

$$
\begin{array}{c}
C @>{p_0}>> E_A \\
\downarrow d_1 \downarrow d_0 \downarrow \downarrow z_1 \\
R[p] @>{r_0}>> A
\end{array}
$$

Let us consider then the coequalizers $q : A \to B$, $\bar{q} : R[p] \to \overline{E_A}$ of $E_A$ and $C$ respectively. Since $C$ is exact, we can apply Barr-Kock Theorem (see [9]) to the following diagram:

$$
\begin{array}{c}
C @>{p_0}>> E_A \\
\downarrow d_1 \downarrow d_0 \downarrow \downarrow z_1 \\
R[p] @>{s_0}>> A \\
\downarrow \bar{q} \downarrow q \\
\overline{A} @>{\bar{p}}>> \overline{B}
\end{array}
$$

and conclude that the lower square is a pullback. This means that $\ker \bar{p} = X$ and $(\overline{B}, \overline{A}, \bar{p}, \bar{s})$ is a split extension on $X$. We want to show that it is faithful, applying Proposition 4.3.

Consider the following composition of morphisms in $\text{SplExt}(X)$:

$$
\begin{array}{c}
X @>{(0, x)}>> R[p] @>{r_0}>> A \\
\downarrow \pi \downarrow \downarrow \downarrow \bar{q} \\
\overline{X} @>{\bar{p}}>> \overline{A} @>{\bar{s}}>> \overline{B} \\
\downarrow \pi \downarrow \downarrow \downarrow \pi \\
X @>{(0, \pi)}>> R[\pi] @>{\tau_0}>> \overline{A}
\end{array}
$$

Let $E_{\overline{A}}$ be the ns-centralizer of $R[\pi]$, and $z_{\overline{A}} : Z_{\overline{A}} \to \overline{A}$ the associated normal subobject, then by Proposition 4.4 $\tau_0 z_{\overline{A}}$ is normal. Now take the following
composition of pullbacks:

\[
\begin{array}{c}
K \xrightarrow{k} A \\
\downarrow q' \quad \downarrow q \\
Z \xrightarrow{z} B \\
\downarrow s' \quad \downarrow s \\
Z \xrightarrow{z} A
\end{array}
\]

then by composition \(s_0k\) results to be the pullback of \(\tilde{z}_0z\) along \(\tilde{q}(\tilde{z},1)\), and then it is normal in \(R[p]\). So, again by Proposition 4.4, the equivalence relation \(E_K\) associated to \(k : K \to A\) commutes with \(R[p]\) and consequently \(E_K \leq E_A\), since \(E_A\) is the ns-centralizer. But this means that \(qk = 0\) and then \(Z_B = 0\) because \((z_B,q')\) is a ( mono, regular epi)-factorization. Finally, by Proposition 4.3, the split extension

\[
\begin{array}{c}
X \xrightarrow{\pi} A \xleftarrow{\pi} B
\end{array}
\]

is faithful and the proof is concluded.

References


