The Skies in your room

A History of Celestial Mechanics

from Ptolemaic Epicycles to Kolmogorov Tori

BRUNO CORDANI
email: bruno.cordani@unimi.it
tel: 0374-344759

August 24, 2015

Preface

The starry heavens above me and the moral law within me.
— I. Kant Critique of practical reason

Why are you there, Moon, in the sky? Tell me
why you are there, silent Moon
— G. Leopardi Night-time chant of a wandering Asian sheep-herder

Supported, guided, it seemed, by calculations which
were invisible at that hour yet ever present,
the stars cleft the ether in those exact trajectories of theirs.
The comets would be appearing as usual, punctual
to the minute, in sight of whoever was observing them.
They were not messengers of catastrophe;
on the contrary, their appearance at the time foreseen
was a triumph of the human mind's capacity to project itself
and to participate in the sublime routine of the skies.
— G. Tomasi di Lampedusa The Leopard

The night-time view of the welkin has always filled every human being with admiration and astonishment, as shown by Kant's "starry heavens above me" and Leopardi's wandering sheepherder's awe in front of the moon's splendor. Nonetheless, the contemplation of the stars can generate another sort of fascination, more subtly intellectual although not immediately perceptible for everyone, as it requires – or at least it appears to require – some not-so-trivial knowledge of physics and mathematics. This feeling is well perceived by the protagonist of "The Leopard", the Prince of Salina, who, while studying the skies, is continuously amazed at the way every celestial body shows up for the date set by the human rationality with a perfect precision.

Is this fascination inevitably perceivable only for a few people? Are we facing, once again, the old and perhaps excessively disputed (so that it even appears hackneyed by now) problem of the two cultures, the philosophic-humanistic and the scientific? The answer is negative. Too many times do we forget, or even ignore, that the history of science (including the physical and mathematical science) is essentially a history of ideas, concepts and visions of the world, which are not necessarily expressible through abstruse formulas that turn out to be unintelligible to non-experts. Clearly, you can not expect to become an expert of celestial mechanics, who, by definition, has to be capable of quantitatively solving actual and specific problems, without resorting to mathematical formulas. However, those who are going to carefully read this book will not eventually fail to feel the same awe as the wandering sheepherder in front of the fact that it is possible to understand and predict the sky motion in the privacy of their own rooms and with the sole assistance of a pen and some paper.

In the same way as every historical treatise – even though this is a history of ideas rather than events – it will be useful to split the considered period, which encompasses almost two millennia, into three subperiods. Two revolutions, related not only to celestial mechanics but also to the vision of the world, mark the transitions from one subperiod to the other.

i) Rise and fall of the geocentric universe: from Ptolemy (2nd century AD) to Newton (1687)

It is well known that the the geocentric vision of the universe was dominant throughout classical antiquity and the Middle Ages. The entire cosmos was believed to have a spherical, symmetric structure where the earth was located in a privileged position, the centre. However, this was not a scientific concept in the modern sense, because neither were questions about natural laws raised nor were experiments contrived for the purpose of validating (or falsifying) such laws. The world and its phenomenons were considered as direct expressions of the divine will; men could describe them and at most try to decipher their meaning.

Years and years of sky observations with the naked eye and without measuring instruments eventually resulted in the Ptolemaic system (2nd century AD), which prevailed until the 16th century, when it was questioned by Copernicus. Its central idea consists in the assumption that a body's natural and spontaneous motions, which were supposed to reflect the perfect and spherical symmetry of the universe, are the uniform circular motions, i.e. the motions running at constant speed along a circumference. By ingeniously combining these motions, it is possible to provide a description that agrees with the observations. Every planet moves along a circle named epycicle, whose centre, in turn, moves along a main circle named deferent. At the centre of the deferent is the Earth. It is possible to refine the representation by adding further epycicles, whose centre moves along the first epycicles.

Perhaps some vague school memories of planets outlining ellipses around the sun will immediately – and erroneously – make this representation appear bizarre and mistaken. On the contrary, its naturalness can be proved by a careful observation of the motions of the skies. Luckily for us, nowadays we do not need to spend innumerable sleepless nights, like the scholars used to do in the past. A simple computer and an astronomical programme that can be downloaded from the Internet for free – one will be explicitly indicated – is sufficient to realize it, thanks to those few commands we will describe. As a side note, this will be an occasion to retrieve that ancient knowledge about the motion of the sky that was well known to any farmers but has now become unknown to the modern inhabitant of big and polluted cities.

Moreover, the fact that the Ptolemaic system, where the Earth is at the centre of the universe, is not even that wrong will certainly appear surprising to many; therefore, the question will be more clearly explained throughout the book. We will now mention very briefly that the centrality of the sun is not a mere experimental verification to be acknowledged (like, for instance, the roundness of the Earth), but an assertion that makes sense only within an extended physical theory – the Newtonian one – which radically restructures our vision of the world. Whereas the Ptolemaic theory is restricted to a description of the existing situation, the Newtonian theory, which was preceded by Copernicus's, Galileo's and Kepler's fundamental studies, gets to the bottom of the matter. Simply starting from two universal laws, it mathematically demonstrates, among other things, the nature of the planetary motions.

i) The golden age: from Newton (1687) to Poincaré (1890)

The golden age of celestial mechanics starts with the discovery of the two fundamental laws of mechanics and the improvement of the infinitesimal calculus introduced by Leibniz and Newton himself. Now the mathematicians have the exact solution to the two-body problem and, consequently, the exact solution to the problem of any number of planets as long as the little interaction between the planets is ignored and only the sun's attraction is considered. In order to take the latter into account, the mathematicians developed the so-called perturbation method. Note that, for the moment, the exact solution to the problem of three or more bodies is out of reach; in fact, as we will see, it will somehow be so forever. The perturbation method is indeed approximate, but through the so-called series development it seems to make it possible to come as close as you want to the exact solution, although at the price of long and complex calculi.

The method works extraordinarily well and, in the hands of masterminds such as Euler, Lagrange and Laplace, it leads within a century to a mathematical theory that calculates the motion of the solar system bodies with great adherence to reality. A real triumph is celebrated in 1846, when Adams and LeVerrier, working independently from each other, discover the existence of planet Neptun with the sole use of perturbation calculation.

At this point, a rigorous mathematician might point out that no one has ever demonstrated the convergence of the perturbative series development. In other words, nobody can be sure that, as you calculate more and more terms, at some point of the series development you will not distance yourself from the exact solution, instead of getting closer and closer.

Moreover, there is still an unsolved question – the three-body problem. No one has found the exact solution to the motion of three or more bodies subject to the newtonian gravitational attraction. From a practical standpoint, these might appear as marginal problems. These two problems instead, proving to be somehow connected with each other, will kick-start the second conceptual revolution, which will banish celestial mechanics from its heaven of absolute precision.

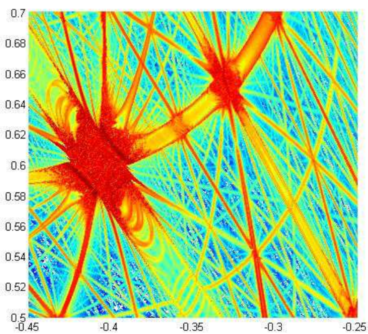
i) The breakthrough of chaos: from Poincaré (1890) to Kolmogorov (1954) until the present day

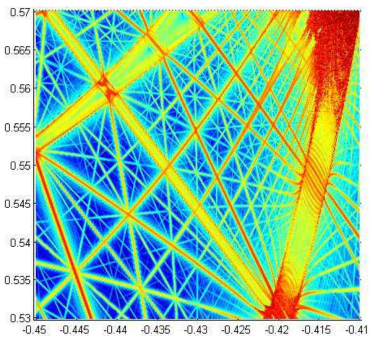
At the beginning of the 20th century, two revolutions shake the notions of physics to their foundations. Einstein's theory of relativity and quantum mechanics demonstrate that time, space, energy and even material reality are profoundly different from the way they appear in everyday life.

A few years before, a similar revolution had happened in the field of celestial mechanics. While studying the three-body problem, Poincaré had come across deterministic chaos for the first time. He succeeded in describing its essence in a few admirable pages, although approximately 70 years will pass before the scientific community can fully comprehend their relevance, thanks to the invention of computers. An immediate consequence of this is that the perturbative series development are probably not convergent.

We are facing a paradoxical situation. On the one hand, two centuries of success of the perturbation method testify to its substantive validity. On the other hand, Poincaré's result can not but deeply undermine our trust in such method. This dilemma will be solved by Kolmogorov, who, in 1954, will enunciate a groundbreaking theorem, along with the draft of its demonstration, which will be later completed by Arnold and Moser. A great deal of this book will be devoted to describing this subtle, deep and fundamental theorem. For the moment, we will only mention that the starting point will be the description of the planetary motion not in the usual physical space but in an abstract space where planets, instead of moving along ellipses, coil around a torus forming a helical shape. By torus, we mean a doughnut or a tire-like shape – such term is clearly borrowed from the architectural jargon.

The result is that some of these tori stay essentially the same, although they end up being slightly deformed by the perturbation's action, whereas others are completely destroyed. In the first case, the series converge; in the second case, they do not. Note that each torus contains infinite physical orbits, all of which share the same fate; that is (simplifying at the cost of some inaccuracies), orderly motions in the first case, chaotic motions in the second case. Here, however, is the biggest surprise: tori that stay the same and tori that get destroyed are inextricably mixed, quite in the same way as rational and irrational numbers. To have an idea of such complexity, consider the two following images (the second is a detail of the first). Each point represents a torus; blue points are tori that stay the same, yellow and red points are tori that get destroyed. Like fractals, these images are self-similar. If you enlarge a small detail, you will obtain a similar image, and so on.





We have finally reached the present day. Celestial mechanics no longer aims to calculate planetary orbits with a higher and higher precision, like it used to do in the 19th century – this problem is now virtually solved for time spans of millions of years. Like in the case of the above-mentioned figures, the purpose is now to determine the distribution of order and chaos within the solar system, which no longer appears as similar to a perfect and eternally immutable clockwork as it was assumed to be in the past. Instead, the solar system appears to be more and more subject to a diffused chaos, although such chaos is well disguised inasmuch as it requires thousands of years to reveal itself. Any hope to dominate its evolution from here to eternity seems to be forever lost.

****

The prerequisites to read this book are not extremely restrictive. A reader who possesses a high school diploma will at least be able to gain a general understanding of its logic. The first two years of a scientific degree will be more than enough for a thorough comprehension.

Bear in mind that technical details have been omitted with the purpose of not making the presentation too heavy going. Readers who wish to deepen their knowledge of the topic can consult the following two monographs by the author himself, where they will also be able to find a comprehensive bibliography:

- *The Kepler Problem – Group Theoretical Aspects, Regularization and Quantization, with Application to the Study of Perturbations*, Birkhäuser-Springer, 2003.
- *Geography of Order and Chaos in Mechanics – Investigations of Quasi-Integrable Systems with Analytical, Numerical, and Graphical Tools*, Birkhäuser-Springer, 2013.

The above-mentioned works are highly specialised treatises aimed at experts in this field. The latter also includes the software that has been used to create all graphs in this book.

CHAPTER 1

Rise and fall of the geocentric universe: from Ptolemy (2nd century AD) to Newton (1687)

1.1 The Earth at the centre of the universe

Life and works of Ptolemy.

1.2 Observing the sky with a computer

How using an electronic planetarium to see the motion of the planets.

1.3 The Ptolemaic model

How reproducing the motion of the planets by composing deferents and epicycles.

1.4 The Sun at the centre of the universe

Life and works of Copernicus.

1.5 Rekindled flame of geocentrism

Life and works of Tycho Brahe.

1.6 Brief excursus on the conics

Definitions and properties of the conics.

1.7 Throw the epicycles, welcome the conics

Life and works of Kepler.

1.8 ”And yet it moves”

Life and works of Galileo.

1.9 ”Let Newton be, and all was light”

Life and works of Newton. A brief excursus on the fundamental laws and ideas of classical mechanics.

1.9.1 Absolute space and time

1.9.2 Vectors, velocity, acceleration

1.9.3 Force, mass, energy

1.9.4 The three laws of mechanics

1.9.5 The gravitation law

CHAPTER 2

The golden age: from Newton (1687) to Poincaré (1890)

2.1 The differential equations

What is a differential equation and methods of integration: analytic, numeric and symplectic.

2.2 The two fixed centres problem

Life and works of Euler. Prelude to the three-body problem: The Euler problem of a body that is actracted gravitationally by two fixed centres and its solutions.

2.3 The inertial forces

The centrifugal and Coriolis forces. The winds in cyclonic and anticyclonic zones. The precession motion of the Earth's axis.

2.4 The circular restricted three-body problem

The problem of an asteroid moving under the gravitational forces of Jupiter and the Sun.

2.4.1 The equilibrium Lagrangian points

The greek and trojan asteroids.

2.4.2 The stability of the Lagrangian points

2.5 Achilles, the tortoise and the convergent series

Why can Achilles reach the tortoise? How to compute the square root of 2 with a convergent series.

2.6 The planetary problem

Life anf works of Lagrange and Laplace,

2.6.1 The perturbation theory

How to study the non-integrable planetary problem with the series development. The convergence problem.

2.6.2 Where is Neptune?

How Adans and Le Verrier discovered Neptune with the perturbation theory.

2.6.3 Climatic changes: whose fault is it?

The frequency analysis of the terrestrial orbit parameters is compared with that of the temperature in the last 800.000 years.

CHAPTER 3

The breakthrough of chaos: from Poincaré (1890) to Kolmogorov (1954) until the present day

3.1 The 30 years that changed the face of physics

3.1.1 The restricted relativity

Life and works of Einstein.

3.1.2 The general relativity

The general relativity and the anomalous precession of the Mercury perihelium.

3.1.3 The quantum mechanics

A crazy theory.

3.2 Poincaré and the three-body problem

3.2.1 A wrong competition?

Life and works of Poincaré. The competition of the king Oscar regarding the solution of the three-body problem: the "wrong" question and the "right" answer of Poincaré.

3.2.2 The chaos and the pendulum

The slightly perturbed pendulum displays the typical chaotic motion around the unstable equilibrium point.

3.2.3 Error and discovery

How firstly Poincaré made a blunder, then discovered the chaos. Resonant and non-resonant tori, the Poincaré map and the homoclinic tangle.

3.2.4 An examplary chaos

The standard map generates the typical chaos.

3.3 A break on numbers, tori and Cantor sets

A brief excursus on some mathematical arguments that will be used in the following.

3.3.1 Rational and irrational numbers

3.3.2 Linear motion on the torus

3.3.3 Cantor sets

3.3.4 Growth and decreasing law

3.4 KAM theory

The section is devoted to the theory of Kolmogorov, Arnold, Moser and Nekhoroshev: what happens to the regular tori foliation when an integrable system is slightly perturbed?

3.4.1 Kolmogorov and perpetual stability

Kolmogorov theorem: only the sufficiently non-resonant tori survive to the perturbation for infinite time. Outline of the proof.

3.4.2 Other proofs

Outline of the proof of Kolmogorov theorem by Arnold and Moser

3.4.3 Nekhoroshev and exponential stabilty

The set of the tori satisfying Kolmogorov theorem has a Cantor structure, which prevents the physical applications. Nekhoroshev theorem and outline of the proof.

3.4.4 Three (non totalitarian) regimes

When the perturbation grows, a mechanical system visits three regimes: KAM (perpetual stability), Nekhoroshev (Arnold web and extremely slow diffusion along the resonances), Chirikov (chaos and rapid diffusion across the resonances).

3.5 Order and chaos in micro and macrocosmos

3.5.1 Chaos detection

Some numerical methods detecting the chaos.

3.5.2 The hydrogen atom in a magnetic and electric field

The Arnold web

3.5.3 The solar system

The Arnold web.

Postface

Some “philosophical” final conclusions.

APPENDIX A

The Smale horseshoe

Mathematical introduction to chaos theory.

APPENDIX B

Action-angle variables

Mathematical introduction.

APPENDIX C

The Kolmogorv theorem

Mathematical introduction.

APPENDIX D

The Nekhoroshev theorem

Mathematical introduction.