Rappresentare un'aperto e una immersione
base in $V$ e una immagine base in $W$
l'aff.: $f: V \rightarrow W \quad V=\mathbb{R}^4 \quad W=\mathbb{R}^3$

$$k\left(\begin{array}{c} x \\ y \\ z \end{array}\right) = \left(\begin{array}{ccc} 1 & 2 & -1 \\ 0 & 4 & 1 \\ 2 & 8 & -1 \end{array}\right) \left(\begin{array}{c} x \\ y \\ z \end{array}\right)$$

in modo che la sua rappresentazione
via del tipo

$$\left(\begin{array}{c} 1 \quad 0 \\ 0 \quad 0 \end{array}\right)$$

ove $0 < r \leq \min(3,4)$

1) Cerco $\ker f$: 

$$\left(\begin{array}{ccc} 1 & 2 & -1 \\ 0 & 4 & 1 \\ 2 & 8 & -1 \end{array}\right) \left(\begin{array}{c} x \\ y \\ z \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right)$$

R**Gauss**

$$\left(\begin{array}{ccc} 1 & 2 & -1 \\ 0 & 4 & 1 \\ 2 & 8 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc} 1 & 2 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & -2 \end{array}\right)$$

$$\left\{ \begin{array}{l} x + 2y - z = 0 \\ 4y + 2 = 0 \\ -12w = 0 \end{array} \right\} \Rightarrow \begin{array}{c} x = -6y \\ z = -4y \\ w = 0 \end{array}$$

$\ker f = \left< \begin{pmatrix} 6 \\ -1 \\ 0 \end{pmatrix} \right>$

$\Rightarrow$ dimensione $\ker f = 4 - 1 = 3$

$\ker f$ non iniettiva
2) Completo la base di $\mathbb{R}^4$:
\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}
\]
sono indipendenti.

\[
\text{det} \begin{pmatrix} 6 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 6 \neq 0 \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2, v_3, v_4
\]
sono indipendenti.

3) Calcolo i vettori immagine delle base

f(v_1) = 0 \quad \text{(FATE LA VERIFICA)} \quad \text{poiché} \quad x_1 = 0

f(v_2) = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad f(v_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f(v_4) = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}

Sono u. d.? Si
sono una base di $\mathbb{R}^3$!

w_1 = f(v_2), \quad w_2 = f(v_3), \quad w_3 = f(v_4)

4) Rappresenta f : V \rightarrow W rispetto alle basi ordinarie (v_1, v_2, v_3, v_4) di $\mathbb{R}^4$ e (w_1, w_2, w_3) di $\mathbb{R}^3$

\[
\begin{pmatrix} f(v_1), f(v_2), f(v_3), f(v_4) \end{pmatrix} = \begin{pmatrix} w_1, w_2, w_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Se lo voglio come richiesto cambio l'ordine delle base in $\mathbb{R}^4$:

\[
\begin{pmatrix} v_2, v_3, v_4, v_1 \end{pmatrix}
\]

\[
\begin{pmatrix} f(v_2), f(v_3), f(v_4), f(v_1) \end{pmatrix} = \begin{pmatrix} w_1, w_2, w_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]
Determine una base di autovettori di \( f \), \( A = A^T \)

\[
\begin{pmatrix}
\ell \frac{x}{y} \\
\end{pmatrix}
\]

Autovettori di \( f \) \( \iff \) autovettori di \( A \)

Cercare \( \exists \neq 0 \) e \( \lambda \in \mathbb{R} : \quad A \mathbf{v} = \lambda \mathbf{v} \)

\( \Rightarrow \) \( A^2 \mathbf{v} = 2 \mathbf{v} \)

\( \Rightarrow \) \( \det (A - \lambda I) = 0 \quad (\text{eq. caratteristica}) \)

\[
\begin{vmatrix}
-\lambda & 0 & 0 & 1 \\
0 & -\lambda & -1 & 0 \\
0 & 1 & -\lambda & 0 \\
1 & 0 & 0 & -\lambda \\
\end{vmatrix}
= \lambda^2 \begin{vmatrix}
-1 & -\lambda & 1 \\
1 & -\lambda & 1 \\
1 & -\lambda & 1 \\
\end{vmatrix}
= (-\lambda^2 - 1) [(-1)^2 - 1] = (\lambda^2 - 1)(-\lambda(2-\lambda)) = \\
\lambda (\lambda-1)(\lambda+1)(\lambda-2) = 0 \quad (\text{EQ C.R.})
\]

\( \Leftrightarrow \) \( \lambda = 0 \quad \lambda = 1 \quad \lambda = -1 \quad \lambda = 2 \)

4 autovettori distinti

\( \Rightarrow \) \( A \) è diagonalizzabile poiché

\( m.a. = m.q. = 1 \) e ci sono 4 di cui \( R^4 \) autovalori \( \Rightarrow \) 4 autozuppi indipendenti.
\[ A - \lambda I = 
\begin{pmatrix}
-\lambda & 0 & 0 & 1 \\
0 & -\lambda & 0 & 0 \\
0 & 0 & -\lambda & 0 \\
1 & 0 & 0 & -\lambda
\end{pmatrix} \]

\[ \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = 2 \]

\[ \lambda_1 = 0 \]
\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[ \begin{array}{c}
\mathbf{w} = 0 \\
y - z = 0 \\
y + z = 0 \\
x = 0
\end{array} \]

\[ \begin{array}{c}
v \in \mathbb{R}^4 \\
k \in \mathbb{R}
\end{array} \]

\[ \mathbf{v} = \begin{pmatrix}
k \\
0 \\
0 \\
k
\end{pmatrix}, \mathbf{v} \in \mathbb{R}^4 \]

\[ \text{autovector, relativo a} \lambda_1 \text{ sono i } \mathbf{v} \in \mathbb{V}_1 \text{ t.c. } k \neq 0 \]

\[ \lambda_2 = 1 \]
\[
\begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[ \begin{array}{c}
-x + w = 0 \\
z = 0 \\
y = 0 \\
x - w = 0
\end{array} \]

\[ \mathbf{v} \in \mathbb{R}^4 \]

\[ \begin{array}{c}
k \neq 0
\end{array} \]

\[ \text{autovector, relativo a} \lambda_2 \text{ sono i } \mathbf{v} \in \mathbb{V}_2 \text{ t.c. } k \neq 0 \]

\[ \lambda_3 = -1 \]
\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 2 & -1 & 0 \\
0 & -1 & 2 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[ \begin{array}{c}
x + w = 0 \\
2y - z = 0 \\
y + 2z = 0 \\
x + w = 0
\end{array} \]

\[ \begin{array}{c}
k \neq 0
\end{array} \]

\[ \mathbf{v} \in \mathbb{R}^4 \]

\[ \text{autovector, relativo a} \lambda_3 \text{ sono i } \mathbf{v} \in \mathbb{V}_3 \text{ t.c. } k \neq 0 \]

\[ \lambda_4 = 2 \]
\[
\begin{pmatrix}
-2 & 0 & 0 & 1 \\
0 & -1 & -1 & 0 \\
0 & -1 & -1 & 0 \\
1 & 0 & 0 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[ \begin{array}{c}
-2x + w = 0 \\
y - z = 0 \\
y - 2z = 0 \\
x - 2w = 0
\end{array} \]

\[ \mathbf{v} \in \mathbb{R}^4 \]

\[ \begin{array}{c}
m \neq 0
\end{array} \]

\[ \mathbf{v} \in \mathbb{R}^4 \]

\[ \text{autovector, relativo a} \lambda_4 \text{ sono i } \mathbf{v} \in \mathbb{V}_4 \text{ t.c. } m \neq 0 \]

\[ \text{base di autovettori:} \]
\[ \left\{ \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix} \right\} \]

\[ \begin{pmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\mathbf{v}_3 \\
\mathbf{v}_4
\end{pmatrix} \]

\[ \mathbf{f}(\mathbf{v}_1) = 0 \mathbf{v}_1 \]

\[ \mathbf{f}(\mathbf{v}_2) = 1 \mathbf{v}_2 \]

\[ \mathbf{f}(\mathbf{v}_3) = -1 \mathbf{v}_3 \]

\[ \mathbf{f}(\mathbf{v}_4) = 2 \mathbf{v}_4 \]

\[ (\mathbf{f}(\mathbf{v}_1), \mathbf{f}(\mathbf{v}_2), \mathbf{f}(\mathbf{v}_3), \mathbf{f}(\mathbf{v}_4)) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix} \]