ANTICYCLOTOMIC p-ADIC L-FUNCTIONS AND ICHINO'S FORMULA

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Abstract. We give a new construction of a p-adic L-function $\mathcal{L}(f,\Xi)$, for f a holomorphic newform and Ξ an anticyclotomic family of Hecke characters of $\mathbb{Q}(\sqrt{-d})$. The construction uses Ichino's triple product formula to express the central values of $L(f,\xi,s)$ in terms of Petersson inner products, and then uses results of Hida to interpolate them. The resulting construction is well-suited for studying what happens when f is replaced by a modular form congruent to it modulo p, and has future applications in the case where f is residually reducible.

1. Introduction

Given a classical holomorphic newform f and a Hecke character ξ of an imaginary quadratic field, we can consider the classical Rankin-Selberg L-function $L(f \times \xi, s)$ and in particular study its special values. If we vary the character ξ in a p-adic family Ξ, we obtain a collection of special values which (one hopes) can be assembled into a p-adic analytic function $\mathcal{L}_p(f,\Xi)$. This paper gives a new construction of a certain type of *anticyclotomic p-adic L-function* of this form. Such p-adic L-functions have been studied fruitfully in recent years: their special values have been related to algebraic cycles on certain varieties ([BDP13], [Bro15], [LZZ15]), and a Main Conjecture of Iwasawa theory was proven for them ([Wan15]). In the case where f is a weight-2 newform corresponding to an elliptic curve E, these results have been used by Skinner and others to obtain progress towards the Birch and Swinnerton-Dyer conjecture for curves that have algebraic rank 1 ([Ski14], [JSW17]).

Even though p-adic L-functions are characterized uniquely by an interpolation property for their special values, merely knowing their *existence* is not enough to be able to prove many theorems involving them. Instead one usually needs to work with the underlying formulas and methods used to construct them. Prior papers (e.g. [BDP13], [Bro15], [Hsi14], [LZZ15]) have usually realized $\mathcal{L}_p(f,\Xi)$ by using formulas of Waldspurger that realize special values of $L(f \times \xi, s)$ as toric integrals. The purpose of this paper is to give a construction instead using a triple-product formula due to Ichino [Ich08].

Having a different construction for $\mathcal{L}_p(f,\Xi)$ will lead to new results about this p-adic L-function. In particular, in future work we will study the case where f is congruent to an Eisenstein series E modulo p, and show that we get a congruence between $\mathcal{L}_p(f,\Xi)$ and a product of simpler p-adic L-functions (arising just from Hecke characters). Having this congruence information will allow us to obtain Diophantine consequences for certain families of elliptic curves. It will also allow us to work with Iwasawa theory in the "residually reducible" case, providing complementary results to the work of [Wan15] in the residually irreducible case.

Notations and conventions. Before stating our results more precisely, we start with some basic conventions for this paper. Throughout, we will be fixing a prime number $p > 2$ and then studying p-adic families (of modular forms, L-values, etc.). Thus we will fix once and for all embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

We will also be working with an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, which again we will fix subject to some hypotheses specified later. We will treat K as being a subfield of $\overline{\mathbb{Q}}$, and thus having distinguished embeddings into $\mathbb{C}$ and into $\overline{\mathbb{Q}}_p$ via ι_∞ and ι_p, respectively. The embedding $\iota_p : K \hookrightarrow \overline{\mathbb{Q}}_p$ is a place of K, and thus corresponds to a prime ideal $\mathfrak{p}$ lying over p. We will always be working in the situation where p splits in K; thus we can always take $\mathfrak{p}$ to denote the prime ideal lying over p that corresponds to ι_p, and $\overline{\mathfrak{p}}$ the other one. We let ∞ denote the unique infinite place of K, coming from composing the embedding $\iota_\infty : K \hookrightarrow \mathbb{C}$ with the complex absolute value.Our conventions on Hecke characters for K (in their many guises) are spelled out in Section 2.1.

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We will work heavily with the theory of classical modular forms and newforms, as developed in e.g. [Miy06] or [Shi94]. If χ is a Dirichlet character modulo N we let $M_k(N,\chi) = M_k(\Gamma_0(N),\chi)$ be the $\mathbb{C}$-vector space of modular forms that transform under $\Gamma_0(N)$ with weight k and character χ. We let $S_k(N,\chi)$ denote the subspace of cusp forms; on this space we have the Petersson inner product, which we always take to be normalized by the volume of the corresponding modular curve:

$$\langle f, g\rangle = \frac{1}{\mathrm{vol}(\mathbb{H}\backslash\Gamma_0(N))}\int_{\mathbb{H}\backslash\Gamma_0(N)} f(z)\overline{g(z)}\,\mathrm{Im}(z)^k\frac{dx\ dy}{y^2}.$$

The anticyclotomic p-adic L-function. Given this setup, we will fix a newform $f \in S_k(N)$ with trivial central character. We will also want to fix an "anticyclotomic family" of Hecke characters ξ_m for our imaginary quadratic field K. The precise meaning of this is defined in Section 5.2, but it amounts to starting with a fixed character ξ_a (normalized to have infinity-type $(a+1, -a+k-1)$) and then constructing closely-related characters ξ_m of infinity-type $(m+1, -m+k-1)$ for each integer $m \equiv a \pmod{p-1}$. With these normalizations, the Rankin-Selberg L-function $L(f\times\xi_m^{-1}, s)$ (which we also denote as $L(f,\xi_m^{-1},s)$, thinking of it as "twisting the L-function of f by ξ_m^{-1}") has central point 0.

The anticyclotomic p-adic L-function $\mathcal{L}_p(f,\Xi^{-1})$ that we want to construct is then essentially a p-adic analytic function $L_p : \mathbb{Z}_p \to \mathbb{C}_p$ such that for $m > k$ satisfying $m \equiv a \pmod{p-1}$, the value of L_p at $s = m$ is the central L-value $L(f,\xi_m^{-1},0)$. Of course, this doesn't make sense as-is because $L(f,\xi_m^{-1},0)$ is a (likely transcendental) complex number. So we need to define an "algebraic part" $L_{\mathrm{alg}}(f,\xi_m^{-1},0) \in \overline{\mathbb{Q}} \subseteq \mathbb{C}$, which we move to $\overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_p$ via our embeddings i_∞ and i_p, and then modify to a "p-adic part" $L_p(f,\xi_m^{-1},0)$. The basic algebraicity result is due to Shimura, and the exact choices we make to define these values are specified in Section 2.3

Also, rather than literally take $\mathcal{L}_p(f,\Xi^{-1})$ an analytic function on $\mathbb{Z}_p$, we instead use an algebraic analogue of this: we construct $\mathcal{L}_p(f,\Xi^{-1})$ as an element of a power series ring $\mathcal{I} \cong \mathbb{Z}_p^{\mathrm{ur}}[\![X]\!]$. Certain continuous functions $P_m : \mathcal{I} \to \mathbb{Z}_p^{\mathrm{ur}}$ serve as "evaluation at m"; this is defined in Section 2.2. With all of these definitions made, we can precisely specify what $\mathcal{L}_p(f,\Xi^{-1})$ should be:

Definition 1.0.1. The anticyclotomic p-adic L-function $\mathcal{L}_p(f,\Xi^{-1})$ is the unique element of $\mathcal{I}$ such that, for integers $m > k$ satisfying $m \equiv a \pmod{p-1}$, we have $P_m(\mathcal{L}_p(f,\Xi^{-1})) = L_p(f,\xi_m^{-1},0)$.

Ichino's formula, classically. The definition of L-values does not lend itself to p-adic interpolation. Instead, p-adic L-functions are constructed by relating L-values to something else that is more readily interpolated. This often comes from the theory of automorphic representations, where there are many formulas relating L-values to integrals of automorphic forms. Our approach is to use *Ichino's triple product formula* [Ich08], which relates a certain global integral (for three automorphic representations π_1,π_2,π_3 on GL_2) to a product of local integrals. The constant relating them is the central value of a triple-product L-function $L(\pi_1\times\pi_2\times\pi_3, s)$.

We will apply this by taking π_1 to correspond to the modular form f in question, and letting π_2 and π_3 be representations induced from Hecke characters ψ and φ on K. Translating Ichino's formula into classical language gives an equation of the form

$$|\langle f(z)g_\varphi(z), g_\psi(cz)\rangle|^2 = C\cdot L(f,\varphi\psi^{-1},0)L(f,\psi^{-1}\varphi^{-1}N^{m-k-1},0),$$

where g_φ and g_ψ are the classical CM newforms associated to the Hecke characters φ,ψ. Our goal will be to set up φ,ψ to vary simultaneously in p-adic families, so that one of our two L-values is a constant and the other realizes $L(f,\xi_m^{-1},0)$. The p-adic theory we develop will allow us to interpolate the Petersson inner product on the left, and the formula will tell us that this realizes $\mathcal{L}(f,\Xi^{-1})$ times some constants.

The fact that we use two characters φ and ψ gives us quite a bit of flexibility in our calculations. This flexibility allows us to appeal to theorems in the literature of the form "for all but finitely many Hecke characters, a certain L-value is a unit mod p" because we can avoid the finitely many bad characters. For instance, we can use results like Theorem C of [Hsi14] to arrange the auxiliary L-value $L_{\mathrm{alg}}(f,\varphi\psi^{-1},0)$ is a p-adic unit and thus doesn't interfere with integrality or congruence statements for the rest of our formula.

Obtaining the constant C in Ichino's formula explicitly is carried out in Chapter 3. The main difficulty is that the formula involves local integrals at each bad prime q. Specifically, the integrals are over a product of three matrix coefficients, one for the newvector in each of the local representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ coming from

f, g_φ, and g_ψ. Evaluating these integrals seems to be a hard problem in general, though several cases have been worked out in the literature (for instance in [Woo12], [NPS14], [Hu17]).

By choosing our hypotheses on f, g_φ, and g_ψ carefully, we place ourselves in a situation where most of the local integrals we need are already calculated in the literature. However, we cannot avoid having to compute one new case: when one local representation is spherical and the other two are ramified principal series of conductor 1. We carry out this computation following ideas from [NPS14]; the result (Proposition 3.2.3) may be of interest to others who want to apply Ichino's formula.

We have performed numerical computations as a check of the correctness of all of the local Ichino integral computations we use, as well as the overall form of the explicit formula; this is carried out in our paper [Col18].

The Λ-adic theory. With an explicit version of Ichino's formula established, we next need to establish that we can p-adically interpolate Petersson inner products of the form $\langle f(z)g_\varphi(z), g_\psi(cz)\rangle$. In Chapter 4 we recall the basics of Hida's theory of Λ-adic modular forms in the form we will need to use them. Chapter 5 gives an explicit construction of Λ-adic families of Hecke characters and then of the associated Λ-adic CM forms, allowing us to construct forms $f\mathbf{g}_\Phi$ and $\mathbf{g}_\Psi$ that interpolate the modular forms fg_φ and g_ψ that appear in our version of Ichino's formula when φ, ψ vary in a suitable family.

Chapter 6 constructs an element $\langle f\mathbf{g}_\Phi, \mathbf{g}_\Psi\rangle$ that interpolates the Petersson inner products $\langle fg_\varphi, g_\psi\rangle$. The construction is due to [Hid88], and is based on the fact that if h is a newform and 1_h is the associated projector in the Hecke algebra, then $1_h g$ is $\langle g, h\rangle/\langle h, h\rangle$ times g. Using this idea, we need to make it completely explicit how to start with the complex number $\langle fg_\varphi, g_\psi\rangle$, associate to it an algebraic part $\langle fg_\varphi, g_\psi\rangle_{\mathrm{alg}}$, and finally a p-adic part $\langle fg_\varphi, g_\psi\rangle_p$. This is carried out throughout Chapter 6 and is summarized in Section 6.6.

The most involved part of the calculation is relating $\langle fg_\varphi, g_\psi\rangle_{\mathrm{alg}}$ (which is directly defined in terms of $\langle fg_\varphi, g_\psi\rangle$) to $\langle fg_\varphi, g_\psi\rangle_p$, which arises as a specialization of $\langle f\mathbf{g}_\Phi, \mathbf{g}_\Psi\rangle$. The difficulty here is that $\langle fg_\varphi, g_\psi\rangle_p$ is actually defined as $\langle fg_\varphi^\sharp, g_\psi^\natural\rangle$, where $g_\varphi^\sharp$ and $g_\psi^\natural$ are modifications of g_φ and g_ψ related to the process of p-stabilization. Relating $\langle fg_\varphi^\sharp, g_\psi^\natural\rangle$ to $\langle fg_\varphi, g_\psi\rangle$ is carried out in Section 6.4 and involves some delicate manipulations of Petersson inner products. The result is that the two values differ by a *removed Euler factor* that is expected to appear in the construction of p-adic L-functions.

Our results. We can now state our main theorem. Our hypotheses are that we begin with:

- A holomorphic newform f of some weight k, level $N = N_0 p^{r_0}$, and trivial central character.
- An imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ with odd fundamental discriminant d.
- A Hecke character $\underline{\xi}_a$ of weight $(a-1, -a+k+1)$ for some integer a satisfying $a - 1 \equiv k \pmod{w_K}$, with trivial central character and with conductor (c) for an integer c coprime to dN. (Here $w_K = |\mathcal{O}_K^\times|$ is 6 if $d = 3$ and 2 in all other cases).
- An odd prime p coprime to $2dcN$.

We also have the following auxiliary data we can freely choose:

- A prime $\ell \nmid 2pdcN$ inert in K, and a power ℓ^{c_ℓ} of it.
- A character ν of $(\mathcal{O}_K/\ell^{c_\ell}\mathcal{O}_K)^\times$ that's trivial on $(\mathbb{Z}/\ell^{c_\ell}\mathbb{Z})^\times$ and $\mathcal{O}_K^\times$.

Given this data we can construct:

- An anticyclotomic family of Hecke characters Ξ that goes through $\underline{\xi}_a$ and specializes to the characters ξ_m mentioned earlier. (Lemma 5.2.1)
- Two families Φ, Ψ of Hecke characters giving rise to families of CM newforms $\mathbf{g}_\Phi, \mathbf{g}_\Psi$. These families are such that $\Phi\Psi N^{-(m-k-1)} = \Xi$ and that $\Phi^{-1}\Psi$ is a constant family for a Hecke character η with its local behavior at ℓ corresponding to ν^2. (Lemma 7.1.1)

Our main theorem is then the following construction of $\mathcal{L}(f, \Xi^{-1})$ as an element of $\mathcal{I}^{\mathrm{ur}}$, where $\mathcal{I}$ is a certain extension of Λ (discussed in detail in Sections 2.2, 4.3, and 5.2).

Theorem 1.0.2. *Under the above hypotheses and notation, the element $\mathcal{L}(f, \Xi^{-1})$ is equal to a product*

$$\frac{1}{(*)_{f,\ell} \cdot L_p^*(f, \eta^{-1}, 0)} \cdot \mathcal{C} \cdot \langle f\mathbf{g}_\Phi, \mathbf{g}_\Psi\rangle\langle f^\rho \mathbf{g}_{\Phi^\rho}, \mathbf{g}_{\Psi^\rho}\rangle.$$

Here $\langle f\mathbf{g}_\Phi, \mathbf{g}_\Psi\rangle$ and $\langle f^\rho\mathbf{g}_{\Phi^\rho}, \mathbf{g}_{\Psi^\rho}\rangle$ are the elements of Chapter 6 discussed above, $\mathcal{C} \in \Lambda^\times$ is a unit satisfying

$$P_m(\mathcal{C}) = \eta(\overline{\mathfrak{p}})^r \frac{2^2\ell^{2c_\ell(k+5)}}{N_0^{m+3}},$$

the term $L_p^*(f,\eta^{-1},0)$ is essentially the algebraic part of the L-value $L(f,\eta^{-1},0)$ (see Section 2.3), and

$$(*)_{f,\ell} = \left(\sum_{i=0}^{c_\ell}\left(\frac{\alpha}{\ell^{(k-1)/2}}\right)^{2i-c_\ell} - \frac{1}{\ell}\sum_{i=i}^{c_\ell-1}\left(\frac{\alpha}{\ell^{(k-1)/2}}\right)^{2i-c_\ell}\right).$$

with $\alpha_{f,\ell}$ one of the roots of the Hecke polynomial for f at ℓ.

Strictly speaking, this is only an equality in $\mathcal{I}^{\mathrm{ur}}[1/p]$ (and could potentially be undefined if $L_p^*(f,\eta^{-1},0)$ and $(*)_{f,\ell}$ are zero). However, the point is that both $L_p^*(f,\eta^{-1},0)$ and $(*)_{f,\ell}$ are terms involving our auxiliary choice of data ℓ, c_ℓ, and ν, and by choosing this data carefully we can arrange for them to be p-adic units (or otherwise suitably controlled) in the situations we want to study.

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2. Prior work on anticyclotomic p-adic L-functions

We begin in this section by recalling previous work on the p-adic L-functions we're interested in studying, and describing the precise framework in which we'll study them. Given a classical modular form f and a Hecke character ξ associated to an imaginary quadratic field K, f determines an automorphic representation π_f for $\mathrm{GL}_2/\mathbb{Q}$ and ξ determines an automorphic representation for GL_1/K which can be induced to π_ξ for $\mathrm{GL}_2/\mathbb{Q}$. We are then interested in studying the central value $L(\pi_f \times \pi_\xi, 1/2)$ of the $\mathrm{GL}_2 \times \mathrm{GL}_2$ L-function corresponding to this pair.

Hida [Hid88] constructed p-adic L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_2$ in a great deal of generality. However, the starting point for our work is the more recent paper of Bertolini, Darmon, and Prasanna [BDP13] which gave a different construction of a particular such p-adic L-function (with f fixed and ξ varying "anticyclotomically", in a way we'll make precise shortly). From the formula they used in their construction, they were able to obtain special value formulas connecting their p-adic L-function to certain algebraic cycles.

In this section we'll recall the results of [BDP13] and other related papers. Along the way we'll set up our conventions and normalizations, and mention how they compare to others in the literature we cite.

2.1. **Conventions about Hecke characters.** To start off, we explicitly describe our conventions for Hecke characters (in all of their guises) associated to the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$. If $\mathfrak{m}$ is a nonzero ideal of $\mathcal{O}_K$ and $I^{S(\mathfrak{m})}$ is the group of fractional ideals coprime to $\mathfrak{m}$, a classical Hecke character is a group homomorphism $\varphi : I^{S(\mathfrak{m})} \to \mathbb{C}^\times$ satisfying

$$\varphi(\alpha\mathcal{O}_K) = \varphi_{\mathrm{fin}}(\alpha)\alpha^a\overline{\alpha}^b$$

for all $\alpha \in \mathcal{O}_K$ that are coprime to $\mathfrak{m}$, where $\varphi_{\mathrm{fin}} : (\mathcal{O}_K/\mathfrak{m})^\times \to \mathbb{C}^\times$ is a character (the *finite part* or *finite-type* of φ) and a, b are complex numbers (with the pair (a, b) called the *infinity type*). With this convention, the norm character determined by $N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$ on prime ideals has trivial conductor (i.e. $\mathfrak{m} = \mathcal{O}_K$), trivial finite part, and infinity-type $(1, 1)$. Also, we sometimes view φ as being defined on all fractional ideals of $\mathcal{O}_K$, implicitly setting $\varphi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not coprime to $\mathfrak{m}$. We caution that many places in the literature take the opposite convention of calling $(-a, -b)$ the infinity-type, so one must be careful when comparing different papers. In particular, our convention matches up with that of Bertolini-Darmon-Prasanna [BDP13], but is opposite of Hsieh [Hsi14] and from most of Hida's papers.

Next, we know that (primitive) classical Hecke characters are in bijection with adelic Hecke characters, i.e. continuous homomorphisms $\mathbb{I}_K/K^\times \to \mathbb{C}^\times$. A straightforward way to describe this bijection is by letting $\mathrm{id}(\alpha) = \prod \mathfrak{p}^{v_\mathfrak{p}(\alpha_\mathfrak{p})}$ denote the ideal associated to an idele $\alpha = (\alpha_v)$; then a classical Hecke character $\varphi : I^{S(\mathfrak{m})} \to \mathbb{C}$ corresponds to a continuous character $\varphi_\mathbb{C} : \mathbb{I}_K/K^\times \to \mathbb{C}^\times$ such that $\varphi_\mathbb{C}(\alpha) = \varphi(id(\alpha))$ for every $\alpha \in \mathbb{I}_K^{S(\mathfrak{m}),\infty}$ (the set of ideles that are trivial at infinite places and places in S). Under this correspondence we find that if φ had infinity-type (a,b) then the local factor $\varphi_{\mathbb{C},\infty}$ at the infinite place is given by $\varphi_{\mathbb{C},\infty}(z) = z^{-a}\overline{z}^{-b}$. Thus our convention for the infinity-type for an adelic Hecke character is that it corresponds to the *negatives* of the exponents of z and $\overline{z}$ in the local component at the infinite place. In particular the adelic absolute value $\|\cdot\|_{\mathbb{A}}$ is a character of infinity type $(-1,-1)$, and it corresponds to the inverse of the norm character.

A classical or adelic Hecke character of K is called *algebraic* if its infinity-type (a,b) consists of integers. Algebraic adelic Hecke characters are in bijection with algebraic p-adic Hecke characters; a p-adic Hecke character is a continuous homomorphism $\psi : \mathbb{I}_K/K^\times \to \overline{\mathbb{Q}}_p^\times$, and ψ is algebraic of weights (a,b) if its local factors $\psi_\mathfrak{p}$ and $\psi_{\overline{\mathfrak{p}}}$ on $K_\mathfrak{p}^\times \cong \mathbb{Q}_p^\times$ and $K_{\overline{\mathfrak{p}}}^\times \cong \mathbb{Q}_p^\times$ are given by $\psi_\mathfrak{p}(x) = x^{-a}$ and $\psi_{\overline{\mathfrak{p}}}(x) = x^{-b}$ on some neighborhoods of the identity in these multiplicative groups of local fields. Then, an algebraic adelic Hecke character $\varphi_\mathbb{C}$ of infinity-type (a,b) corresponds to a p-adic Hecke character $\varphi_{\mathbb{Q}_p}$ of weight (a,b) by the formula

$$\varphi_{\mathbb{Q}_p}(\alpha) = (\iota_p \circ \iota_\infty^{-1})\big(\varphi_\mathbb{C}(\alpha)\alpha_\infty^a\overline{\alpha}_\infty^b\big)\alpha_\mathfrak{p}^{-a}\alpha_{\overline{\mathfrak{p}}}^{-b}$$

for any idele $\alpha = (\alpha_v)$. It is straightforward that this defines a continuous character $\mathbb{I}_K \to \overline{\mathbb{Q}}_p^\times$, and it's trivial on $K^\times$ because if $\alpha \in K^\times$ is treated as a principal idele then $\iota_\infty^{-1}(\alpha_\infty) = \iota_p^{-1}(\alpha_\mathfrak{p})$ and $\iota_\infty^{-1}(\overline{\alpha}_\infty) = \iota_p^{-1}(\alpha_{\overline{\mathfrak{p}}})$ via how we set up our embeddings.

So, whenever we have an algebraic Hecke character φ we can consider any of the three types of realizations of it discussed above. We will pass between them fairly freely, only being as explicit as we need to be clear and to make precise computations. In particular we'll often abuse notation and allow φ to denote whichever of the three associated Hecke characters that's most convenient for our purposes at any given time. Also, we'll use the terminology "weights" and "infinity-type" interchangeably for the pair of parameters (a,b).

2.2. **The L-function of Bertolini-Darmon-Prasanna.** To set up the construction of the BDP L-function precisely, let us fix a classical newform $f = \sum a_n q^n \in S_k(N, \chi_f)$, along with our imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ with fundamental discriminant $-d$. We then look at the classical Rankin-Selberg L-function associated to f and the theta function of ξ; at good primes q the Euler factor is

$$L_{(q)}(f,\xi,s)^{-1} = \prod_{i,j=1}^{2}(1 - \alpha_{f,i}(q)\alpha_{\xi,i}(q)q^{-s})$$

where the coefficients α are the roots of the appropriate Hecke polynomials:

$$L_{(q)}(f,s)^{-1} = (1 - a_q q^{-s} + \chi_f(q)q^{k-1}q^{-2s}) = \textstyle\prod_{i=1}^2(1-\alpha_{f,i}(q)q^{-s}),$$
$$L_{(q)}(\xi,s)^{-1} = \textstyle\prod_{\mathfrak{q}|q}(1-\xi(\mathfrak{q})N(\mathfrak{q})^{-s}) = \prod_{i=1}^2(1-\alpha_{\xi,i}(q)q^{-s}).$$

(We use the notation $L_{(q)}$ to denote Euler factors, since L_p is already our notation for p-adic L-functions). At primes where f, ξ, or K has ramification the correct local factor comes from the local representation theory and is more difficult to write down directly from the coefficients of f and the values of ξ; such local factors are worked out case-by-case in [Jac72].

The paper [BDP13] studies the special value $L(f,\xi^{-1},0)$ for certain characters ξ. They study the set $\Sigma_{cc}^{(2)}$ of all "central critical characters" of infinity-type $(j+k,-j)$ for $j \geq k$, and then focus on a subset $\Sigma_{cc}^{(2)}(\mathfrak{N},c)$ of characters with a certain prescribed ramification. For characters ξ in this set they suitably define $L_{\text{alg}}^{BDP}(f,\xi^{-1},0) \in \overline{\mathbb{Q}}$ and then $L_p^{BDP}(f,\xi^{-1}) \in \mathbb{C}_p$ from the resulting L-value, and prove the following interpolation result:

Theorem 2.2.1. *Given the data above (under some hypotheses), the map $\Sigma_{cc}^{(2)}(\mathfrak{N},c) \to \mathbb{C}_p$ defined by $\xi \mapsto L_p^{BDP}(f,\xi^{-1})$ is uniformly continuous (relative to a certain function space topology on the domain) and thus extends to the completion of $\Sigma_{cc}^{(2)}(\mathfrak{N})$ (which includes other functions such as the characters in $\Sigma_{cc}^{(1)}(\mathfrak{N})$).*

We would now like to rephrase the results to view $L_p^{BDP}(f,\xi)$ as an analytic function of the parameter ξ, rather than just a continuous one. Of course, doing this requires putting some sort of analytic structure on the domain $\Sigma_{cc}^{(2)}(\mathfrak{N})$. The correct general framework for formalizing this is rigid analytic geometry.

For our purposes, we can get away working entirely algebraically, by focusing on the formal power series ring $\Lambda = \mathcal{O}_F[\![X]\!]$ (where $\mathcal{O}_F$ is the ring of integers of a finite extension $F/\mathbb{Q}_p$). In rigid geometry, Λ is the coordinate ring of the open unit disc, and the points of the disc correspond to homomorphisms $P : \Lambda \to \mathcal{O}_F$. So our goal will be to construct our p-adic L-function $\mathcal{L}$ as an element of Λ, which is a "function" in the sense that it associates a value $P(\mathcal{L})$ to each point P. For technical reasons, we'll actually need to work with rings $\mathcal{I}$ that are extensions of Λ - so geometrically, with certain covers of the open disk - and then extend our scalars to the maximal unramified extension F^{ur}/F. This will leave us with the ring $\mathcal{I}^{\mathrm{ur}} = \mathcal{O}_{F^{\mathrm{ur}}} \otimes_{\mathcal{O}_F} \mathcal{I}$.

To formulate the interpolation properties used to characterize the p-adic L-function, for each integer m we define a distinguished point $P_m : \Lambda \to \mathcal{O}_F$ by $P_m(X) = (1+p)^m - 1$ (and lift these to points $P_m : \mathcal{I} \to \mathcal{O}_F$, as discussed in Section 4.3, and then further extend linearly to $P_m : \mathcal{I}^{\mathrm{ur}} \to \mathcal{O}_{F^{\mathrm{ur}}}$). We then want our element $\mathcal{L}$ to satisfy $P_m(\mathcal{L}) = L_p^{BDP}(f, \xi_m^{-1})$ for m in a certain arithmetic progression, where ξ_m "varies Λ-adically". We note that specifying values of $\mathcal{L}$ on any infinite set of points is enough to specify it uniquely by the following lemma.

Lemma 2.2.2. *If $A, B \in \mathcal{I}^{\mathrm{ur}}$ are two elements such that $P(A) = P(B)$ for infinitely many points $P \in \mathcal{X}(\mathcal{I}^{\mathrm{ur}}; \mathcal{O}_F^{\mathrm{ur}}) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{I}^{\mathrm{ur}}, \mathcal{O}_F^{\mathrm{ur}})$, then $A = B$.*

Proof. In our case this will follow from the Weierstrass Preparation Theorem because the rings $\mathcal{I}^{\mathrm{ur}}$ we use will be abstractly isomorphic to $\mathcal{O}_F^{\mathrm{ur}}[\![X]\!]$. Alternatively it can be proven by commutative algebra using that $\mathcal{I}^{\mathrm{ur}}$ is Noetherian of dimension 2 and that the points P correspond to prime ideals of height 1. ☐

To formulate the idea of Λ-adically varying the characters ξ_m, we define a Λ-adic family of Hecke characters to be a continuous homomorphism $\Xi : \mathbb{I}_K/K^\times \to \Lambda^\times$ such that $\xi_m = P_m \circ \Xi$ has weight $(m-1, -m+k+1)$; in Section 5.2 we'll construct such families, and see that any $\xi \in \Sigma_{cc}(\mathfrak{N})$ fits into a family Ξ with infinitely many specializations also lying in $\Sigma_{cc}(\mathfrak{N})$. With this formalism, we can reinterpret the BDP p-adic L-function as follows.

Theorem 2.2.3. *Fix a Hecke character $\xi_m \in \Sigma_{cc}(\mathfrak{N}, c)$ of infinity-type $(a-1, -a+k+1)$, and put it into a family $\Xi : \mathbb{I}_K/K^\times \to \mathcal{I}^\times$. Then there exists an element $\mathcal{L}_p^{BDP}(f, \Xi^{-1}) \in \mathcal{I} \otimes \mathcal{O}_F^{\mathrm{ur}}$ that satisfies*

$$P_m(\mathcal{L}_p^{BDP}(f, \Xi^{-1})) = L_p^{BDP}(f, \xi_m^{-1})$$

for all $m \geq k$ satisfying $m \equiv a \pmod{p-1}$.

This does not follow directly from the statement of the theorems in [BDP13], but it can be obtained from working with the proofs. A more general result of this form, under a different set of hypotheses (and extending the situation of an imaginary quadratic field $K/\mathbb{Q}$ to a CM extension of a totally real field K/F) is proven by Hsieh [Hsi14].

2.3. **Our normalizations of L-values.** In this section we describe the constants and other factors used in constructing our p-adic L-functions. We start with a central critical L-value $L(f, \xi^{-1}, 0)$ which is (presumably) a transcendental complex number, and define the corresponding "algebraic value" and then the "p-adic value", which is what will actually be interpolated by the L-function. There is no canonical choice for this (at best everything is only defined up to a p-adic unit anyway), so we choose our precise definitions to be what's most convenient for our formulas. Thus our definitions of $L_{\mathrm{alg}}(f, \xi^{-1}, 0)$ and $L_p(f, \xi^{-1}, 0)$ are not exactly the same as those in other papers, but they are of a very similar form.

So suppose f is a weight k modular form with character χ_f, and let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field of odd fundamental discriminant $-d$. We will consider Hecke characters ξ on K with central character equal to χ_f, of weight $(m-1, k+1-m)$ (matching the convention we'll be using later in the paper). Let c denote the norm of the conductor of ξ, which we assume is coprime to p. We define the *algebraic part* of the L-value $L(f, \xi^{-1}, 0)$ as

$$L_{\mathrm{alg}}(f, \xi^{-1}, 0) = \frac{L(f, \xi^{-1}, 0)}{\Omega_\infty^{4m-2k-4}} w_K (m-2)!(m-k-1)! \frac{c^{2m-k+2}\pi^{2m-k-3}}{\sqrt{d}^{2m-k-3} 2^{2m-k-3}}.$$

Here w_K is the order of the group of units in $\mathcal{O}_K$ (so is 6 if $d = 3$ and 2 otherwise) and Ω_∞ is a real period which we'll define explicitly in Section 6.5. An important case of this is when η is a Hecke character of infinity-type $(k, 0)$, so $m = k+1$ and thus

$$L_{\mathrm{alg}}(f, \eta^{-1}, 0) = \frac{L(f, \eta^{-1}, 0)}{\Omega_\infty^{2k}} w_K (k-1)! \frac{c^{k+4}\pi^{k-1}}{\sqrt{d}^{k-1} 2^{k-1}} \in \overline{\mathbb{Q}}.$$

To p-adically interpolate these values as ξ varies in an anticyclotomic family as described before, we embed the algebraic value $L(f,\xi^{-1},0)$ into $\overline{\mathbb{Q}}_p$ via our embeddings ι_∞ and ι_p, and define a modified *p-adic part* as

$$L_p(f,\xi^{-1},0) = e_p(f,\xi^{-1})\Omega_p^{4m-2k-4} L_{\mathrm{alg}}(f,\xi^{-1},0)$$

where Ω_p is a p-adic period defined in parallel with Ω_∞ in Section 6.5, and $e_p(f,\xi^{-1})$ is an Euler factor at p that modified the L-value so it interpolates p-adically. The specific form of $e_p(f,\eta^{-1})$ depends on the local behavior of f at the prime p; for this paper (where f has trivial central character) it will be one of the following three cases based on how p divides the level N of f and involving the p-th Fourier coefficient a_p of f:

- If $p \nmid N$ then $e_p(f,\xi^{-1}) = (1 - a_p\xi^{-1}(\overline{\mathfrak{p}}) + \xi^{-2}(\overline{\mathfrak{p}})p^{k-1})^2$.
- If $p \| N$ then $e_p(f,\xi^{-1}) = p^{k/2}\xi_m(\overline{\mathfrak{p}})^{-1}(1 - a_p\xi^{-1}(\overline{\mathfrak{p}}))^2$
- If $p^r \| N$ with $r \geq 2$ then $e_p(f,\xi^{-1}) = (p^{k/2}\xi_m(\overline{\mathfrak{p}})^{-1})^r$.

This modified Euler factor is only relevant when ξ is varying in a family and we want to interpolate it; for fixed values (which will occur for a character η of weight $(k,0)$ in our ultimate formula) it's useful to simply define

$$L_p^*(f,\xi^{-1},0) = \Omega_p^{4m-2k-4} L_{\mathrm{alg}}(f,\xi^{-1},0).$$

Since Ω_p is generally taken to be a p-adic unit, L_p^* has the same p-adic valuation as L_{alg}.

3. An explicit version of Ichino's formula

In this chapter we obtain an explicit version of Ichino's triple-product formula [Ich08] for classical holomorphic newforms. Ichino's formula is stated abstractly in terms of automorphic representations; we will use the case where the quaternion algebra is GL_2 and the étale cubic algebra is $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$ over $\mathbb{Q}$. In this case the formula can be written:

Theorem 3.0.1 (Ichino's Formula)**.** *Let π_1,π_2,π_3 be irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{Q})$ with the product of their central characters trivial, If we set $\Pi = \pi_1\otimes\pi_2\otimes\pi_3$, we have an equality of $\mathrm{GL}_2^\times(\mathbb{A}_\mathbb{Q})$-linear functionals $\Pi\otimes\widetilde{\Pi}\to\mathbb{C}$ of the form*

$$\frac{I(\varphi,\widetilde{\varphi})}{\langle\varphi,\widetilde{\varphi}\rangle} = \frac{(6/\pi^2)}{8}\frac{\zeta^*(2)^2 L^*(\pi_1\times\pi_2\times\pi_3,1/2)}{L^*(\mathrm{ad}\,\pi_1,1)L^*(\mathrm{ad}\,\pi_2,1)L^*(\mathrm{ad}\,\pi_3,1)}\prod_v \frac{I_v^*(\varphi_v,\widetilde{\varphi}_v)}{\langle\varphi_v,\widetilde{\varphi}_v\rangle_v}.$$

The notation used in this theorem is as follows. The left hand side involves global integrals on the quotient set $[\mathrm{GL}_2^\times(\mathbb{A})] = \mathbb{A}_\mathbb{Q}^\times\mathrm{GL}_2^\times(\mathbb{Q})\backslash\mathrm{GL}_2^\times(\mathbb{A}_\mathbb{Q})$. In particular, I is the global integration functional given on simple tensors $\varphi = \varphi_1\otimes\varphi_2\otimes\varphi_3$ and $\widetilde{\varphi} = \widetilde{\varphi}_1\otimes\widetilde{\varphi}_2\otimes\widetilde{\varphi}_3$ by

$$I(\varphi,\widetilde{\varphi}) = \left(\int_{[\mathrm{GL}_2^\times(A)]}\varphi_1(g)\varphi_2(g)\varphi_3(g)dx_T\right)\left(\int_{[\mathrm{GL}_2^\times(A)]}\widetilde{\varphi}_1(g)\widetilde{\varphi}_2(g)\widetilde{\varphi}_3(g)dx_T\right).$$

The global pairing $\langle\varphi,\widetilde{\varphi}\rangle$ is given on simple tensors by

$$\langle\varphi,\widetilde{\varphi}\rangle = \prod_{i=1}^3\left(\int_{[\mathrm{GL}_2^\times(A)]}\varphi_i(g)\widetilde{\varphi}_i(g)dx_T\right).$$

All of the L-functions (and the ζ-value) are written as L^* and ζ^* to denote that these are taken to include their Γ-factors at infinity. The triple-product L-function is the one studied by Garrett [Gar87] and Piatetski-Shapiro and Rallis [PSR87], and $L(\mathrm{ad}\,\pi,s)$ is the trace-zero adjoint L-function associated to π (originally constructed by Gelbart and Jacquet in [GJ78], and closely related to symmetric square L-functions).

The right-hand side involves a product of local functionals I_v^*, each of which is a $\mathrm{GL}_2^\times(\mathbb{Q}_v)$-invariant functional on $\Pi_v\otimes\widetilde{\Pi}_v$. To set this up we fix an invariant bilinear local pairing $\langle\ ,\ \rangle_{i,v}$ on $\pi_{v,i}\otimes\widetilde{\pi}_{v,i}$ for each place v and each $i=1,2,3$, and use this to define a pairing $\langle\ ,\ \rangle_v$ on $\Pi_v\otimes\widetilde{\Pi}_v$ determined on simple tensors $\varphi_v = \varphi_{1,v}\otimes\varphi_{2,v}\otimes\varphi_{3,v}$ and $\widetilde{\varphi}_v = \widetilde{\varphi}_{1,v}\otimes\widetilde{\varphi}_{2,v}\otimes\widetilde{\varphi}_{3,v}$ by

$$\langle\varphi_v,\widetilde{\varphi}_v\rangle_v = \langle\varphi_{1,v},\widetilde{\varphi}_{1,v}\rangle_{1,v}\langle\varphi_{2,v},\widetilde{\varphi}_{2,v}\rangle_{2,v}\langle\varphi_{3,v},\widetilde{\varphi}_{3,v}\rangle_{3,v}.$$

We then define a functional I_v on $\Pi_v \otimes \widetilde{\Pi}_v$ by

$$I_v(\varphi_v, \widetilde{\varphi}_v) = \int_{\mathbb{Q}_v^\times \backslash \mathrm{GL}_2(\mathbb{Q}_v)} \langle \pi(g_v)\varphi_v, \widetilde{\varphi}_v \rangle_v dx_v$$

We then normalize this functional with (the reciprocal of) the local factors of the L-functions that show up in the global equation to get I_v^*:

$$I_v^*(\varphi_v, \widetilde{\varphi}_v) = \frac{L_v(\mathrm{ad}\,\pi_1, 1)L_v(\mathrm{ad}\,\pi_2, 1)L_v(\mathrm{ad}\,\pi_3, 1)}{\zeta(2)^2 L_v(\pi_1 \times \pi_2 \times \pi_3, 1/2)} I_v(\varphi_v, \widetilde{\varphi}_v).$$

Finally, the global measure dx_T is taken to be the Tamagawa measure for $\mathrm{PGL}_2(\mathbb{A})$, which has volume 2 (see e.g. Theorem 3.2.1 of [Wei82]). The local Haar measures dx_p on $\mathrm{PGL}_2(\mathbb{Q}_p)$ are chosen so that $\mathrm{PGL}_2(\mathbb{Z}_p)$ has volume 1, and the local Haar measure dx_∞ on $\mathrm{PGL}_2(\mathbb{R})$ is chosen as the quotient of the measure

$$x_\infty = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \qquad dx_\infty = \frac{d\alpha\ d\beta\ d\gamma\ d\delta}{|\det(x_\infty)|^2}.$$

on $\mathrm{GL}_2(\mathbb{R})$ by the usual multiplicative Haar measure $dx_{Lebesgue}/|x|$ on the center $\mathbb{Z}(\mathrm{GL}_2(\mathbb{R})) \cong \mathbb{R}^\times$. With these normalizations we can check $\prod dx_v$ has volume $\pi^2/3$, so $dx_T = (6/\pi^2)\prod dx_v$, hence the factor of $6/\pi^2$ in our formula.

With this setup, Ichino showed that $I_v^*(\varphi_v, \widetilde{\varphi}_v)/\langle \varphi_v, \widetilde{\varphi}_v \rangle_v = 1$ whenever v is a place such that all of the π_v's are unramified, φ_v and $\widetilde{\varphi}_v$ are spherical vectors, and $\mathrm{PGL}_2(\mathcal{O}_v)$ has volume 1. Because of this, the product over all places in Ichino's formula is actually a finite product.

3.1. **Ichino's formula in the classical case.** The situation we want to apply Ichino's formula in is the following: We will fix integers $m > k > 0$ and take classical newforms f, g, h of weights k, $m-k$, and m, respectively. We set notation that N_f and χ_f denote the level and character of f, and similarly for g and h; let $N_{fgh} = \mathrm{lcm}(N_f, N_g, N_h)$. We ultimately will want to use Ichino's formula to relate the triple product L-value a classical Petersson inner product pairing h with a product of f and g.

The naive idea is to work with the Petersson inner product $\langle f(z)g(z), h(z)\rangle$. However, this may not quite work if the levels of the newforms don't match up - if the LCM of N_f and N_g is a proper divisor of N_h, for instance, then certainly $f(z)g(z)$ is old at level N_h and thus $\langle f(z)g(z), h(z)\rangle = 0$. We can fix this issue by replacing the newforms with oldforms of higher level associated to them. We will consider a pairing $\langle f(M_f z)g(M_g z), h(M_h z)\rangle$ where these integers are chosen so that, for each prime q, q divides at most one of M_f, M_g, M_h and the largest power of q to divide any of the products $M_f N_f$, $M_g N_g$, and $M_h N_h$ actually divides two of them.

To apply Ichino's formula, we let π_f be the unitary automorphic representation associated to f. The classical newforms f, g, h correspond to specific vectors in the automorphic representation, namely F given by

$$F(x) = \big((y^{k/2}f)|[x_\infty]_k\big)(i)\widetilde{\chi}_f(k_0),$$

Here we decompose $x = \gamma x_\infty k_0$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $x_\infty \in \mathrm{GL}_2^+(\mathbb{R})$, and $k_0 \in K_0(N_f)$, and we let $\widetilde{\chi}_f$ be the character of $K_0(N_f)$ given by applying χ_f to the lower-right entry.

Of course, since this vector F corresponds to the newform $f(z)$, we need to suitably modify it to get something that will correspond to $f(M_f z)$ instead. To do this we take the notation that if M is an integer we let

$$\delta_v(M) = \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}_v) \qquad \delta(M) = (\delta_v(M)) \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}).$$

Moreover, if v is a finite place and we've fixed a uniformizer ϖ_v of $\mathbb{Q}_v$ we let $\delta_v^0(M) = \delta_v(\varpi^{v(M)})$, and we let $\delta^0(M) \in \mathrm{GL}_2(\mathbb{A}^{\mathrm{fin}})$ have coordinates $\delta_v^0(M)$. Then, the adelic lift of $f_{M_f}(z) = f(M_f z)$ is given by

$$x \mapsto \big((y^{k/2}f_{M_f})|[x_\infty]_k\big)(i)\widetilde{\chi}_f(k_0) = M_f^{-k}\big((y^{k/2}f)|[\delta_\infty(M_f)x_\infty]_k\big)(i)\widetilde{\chi}_f(k_0)$$

for a decomposition $x = \gamma x_\infty k_0 \in \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2^+(\mathbb{R})K_0(M_f N_f)$. A straightforward computation shows that if we take $y = \delta^0(M_f^{-1}) \in \mathrm{GL}_2(\mathbb{A}^{\mathrm{fin}})$, then the vector $F_{M_f} = \pi(y)F \in \pi_f$ is a multiple of the adelic lift of f_{M_f}.

Similarly, we can take shifts of adelic lifts of g and h, and come up with an input vector

$$\varphi = \delta^0(M_f)F \otimes \delta^0(M_g)G \otimes \delta^0(M_h)\overline{H} = F_{M_f} \otimes G_{M_g} \otimes \overline{H}_{M_h}$$

in $\pi_f \otimes \pi_g \otimes \pi_h$. We let $\widetilde{\varphi}$ to be the vector of complex conjugates of these in the contragredients. The automorphic condition for the central characters being trivial is equivalent to asking $\chi_f \chi_g = \chi_h$ as an equality of Dirichlet characters; assuming this Ichino's formula states

$$\frac{I(\varphi, \widetilde{\varphi})}{\langle \varphi, \widetilde{\varphi} \rangle} = \frac{(6/\pi^2)}{8} \frac{\zeta_F^*(2)^2 L^*(\pi_f \times \pi_g \times \widetilde{\pi}_h, 1/2)}{L^*(\operatorname{ad} \pi_f, 1) L^*(\operatorname{ad} \pi_g, 1) L^*(\operatorname{ad} \widetilde{\pi}_h, 1)} \prod_v \frac{I_v^*(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v}.$$

Next, we want to interpret the global integrals $I(\varphi, \widetilde{\varphi})$ and $\langle \varphi, \widetilde{\varphi} \rangle$ in terms of Petersson inner products. In general, if Ψ, Ψ' are adelic lifts of modular forms ψ, ψ' then computing $\int \Psi \Psi' dx_T$ on $\mathbb{A}^\times \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A})$ may be done by passing to a fundamental domain of the form $D_\infty K_0(N)$ for D_∞ a fundamental domain of $\Gamma_0(N) \backslash \mathrm{PGL}_2^+(\mathbb{R})$. This can then be reinterpreted as an integral over $\Gamma_0(N) \backslash \mathbb{H}$; keeping track of all of our normalizations (including that Petersson inner products are normalized by the volume of $\Gamma_0(N) \backslash \mathbb{H}$) we obtain

$$\int_{\mathrm{PGL}_2(\mathbb{Q}) \backslash \mathrm{PGL}_2(\mathbb{A})} \Psi(g) \overline{\Psi}'(x) dx_T = 2 \langle \psi, \psi' \rangle.$$

Since F_{M_f} is M_f^k times the adelic lift of f_{M_f} and likewise similarly for G_{M_g} and H_{M_h}; the left-hand side of Ichino's formula becomes

$$\frac{I(\varphi, \widetilde{\varphi})}{\langle \varphi, \widetilde{\varphi} \rangle} = \frac{2^2 M_f^{2k} M_g^{2(m-k)} M_h^{2m}}{2^3 M_f^{2k} M_g^{2(m-k)} M_h^{2m}} \frac{|\langle f_{M_f} g_{M_g}, h_{M_h} \rangle|^2}{\langle f_{M_f}, f_{M_f} \rangle \langle g_{M_g}, g_{M_g} \rangle \langle h_{M_h}, h_{M_h} \rangle}.$$

We can further see $\langle f_{M_f}, f_{M_f} \rangle = M_f^{-k} \langle f, f \rangle$ by a simple change-of-variables (similar to Lemma 6.2.3), this becomes

$$\frac{|\langle f_{M_f} g_{M_g}, h_{M_h} \rangle|^2}{2 M_f^{-k} M_g^{k-m} M_h^{-m} \langle f, f \rangle \langle g, g \rangle \langle h, h \rangle}.$$

Also, the value of $\zeta^*(2)$ is $\pi^{-2/2} \Gamma(2/2) \zeta(2) = \pi/6$, so at this point we've simplified Ichino's formula to

$$\frac{|\langle f_{M_f} g_{M_g}, h_{M_h} \rangle|^2}{\langle f, f \rangle \langle g, g \rangle \langle h, h \rangle} = \frac{M_f^{-k} M_g^{k-m} M_h^{-m} \cdot L^*(\pi_f \times \pi_g \times \widetilde{\pi}_h, 1/2)}{2^3 \cdot 3 \cdot L^*(\operatorname{ad} \pi_f, 1) L^*(\operatorname{ad} \pi_g, 1) L^*(\operatorname{ad} \widetilde{\pi}_h, 1)} \prod_v \frac{I_v^*(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v}.$$

There are a few more simplifications to make as well. First of all, a formula of Shimura and Hida (see [Shi76], Section 5 of [Hid81], and Section 10 of [Hid86a]) tells us that the Petersson inner product $\langle f, f \rangle$ is equal to $L^*(\operatorname{ad} \pi_f, 1)$ up to an explicit factor, so we can can remove those terms from our formula. Specifically, we can formulate the result as follows, where we define a modified version of the adjoint L-value to absorb some factors at bad places (where we'll deal with them on a prime-by-prime basis later).

Theorem 3.1.1. *Let $\psi \in S_\kappa(N, \chi)$ be a newform, and let N_χ be the conductor of the Dirichlet character χ (which we take to be primitive). Then we have an equality*

$$L^H(\operatorname{ad} \psi, 1) = \frac{\pi^2}{6} \frac{(4\pi)^\kappa}{(\kappa - 1)!} \langle \psi, \psi \rangle.$$

Here, $L^H(\operatorname{ad} \psi, 1)$ is defined by starting from a shift of the "naive" twisted symmetric square L-function:

$$L_q^{\mathrm{naive}}(\operatorname{ad} \psi, s)^{-1} = \left(1 - \frac{\overline{\chi}(q) \alpha_q^2}{q^{\kappa-1}} q^{-s}\right) \left(1 - \frac{\overline{\chi}(q) \alpha_q \beta_q}{q^{\kappa-1}} q^{-s}\right) \left(1 - \frac{\overline{\chi}(q) \beta_q^2}{q^{\kappa-1}} q^{-s}\right).$$

where $L_q(\psi, s)^{-1} = (1 - \alpha_q q^{-s})(1 - \beta_q q^{-s})$, and then setting

$$L^H(\operatorname{ad} \psi, 1) = \begin{cases} L_q^{\mathrm{naive}}(\operatorname{ad} \psi, 1) & q \nmid N \\ (1 - q^{-2})^{-1} (1 + q^{-1})^{-1} & q \| N, q \nmid N_\chi \\ (1 - q^{-2})^{-1} & q | N, q \nmid (N/N_\chi) \\ (1 + q^{-1})^{-1} & \text{otherwise} \end{cases}.$$

We note that $L(\operatorname{ad} \psi, 1)$ equals $L(\operatorname{ad} \pi_\psi, 1)$ without a shift, and also equals $L(\operatorname{ad} \widetilde{\pi}_\psi, 1)$ by self-duality. Also, newforms have discrete series representations at infinity, so the archimedean L-factor is worked out directly to be

$$L_\infty(\operatorname{ad} \pi_\psi, s) = 2(2\pi)^{-(s+\kappa-1)} \Gamma(s + \kappa - 1) \pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right);$$

ultimately we conclude

$$L^*(\mathrm{ad}\,\pi_\psi, 1) = \frac{2^\kappa \pi}{3}\langle f, f\rangle \prod_{q|N} \frac{L_q(\mathrm{ad}\,\psi, 1)}{L_q^H(\mathrm{ad}\,\psi, 1)}.$$

In the context of Ichino's formula we see we can write

$$\frac{1}{L^*(\mathrm{ad}\,\pi_f, 1)L^*(\mathrm{ad}\,\pi_g, 1)L^*(\mathrm{ad}\,\widetilde{\pi}_h, 1)} = \frac{1}{\langle f, f\rangle\langle g, g\rangle\langle h, h\rangle} \frac{3^3}{\pi^3 2^k 2^{m-k} 2^m} \prod_q \mathcal{E}_q$$

where

$$\mathcal{E}_q = \frac{L_q^H(\mathrm{ad}\,f, 1)}{L_q(\mathrm{ad}\,f, 1)} \frac{L_q^H(\mathrm{ad}\,g, 1)}{L_q(\mathrm{ad}\,g, 1)} \frac{L_q^H(\mathrm{ad}\,h, 1)}{L_q(\mathrm{ad}\,h, 1)}.$$

We can also look at the L-factor $L^*(\pi_f \times \pi_g \times \widetilde{\pi}_h, 1/2)$. The archimedean factor can be computed to be

$$L_\infty(1/2, \pi_f \otimes \pi_g \otimes \widetilde{\pi}_h) = 2^4(2\pi)^{-2m}(m-2)!(k-1)!(m-k-1)!,$$

and we can also check that our normalizations are such that the non-complete central L-value $L(\pi_f \times \pi_g \times \widetilde{\pi}_h, 1/2)$ equals $L(f \times g \times \overline{h}, m-1)$ when written classically. Finally, the local integral I_∞^* at the archimedean place is known by results of Ichino-Ikeda ([II10] Proposition 7.2) or Woodbury ([Woo12] Proposition 4.6), and is 2π with our normalizations. So we conclude:

Theorem 3.1.2 (Ichino's formula, classical version)**.** *Fix integers $m > k > 0$, and let $f \in S_k(N_f, \chi_f)$, $g \in S_{m-k}(N_g, \chi_g)$, and $h \in S_m(N_h, \chi_h)$ be classical newforms such that the characters satisfy $\chi_f \chi_g = \chi_h$. Take $N_{fgh} = \mathrm{lcm}(N_f, N_g, N_h)$ and choose positive integers M_f, M_g, M_h such that the three numbers $M_f N_f$, $M_g N_g$, $M_h N_h$ divide N_{fgh} and moreover none of the three is divisible by a larger power of any prime q than both of the others. Then we have*

$$|\langle f_{M_f} g_{M_g}, h_{M_h}\rangle|^2 = \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2}2^{4m-2}M_f^k M_g^{m-k} M_h^m} L(f \times g \times \overline{h}, m-1) \prod_{q|N_{fgh}} \mathcal{E}_q I_q^*,$$

where I_q^* are the Ichino local integrals and $\mathcal{E}_q$ is the term coming from our modified adjoint L-value.

Here we use that I_q^* is known to be 1 at unramified primes, and that $\mathcal{E}_q$ is trivially 1 at such primes as well.

3.2. **Known results on the local integrals.** The difficult part of making Ichino's formula completely explicit is evaluating the local integrals I_q at each ramified prime, which has to be done on a case-by-case basis. Upon decomposing π_f as a product of local representations $\bigotimes' \pi_{f,v}$, a result of Casselman [Cas73] tells us that the vector F corresponds to a simple tensor $\otimes F_v$ where each F_v is a "newvector" in $\pi_{f,v}$. Thus $F_{M_f} = \delta^0(M_f)F$ has local components $\delta_v^0(M_f)F$. Similarly the local components of G_{M_g} and H_{M_h} are newvectors shifted by an appropriate matrix $\delta_v^0(M)$. So I_q^* only depends on the isomorphism types of $\pi_{f,v}$, $\pi_{g,v}$, and $\widetilde{\pi}_{h,v}$, plus perhaps a choice of which newvector to apply a matrix $\delta_v(\varpi_q^m)$ to.

To deal with these integrals abstractly, let π_1, π_2, π_3 be local representations of $G = \mathrm{GL}_2(\mathbb{Q}_q)$, always assumed to have the product of their central characters trivial. We let c_i denote the conductor of π_i and let x_i be a *newvector*: a vector in the one-dimensional invariant subspace for the group

$$K_2(\mathfrak{a}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \in \mathfrak{a}, d \in 1 + \mathfrak{a}, a \in \mathbb{Z}_q^\times, b \in \mathbb{Z}_q \right\}$$

where $\mathfrak{a} = (p^{c_i})$ (so if π_i is unramified then x_i is a spherical vector). We note that we look at newvectors invariant under K_2 rather than K_1 (as in [Cas73]) in accordance with our convention that we extend χ_f to a character $\widetilde{\chi}_f$ of K_0 by applying χ_f to the lower-right entry rather than the upper-left.

We assume without loss of generality that π_3 has the largest conductor, i.e. $c_3 \geq c_1, c_2$. Then we set

$$I(\pi_1, \pi_2, \pi_3) = \int_{Z\backslash G} \frac{\langle gx_1, x_1\rangle}{\langle x_1, x_1\rangle} \frac{\langle gy_2, y_2\rangle}{\langle y_2, y_2\rangle} \frac{\langle gx_3, x_3\rangle}{\langle x_3, x_3\rangle} dg,$$

where y_2 is the translate $\delta_v(\varpi^{c_3 - c_2})x_2$ of our newvector, and we normalize by setting

$$I^*(\pi_1, \pi_2, \pi_3) = \frac{L(\mathrm{ad}\,\pi_1, 1)L(\mathrm{ad}\,\pi_2, 1)L(\mathrm{ad}\,\pi_3, 1)}{L(\pi_1 \times \pi_2 \times \pi_3, 1/2)\zeta_q(2)^2} I(\pi_1, \pi_2, \pi_3).$$

Then every local integral I_q^* from Ichino's formula is of the form $I(\pi_1, \pi_2, \pi_3)$.

The values of the local integrals $I^*(\pi_1, \pi_2, \pi_3)$ are not known in general. Instead they have been computed in various special cases, as needed for various applications of Ichino's formula. We will quote some of these special cases that we need, and then make a computation in one new case, in order to deal with the choices of newforms f, g, h we will need for this paper. We start by stating the following easy lemma, which is useful for simplifying computations (for instance, letting us assume our unramified principal series are of the form $\pi(\chi, \chi^{-1})$).

Lemma 3.2.1. *Suppose π_1, π_2, π_3 are as above and χ_1, χ_2, χ_3 are unramified characters satisfying $\chi_1\chi_2\chi_3 = 1$. Let $\chi_i\pi_i = \chi_i \otimes \pi_i$ be the associated twists. Then we have $I(\chi_1\pi_1, \chi_2\pi_2, \chi_3\pi_3) = I(\pi_1, \pi_2, \pi_3)$ and similarly for I^*.*

The simplest case is when π_1, π_2, π_3 have conductors $c_i \leq 1$ (i.e. all of the original modular forms have squarefree level at q). In the case of trivial central characters (and thus for unramified central characters via the above lemma), this is worked out explicitly by Woodbury [Woo12], and is implicit in the computations of Watson [Wat02]. In particular this covers the case where two of the representations are unramified and the third is special, which we will need.

Another case where $I(\pi_1, \pi_2, \pi_3)$ can be computed in a fairly uniform way is when π_3 has a much larger conductor than π_1 or π_2; this is carried out by Hu [Hu17]. In particular it applies to the case where two of the representations are unramified and the third has conductor at least 2. Including the factor $\mathcal{E}_q$ that appears in our formula, we have the following uniform result.

Corollary 3.2.2 ([Woo12], [Hu17])**.** *Suppose that π_1, π_2 are unramified and π_3 is any ramified representation (necessarily having an unramified central character). Then we have*

$$\mathcal{E}_q I^*(\pi_1, \pi_2, \pi_3) = q^{-c_3}(1 + q^{-1})^{-2}.$$

If two of the three representations are ramified, the formulas become more complicated, and can start to involve factors that more heavily depend on the parameters of the local representations being studied. These have been computed in the literature in some cases; in particular, [NPS14] computes $I^*(\pi_1, \pi_2, \pi_3)$ in the cases where all three representations have trivial central character, π_1 is unramified, and $\pi_2 \cong \pi_3$. We discuss their method in the next section, where we use it to prove one new identity; it applies whenever π_1 is unramified, though it's unclear whether the computations would be tractable for all choices of π_2 and π_3.

For our purposes we only need such computations in two cases. The first one is the case where π_2, π_3 are ramified principal series of conductor 1. These representations have nontrivial central character and this computation does not seem to have been done in the literature; we carry it out in Section 3.3.

Proposition 3.2.3. *Suppose π_1 an unramified principal series, and π_2, π_3 both principal series of conductor 1 (so both are of the form $\pi(\chi_1, \chi_2)$ with χ_1 having conductor 1 and χ_2 unramified, or vice-versa), such that the product of central characters $\omega_1\omega_2\omega_3$ is trivial. Then we have*

$$I^*(\pi_1, \pi_2, \pi_3) = q^{-1} \qquad \mathcal{E}_q = (1 + q^{-1})^{-2}.$$

The second is when π_2, π_3 are both supercuspidal representations, in particular ones of "type 1" in the notation of [NPS14]: these are invariant under twisting by the nontrivial unramified quadratic character of $\mathbb{Q}_q^\times$. For simplicity we state the result in the case where the supercuspidal π has the same conductor as $\pi \times \pi$. (A type 1 representation π must be dihedral corresponding to a character ξ of the unramified quadratic extension of $\mathbb{Q}_q$, and the conductors of π and $\pi \times \pi$ are two times the conductors of ξ and ξ^2, respectively. So if q is odd these conductors are automatically equal as long as ξ is not a quadratic character.)

Proposition 3.2.4. *Suppose $\pi_1 = \pi(\chi, \chi^{-1})$ is an unramified principal series (for χ an unramified unitary character) and $\pi_2 \cong \pi_3 \cong \pi$ is supercuspidal of Type 1 with conductor n (necessarily even) and trivial central character. If we assume that $\pi \times \pi$ also has conductor n, then*

$$\mathcal{E}_q I^*(\pi_1, \pi_2, \pi_3) = q^{-n}(1 + q^{-1})^{-2} \cdot (*)$$

where we set $\alpha = \chi(q)$ and define

$$(*) = \left(\frac{(\alpha^{n/2+1} - \alpha^{-n/2-1}) - q^{-1}(\alpha^{n/2-1} - \alpha^{-n/2+1})}{\alpha - \alpha^{-1}} \right)^2.$$

In Section 3.4, we will use these local integral computations to give a totally explicit version of Ichino's formula in certain cases where f is a newform and g, h are CM newforms. Also, we remark that we have performed numerical computations to provide evidence for the correctness of all of the factors in the various cases of local integral computations described above (as well as the global constant in our explicit Ichino's formula); this is described in detail in [Col18].

3.3. **A new local integral computation.** The method of [NPS14] is based on the following result, which is a key lemma from [MV10].

Proposition 3.3.1 (Michel-Venkatesh, [MV10] Lemma 3.4.2)**.** *If π_1, π_2, π_3 are tempered smooth representations of $\mathrm{GL}_2(\mathbb{Q}_q)$, with $\pi_1 \cong \pi(\chi, \chi^{-1})$ unramified and satisfying $\chi(q) = q^s$, satisfy $\omega_1\omega_2\omega_3 = 1$, then we have*

$$I^*(\pi_2, \pi_3; s) = (1 + q^{-1})^2 L(\mathrm{ad}\,\pi_2, 1)L(\mathrm{ad}\,\pi_3, 1)J^*(\pi_2, \pi_3; s)J^*(\widetilde{\pi}_2, \widetilde{\pi}_3; -s).$$

Here J^* comes from a certain *local Rankin-Selberg integral* associated to the two representations, namely

$$J(\pi_2, \pi_3; s) = \int_{NZ\backslash G} f_s^\circ(g)W_2^{\psi}(g)W_3^{\overline{\psi}}(g)dg$$

which is then normalized by

$$J^*(\pi_2, \pi_3; s) = \frac{\zeta_q(1 + 2s)}{L(\pi_2 \times \pi_3, 1/2 + s)} J(\pi_2, \pi_3; s).$$

Here f_s° is the normalized spherical vector of $\pi(\chi, \chi^{-1})$ given by

$$f_s^\circ\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} k\right) = \left|\frac{a}{d}\right|^{s+1/2},$$

and W^ψ denotes the Whittaker newvector (in the Whittaker model $W(\pi, \psi)$ of π) normalized by requiring $W^\psi(1) > 0$ and $\langle W^\psi, W^\psi\rangle = 1$ under the natural pairing

$$\langle W_1, W_2\rangle = \int_{\mathbb{Q}_q^\times} W_1\left(\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\right)\overline{W_2\left(\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\right)}d^\times y.$$

Our integral is over a quotient of $G = \mathrm{GL}_2(\mathbb{Q}_q)$ by a product of the center $Z \cong \mathbb{Q}_q^\times$ and the standard unipotent radical $N \cong (\mathbb{Q}_q, +)$.

A decomposition for our integral. To use this result, we need to evaluate the integrals $J(\pi_2, \pi_3; s)$. The first step is to expand out our integral over domains we understand how to integrate over. We start by setting up a bit of notation (following [NPS14]); we set

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad a(y) = \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} \quad z(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \quad n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

for $y, t \in \mathbb{Q}_q^\times$ and $x \in \mathbb{Q}_q$, and accordingly we set $A = \{a(y) : y \in \mathbb{Q}_q^\times\}$, $Z = \{z(t) : t \in \mathbb{Q}_q^\times\}$ and $N = \{n(x) : x \in \mathbb{Q}_q\}$. With this notation, the usual upper-triangular Borel subgroup is $B = ZNA$. The normalized Haar measures on $\mathbb{Q}_q$ and $\mathbb{Q}_q^\times$ (giving $\mathbb{Z}_q$ and $\mathbb{Z}_q^\times$ volumes 1, respectively) pass to Haar measures on Z, N, and A.

We then use the following decomposition of our group G, extending the Iwasawa decomposition. We first decompose

$$K = \coprod_{i=0}^{n}(B \cap K)\gamma_i K_2(\varpi^n) \qquad \gamma_i = \begin{bmatrix} 1 & 0 \\ \varpi^i & 1 \end{bmatrix}$$

and then conclude $G = \coprod_{i=0}^n B\gamma_i K$. Note in particular in the extreme cases of $i = 0$ and $i = n$ we have $B\gamma_0 K_2(\varpi^n) = BwK_2(\mathfrak{p}^n)$ and $B\gamma_n K_2(\varpi^n) = BK_2(\varpi^n)$. This decomposition is discussed in Section 2.1 of [Sch02] and in Appendix A of [Hu16]. For a function g invariant by $K_2(\varpi^n)$ on the right, it leads to us being able to write an integral over G as

$$\int_G f(g)dg = \sum_{0\le i\le n} v_i \int_B f(b\gamma_i)db \qquad v_i = \begin{cases} \frac{1}{(1+q^{-1})} & i = 0 \\ \frac{(1-q^{-1})}{(1+q^{-1})}q^{-i} & 0 < i < n \\ \frac{1}{(1+q^{-1})}q^{-n} & i = n \end{cases},$$

where db is the usual Haar measure on the Borel subgroup $B = ZNA$, given by $|a|^{-1}d^\times z\, dn\, d^\times a$ on this decomposition. We have a similar expression for integrals over $Z\backslash G$ or $ZN\backslash G$. Thus, if π_2 and π_3 have conductor ϖ^n (so their Whittaker model is right $K_2(\varpi^n)$-invariant) we can write

$$J(\pi_2,\pi_3;s) = \sum_{0\le i\le n} v_i \int_{\mathbb{Q}_q^\times} |y|^{s-1/2} W_2^{\psi}(a(y)\gamma_i) W_3^{\overline{\psi}}(a(y)\gamma_i) d^\times y,$$

using that $f_s^\circ(a(y)\gamma_i) = |y|^{s+1/2}$ by definition. So, if we can come up with an explicit enough expression for these values of the Whittaker function, we can compute this integral directly via this decomposition.

Whittaker newvectors of principal series. We now want to compute the Whittaker newvector W^ψ for a principal series representation of conductor 1 and with central character satisfying $\chi_\pi(\varpi) = 1$, so $\pi = \pi(\mu\chi, \mu^{-1})$, where μ is unramified and χ has conductor 1 and $\chi(\varpi) = 1$. We note that a general version of this computation is also carried out in Section 4 of [Tem14].

We start by recalling that $\pi(\mu\chi, \mu^{-1})$ can first be realized in its *induced model* consisting of all smooth functions $f : G \to \mathbb{C}$ satisfying

$$f\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} g\right) = |a/d|^{1/2}(\mu\chi)(a)\mu^{-1}(d)f(g).$$

In the induced model, the computation of the newvector is straightforward (see section 2.1 of [Sch02], for instance). It is the following function f defined in terms of the decomposition $G = B\gamma_0 K_2(\varpi) \sqcup B\gamma_1 K_2(\varpi)$:

$$f\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \gamma_i k\right) = \begin{cases} \chi(a)\mu(ad^{-1})|ad^{-1}|^{1/2} & i = 1 \\ 0 & i = 0 \end{cases}.$$

Next, we need to transfer this to the Whittaker model $W(\pi,\psi)$. The isomorphism from the induced model is given by $h \mapsto \int_{\mathbb{Q}_q} \psi(-x)h(wn(x)g)dx$. Thus the Whittaker newvector $W^\psi \in W(\pi,\psi)$ is the function $G \to \mathbb{C}$ determined by

$$W^\psi(g) = \int_{\mathbb{Q}_q} \psi(-x) f(wn(x)g)dx$$

for our induced model newvector $f(x)$ written above.

To evaluate this integral for $g = a(y)\gamma_i$, we need to compute $f(wn(z)\gamma_i)$ for all z and all $i = 0, 1$. To do this, we start by writing explicitly that if $z \in \mathbb{Z}_q$ then

$$wn(z)\gamma_i = \begin{bmatrix} 0 & 1 \\ -1 & -z \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \varpi^i & 1 \end{bmatrix} = \begin{bmatrix} \varpi^i & 1 \\ -1 - z\varpi^i & -z \end{bmatrix} \in K,$$

and we can then see that this lies in $B\gamma_0 K$ unless $i = 0$ and $z \in -1 + \varpi\mathbb{Z}_q$, and in that case the resulting matrix lies in K_2 so $f(wn(z)\gamma_i) = 1$. If $z \notin \mathbb{Z}_q$ we compute

$$wn(z)\gamma_i = \begin{bmatrix} -z^{-1} & 1 \\ 0 & -z \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \varpi^i + z^{-1} & 1 \end{bmatrix} \in B \cdot K,$$

and we find this decomposition lies in $B\gamma_0 K_2$ if $i = 0$ and in $B\gamma_1 K_2$ if $i = 1$. In fact, in the $i = 1$ case the second matrix is in K_2 already, so $f(wn(z)\gamma_i) = \chi(-z^{-1})\mu(z^{-2})|z|^{-1}$. Combining these facts we conclude

$$f(wn(z)\gamma_i) = \begin{cases} 1 & i = 0, z \in -1 + \varpi\mathbb{Z}_q \\ \chi(-z^{-1})\mu(z^{-2})|z|^{-1} & i = 1, z \notin \mathbb{Z}_q \\ 0 & \text{otherwise} \end{cases}.$$

We can then go back to the integral $\int \psi(-x)f(wn(x/y)\gamma_i)dx$ we needed to evaluate to compute $W^\psi(a(y)\gamma_i)$. If $i = 0$ we know that the integrand is nonzero only when $x/y \in -1 + \varpi\mathbb{Z}_q$, and the integral becomes the integral of $\psi(-x)$ over $x \in -y + y\varpi\mathbb{Z}_q = y + \varpi^{v+1}\mathbb{Z}_q$ for $v = v(y)$. Taking the substitution $x' = -x - y$ we conclude the integral is

$$\psi(y)\int_{\varpi^{v+1}\mathbb{Z}_q} \psi(x')dx' = \psi(y)\begin{cases} q^{-v-1} & v + 1 \ge 0 \\ 0 & v + 1 < 0 \end{cases}.$$

Noting that $|y| = q^{-v}$ by definition, we conclude that we have

$$W^\psi(a(y)\gamma_0) = \begin{cases} \mu(y)^{-1}|y|^{1/2}\psi(y)q^{-1} & v(y) \geq -1 \\ 0 & v(y) < -1 \end{cases}.$$

Similarly, for $i = 1$ our computations tell us that $f(wn(x/y)\gamma_1)$ is nonzero exactly when $x/y \notin \mathbb{Z}_q$, i.e. $v(x) < v = v(y)$. For x satisfying $v(x) = u < v$ we have

$$f(wn(x/y)\gamma_1) = \chi(-y/x)\mu(\varpi)^{2v-2u}q^{u-v}$$

and thus we have that $\int \psi(-x)f(wn(x/y)\gamma_1)dx$ expands as

$$\sum_{u=-\infty}^{v-1} \chi(y)\mu(\varpi)^{2v-2u}q^{u-v}\int_{\varpi^i\mathbb{Z}_q^\times} \psi(-x)\chi^{-1}(-x)dx.$$

Now, the integral in the sum is zero except for the case $u = -1$, when it gives the ε-factor $q^{1/2}\varepsilon(1/2, \chi^{-1}, \psi)$. Thus we find

$$W^\psi(a(y)\gamma_1) = \begin{cases} \chi(y)\mu(y\varpi^2)|y|^{1/2}q^{-1/2}\varepsilon(1/2, \chi^{-1}, \psi) & v(y) \geq 0 \\ 0 & v(y) < 0 \end{cases}.$$

So we have a formula for a newvector W^ψ; recall that we want to normalize it by requiring $\langle W^\psi, W^\psi\rangle = 1$ and $W^\psi(1) > 0$. First we note that

$$W^\psi(1) = W^\psi(a(1)\gamma_1) = \mu(\varpi^2)\varepsilon(1/2, \chi^{-1}, \psi)q^{-1/2}$$

so we can multiply by $\mu(\varpi)^{-2}\varepsilon(1/2, \chi, \overline{\psi})$ to guarantee that this is positive. Then since $W^\psi(a(y)\gamma_1) = W^\psi(a(y))$ we compute

$$\langle W^\psi, W^\psi\rangle = \int |W^\psi(a(y))|^2 d^\times y = \int_{v(y)\geq 0} |y|q^{-1}d^\times y = (1-q^{-1})^{-1}\int_{v(y)\geq 0} q^{-1}dy = (1-q^{-1})^{-1}q^{-1},$$

so we need to multiply by $(1-q^{-1})^{1/2}q^{1/2}$ to normalize the absolute value. We conclude that the normalized Whittaker newvector is given by:

$$W^\psi(a(y)\gamma_i) = \begin{cases} \chi(y)\mu(y)|y|^{1/2}(1-q^{-1})^{1/2} & v(y) \geq 0, i = 1 \\ \mu^{-1}(y\varpi^2)|y|^{1/2}(1-q^{-1})^{1/2}q^{-1/2}\psi(y)\varepsilon(1/2, \chi, \overline{\psi}) & v(y) \geq -1, i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

The local integral for two representations of this type. Now, we want to compute $J(\pi_2, \pi_3; s)$ for $\pi_2 = \pi(\mu\chi, \mu^{-1})$ and $\pi_3 = \pi(\nu\chi^{-1}, \nu^{-1})$ are two representations of the type just considered (with μ, ν unramified and χ of conductor 1). For convenience we let ξ be the unramified representation $\xi = |\cdot|^s$ (since ultimately our parameter s corresponds to the spherical representation $\pi(\xi, \xi^{-1})$).

Applying our computation of the Whittaker newvectors in the previous section we get the following formula:

$$W_2^\psi(a(y)\gamma_i)W_3^{\overline{\psi}}(a(y)\gamma_i) = \begin{cases} (\mu\nu)(y)|y|(1-q^{-1}) & v(y) \geq 0, i = 1 \\ (\mu^{-1}\nu^{-1})(y\varpi^2)|y|q^{-1}(1-q^{-1}) & v(y) \geq -1, i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Using our expression for $J(\pi_2, \pi_3; s)$ from the decomposition in terms of double cosets $B\gamma_i K_2$ we can write

$$J(\pi_2, \pi_3; s) = (1+q^{-1})^{-1}\sum_{i=0}^{1} q^{-i}\int_{\mathbb{Q}_q^\times} \xi(y)|y|^{-1/2}W_2^\psi(a(y)\gamma_i)W_3^{\overline{\psi}}(a(y)\gamma_i)d^\times y.$$

Then the $i = 1$ term is

$$q^{-1}(1-q^{-1})\int_{v(y)\geq 0} (\xi\mu\nu)(y)|y|^{1/2}d^\times y = q^{-1}(1-q^{-1})\sum_{i=0}^{\infty}(\xi\mu\nu)(\varpi^i)q^{-i/2},$$

which is a geometric series summing to $q^{-1}(1-(\xi\mu\nu)(\varpi)q^{-1/2})^{-1}$. Similarly, the $i = 0$ term becomes

$$q^{-1}(1-q^{-1})(\mu^{-1}\nu^{-1})(\varpi)^2\sum_{i=-1}^{\infty}(\xi\mu^{-1}\nu^{-1})(\varpi^i)q^{-i/2}.$$

which sums to

$$q^{-1}(1-q^{-1})\frac{(\mu^{-1}\nu^{-1})(\varpi)^2\cdot(\xi\mu^{-1}\nu^{-1})(\varpi^{-1})q^{1/2}}{1-(\xi\mu^{-1}\nu^{-1})(\varpi)q^{-1/2}}.$$

So, we conclude

$$J(\pi_2,\pi_3;s)=\frac{q^{-1}(1-q^{-1})}{(1+q^{-1})}\left(\frac{1}{1-(\xi\mu\nu)(\varpi)q^{-1/2}}+\frac{(\xi\mu\nu)(\varpi^{-1})q^{1/2}}{1-(\xi\mu^{-1}\nu^{-1})(\varpi)q^{-1/2}}\right).$$

Collecting terms we find we get

$$J(\pi_2,\pi_3;s)=(1+q^{-1})^{-1}(1-q^{-1})q^{-1}\frac{(\xi\mu\nu)(\varpi^{-1})q^{1/2}\cdot(1-\xi^2(\varpi)q^{-1})}{(1-(\xi\mu^{-1}\nu^{-1})(\varpi)q^{-1/2})(1-(\xi\mu\nu)(\varpi)q^{-1/2})}.$$

Next, we recall that we get $J^*(\pi_2,\pi_3;s)$ by multiplying this quantity by $\zeta_q(1+2s)/L(\pi_2\times\pi_3,1/2+s)$. But

$$\zeta_q(1+2s)=(1-q^{-1-2s})^{-1}=(1-\xi^2(\varpi)q^{-1})^{-1}$$

cancels a term on the top of our expression above, and similarly

$$L(\pi_2\times\pi_3,1/2+s)=(1-(\mu\nu)(\varpi)q^{-1/2-s})^{-1}(1-(\mu^{-1}\nu^{-1})(\varpi)q^{-1/2-s})^{-1}$$

cancels the bottom. So we conclude

$$J^*(\pi_2,\pi_3;s)=(1+q^{-1})^{-1}(1-q^{-1})q^{-1/2}(\xi^{-1}\mu\nu)(\varpi).$$

Finally, we recall that the ultimate local integral we want is given by

$$I^*(\pi_1,\pi_2,\pi_3)=(1+q^{-1})^2L(\mathrm{ad}\,\pi_2,1)L(\mathrm{ad}\,\pi_3,1)J^*(\pi_2,\pi_3;s)J^*(\widetilde{\pi}_2,\widetilde{\pi}_3;-s).$$

Since $\widetilde{\pi}_2=\pi(\mu^{-1},\mu\chi)$ and $\widetilde{\pi}_3=\pi(\nu^{-1},\nu\chi)$ our computation above gives us

$$J^*(\widetilde{\pi}_2,\widetilde{\pi}_3;-s)=(1+q^{-1})^{-1}(1-q^{-1})q^{-1/2}(\xi\mu^{-1}\nu^{-1})(\varpi).$$

Thus we have

$$J^*(\pi_2,\pi_3;s)J^*(\widetilde{\pi}_2,\widetilde{\pi}_3;-s)=(1+q^{-1})^{-2}(1-q^{-1})^2q^{-1};$$

since we can easily check $L(\mathrm{ad}\,\pi_2,1)=L(\mathrm{ad}\,\pi_3,1)=(1-q^{-1})^{-1}$ we conclude:

Proposition 3.3.2. *Let $\pi_1=\pi(\xi,\xi^{-1})$, $\pi_2=\pi(\mu\chi,\mu^{-1})$, and $\pi_3(\nu\chi^{-1},\nu^{-1})$ be three principal series representations, with ξ,μ,ν unramified characters and χ a ramified character of conductor 1 satisfying $\chi(\varpi)=1$. Then we have*

$$I^*(\pi_1,\pi_2,\pi_3)=q^{-1}.$$

To deduce Proposition 3.2.3 from this, we can use Lemma 3.2.1 to twist each principal series so that the central character has value 1 at ϖ (note the product of twists is trivial because of the initial assumption that the product of central characters is trivial!). Then we can write π_1 and π_2 in the desired form, and note that their central characters are 1 and χ, respectively; this forces the central character of π_3 to be χ^{-1} and thus π_3 to have the desired form as well.

3.4. **Specialization to the case of CM forms.** Finally, we want to use the local integral computations in Section 3.2 to obtain a completely explicit version of Ichino's formula (Theorem 3.1.2) for certain choices of modular forms f,g,h. In particular, we will assume that g,h are both *CM forms*: they come from Hecke characters ψ of imaginary quadratic field. Given such a Hecke character, the associated CM form g_ψ should be defined by

$$g_\psi(z)=\sum_{\mathfrak{a}\subseteq\mathcal{O}_K}\psi(\mathfrak{a})e^{2\pi iN(\mathfrak{a})z}=\sum_{n=1}^{\infty}\left(\sum_{\mathfrak{a}:N(\mathfrak{a})=n}\psi(\mathfrak{a})\right)q^{2\pi inz},$$

to guarantee $L(\psi,s)=L(g_\psi,s)$. If ψ has infinity-type $(m,0)$ or $(0,m)$ then this does indeed define a newform (see e.g. Section 4.8 of [Miy06]).

Proposition 3.4.1. *Let ψ be an algebraic Hecke character of infinity-type $(m,0)$ (for an integer $m\geq 0$) for an imaginary quadratic field $K=\mathbb{Q}(\sqrt{-d})$. Then the function g_ψ defined above is a newform of weight $m+1$, level $d\cdot N(\mathfrak{m}_\psi)$, and character $\chi_K\cdot\chi_\psi$.*

We then take the following setup to guarantee we get an instance of Ichino's formula where we know all of the local integrals.

- f is a newform of some weight k, level N, and with trivial character.
- $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field of odd fundamental discriminant $-d$, such that d is coprime to N.
- φ, ψ are Hecke characters of K of weights $(m-k-1, 0)$ and $(m-1, 0)$, respectively, for some integer $m > k$.
- The central characters χ_φ, χ_ψ (the finite-type parts of φ and ψ, restricted to $\mathbb{Z}$) are trivial. This forces the conductors of φ and ψ to be ideals generated by integers in $\mathbb{Z}$.
- The conductors of φ and ψ are coprime to N and d. Moreover, they are given by $c\ell^{c_\ell}$ and ℓ^{c_ℓ}, respectively, and we have
  - c is coprime to Nd.
  - $\ell \nmid 2Ndc$ is a prime inert in K, and the local components of φ and ψ at ℓ are inverse to each other and not quadratic characters.

Ichino's formula can then be written

$$|\langle f(z)g_\varphi(z), g_\psi(c^2Nz)\rangle|^2 = \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2}2^{4m-2}(c^2N)^m} L(f \times g_\varphi \times \overline{g}_\psi, m-1) \prod_{q|dNc\ell} \mathcal{E}_q I_q.$$

We also note that the triple-product L-value $L(f \times g_\varphi \times \overline{g}_\psi, m-1)$ factors as a product of $L(f, \varphi\psi^{-1}, 0)$ and $L(f, \psi^{-1}\varphi^{-1}N^{m-k-1}, 0)$ due to a decomposition of the corresponding Weil-Deligne representations:

$$(\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \varphi) \otimes (\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \psi) \cong \big(\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \psi\varphi\big) \oplus \big(\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \psi\varphi^c\big).$$

We then consider the factors $\mathcal{E}_q I_q$ at the primes dividing $dNc\ell$:

(1) $q|N$: In this case, $\pi_{f,q}$ is ramified (and we can't say much else about it since we aren't putting many assumptions on f) and the other two local representations are unramified. Thus we're in the situation of Corollary 3.2.2, so $\mathcal{E}_q I_q^* = q^{-n_q}(1+q^{-1})^{-2}$ where q^{n_q} is the power of q dividing N.
(2) $q|c$: In this case only $\pi_{\varphi,q}$ is ramified (either a ramified principal series or supercuspidal) and the other two local representations are unramified. So again we're in the situation of Corollary 3.2.2 and $\mathcal{E}_q I_q^* = q^{-2n_q}(1+q^{-1})^{-2}$ where q^{n_q} is the power of q dividing c (and thus q^{2n_q} is the power of q dividing the conductor of $\pi_{\varphi,q}$).
(3) $q|d$: Here q is odd, $\pi_{f,q}$ is unramified, and the local representations $\pi_{\varphi,q}$ and $\widetilde{\pi}_{\psi,q}$ are each principal series associated to a pair of an unramified character and a character of conductor q. By Proposition 3.2.3, we have $\mathcal{E}_q I_q^* = q^{-1}$.
(4) $q = \ell$: In this case $\pi_{f,q}$ is an unramified principal series, and $\pi_{\varphi,q}$ and $\widetilde{\pi}_{\psi,q}$ are supercuspidal representations of "Type 1". More specifically, since ψ and φ have inverse local components at ℓ (including their values on ϖ_ℓ, which are $\chi_K(\ell) = -1$), these local components $\pi_{\varphi,q}$ and $\widetilde{\pi}_{\psi,q}$ are isomorphic. By Lemma 3.2.1 we can twist both to have trivial central character (without twisting $\pi_{f,q}$), and thus we're in the situation of Proposition 3.2.4 and we get $\mathcal{E}_q I_q^* = q^{-n_q}(1+q^{-1})^{-2}(*)$ where $n_q = 2c_\ell$ and the term $(*)$ has parameter $\alpha = \alpha_\ell(f)/\ell^{(k-1)/2}$. Note that our hypothesis that the local representations are not quadratic characters is exactly what we need to guarantee the condition on conductors stated in that proposition.

Putting this together we conclude:

Theorem 3.4.2 (Explicit Ichino's formula, CM case)**.** *Let f, $g = g_\varphi$, and $h = g_\psi$ satisfy the hypotheses listed earlier in this section. Then we have*

$$\begin{aligned}&|\langle f(z)g_\varphi(z), g_\psi(c^2Nz)\rangle|^2 \\ &\quad= \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2}2^{4m-2}d\ell^{2c_\ell}(c^2N)^{m+1}} \cdot \prod_{q|cNd\ell}(1+q^{-1})^{-2} \cdot (*)_{f,\ell} \cdot L(f, \varphi\psi^{-1}, 0)L(f, \psi^{-1}\varphi^{-1}N^{m-k-1}, 0)\end{aligned}$$

where $(*)_{f,\ell}$ *is determined in terms of the root* $\alpha = \alpha_\ell(f)$ *of the Hecke polynomial for* f *at* ℓ *and is given by*

$$(*)_{f,\ell} = \left(\sum_{i=0}^{c_\ell} \left(\frac{\alpha}{\ell^{(k-1)/2}} \right)^{2i-c_\ell} - \frac{1}{\ell} \sum_{i=i}^{c_\ell - 1} \left(\frac{\alpha}{\ell^{(k-1)/2}} \right)^{2i-c_\ell} \right).$$

4. Background from Hida theory

In this section we collect the results from Hida's theory of Λ-adic modular forms that we'll need. Ultimately, we want to establish that if f is a fixed modular form and φ, ψ are Hecke characters varying suitably in families Φ, Ψ, we can construct a p-adic analytic function $\langle f\mathbf{g}_\Phi, \mathbf{g}_\Psi \rangle$ that explicitly interpolates the family of Petersson inner products $\langle fg_\varphi, g_\psi \rangle$. After recalling the basic setup of Hida theory in this section we will proceed to constructing the Λ-adic families $\mathbf{g}_\Phi$ and $\mathbf{g}_\Psi$ in Chapter 5, and then to working with $\langle f\mathbf{g}_\Phi, \mathbf{g}_\Psi \rangle$ in Chapter 6 (and in particular finding the removed Euler factors at the prime p).

Throughout this section, we will always assume p is an odd prime. In some cases this is for simplicity, but in others it's because important parts of the theory have not been worked out for the case $p = 2$. We largely follow Hida's writings, especially [Hid88] and [Hid93], but also [Hid85], [Hid86a] and [Hid86b], as well as Wiles' paper [Wil88]. However we remark that many of these papers predate the formalism of Λ-adic forms that we use, so results need to be translated over; unfortunately we do not know of any comprehensive references for this theory written in the more modern language we use.

4.1. p-adic modular forms.

Consider the spaces of classical modular forms $M_k(\Gamma, \chi) = M_k(\Gamma, \chi; \mathbb{C})$ or cusp forms $S_k(\Gamma, \chi) = S_k(\Gamma, \chi; \mathbb{C})$ for a weight k, a congruence subgroup Γ, and character χ. For any subalgebra $A \subseteq \mathbb{C}$ we can define A-submodules $M_k(\Gamma, \chi; A)$ consisting of the forms with Fourier coefficients lying in A, and similarly $S_k(\Gamma, \chi; A)$ for cusp forms; we view both as subspaces of a formal power series ring $A[\![q^{1/M}]\!]$. Standard results on integrality of newform coefficients tell us that for any ring A containing the image of χ we have bases of M_k and S_k with coefficients in A and thus

$$M_k(\Gamma, \chi; A) \otimes_A \mathbb{C} \cong M_k(\Gamma, \chi; \mathbb{C}) \qquad S_k(\Gamma, \chi; A) \otimes_A \mathbb{C} \cong S_k(\Gamma, \chi; \mathbb{C}).$$

Applying this, we can change our scalars to (the valuation ring of) a p-adic field F:

Definition 4.1.1. Fix a weight k, a congruence subgroup Γ, and a character χ. If $F/\mathbb{Q}_p$ is a p-adic field containing the image of χ and $F_0 \subseteq F$ is a number field with F as its completion (which also contains the image of χ), and we let $\mathcal{O}_F$ and $\mathcal{O}_{F_0}$ be the integer rings of F and F_0, respectively, then we can define

$$M_k(\Gamma, \chi; \mathcal{O}_F) = M_k(\Gamma, \chi; \mathcal{O}_{F_0}) \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F$$

and similarly for S_k.

One can then check that this space is independent of the choice of field F_0. The ring $\mathcal{O}_F[\![q^{1/M}]\!]$ is naturally equipped with a norm $|\sum a_n q^{n/M}| = \sup\{|a_n|_p\}$, making it into a p-adic Banach space. We will define the space of p-adic modular forms (over F) as a certain closed subspace of $\mathcal{O}_F[\![q^{1/M}]\!]$, which will thus be a p-adic Banach space. Any individual space $M_k(\Gamma, \chi; \mathcal{O}_F)$ is finite-rank and thus already closed, but we can define

$$M(\Gamma, \chi; \mathcal{O}_F) = M_{\leq\infty}(\Gamma, \chi; \mathcal{O}_F) = \bigoplus_{j=0}^{\infty} M_j(\Gamma, \chi; \mathcal{O}_F)$$

and take its closure:

Definition 4.1.2. Fix a congruence subgroup Γ, a character, and a p-adic field $F/\mathbb{Q}_p$ containing the values of χ. We define the spaces $\overline{M}(\Gamma, \chi; \mathcal{O}_F)$ of *p-adic modular forms* and $\overline{S}(\Gamma, \chi; \mathcal{O}_F)$ of *p-adic cusp forms* with coefficients in $\mathcal{O}_F$ as the closures of the space $M(\Gamma, \chi; \mathcal{O}_F)$ or $S(\Gamma, \chi; \mathcal{O}_F)$, respectively, in $\mathcal{O}_F[\![q^{1/M}]\!]$ with the Banach space topology given above.

Equivalently, $\overline{M}(\Gamma, \chi; \mathcal{O}_F)$ is the completion of $M(\Gamma, \chi; \mathcal{O}_F)$ with respect to the given norm on q-expansions. We are most interested in the case of $\Gamma = \Gamma_1(N)$, and we write $\overline{M}(N; \mathcal{O}_F)$ to denote $\overline{M}(\Gamma_1(N); \mathcal{O}_F)$ and similarly for $\overline{S}$. We will also occasionally need to work with the larger space $\overline{M}(\Gamma_1(N, M); \mathcal{O}_F)$. We also recall that by a theorem of Katz [Kat76] (using the theory of geometric modular forms), the spaces we've defined actually have "p^∞-level":

Theorem 4.1.3. *As subspaces of* $\mathcal{O}_F[\![q^{1/M}]\!]$*, we have*

$$\overline{S}(\Gamma \cap \Gamma_1(p^r), \mathcal{O}_F) = \overline{S}(\Gamma, \mathcal{O}_F) \qquad \overline{M}(\Gamma \cap \Gamma_1(p^r), \mathcal{O}_F) = \overline{M}(\Gamma, \mathcal{O}_F)$$

for $\Gamma = \Gamma(N_0), \Gamma_1(N_0)$*, or* $\Gamma_1(N_0, M_0)$ *with* N_0, M_0 *prime to* p*. In particular*

$$M_k(N_0 p^\infty; \mathcal{O}_F) = M_k(\Gamma_1(N_0 p^\infty); \mathcal{O}_F) = \bigcup_r M_k(\Gamma_1(N_0 p^r); \mathcal{O}_F)$$

is a subspace of $\overline{M}(N_0; \mathcal{O}_F)$*.*

Now that we've defined the spaces $\overline{M}(N_0; \mathcal{O}_F)$ and $\overline{S}(N_0; \mathcal{O}_F)$ we want to put a Hecke action on them. To do this we actually first need to define an action by a profinite group

$$\widehat{Z}_{N_0} = \varprojlim_r (\mathbb{Z}/N_0 p^r \mathbb{Z})^\times \cong (\mathbb{Z}/N_0\mathbb{Z})^\times \times \mathbb{Z}_p^\times \cong (\mathbb{Z}/N_0 p\mathbb{Z})^\times \times (1+p)^{\mathbb{Z}_p}.$$

We first define an action on $M_k(\Gamma_1(N_0 p^r); \mathcal{O}_F)$ for any k and any $r \geq 0$ by

$$\langle z \rangle f = z_p^k f|_k[\sigma_z] \qquad \sigma_z \in \mathrm{SL}_2(\mathbb{Z}), \sigma_z \equiv \begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix} \pmod{N_0 p^r},$$

where $z \mapsto z_p$ under the projection $\widehat{Z}_{N_0} \to \mathbb{Z}_p^\times$; so this is a slightly modified version of the classical action of $(\mathbb{Z}/N_0 p^r \mathbb{Z})^\times$ by diamond operators (hence the notation). We can then check that these actions are all compatible:

Proposition 4.1.4. *The action of* $\widehat{Z}_{N_0}$ *on the spaces* $M_k(\Gamma_1(N_0 p^r); \mathcal{O}_F)$ *are compatible, i.e. they extend to a unique action on* $\sum_{k,r} M_k(\Gamma_1(N_0 p^r); \mathcal{O}_F)$*. Moreover, this extends to a continuous action of* $\widehat{Z}_{N_0}$ *on* $\overline{M}(N_0; \mathcal{O}_F)$*, and* $\overline{S}(N_0; \mathcal{O}_F)$ *is invariant under this action.*

Since $\overline{M}(N_0; \mathcal{O}_F)$ is a $\mathcal{O}_F$-module, this group action naturally gives it a $\mathcal{O}_F[\widehat{Z}_{N_0}]$-module structure, which in fact extends to a $\mathcal{O}_F[\![\widehat{Z}_{N_0}]\!]$-module structure. By just considering the direct factor $(1+p)^{\mathbb{Z}_p}$ of $\widehat{Z}_{N_0}$, we get that $\overline{M}(N_0; F)$ has a Λ-module structure for

$$\Lambda = \mathcal{O}_F[\![(1+p)^{\mathbb{Z}_p}]\!] \cong \mathcal{O}_F[\![\mathbb{Z}_p]\!] \cong \mathcal{O}_F[\![X]\!].$$

This module structure will be fundamental for the definition we will give of families of p-adic modular forms! We note that any integer d coprime to $N_0 p$ maps into Λ via the inclusion $(\mathbb{Z}/N_0 p\mathbb{Z})^\times \hookrightarrow \widehat{Z}_{N_0}$ and then the projection $\widehat{Z}_{N_0} \twoheadrightarrow (1+p)^{\mathbb{Z}_p}$. We denote the image of such an operator as $\langle d \rangle_\Lambda$; note that this is *not* the same as $\langle d \rangle \in \widehat{Z}_{N_0}$. In fact, if $\widetilde{\omega} : \widehat{Z}_{N_0} \twoheadrightarrow (\mathbb{Z}/N_0 p^r \mathbb{Z})^\times \hookrightarrow \mathcal{O}_F^\times$ is the natural character (serving the same purpose as the Teichmüller character for $N_0 = 1$), we have $\langle d \rangle = \widetilde{\omega}(d) \langle d \rangle_\Lambda$.

One thing that the action of $\widehat{Z}_{N_0}$ does is lets us recover the character and weight of the modular form. In particular, if χ is a character of $(\mathbb{Z}/p^r N_0 \mathbb{Z})^\times$ for some $r \geq 1$, then modular form $f \in M_k(N_0 p^r, \chi)$ satisfies

$$\langle z \rangle f = z_p^k (\widetilde{\omega}^{-k} \chi)(z) f$$

for all $z \in \widehat{Z}_{N_0}$. Accordingly, we say that a p-adic modular form $f \in \overline{M}(N_0; \mathcal{O}_F)$ has *weight* k *and character* χ if it satisfies this identify for all z. Clearly this is a necessary condition for the form to actually lie in $M_k(N_0 p^r, \chi)$, but in general it is not sufficient - a p-adic modular form of weight k and character χ need not be classical of weight k and character χ. If χ is a character of $(\mathbb{Z}/N_0 p\mathbb{Z})^\times$ we can define the space $\overline{M}(N_0; \mathcal{O}_F)[\chi]$ as the subspace of $\overline{M}(N_0; \mathcal{O}_F)$ of forms that have tame-at-p character χ, i.e. such that $\langle z \rangle f = \chi(z) f$ for all $z \in (\mathbb{Z}/pN_0\mathbb{Z})^\times$.

Finally, we define Hecke operators and their associated Hecke algebras for $\overline{M}(N_0; \mathcal{O}_F)$ and its subspaces. The Hecke operators themselves can be defined from the usual ones of classical modular forms $M_k(\Gamma_1(N_0 p); \mathcal{O}_F)$, or equivalently by using the usual formula on q-expansions:

Proposition 4.1.5. *Fix an integer* n*. Then we can define a Hecke operator* $T(n)$ *on* $\overline{M}(N_0; \mathcal{O}_F)$ *as the unique continuous extension of the usual Hecke operator* $T(n)$ *on the sum of all subspaces* $M_k(\Gamma_1(N_0 p^r), F)$ *for* $r > 0$*. This can be equivalently described in terms of its Fourier coefficients by*

$$a(m, f|T(n)) = \sum_{d|(m,n),(d,N_0 p)=1} d^{-1} a(mn/d^2, \langle d \rangle_\Lambda f).$$

where $f|d$ denotes the action of $d \in \mathbb{Z}_p^\times \subseteq \widehat{Z}_{N_0}$ discussed in the previous section. The subspaces $\overline{S}(N_0; \mathcal{O}_F)$, $\overline{M}(N_0; \mathcal{O}_F)$, and $\overline{S}(N_0; \mathcal{O}_F)$ are invariant under this action.

We then define the Hecke algebra $\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F)) \subseteq \mathrm{End}_{F\text{-cont}}(\overline{M}(N_0; \mathcal{O}_F))$ as the F-subalgebra generated by the Hecke operators. Evidently the restriction map from $\overline{M}(N_0; \mathcal{O}_F)$ to any space $M_k(\Gamma_1(N_0p^r); \mathcal{O}_F)$ induces a surjection of $\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F))$ onto $\mathbb{T}(M_k(\Gamma_1(N_0p^r); \mathcal{O}_F))$ taking $T(n)$ to $T(n)$ for all n, and we can in fact check that $\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F))$ is the inverse limit of finite-dimensional algebras $\mathbb{T}(M_{\leq k}(\Gamma_1(N_0p^r); \mathcal{O}_F))$ (varying k and/or r). The same statements hold for Hecke algebras of $\overline{S}(N_0; \mathcal{O}_F)$, and also for replacing F with $\mathcal{O}_F$ in each case.

Next, we recall that for finite-dimensional spaces we have a perfect pairing

$$M_k(\Gamma_1(N_0p^r); \mathcal{O}_F) \times \mathbb{T}(M_k(\Gamma_1(N_0p^r), \mathcal{O}_F)) \to \mathcal{O}_F$$

given by $(f, t) \mapsto a(1, f|t)$. This formula induces a perfect pairing $\overline{M}(N_0; \mathcal{O}_F) \times \mathbb{T}(\overline{M}(N_0; \mathcal{O}_F)) \to \mathcal{O}_F$, and thus we have isomorphisms

$$\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F)) \cong \mathrm{Hom}_{\mathcal{O}_F}(\overline{M}(N_0; \mathcal{O}_F), \mathcal{O}_F),$$

$$\overline{M}(N_0; \mathcal{O}_F) \cong \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F)), \mathcal{O}_F);$$

and similarly for cusp forms; see Theorem 1.3 of [Hid88]. We will use this duality repeatedly in the next section.

Finally, we note that the automorphisms of $\overline{M}(N_0; \mathcal{O}_F)$ arising from the action of $\widehat{Z}_{N_0}$ defined in the previous section all lie in $\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F))$; this can be deduced from checking that if $\ell \nmid N_0p$ is a prime then our formula for Hecke operators gives that the action of $\ell \in \widehat{Z}_{N_0}$ is given by the Hecke operator $\ell(T(\ell)^2 - T(\ell^2))$. Thus we have a natural map $\widehat{Z}_{N_0} \to \mathbb{T}(\overline{M}(N_0; \mathcal{O}_F))$, which extends to a homomorphism $\mathcal{O}_F[\![\widehat{Z}_{N_0}]\!] \to \mathbb{T}(\overline{M}(N_0; \mathcal{O}_F))$. In particular this makes $\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F))$ into a Λ-algebra.

4.2. **Λ-adic families of modular forms.** Now that we've set up the basic theory of p-adic modular forms, we develop the theory of Λ-adic modular forms, which are "p-adic families of p-adic of modular forms." Recall that, given a finite extension $F/\mathbb{Q}_p$ we're working over, Λ was defined as $\mathcal{O}_F[\![\Gamma]\!]$ for $\Gamma = (1+p)^{\mathbb{Z}_p} \subseteq \mathbb{Z}_p^\times$ abstractly isomorphic to $\mathbb{Z}_p$. Moreover we know Λ is abstractly isomorphic to the formal power series ring $\mathcal{O}_F[\![X]\!]$; if we pick a topological generator γ of Γ (usually $\gamma = 1+p$) then the isomorphism is determined by $\gamma \leftrightarrow 1 + X$.

Using the description of Λ as a power series ring, we know that the set of continuous $\mathcal{O}_F$-algebra homomorphisms $\mathrm{Hom}(\Lambda, \mathcal{O}_F)$ is in bijection with elements of the maximal ideal $\mathfrak{m}_F \subseteq \mathcal{O}_F$, by associating $x \in \mathfrak{m}_F$ to the homomorphism $\Lambda \to \mathcal{O}_F$ characterized by $X \mapsto x$. In the framework of rigid geometry, this means that Λ is the coordinate ring of the open unit disc, and its F-valued points are the homomorphisms $\Lambda \to \mathcal{O}_F$. Motivated by this, we think of of an element $f \in \Lambda$ as an analytic function on the open unit disc, which we can evaluate at a point $P \in \mathrm{Hom}(\Lambda, \mathcal{O}_F)$ by taking $P(f)$.

Given this setup, for an integer k we define a distinguished point $P_k \in \mathrm{Hom}(\Lambda, \mathcal{O}_F)$ by $P_k(X) = (1+p)^k - 1$, or equivalently $P_k(\gamma) = (1+p)^k$. Then, following the formalism of Wiles [Wil88], we define a Λ-adic modular form to be a formal q-expansion with coefficients in Λ, such that evaluating it at a point P_k gives a classical modular form of weight k.

Definition 4.2.1. A Λ-*adic modular form* of level N_0p^r (for $p \nmid N_0$ and $r \geq 1$) and tame character χ (a Dirichlet character modulo N_0p) is a formal power series $\mathbf{f} = \sum A_n q^n \in \Lambda[\![q]\!]$ such that, for all but finitely many $k \geq 2$, the following is satisfied:

- The formal power series $\mathbf{f}_k = P_k(\mathbf{f}) = \sum P_k(A_n) q^n$ is in fact a classical modular form lying in the space $M_k(N_0p^r, \chi\omega^{-k}; \mathcal{O}_F)$.

If all but finitely many $\mathbf{f}_k$'s are actually cusp forms, we say $\mathbf{f}$ is a Λ-adic cusp form. We let $\mathbf{M}(N_0p^r, \chi; \Lambda)$ denote the set of all Λ-adic modular forms of level N_0p^r and character χ, and $\mathbf{S}(N_0p^r, \chi; \Lambda)$ the set of Λ-adic cusp forms; these are evidently sub-Λ-modules of $\Lambda[\![q]\!]$.

An alternative way to formalize this concept is through the idea of *measures.* This requires a bit of setup:

Definition 4.2.2. Let X be a (compact) topological space, and let $C(X; \mathcal{O}_F)$ be the compact p-adic Banach space of all continuous functions $X \to \mathcal{O}_F$ with the sup-norm. If M is a $\mathcal{O}_F$-Banach space, we define the space of *M-valued measures on X* as the space

$$\mathrm{Meas}(X; \mathcal{O}_F) = \mathrm{Hom}_{\mathcal{O}_F\text{-cont}}(C(X; \mathcal{O}_F), M).$$

This definition is by formal analogy with real-valued measure theory; a measure (in the classical sense) on a compact space is equivalently determined by the continuous $\mathbb{R}$-linear integration functional $C(X, \mathbb{R}) \to \mathbb{R}$. In the literature this analogy is sometimes emphasized by writing measures (in our sense) as $f \mapsto \int f d\mu$, but we'll just use $f \mapsto \mu(f)$ to denote the continuous homomorphism we're calling a "measure".

The reason that measures come up naturally in our context is that the ring $\Lambda = \mathcal{O}_F[\![\Gamma]\!]$ itself can be viewed as a space of them. We let $\log_\Gamma : (1+p)^{\mathbb{Z}_p} \to \mathbb{Z}_p$ be the isomorphism $(1+p)^x \mapsto x$; this is not equal to the usual p-adic logarithm but is a scalar multiple of it. Then, the following result is an easy consequence of Mahler's theorem (which says that all continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ can be written as a series $x \mapsto \sum a_k \binom{x}{k}$).

Lemma 4.2.3. *We have* $\mathrm{Meas}(\Gamma, \mathcal{O}_F) \cong \Lambda$*, via the map sending a power series* $A = \sum a_n X^n \in \mathcal{O}_F[\![X]\!] \cong \Lambda$ *to the measure* μ_A *that takes the function* $x \mapsto \log_\Gamma(x) \mapsto \binom{\log_\Gamma(x)}{n}$ *to the value* a_n*. Under this isomorphism, the action of* $\gamma \in \Gamma$ *by multiplication on* Λ *corresponds to the action of* Γ *on* $\mathrm{Meas}(\Gamma, \mathcal{O}_F)$ *described by* $(\gamma \cdot \mu)(f) = \mu(x \mapsto f(\gamma x))$*.*

Using this isomorphism, we can then see that if A is an element of Λ, taking the specialization $P_m(A)$ is the same as evaluating the measure μ_A under the continuous function $\Gamma \to \mathcal{O}_F$ given by $x \mapsto x^m$. Furthermore, we can also check that the above isomorphism extends to an isomorphism

$$\mathrm{Meas}(\Gamma, \mathcal{O}_F[\![q]\!]) \cong \Lambda[\![q]\!],$$

again such that if $A \leftrightarrow \mu_A$ then the specialization $P_k(M) \in \mathcal{O}_F[\![q]\!]$ is equal to $\mu_A(x \mapsto x^m)$.

Thus, a Λ-adic modular form $\mathbf{f}$ (which is by definition an element of $\Lambda[\![q]\!]$) naturally corresponds to a $\Lambda[\![q]\!]$-valued measure $\mu_{\mathbf{f}}$ on Γ. Moreover, we know that the specializations $\mu_{\mathbf{f}}(x \mapsto x^k)$ actually lie in $\overline{M}(N_0; \mathcal{O}_F)$ for all $k \gg 0$. Using the following "density" lemma (for which we omit the elementary proof) we can check that this means the whole domain maps into $\overline{M}(N_0; \mathcal{O}_F)$:

Lemma 4.2.4. *Let* M *be a Banach space over* $\mathcal{O}_F$*, and* $M' \subseteq M$ *a closed subspace that's saturated in the sense that if* $m \in M$ *satisfies* $pm \in M'$*, then we have* $m \in M'$*. If* $\mu \in \mathrm{Meas}(\Gamma, M)$ *is a measure such that for all* $k \geq k_0$*, we have* $\mu(x \mapsto x^k) \in M'$*, then in fact the image of* μ *lies in* M' *and thus* $\mu \in \mathrm{Meas}(\Gamma, M')$*.*

We can then apply this with $M = \mathcal{O}_F[\![q]\!]$ and $M' = \overline{M}(N_0; \mathcal{O}_F)$; we note that this space of p-adic modular forms is saturated because we can realize it as an intersection of a F-vector space $\overline{M}(N_0; F)$ with M. So if $\mathbf{f}$ is a Λ-adic form, we conclude that $\mu_{\mathbf{f}}$ is actually a $\overline{M}(N_0; \mathcal{O}_F)$-valued measure on Γ by the lemma. We can further analyze it by defining a $(\Lambda \times \Lambda)$-module structure on the module

$$\mathrm{Meas}(\Gamma, \overline{M}(N_0; \mathcal{O}_F)) \cong \mathrm{Hom}_{\mathcal{O}_F\text{-cont}}(C(\Gamma; \mathcal{O}_F), \overline{M}(N_0; \mathcal{O}_F))$$

induced by our Λ-actions on the spaces $C(\Gamma; \mathcal{O}_F)$ and $\overline{M}(N_0; \mathcal{O}_F)$; in particular for $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$ and a measure μ we define

$$((\gamma_1, \gamma_2) \cdot \mu)(x \mapsto f(x)) = \langle \gamma_2 \rangle \cdot \mu(x \mapsto f(\gamma_1 x)) \in \overline{M}(N_0; \mathcal{O}_F).$$

Then, if $\mu = \mu_{\mathbf{f}}$ for a Λ-adic modular form $\mathbf{f}$, we claim that the action of an element (γ, γ^{-1}) is trivial; to check this, note that evaluating $\mathbf{f}$ at $x \mapsto x^k$ gives us a classical modular form $\mathbf{f}_k$ for $k \gg 0$, and then evaluating $(\gamma, \gamma^{-1}) \cdot \mathbf{f}$ gives us

$$((\gamma, \gamma^{-1}) \cdot \mathbf{f})(x \mapsto x^k) = \langle \gamma^{-1} \rangle \mathbf{f}(x \mapsto \gamma^k x^k) = \gamma^k \langle \gamma^{-1} \rangle \mathbf{f}_k = \mathbf{f}_k$$

using linearity of both $\mathbf{f}$ and $\langle \gamma^{-1} \rangle$. So $(\gamma, \gamma^{-1}) \cdot \mathbf{f} = \mathbf{f}$ when evaluated at $x \mapsto x^k$ for $k \gg 0$, and because such functions span a dense subspace we can conclude $(\gamma, \gamma^{-1}) \cdot \mathbf{f} = \mathbf{f}$ as measures and thus Λ-adic modular forms. Since this is true for all $\gamma \in \Gamma$, we conclude that such an $\mathbf{f}$ is invariant under the antidiagonal copy of Λ in $\Lambda \times \Lambda$; we say it's "Λ-invariant" for short. Summing up:

Proposition 4.2.5. *If $\mathbf{f} \in \mathbf{M}(N_0p^r, \chi; \Lambda)$ is a Λ-adic modular form, then the associated measure $\mu_{\mathbf{f}}$ is valued in $\overline{M}(N_0; \mathcal{O}_F)$ and is Λ-invariant. Thus we could equivalently define Λ-adic modular forms of this level and character as being Λ-invariant $\overline{M}(N_0; \mathcal{O}_F)$-valued measures μ such that the specializations $\mu(x \mapsto x^k)$ lie in $M_k(N_0p^r, \chi\omega^{-k}; \mathcal{O}_F)$ for all but finitely many $k \geq 2$.*

The space of Λ-invariant measures still naturally has an action of Λ (coming from the quotient of $\Lambda \times \Lambda$ by the antidiagonal Λ where the action is invariant); this is equivalently described by

$$(\gamma \cdot \mu)(x \mapsto f(x)) = \langle \gamma \rangle \mu(x \mapsto f(x)) = \mu(x \mapsto f(\gamma x)).$$

This resulting Λ-action on Λ-invariant measures corresponds to the natural Λ-action on Λ-adic modular forms coming from scalar multiplication.

The point of view of measures makes it clear that if $\mathbf{f}$ is a Λ-adic form, all of the specializations $\mathbf{f}_k = P_k(\mathbf{f}) = \mu_{\mathbf{f}}(x \mapsto x^k)$ satisfy the appropriate transformation property to be p-adic modular forms of weight k and character $\chi\omega^{-k}$. However they may not be classical forms!

Finally, we note that using measures makes it easy to define Hecke algebras for Λ-adic forms. In fact, the Hecke algebra $\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F))$ naturally acts on $\mathbf{M}(N_0p^r, \chi; \Lambda)$! We can define a pairing

$$\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F)) \times \mathbf{M}(N_0p^r, \chi; \Lambda) \to \Lambda$$

by mapping $(T, \mathbf{f})$ to $T\mathbf{f}$ where $\mu_{T\mathbf{f}}$ is determined in terms of $\mu_{\mathbf{f}}$ by $\mu_{T\mathbf{f}}(f) = T \cdot \mu_{\mathbf{f}}(f)$. This is evidently a Λ-invariant pairing and thus induces a map $\mathbb{T}(\overline{M}(N_0; \mathcal{O}_F)) \to \mathrm{End}_\Lambda(\mathbf{M}(N_0p^r, \chi; \Lambda))$. We define the image of this map to be $\mathbb{T}(\mathbf{M}(N_0p^r, \chi; \Lambda))$; it's generated by the operators $T(n)$ which we can check act on the q-expansions in $\Lambda[\![q]\!]$ by

$$a(m, \mathbf{f}|T(n)) = \sum_{d|(m,n),(d,N_0p)=1} \langle d \rangle_\Lambda d^{-1} a(mn/d^2, \mathbf{f}),$$

where now $\langle d \rangle_\Lambda$ is just treated as a scalar in Λ.

4.3. **$\mathcal{I}$-adic modular forms.** We now want to expand our discussion of Λ-adic forms by allowing coefficients to lie in certain extensions $\mathcal{I} \supseteq \Lambda$ (which will ultimately be needed for what we want to construct). To do this, we start by setting up a slightly more sophisticated notation for dealing with specializations of Λ-adic forms. Recall that if we fix a topological generator γ of Γ, homomorphisms $\Lambda \to \mathcal{O}_F$ are in bijection with elements of $\mathfrak{m}_F$ by having an element $x \in \mathfrak{m}_F$ correspond to the unique homomorphism $\Lambda \to \mathcal{O}_F$ given by $\gamma \mapsto 1 + x$. We set up some notation:

Definition 4.3.1. We let $\mathcal{X}(\Lambda, \mathcal{O}_F)$ denote the set of all homomorphisms $P : \Lambda \to \mathcal{O}_F$, which is naturally in bijection with $\mathfrak{m}_F$ as above. Actually, any such point P can be thought of in three different ways that we'll pass between freely:

- As a $\mathcal{O}_F$-linear homomorphism $\Lambda \to \mathcal{O}_F$, characterized by $\gamma \mapsto x$
- As the kernel of such a homomorphism, which is a height-one prime ideal of Λ.
- As a generator of such a kernel, namely $\gamma - x$ (where x is the image of γ under the homomorphism).

With this set up, we can define a distinguished subset of $\mathcal{X}(\Lambda, \mathcal{O}_F)$.

Definition 4.3.2. We define $\mathcal{X}_{\mathrm{alg}}(\Lambda, \mathcal{O}_F)$ as the subset of $\mathcal{X}(\Lambda, \mathcal{O}_F)$ consisting of all points $P_{k,\varepsilon}$ specified by $P_{k,\varepsilon}(\gamma) = \varepsilon(\gamma) \cdot (1+p)^k$ for $k \geq 2$ and $\varepsilon : \Gamma \to \mathcal{O}_F^\times$ a finite-order character. Given such a point $P = P_{k,\varepsilon}$, we write $k(P) = k$, $\varepsilon_P = \varepsilon$, and $r(P) = r$ where r is the conductor of ε (i.e. the kernel of ε is $\gamma^{p^{r-1}}\Gamma$).

Our definition of Λ-adic modular forms only mentioned the specializations at $P_k = P_{k,\varepsilon_0}$, for the trivial character ε_0. The Λ-invariance of the associated measure tells us that specializations at other points $P_{k,\varepsilon}$ would have the appropriate transformation property to be a p-adic modular form of the weight and character we'd expect, but it wouldn't let us conclude that these forms are classical. One could give a similar definition requiring classical behavior at some larger subset of the algebraic points; this more restrictive definition would give subspace of Λ-adic forms, and one can analyze its relation to the original space. However, this point is not particularly important for out purposes, so we will continue working with our original definition (only requiring classicality at all but finitely many of the points P_k). Moreover, for *ordinary* forms the two definitions are known to agree.

Next, we consider enlarging our base ring Λ. One way is to expand from $\Lambda_F = \mathcal{O}_F[\![\Gamma]\!]$ to $\Lambda_L = \mathcal{O}_L[\![\Gamma]\!]$ for L/F a finite extension; this is not particularly interesting since everything we've done before works just as well over a different base field from F. More interesting is considering other sorts of extensions of Λ; the most general case one could reasonably work with would be to take finite flat extensions $\mathcal{I}/\Lambda$. For our purposes, it's sufficient to consider extensions $\mathcal{I} = \mathcal{O}_F[\![\Gamma']\!]$ where Γ' is a group containing Γ with finite index; throughout this paper we consider only extensions $\mathcal{I}$ of this type.

Definition 4.3.3. We define $\mathcal{X}_{\text{alg}}(\mathcal{I};\mathcal{O}_F)$ as the set of points $P \in \mathcal{X}(\mathcal{I};\mathcal{O}_F) = \text{Hom}(\mathcal{I},\mathcal{O}_F)$ such that the restriction $P|_\Lambda$ lies in $\mathcal{X}(\Lambda,\mathcal{O}_F)$. We often abuse notation and let $P_{k,\varepsilon}$ denote any point of $\mathcal{X}(\mathcal{I};\mathcal{O}_F)$ lying over the point $P_{k,\varepsilon}$ in $\mathcal{X}_{\text{alg}}(\Lambda,\mathcal{O}_F)$.

For the type of extensions $\mathcal{I}$ we're considering, if the field F is large enough (e.g. contains all e-th roots of unity and e-th roots of $(1+p)$), where e is the exponent of the finite abelian group Γ'/Γ), then there are $[\Gamma' : \Gamma]$ points in $\mathcal{X}_{\text{alg}}(\mathcal{I};\mathcal{O}_F)$ lying over each point $P_{k,\varepsilon}$, which differ from each other by the characters of Γ/Γ'. Therefore, such extensions satisfy conditions (3.1a) and (3.1b) of [Hid88], so the results of that paper apply directly to our context.

Definition 4.3.4. A *$\mathcal{I}$-adic modular form* of level N_0p^r (for $p \nmid N_0$ and $r \geq 1$) and tame character χ (a Dirichlet character modulo N_0p^r) is a formal power series $\mathbf{f} = \sum A_n q^n \in \mathcal{I}[\![q]\!]$ such that, for all but finitely many $k \geq 1$, the following is satisfied:

- For any point $P'_k \in \mathcal{X}_{\text{alg}}(\mathcal{I},\mathcal{O}_F)$ lying over $P_k \in \mathcal{X}_{\text{alg}}(\Lambda,\mathcal{O}_F)$, the formal power series $P_k(\mathbf{f}) = \sum P_k(A_n)q^n$ is in fact a classical modular form lying in the space $M_k(N_0p^r,\chi\omega^{-k};\mathcal{O}_F)$.

If all but finitely many $\mathbf{f}_k$'s are actually cusp forms, we say $\mathbf{f}$ is a $\mathcal{I}$-adic cusp form. We let $\mathbf{M}(N_0p^r,\chi;\mathcal{I})$ denote the set of all $\mathcal{I}$-adic modular forms of level N_0p^r and character χ, and $\mathbf{S}(N_0p^r,\chi;\mathcal{I})$ the set of $\mathcal{I}$-adic cusp forms.

By similar arguments as in the previous section, we could equivalently view $\mathcal{I}$-adic modular forms as $\mathcal{I}$-invariant $\overline{M}(N_0;\mathcal{O}_F)$-valued measures on Γ' (such that all but finitely many of the specializations $\mu_{\mathbf{f}}(P_k)$ are classical of the appropriate weight and character). Also as in the last section, we can define a Hecke algebra $\mathbb{T}(\mathbf{M}(N_0p^r,\chi;\mathcal{I}))$ arising from the action of $\mathbb{T}(\overline{M}(N_0;\mathcal{O}_F))$ on measures, or equivalently from the formula for $T(n)$ on formal q-expansions written there.

4.4. **Ordinary $\mathcal{I}$-adic newforms.** Hida's theory of p-adic modular forms and their families largely focuses on *ordinary* forms. If f is a classical modular form which is an eigenform of the $T(p)$ operator with eigenvalue λ_p, we say f is *ordinary* (or *p-ordinary* to emphasize the prime) if λ_p is a p-adic unit. We work with this idea by using the *ordinary idempotent* operator.

Definition 4.4.1. If $\mathbb{T} = \mathbb{T}(M)$ is the Hecke algebra associated to a space of modular forms M over $\mathcal{O}_F$ (for F a p-adic field), we define its *ordinary idempotent* e as the unique idempotent $e \in \mathbb{T}$ such that $eT(p)$ is a unit in $e\mathbb{T}$ and $(1-e)T(p)$ is topologically nilpotent in $(1-e)T(p)$. We define the ordinary Hecke algebra $\mathbb{T}^{\text{ord}}$ to be the direct factor $e\mathbb{T}$, and the ordinary subspace M^{ord} of M to be the image $e[M]$.

For classical spaces $M_k(N_0p^r,\chi;\mathcal{O}_F)$, one can construct this e by noting that the Hecke algebra is finite-dimensional over F and thus decomposes as a finite product of local rings; thus e can just be the projection onto those local rings in which $T(p)$ acts as a unit. By taking inverse limits we can obtain an e for $\overline{M}(N_0;\mathcal{O}_F)$ and then we can further get one for $\mathbf{M}(N_0p^r,\chi;\mathcal{I})$ by using the surjection of Hecke algebras we have. Alternatively, we can define $e = \lim_{n\to\infty} T(p)^{n!}$ (interpreted in an appropriate way in each of our contexts). However we define it, we note that the e's are compatible between inclusions of different spaces of p-adic modular forms, and commute with specializations of $\mathcal{I}$-adic forms (i.e. $P(e\mathbf{f}) = e \cdot P(\mathbf{f})$). Also, all of these statements are the same cusp forms in place of holomorphic modular forms.

An important type of ordinary $\mathcal{I}$-adic modular form is one such that the specializations are classical newforms; we quote Hida's results on such forms from [Hid88] (translated into our setup). Naively, we might want to say $\mathbf{f}$ is a $\mathcal{I}$-adic newform if all of the specializations $P_{k,\varepsilon}(\mathbf{f})$ are actual newforms. This is sufficient for the case where all of the newforms truly do have p dividing their level; however, we will be interested in p-adic families related to newforms of a level N_0 prime to p. Here there's evidently a problem: a newform of level N_0 is an eigenform of the $T(p)$ operator for such a prime-to-p level, which differ from the $T(p)$ operators on $\overline{M}(N_0;\mathcal{O}_F)$ induced from $M_k(N_0p^r;\mathcal{O}_K)$. However, if we have a p-ordinary eigenform of level N_0, it turns out there's a canonical way to associate to it a form of level N_0p that's an eigenform for the Hecke algebra of that level.

Lemma 4.4.2. *Suppose $p \nmid N_0$ and $f \in S_k(N_0,\chi)$ is a p-ordinary eigenform of the prime-to-p Hecke operator $T(p) \in \mathbb{T}(S_k(N_0,\chi))$. Then the polynomial*

$$x^2 - a_p(f)x + p^{k-1}\chi(p)$$

has roots α and β with $|\alpha|_p = 1$ and $|\beta|_p < 1$, respectively, and the space $U(f) = \mathbb{C}f(z) \oplus \mathbb{C}f(pz) \subseteq S_k(N_0p,\chi)$ contains two forms

$$f^\sharp(z) = f(z) - \beta f(pz) \qquad\qquad f^\flat(z) = f(z) - \alpha f(pz)$$

which are eigenforms of the $T(p) \in \mathbb{T}(S_k(N_0p,\chi))$ with eigenvalues α and β, respectively.

In particular, the form $f^\sharp$ is p-ordinary; we call it the *p-stabilization* of the p-ordinary eigenform f.

Proof. By definition of f being an eigenform of the operator $T(p)$ of level N_0, we have

$$a_pf(z) = (T(p)f)(z) = p^{k-1}\chi(p)f(pz) + \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{z+i}{p}\right).$$

We want to solve for what constants γ the linear combination $g(z) = f(z) - \gamma f(pz)$ is an eigenform of the Hecke operator $T(p)$ of level N_0p given by $(T(p)g)(z) = \frac{1}{p}\sum_{i=0}^{p-1} g\left(\frac{z+i}{p}\right)$. Thus we want to solve for λ, γ such that

$$\lambda\big(f(z) - \alpha f(pz)\big) = \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{z+i}{p}\right) - \gamma\frac{1}{p}\sum_{i=0}^{p-1} f\left(p\frac{z+i}{p}\right).$$

Note that $f(p(z+b)/p) = f(z+i) = f(z)$, so the latter sum just reduces to $pf(z)$. Meanwhile, our original formula resulting from f being an eigenform tells us that the former sum is equal to $a_pf(z) - p^{k-1}\chi(p)f(pz)$. Thus we conclude $f(z) - \gamma f(pz)$ has eigenvalue λ iff we have the equality

$$\lambda f(z) - \lambda\gamma f(pz) = a_pf(z) - p^{k-1}\chi(p)f(pz) - \gamma f(z).$$

Since $f(z)$ and $f(pz)$ are linearly independent as functions $\mathbb{H} \to \mathbb{C}$, this holds iff λ, α satisfy the equations $\lambda = a_p - \gamma$ and $\lambda\gamma = p^{k-1}\chi(p)$. The solutions are exactly γ satisfying $\gamma^2 - a_p\gamma + p^{k-1}\chi(p) = 0$. This quadratic equation in γ has two solutions, and we know one must not be a p-adic unit (since the product $p^{k-1}\chi(p)$ isn't) but the other one must be (since the sum a_p is); let β denote the non-unit root and α denote the unit root. Then we find that the forms $f^\sharp$ and $f^\flat$ we've defined are eigenforms with eigenvalues α and β, respectively, and these are the only possible ones (up to scalars) since we've found two distinct eigenvalues for a two-dimensional space, as desired. ☐

We can then quote Hida's characterization of $\mathcal{I}$-adic newforms, translated into our setup.

Theorem 4.4.3. *For an ordinary $\mathcal{I}$-adic eigenform $\mathbf{f} \in \mathbf{S}^{\mathrm{ord}}(N_0p,\chi;\mathcal{I})$, the following are equivalent:*

- *$P_{k,\varepsilon}(\mathbf{f})$ is a newform of level $N_0p^{r(P)}$ for any element $P_{k,\varepsilon} \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_F)$ such that the p-part of $\varepsilon_P\chi\omega^{-k(P)}$ is nontrivial.*
- *$P_{k,\varepsilon}(\mathbf{f})$ is a newform of level $N_0p^{r(P)}$ for every element $P_{k,\varepsilon} \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I};\mathcal{O}_F)$ such that the p-part of $\varepsilon_P\chi\omega^{-k(P)}$ is nontrivial.*

Moreover, if these conditions are satisfied and P is a point such that $\varepsilon_P\chi\omega^{-k(P)}$ is trivial (which forces $r(P) = 1$), then either $P(\mathbf{f})$ is actually a newform and $k(P) = 2$, or $P(\mathbf{f}) = f^\sharp$ is the p-stabilization of a p-ordinary newform f of level N_0. Such a $\mathcal{I}$-adic newform induces a $\mathcal{I}$-algebra homomorphism

$$\lambda_{\mathbf{f}} : \mathbb{T}(\mathbf{S}^{\mathrm{ord}}(N_0p,\chi;\mathcal{I})) \to \mathcal{I}$$

given by $T \mapsto a(1,f|T)$.

Given this data, we say that $\mathbf{f}$ is a *$\mathcal{I}$-adic newform* of level N_0 and character χ. (We emphasize that the level is N_0, even if we have that $\mathbf{f}$ is an element of $\mathbf{S}^{\mathrm{ord}}(N_0p,\chi;\mathcal{I})$, to emphasize that some of the specializations actually come from newforms of level N_0).

4.5. **Modules of congruence.** Next, we recall the concept of the *module of congruence* of a newform (both classical and $\mathcal{I}$-adic), translating the results of Chapter 4 of [Hid88]. We start off with the classical case. Suppose that $f \in S_k^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{O}_F)$ is a p-stabilized newform (i.e. a newform of level divisible by p, or the p-stabilization of a newform of level not divisible by p). The classical theory of newforms lets us recover:

Proposition 4.5.1. *Given a p-stabilized newform $f \in S_k^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{O}_F)$ as above, let $\lambda_f : \mathbb{T}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{O}_F) \to \mathcal{O}_F$ be the associated algebra homomorphism. Then we have a F-algebra decomposition*

$$\mathbb{T}_k^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{O}_F) \otimes F = \mathbb{T}(f)_F \oplus \mathbb{T}(f)_F^{\perp}$$

where $\mathbb{T}(f)_F^{\perp}$ is the kernel of $\lambda_f \otimes F$, the other direct factor satisfies $\mathbb{T}(f)_F \cong F$, and the projection $\mathbb{T}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{O}_F) \to \mathbb{T}(f)$ corresponds to $\lambda_f \otimes F$ under this isomorphism.

The multiplicative identity in $\mathbb{T}(f)_F$ is an idempotent in $\mathbb{T}_k^{\mathrm{ord}}(N_0p^r, \chi; F)$ that we denote 1_f. We also define:

Definition 4.5.2. Let f be a p-stabilized newform as above. Let $\mathbb{T}(f)_{\mathcal{O}}$ and $\mathbb{T}(\mathbf{f})_{\mathcal{O}}^{\perp}$ be the projections of $\mathbb{T}_k^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{O}_F)$ onto $\mathbb{T}(f)_F$ and $\mathbb{T}(f)_F^{\perp}$, respectively. Then define the *module of congruences* for f as the quotient $\mathcal{O}_F$-module

$$C(f) = \frac{\mathbb{T}(f)_{\mathcal{O}} \oplus \mathbb{T}(f)_{\mathcal{O}}^{\perp}}{\mathbb{T}_k^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{O}_F)}.$$

We can check that $C(f) \cong \mathcal{O}_F/H_f\mathcal{O}_F$ for some element $H_f \in \mathcal{O}_F$ (unique up to units), which we call the *congruence number* of f.

Next, we carry out the same process for $\mathcal{I}$-adic forms. Suppose $\mathbf{f}$ is an ordinary $\mathcal{I}$-adic newform of level N_0 and character χ; for any level N_0p^r we can obtain an associated algebra homomorphism $\lambda_{\mathbf{f}} : \mathbb{T}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I}) \to \mathcal{I}$. Let $Q(\mathcal{I})$ be the quotient field of $\mathcal{I}$, and by abuse of notation also let $\lambda_{\mathbf{f}}$ denote the extended homomorphism $\mathbb{T}^{\mathrm{ord}}(N_0p, \chi; \mathcal{I}) \otimes_{\mathcal{I}} Q(\mathcal{I}) \to Q(\mathcal{I})$. Hida proves a direct sum decomposition of this algebra (really of $\mathbb{T}(\mathbf{S}^{\mathrm{ord}}(N_0; \mathcal{O}_F)) \otimes_{\Lambda} Q(\mathcal{I})$, but it descends to the one we want):

Theorem 4.5.3. *Given a $\mathcal{I}$-adic newform $\mathbf{f}$ and the associated homomorphism $\lambda_{\mathbf{f}}$ as above, and for any r, we have a $Q(\mathcal{I})$-algebra decomposition*

$$\mathbb{T}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I}) \otimes Q(\mathcal{I}) = \mathbb{T}(\mathbf{f})_Q \oplus \mathbb{T}(\mathbf{f})_Q^{\perp}$$

where $\mathbb{T}(\mathbf{f})_Q^{\perp}$ is the kernel of $\lambda_{\mathbf{f}}$, the other direct factor satisfies $\mathbb{T}(\mathbf{f})_Q \cong Q(\mathcal{I})$, and the projection $\mathbb{T}^{\mathrm{ord}}(N_0p, \chi; \mathcal{I}) \otimes Q(\mathcal{I}) \to \mathbb{T}(\mathbf{f})_Q$ corresponds to $\lambda_{\mathbf{f}}$ under this isomorphism.

As before, we let $1_{\mathbf{f}}$ denote the idempotent of $\mathbb{T}(\mathbf{f})_F$, and we also let $\mathbb{T}(\mathbf{f})_{\mathcal{I}}$ and $\mathbb{T}(\mathbf{f})_{\mathcal{I}}^{\perp}$ denote the images of $\mathbb{T}^{\mathrm{ord}}(N_0p, \chi; \mathcal{I})$ under the projections to $\mathbb{T}(\mathbf{f})_Q$ and $\mathbb{T}(\mathbf{f})_Q^{\perp}$, respectively. Furthermore, we can check that these definitions process are compatible with specialization:

Proposition 4.5.4. *Suppose that $\mathbf{f}$ is a $\mathcal{I}$-adic newform as above, and that we fix a point $P \in \mathcal{X}(\mathcal{I}; \mathcal{O}_F)_{\mathrm{alg}}$. Then the inclusion*

$$\mathbb{T}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I}) \hookrightarrow \mathbb{T}(\mathbf{f})_{\mathcal{I}} \oplus \mathbb{T}(\mathbf{f})_{\mathcal{I}}^{\perp}$$

induces an isomorphism when we localize at the prime ideal generated by P which, when we take a quotient by P, passes to the decomposition associated to $\mathbf{f}_P$ by Proposition 4.5.1. Thus $1_{\mathbf{f}}$ projects to $1_{\mathbf{f}_P}$ under the surjection from $\mathbb{T}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I})$ to the appropriate Hecke algebra for $\mathbf{f}_P$.

We can then define congruence modules for $\mathbf{f}$, and state Hida's theorem that they are compatible with the ones for the specializations $\mathbf{f}_P$. For technical reasons, we introduce the notation that

$$\widetilde{\mathbb{T}}(\mathbf{f})_{\mathcal{I}}^{\perp} = \bigcap_{\mathfrak{p}} (\mathbb{T}(\mathbf{f})_{\mathcal{I}}^{\perp})_{\mathfrak{p}}$$

where $\mathfrak{p}$ runs over all prime ideals of height 1 in $\mathcal{I}$, with this intersection taken inside of $\mathbb{T}(\mathbf{f})_Q^{\perp}$. Clearly $\mathbb{T}(\mathbf{f})_{\mathcal{I}}^{\perp} \subseteq \widetilde{\mathbb{T}}(\mathbf{f})_{\mathcal{I}}^{\perp}$.

Definition 4.5.5. Given a $\mathcal{I}$-adic newform $\mathbf{f}$ as above, define the *modules of congruences* for $\mathbf{f}$ as

$$\mathcal{C}_0(\mathbf{f}; \mathcal{I}) = \frac{\mathbb{T}(\mathbf{f})_{\mathcal{I}} \oplus \mathbb{T}(\mathbf{f})_{\mathcal{I}}^{\perp}}{\mathbb{T}^{\mathrm{ord}}(N_0p, \chi; \mathcal{I})} \qquad \mathcal{C}(\mathbf{f}; \mathcal{I}) = \frac{\mathbb{T}(\mathbf{f})_{\mathcal{I}} \oplus \widetilde{\mathbb{T}}(\mathbf{f})_{\mathcal{I}}^{\perp}}{\mathbb{T}^{\mathrm{ord}}(N_0p, \chi; \mathcal{I})}.$$

By the second isomorphism theorem, our modules of congruence (which are defined in terms of a Hecke algebra that's a quotient of the one Hida uses) are isomorphic to Hida's. Then, translating [Hid88] Theorem 4.6 into our setup gives:

Theorem 4.5.6. *Fix a $\mathcal{I}$-adic newform* $\mathbf{f}$ *as above, and let R be a local ring of $\mathbb{T}(\overline{S}^{\mathrm{ord}}(N_0p;\mathcal{O}_F))$ through which $\lambda_{\mathbf{f}}$ factors. Suppose that R is Gorenstein, i.e. that $R \cong \mathrm{Hom}_{\mathcal{I}}(R,\mathcal{I})$ as an R-module. Then we have $\mathcal{C}_0(\mathbf{f};\mathcal{I}) = \mathcal{C}(\mathbf{f};\mathcal{I}) \cong \mathcal{I}/H_{\mathbf{f}}\mathcal{I}$ for a nonzero element $H_{\mathbf{f}} \in \mathcal{I}$, and for any $P \in \mathcal{X}_{alg}(\mathcal{I},\mathcal{O}_F)$ with $k(P) \geq 2$ we have a canonical isomorphism*

$$\mathcal{C}_0(\mathbf{f};\mathcal{I}) \otimes_{\mathcal{I}} (\mathcal{I}/P\mathcal{I}) \cong C(\mathbf{f}_P).$$

So, from now on, if we're given a $\mathcal{I}$-adic modular form $\mathbf{f}$ (for which the Gorenstein condition above is satisfied), we'll let $H_{\mathbf{f}}$ denote any element $\mathcal{I}$ such that $\mathcal{C}_0(\mathbf{f};\mathcal{I}) \cong \mathcal{I}/H_{\mathbf{f}}\mathcal{I}$ and call it a *congruence number* for $\mathbf{f}$. The theorem above says that for any P, the specialization $H_{\mathbf{f},P}$ we get by projecting $H_{\mathbf{f}}$ to $\mathcal{I}/P\mathcal{I}$ serves as a congruence number for $\mathbf{f}_P$, i.e. $H_{\mathbf{f},P} = H_{\mathbf{f}_P}$. However, there is some subtlety here: even though Hida's work gives us a way to realize $H_{\mathbf{f}_P}$ from a special value of an adjoint L-function for any given P, it's only determined up to a unit, and it's not immediately clear how to choose the units to fit the special values into a p-adic family. Instead, we just know that a family $H_{\mathbf{f}}$ exists and that $H_{\mathbf{f},P}$ is a p-adic unit times Hida's L-value formula. To be able to write $H_{\mathbf{f}}$ explicitly in terms of L-values amounts to showing we can construct a p-adic L-function interpolating the adjoint L-values in question. In general this should be able to be recovered as a consequence of the modularity lifting apparatus developed by Wiles. In the special case we'll deal with (where $\mathbf{f}$ comes from a family of CM modular forms) we will show that the main conjecture of Iwasawa theory for imaginary quadratic fields (proven by Rubin) is enough to write $H_{\mathbf{f}}$ as an explicit p-adic L-function associated to a Hecke character.

5. Families of Hecke characters and CM forms

In this section we discuss how to construct $\mathcal{I}$-adic families of p-adic Hecke characters we'll be concerned with, as well as the associated $\mathcal{I}$-adic CM forms. Here is where we may be forced to actually use an extension $\mathcal{I}$ rather than Λ itself, due to the p-part of the class group of our imaginary quadratic field K.

5.1. **A p-adic Hecke character.** To build our $\mathcal{I}$-adic families of p-adic Hecke characters, we start by constructing a single p-adic Hecke character $\alpha : \mathbb{I}_K/K^\times \to \overline{\mathbb{Q}}_p^\times$ with certain properties. To start, we want this character to be unramified outside of p; this means α should be trivial on a large quotient of the ideles, in particular the one fitting into the short exact sequence

$$1 \to (\mathcal{O}_{\mathfrak{p}}^\times \times \mathcal{O}_{\overline{\mathfrak{p}}}^\times)/\mathcal{O}_K^\times \to \mathbb{I}_K/C \to \mathrm{Cl}_K \to 1$$

coming from the usual inclusion of $\mathcal{O}_{\mathfrak{p}}^\times \times \mathcal{O}_{\overline{\mathfrak{p}}}^\times$ into $\mathbb{I}_K$, where C is some closed subgroup of $\mathbb{I}_K$ containing $K^\times$. Now, we have

$$\frac{\mathcal{O}_{\mathfrak{p}}^\times \times \mathcal{O}_{\overline{\mathfrak{p}}}^\times}{\mathcal{O}_K^\times} \cong \frac{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times}{\mathcal{O}_K^\times} \cong \frac{(\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times}{\delta[\mathcal{O}_K^\times]} \times (1+p)^{\mathbb{Z}_p} \times (1+p)^{\mathbb{Z}_p}.$$

So by passing to a larger quotient of $\mathbb{I}_K$, we get a short exact sequence

$$1 \to (1+p)^{\mathbb{Z}_p} \to \mathbb{I}_K/C' \to \mathrm{Cl}_K \to 1$$

where the factor of $(1+p)^{\mathbb{Z}_p}$ includes into $\mathcal{O}_{\mathfrak{p}}^\times \subseteq \mathbb{I}_K$ in the usual way. Thus $\mathbb{I}_K/C'$ is an abelian group that's an extension of the finite abelian group Cl_K by something isomorphic to $\mathbb{Z}_p$; this means $\mathbb{I}_K/C'$ is isomorphic to a direct product of a copy of $\mathbb{Z}_p$ (perhaps properly containing our original one) and some finite quotient of Cl_K. Taking a further quotient of $\mathbb{I}_K/C'$ to kill this finite factor, we can finally get to a short exact sequence

$$1 \to (1+p)^{\mathbb{Z}_p} \to \mathbb{I}_K/C'' \to (\mathrm{Cl}_K)_0 \to 1$$

where $(\mathcal{L}_K)_0$ is a cyclic group Z_{p^e} and we extend it by $(1+p)^{\mathbb{Z}_p} \cong \mathbb{Z}_p$ to obtain $\mathbb{I}_K/C'' \cong \mathbb{Z}_p$.

The inverse map $(1+p)^{\mathbb{Z}_p} \to (1+p)^{\mathbb{Z}_p} \hookrightarrow \overline{\mathbb{Q}}_p^\times$ defines a continuous character. Using our short exact sequence we can see we can abstractly extend this to characters $\mathbb{I}_K/C'' \to \overline{\mathbb{Q}}_p^\times$ in p^e different ways, which differ by the p^e characters of $(\mathrm{Cl}_K)_0$. There's no "best" extension to pick, but for our purposes it doesn't matter; we simply take α to be any such extension. Now, $\mathbb{I}_K/C_{K_\infty} \cong \mathbb{Z}_p$ and it contains the original

$(1+p)^{\mathbb{Z}_p} \cong \mathbb{Z}_p$ with index p^c; restricted to this subgroup α is continuous and injective. Thus α itself is continuous (since it's continuous on an open subgroup) and injective (because the kernel needs to intersect $(1+p)^{\mathbb{Z}_p}$ trivially, and no nontrivial subgroup does that). Thus we can conclude it maps isomorphically onto a subgroup $(1+\pi)^{\mathbb{Z}_p} \subseteq \overline{\mathbb{Q}}_p^\times$ for some element π such that $(1+\pi)^{p^e} = (1+p)$. By picking a finite extension $F/\mathbb{Q}_p$ containing this element π (as well as any other things we might want), we can view α and as a character $\mathbb{I}_K/K^\times \to \mathcal{O}_F^\times$.

So we have a Hecke character $\alpha : \mathbb{I}_K/K^\times \to \overline{\mathbb{Q}}_p^\times$ which is algebraic with weight $(1,0)$ (recall our convention is that weight (a,b) means the local factors at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ are $x \mapsto x^{-a}$ and $x \mapsto x^{-b}$, which is why we needed to use the inverse map $(1+p)^{\mathbb{Z}_p} \to \overline{\mathbb{Q}}_p^\times$ rather than the inclusion). By definition, if $x = (x_v) \in \prod \mathcal{O}_K^\times$ we have $\alpha(x) = x_{\mathfrak{p}}^{-1}\omega_{\mathfrak{p}}(x_{\mathfrak{p}})$ where $\omega_{\mathfrak{p}} : (\mathcal{O}_K/\mathfrak{p})^\times \to \mathbb{Z}_p^\times$ is the canonical Teichmüller character for $\mathcal{O}_K$ (which we can then compose with $\iota_\infty \circ \iota_p^{-1}$ to get a canonical Teichmüller character $\omega_{\mathfrak{p}} : (\mathcal{O}_K/\mathfrak{p})^\times \to \mathbb{C}$). It's clear from our construction that α should have conductor $\mathfrak{p}$ and finite-type $\omega_{\mathfrak{p}}^{-1}$, and one can trace back through the definitions to fully verify this. We note that by construction of the Teichmüller character, if ζ is any root of unity in K we have $\omega_{\mathfrak{p}}(\zeta) = \zeta$.

We remark that one could have just as well done this construction with the factor of $(1+p)^{\mathbb{Z}_p}$ sitting inside $\mathcal{O}_{\overline{\mathfrak{p}}}^\times$ instead, and obtained a character with infinity-type $(0,1)$, conductor $\overline{\mathfrak{p}}$, and finite-type $\omega_{\overline{\mathfrak{p}}}^{-1}$. Actually we can recover such a character from α itself! We simply define $\alpha^c : \mathbb{I}_K/K^\times \to (1+\pi)^{\mathbb{Z}_p}$ defined by $\alpha^c = \alpha \circ \widetilde{c}$, where $\widetilde{c} : \mathbb{I}_K/K^\times \to \mathbb{I}_K/K^\times$ is the (evidently continuous) automorphism on ideles induced by the nontrivial automorphism c of the imaginary quadratic field K. On the level of ideals, α^c is given by $\alpha^c(\mathfrak{a}) = \alpha(\overline{\mathfrak{a}})$. We will need to make use of this character α^c in what follows.

5.2. **Families of Hecke characters.** What we mean by a $\mathcal{I}$-adic family of Hecke characters is a continuous homomorphism $\Psi : \mathbb{I}_K/K^\times \to \mathcal{I}^\times$; given this, if we take a point $P \in \mathcal{X}(\mathcal{I}; \mathcal{O}_F)$ and view it as a homomorphism $\mathcal{I} \to \mathcal{O}_F$, the composition $P \circ \Psi$ (the "specialization at P") becomes a p-adic Hecke character $\mathbb{I}_K/K^\times \to \mathcal{O}_F^\times \subseteq \overline{\mathbb{Q}}_p^\times$. In particular, we'll want Ψ to be such that if we specialize at $P_{m,\varepsilon} \in \mathcal{X}(\mathcal{I}; \mathcal{O}_F)$ we end up with an algebraic Hecke character with an infinity type determined by m (say, $(m,0)$ or $(m,-m)$).

We can use the character $\alpha : \mathbb{I}_K/K^\times \to (1+\pi)^{\mathbb{Z}_p} \subseteq \mathcal{O}_F^\times$ constructed in the previous section to define a $\mathcal{I}$-adic family of characters, where we take $\mathcal{I} = \mathcal{O}_F[\![\Gamma']\!]$ for $\Gamma' = (1+\pi)^{\mathbb{Z}_p}$. We simply define a family $\mathcal{A} : \mathbb{I}_K/K^\times \to \mathcal{I}^\times$ as the composition of $\alpha : \mathbb{I}_K/K^\times \to \Gamma'$ with the tautological embedding $\Gamma' \hookrightarrow \mathcal{O}_F[\![\Gamma']\!]$. Continuity is immediate, as both α and the tautological embedding are continuous by construction. Similarly we can define $\mathcal{A}^c$ as the composition of $\alpha^c : \mathbb{I}_K/K^\times$ with the tautological embedding, or equivalently the precomposition of $\mathcal{A}$ with the automorphism $\widetilde{c}$ of $\mathbb{I}_K/K^\times$ induced by the nontrivial automorphism of K.

Given that we've picked a p^e-th root $1+\pi$ of $1+p$, we can define a canonical extension of P_m to $\mathcal{I}$ by requiring $P_m(1+\pi) = (1+\pi)^m$. For this P_m, we have the specialization property $P_m \circ \mathcal{A} = \alpha^m$ because P_m composed with the tautological embedding gives the map $\Gamma' \to \Gamma'$ defined by $x \mapsto x^m$. In addition to working with this canonical extension of P_m to $\mathcal{X}(\mathcal{I}, \mathcal{O}_F)$ defined above, for the purposes of the theory of $\mathcal{I}$-adic modular forms we'll have to work with *all* such extensions. It's easy to see that an arbitrary such extension is of the form $P_{m,\zeta} : \gamma_\pi \mapsto \zeta(1+\pi)^m$ for ζ a p^c-th root of unity. Then if we let $\varepsilon_\zeta : \Gamma' \to \overline{\mathbb{Q}}_p^\times$ be the character taking $\varepsilon_\zeta(\gamma_\pi) = \zeta$ we have $P_{m,\zeta} \circ \mathcal{A} = \varepsilon'_\zeta \alpha^m$ for $\varepsilon'_\zeta = \varepsilon_\zeta \circ \alpha$. Note that the characters ε_ζ are exactly the extensions of the trivial character on Γ to Γ', and that ε'_ζ is an unramified finite-order Hecke character (because α takes the embedded copy of $\widehat{\mathcal{O}}_K \subseteq \mathbb{I}_K$ to Γ, which is killed by ε_ζ).

One type of family of Hecke characters we'll want to use will be translates of $\mathcal{A}$ by a fixed Hecke character. For the purposes of defining $\mathcal{I}$-adic CM forms, we'll be interested in fixing a finite-order character ψ and defining families along the lines of $\Psi = \psi\alpha^{-1}\mathcal{A}$ with specializations $P_m \circ \Psi = \psi\alpha^{m-1}$ of infinity-type $(m-1, 0)$ and finite part $\psi\omega_{\mathfrak{p}}^{1-m}$. We can similarly define families $\Psi' = \psi'(\alpha^c)^{-1}\mathcal{A}^c$, with specializations $P_m \circ \Psi'$ of infinity-type $(0, m-1)$.

A third type of family that will be relevant will be translates of the product family $\mathcal{A}(\mathcal{A}^c)^{-1}$; we see that this has specializations $\alpha^m(\alpha^c)^{-m}$ of weight $(m,-m)$. Translates of $\mathcal{A}(\mathcal{A}^c)^{-1}$ will be the "anticyclotomic families" Ξ of characters that appear in our interpretation of the BDP L-function (as in Theorem 2.2.3). The following lemma tells us that any character ξ in the set $\Sigma_{cc}(\mathfrak{N})$ considered by [BDP13] fits into a family Ξ where an arithmetic progression of specializations also lie in that set.

Lemma 5.2.1. *Let a be an integer, and $\underline{\xi}_a : \mathbb{I}_K/K^\times \to \mathcal{O}_F^\times$ an algebraic Hecke character for K of weight $(a-1, -a+k+1)$. Then there is a family $\Xi : \mathbb{I}_K/K^\times \to \mathcal{I}^\times$ such that if we set $\xi_m = P_m \circ \Xi$ we have:*

- *ξ_a is the character $\underline{\xi}_a$ we started with.*
- *For every m, ξ_m is a character of weight $(m-1, -m+k+1)$.*
- *For $m \equiv a \pmod{p-1}$, ξ_m has the same conductor and same finite-type as ξ_a; for $m \not\equiv a \pmod{p-1}$ the finite-type differs by powers of the Teichmüller characters $\omega_\mathfrak{p}$ and $\omega_{\overline{\mathfrak{p}}}$.*

Of course, the particular infinity-type we've chosen for ξ_m is entirely a convention that will be convenient for us later.

Proof. We simply set

$$\Xi = \underline{\xi}_a \alpha^{-a} (\alpha^c)^a \mathcal{A} (\mathcal{A}^c)^{-1};$$

then by construction $P_a \circ \Xi = \underline{\xi}_a$ and more generally $P_m \circ \Xi = \underline{\xi}_a \alpha^{m-a} (\alpha^c)^{a-m}$. The rest of the statements follow from what we already know about the Hecke characters α and α^c. □

5.3. **Families of CM forms.** Now that we've shown how to define families of Hecke characters for our imaginary quadratic field K, we want to build from them an associated family of CM forms. So, suppose we start with an ideal $\mathfrak{m}$ prime to p, and a finite-order character ψ of conductor either $\mathfrak{m}$ or $\mathfrak{m}\mathfrak{p}$. This forces the $\mathfrak{p}$-part of the finite-type of ψ to be of the form $\omega_\mathfrak{p}^{a-1}$ for some residue class a modulo $p-1$. Then we define $\Psi = \psi \alpha^{-1} \mathcal{A}$, so $\psi_m = P_m \circ \Psi$ is given by $\psi \alpha^{m-1}$, and find that ψ_m has conductor $\mathfrak{m}$ when $m \equiv a \pmod{p-1}$, and conductor $\mathfrak{m}\mathfrak{p}$ otherwise.

Next we need to write down the CM newform associated to the Hecke character ψ_m is. A Hecke character of type $(m-1, 0)$ gives rise to a CM newform of weight m as discussed in Section 3.4. There's a slight wrinkle, though: we've constructed ψ_m as a p-adic Hecke character, while the formula above for defining classical CM forms is done in terms of complex Hecke characters. So we need to transfer the algebraic p-adic character $\psi_m : \mathbb{I}_K/K^\times \to \overline{\mathbb{Q}}_p^\times$ to an algebraic complex character $\psi_{m,\mathbb{C}} : \mathbb{I}_I/K^\times \to \mathbb{C}^\times$. We recall from Section 2.1 that if φ is a p-adic character of weight (a, b) the relation between φ and $\varphi_\mathbb{C}$ is

$$\varphi_\mathbb{C}(x) = (\iota_\infty \circ \iota_p^{-1})\big(\varphi(x) x_\mathfrak{p}^a x_{\overline{\mathfrak{p}}}^b\big) x_\infty^{-a} \overline{x}_\infty^{-b}.$$

It's straightforward to see that if we're given two p-adic Hecke characters φ, φ' then $(\varphi\varphi')_\mathbb{C} = \varphi_\mathbb{C}\varphi'_\mathbb{C}$; thus $\psi_{m,\mathbb{C}}$ will be $\psi_\mathbb{C} \alpha_\mathbb{C}^{m-1}$.

The way that $\psi_{m,\mathbb{C}}$ appears in the construction of the corresponding CM newform $g_{\psi_{m,\mathbb{C}}}$ is that values $\psi_{m,\mathbb{C}}(\mathfrak{a})$ appear in the q-expansion, where $\mathfrak{a}$ runs over ideals prime to the conductor of $\psi_{m,\mathbb{C}}$. Here we interpret $\psi_{m,\mathbb{C}}(\mathfrak{a})$ to mean $\psi_{m,\mathbb{C}}$ evaluated on the idele $x = (x_v)$ with $x_\infty = 1$ and $x_\mathfrak{q} = \pi_\mathfrak{q}^{\mathrm{ord}_\mathfrak{q}(\mathfrak{a})}$ at finite places, where $\pi_\mathfrak{q}$ is a fixed uniformizer of the local field $(\mathcal{O}_K)_\mathfrak{q}$. Then we have

$$\psi_{m,\mathbb{C}}(\mathfrak{a}) = \psi_\mathbb{C}(\mathfrak{a}) \alpha_\mathbb{C}(\mathfrak{a})^{m-1}.$$

Now, ψ is a finite-order character so, if we suppress the embeddings ι_∞ and ι_p, we can just write $\psi_\mathbb{C}(\mathfrak{a}) = \psi(\mathfrak{a})$. For $\alpha_\mathbb{C}$, by definition we have

$$\alpha_\mathbb{C}(\mathfrak{a}) = (\iota_\infty \circ \iota_p^{-1})\big(\alpha(\mathfrak{a}) \pi_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{a})}\big).$$

Since K is unramified at p, we can choose p as our uniformizer $\pi_\mathfrak{p}$; this choice then forces $\alpha(\mathfrak{a})$ to be algebraic so (again suppressing the embeddings) we can write $\alpha_\mathbb{C}(\mathfrak{a}) = \alpha(\mathfrak{a}) p^{\mathrm{ord}_\mathfrak{p}(\mathfrak{a})}$. Thus we conclude

$$\varphi_{m,\mathbb{C}}(\mathfrak{a}) = \psi(\mathfrak{a}) \alpha(\mathfrak{a})^{m-1} p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})}.$$

Thus, the CM newform associated to $\psi_{m,\mathbb{C}}$ is given by

$$g_{\psi_m} = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m})=1} \psi(\mathfrak{a}) \alpha(\mathfrak{a})^{m-1} p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})} q^{N(\mathfrak{a})} \in S_m(dN(\mathfrak{m}), \chi_K \chi_\psi \omega^{1-m})$$

in the case when $m \equiv a \pmod{p-1}$ and

$$g_{\psi_m} = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \psi(\mathfrak{a}) \alpha(\mathfrak{a})^{m-1} q^{N(\mathfrak{a})} \in S_m(dN(\mathfrak{m})p, \chi_K \chi_\psi \omega^{1-m})$$

when $m \not\equiv a \pmod{p-1}$. We note that in both cases the form is p-ordinary (with respect to $\iota_p \circ \iota_\infty^{-1}$), as the coefficient of q^p is $\psi_m(\overline{\mathfrak{p}})$ (a p-adic unit by construction) or $\psi_m(\overline{\mathfrak{p}}) + \psi_m(\mathfrak{p}) p^{m-1} \equiv \psi_m(\overline{\mathfrak{p}})$ in the two cases.

Looking at the formula for g_{ψ_m} when $m \not\equiv a \pmod{p-1}$, it's easy to see how to interpolate this: if we define

$$\mathbf{g}_\Psi = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \Psi(\mathfrak{a}) q^{N(\mathfrak{a})} \in \mathcal{I}[\![q]\!]$$

then $P_m(\mathbf{g}_\Psi) = g_{m,\psi}$ (under our embeddings) for all such m. For a general extension $P_{m,\zeta}$ of P_m, we can similarly see that $P_{m,\zeta}(\mathbf{g}_\Psi)$ is the p-ordinary CM modular form associated to the Hecke character $\varepsilon'_\zeta \psi \alpha^{m-1}$.

It remains to check what $P_m(\mathbf{g}_\Psi)$ is when $m \equiv a \pmod{p-1}$; Theorem 4.4.3 suggests it should be the p-stabilization $g^\sharp_{\psi_m}$ of the CM modular form g_{ψ_m} of prime-to-p level. Indeed this is true, and can be calculated directly. The roots of the Hecke polynomial at p for g_{ψ_m} are $\psi_m(\mathfrak{p})p^{m-1}$ and $\psi_m(\overline{\mathfrak{p}})$, respectively, and the latter one is clearly a unit while the former isn't. So we have

$$g^\sharp_{\psi_m} = g_{\psi,m}(z) - \psi_m(\mathfrak{p})p^{m-1} g_{\psi,m}(pz).$$

But the q-series for the latter term is

$$\psi_m(\mathfrak{p})p^{m-1} \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m})=1} \psi_m(\mathfrak{a}) p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})} q^{N(\mathfrak{a})p} = \sum_{\mathfrak{a}\mathfrak{p}:(\mathfrak{a},\mathfrak{m})=1} \psi_m(\mathfrak{a}\mathfrak{p}) p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a}\mathfrak{p})} q^{N(\mathfrak{a}\mathfrak{p})},$$

so subtracting it off from the sum defining g_{ψ_m} kills all of the terms corresponding to ideals divisible by $\mathfrak{p}$ and leaves us with

$$g^\sharp_{\psi_m} = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \psi_m(\mathfrak{a}) p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})} q^{N(\mathfrak{a})}.$$

But this is $P_m(\mathbf{g}_\Psi)$ by definition. Similarly, we can see that $P_{m,\zeta}(\mathbf{g}_\Psi) = g^\sharp_{\varepsilon'_\zeta \psi_m}$, which is also the p-stabilization of a newform.

Thus every specialization $P_{m,\zeta}(\mathbf{g}_\Psi)$ is a classical modular form, and in fact either an ordinary newform or a p-stabilization of an ordinary newform. So $\mathbf{g}_\Psi$ is a $\mathcal{I}$-adic modular form, and Theorem 4.4.3 lets us conclude:

Proposition 5.3.1. *Let $\mathfrak{m}$ be an ideal coprime to p, and let ψ be a finite-order Hecke character of conductor $\mathfrak{m}$ or $\mathfrak{m}\mathfrak{p}$ such that the $\mathfrak{p}$-part of the finite-type of ψ is $\omega_\mathfrak{p}^{a-1}$. Then we can construct a family of Hecke characters $\Psi : \mathbb{I}_K/K^\times \to \mathcal{I}$ such that $P_m \circ \Psi = \psi\alpha^{m-1}$, and an associated $\mathcal{I}$-adic CM newform*

$$\mathbf{g}_\Psi = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \Psi(\mathfrak{a}) p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})} q^{N(\mathfrak{a})} \in \mathbf{S}^{\mathrm{ord}}(dN(\mathfrak{m}), \chi_K\chi_\psi\omega; \mathcal{I})$$

such that $P_m(\mathbf{g}_\Psi) = g^\sharp_{\psi_m}$ for $m \equiv a \pmod{p-1}$ and $P_m(\mathbf{g}_\Psi) = g_{\psi_m}$ for $m \not\equiv a \pmod{p-1}$. If we view ψ_m as a classical (complex-valued) Hecke character, in the $m \equiv a \pmod{p-1}$ case the roots of the Hecke polynomial are $\alpha_g = \psi_{m,\mathbb{C}}(\overline{\mathfrak{p}})$ and $\beta_g = \psi_{m,\mathbb{C}}(\mathfrak{p})$.

5.4. **Complex conjugates of p-adic Hecke characters.** For the purposes of interpolating Petersson inner products of the form $\langle fg_\varphi, g_\psi\rangle$ (with φ and ψ varying simultaneously in a p-adic family) we will work with the family $\mathbf{g}_\Psi$ constructed above and a "shifted" variant $f\mathbf{g}_\Phi$ we'll define soon. However, Ichino's formula involves the absolute value $|\langle fg_\varphi, g_\psi\rangle|^2$, so in addition to interpolating $\langle fg_\varphi, g_\psi\rangle$ we'll also need to interpolate the complex conjugate

$$\overline{\langle fg_\varphi, g_\psi\rangle} = \langle f^\rho g^\rho_\varphi, g^\rho_\psi\rangle$$

where if $f = \sum a_n q^n$ then $f^\rho = \sum \overline{a}_n q^n$. Recall that if $f \in S_k(N,\chi)$ then $f^\rho \in S_k(N,\overline{\chi})$.

For a fixed p-adic modular form f with algebraic coefficients, we can directly define a p-adic modular form f^ρ using our chosen embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \overline{\mathbb{Q}}_p$. If f is an eigenform for a Hecke operator $T(n)$ with eigenvalue a_n, then f^ρ is an eigenform with eigenvalue $\overline{a}_n$. However, if f is an eigenform for the prime-to-p Hecke operator $T(p)$, we need to be a little careful with about the roots α_f and β_f of the Hecke polynomial (as in Lemma 4.4.2):

Lemma 5.4.1. *Let $f \in S_k(N_0,\chi)$ be a p-ordinary eigenform of the prime-to-p Hecke operator $T(p)$, and let α_f, β_f be the roots of the Hecke polynomial with $|\alpha_f|_p = 1$ and $|\beta_f|_p = 1$. Then f^ρ is also a p-ordinary eigenform of $T(p)$, with $\alpha_{f^\rho} = \overline{\beta}_f$ and $\beta_{f^\rho} = \overline{\alpha}_f$.*

In other words, when we replace f by f^ρ the roots α_f, β_f are replaced by $\overline{\alpha}_f, \overline{\beta}_f$ as we'd expect, but we need to swap them because $\overline{\beta}_f$ is now the one that's a p-adic unit.

Proof. Recall that we know $\alpha_f\beta_f = \chi(p)p^{k-1}$, and also that we have $\alpha_f\overline{\alpha}_f = p^{k-1}$ and $\beta_f\overline{\beta}_f = p^{k-1}$ by the Ramanujan conjecture for newforms (proven in general by [Del71], [Del74], though for our applications where f is a CM form it can be obtained from basic properties of Hecke characters). Since α_f and $\chi(p)$ are p-adic units we obtain $|\beta_f|_p = |\overline{\alpha}_f|_p = |p^{k-1}|_p < 1$, and also $|\overline{\beta}_f|_p = |p^{k-1}|_p/|\beta_f|_p = 1$. Thus $\overline{\beta}_f$ is a p-adic unit while $\overline{\alpha}_f$ is not, and f^ρ is p-ordinary because we can also conclude $\overline{a}_f = \overline{\alpha}_f + \overline{\beta}_f$ is a p-adic unit. ☐

However, for a $\mathcal{I}$-adic family $\mathbf{f}$, there is no way to "conjugate" the coefficients directly; we need to instead conjugate the specializations $P_m(\mathbf{f})$ and come up with a $\mathbf{f}^\rho$ that best interpolates them. For a CM form, we have $g_\psi^\rho = g_{\overline{\psi}}$ where $\overline{\psi}$ is the complex conjugate of the complex Hecke character ψ. So to interpolate this quantity, we need to analyze how complex conjugates of our complex Hecke characters pass back to the p-adic side.

So suppose φ is a p-adic character of weight (a, b), so as above we define the associated complex Hecke character is

$$\varphi_{\mathbb{C}}(x) = (\iota_\infty \circ \iota_p^{-1})\big(\varphi(x)x_{\mathfrak{p}}^a x_{\overline{\mathfrak{p}}}^b\big)x_\infty^{-a}\overline{x}_\infty^{-b}.$$

Since this is a complex character we can consider its complex conjugate

$$\overline{\varphi_{\mathbb{C}}(x)} = (c \circ \iota_\infty \circ \iota_p^{-1})\big(\varphi(x)x_{\mathfrak{p}}^a x_{\overline{\mathfrak{p}}}^b\big)x_\infty^{-b}\overline{x}_\infty^{-a},$$

which is thus a Hecke character of weight (b, a) (where $c : \mathbb{C} \to \mathbb{C}$ is complex conjugation). Transferring back we get an associated p-adic character φ^ρ defined by

$$\varphi^\rho(x) = (\iota_p \circ c \circ \iota_p^{-1})\big(\varphi(x)x_{\mathfrak{p}}^a x_{\overline{\mathfrak{p}}}^b\big)x_{\mathfrak{p}}^{-b}x_{\overline{\mathfrak{p}}}^{-a},$$

where here we abuse notation and let $c : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ be the complex conjugation induced by ι_∞. This φ^ρ satisfies $\varphi_{\mathbb{C}}^\rho = \overline{\varphi_{\mathbb{C}}}$, as we wanted.

It's straightforward to check that $(\varphi_1\varphi_2)^\rho = \varphi_1^\rho\varphi_2^\rho$ for any characters φ_1, φ_2, so therefore if we go to our family of characters ψ_m from above we have

$$\psi_m^\rho = (\psi\alpha^{m-1})^\rho = \psi^\rho(\alpha^\rho)^{m-1}.$$

So to construct a family with specializations ψ_m^ρ, we need to interpolate powers of α^ρ. Unfortunately we cannot do this directly - to mimic the construction of $\mathcal{A}$ from α we need a character which maps onto a subgroup $(1+\pi)^{\mathbb{Z}_p} \subseteq \mathcal{O}_F^\times$, and there's no reason that α^ρ should satisfy this. So we need to compare the character α^c we can interpolate powers of to the character α^ρ we need. Since α has infinity-type $(1, 0)$ and finite-type character $\omega_{\mathfrak{p}}^{-1}$, we see for any ideal we have

$$\alpha(\mathfrak{a})\alpha^c(\mathfrak{a}) = \alpha(\mathfrak{a}\overline{\mathfrak{a}}) = \alpha(N(\mathfrak{a})\mathcal{O}_K) = \omega_{\mathfrak{p}}^{-1}(N(\mathfrak{a}))N(\mathfrak{a}),$$

and thus $\alpha\alpha^c = (\omega^{-1} \circ N_{\mathbb{Q}}^K) \cdot N$ as p-adic Hecke characters (with ω the Teichmüller character and N the p-adic norm character). On the other hand, the complex conjugate satisfies

$$\alpha_{\mathbb{C}}(\mathfrak{a})\alpha_{\mathbb{C}}^\rho(\mathfrak{a}) = |\alpha_{\mathbb{C}}(\mathfrak{a})|^2 = N_{\mathbb{C}}(\mathfrak{a})$$

because $N_{\mathbb{C}}$ is the unique Hecke character of infinity-type $(1, 1)$ that takes values in positive real numbers. So $\alpha_{\mathbb{C}}\alpha_{\mathbb{C}}^\rho = N_{\mathbb{C}}$, and converting back to p-adic characters we have $\alpha\alpha^\rho = N$. Comparing these two equations we find

$$\alpha^c = (\omega^{-1} \circ N_{\mathbb{Q}}^K)\alpha^\rho.$$

Since ω^{p-1} is trivial, we can conclude that $P_m(\mathcal{A}^c)$ is actually equal to $(\alpha^\rho)^m$ in the arithmetic progression $m \equiv 0 \pmod{p-1}$. By multiplying by an appropriate fixed character we can of course shift which arithmetic progression we get:

Lemma 5.4.2. *Suppose ψ is a fixed finite-order character, and Ψ is a family with specializations $\psi_m = P_m \circ \Psi = \psi\alpha^{m-1}$ as considered before. Fix a residue class $a \pmod{p-1}$. Then if we define the family*

$$\Psi^\rho = \psi^\rho(\alpha^\rho)^{a-1}(\alpha^c)^{-a}\mathcal{A}^c$$

it satisfies $P_m \circ \Psi^\rho = \psi_m^\rho$ for $m \equiv a \pmod{p-1}$. Outside of this arithmetic progression, $P_m \circ \Psi^\rho$ is $\psi_m^\rho \otimes \omega^{a-m}$.

Our construction of CM forms from the previous section goes through nearly identically (keeping in mind that α^c has conductor $\overline{\mathfrak{p}}$ rather than $\mathfrak{p}$), as long as we take care with the Teichmüller twists we needed. For our purposes we'll want to choose the arithmetic progression where the twist is trivial to match up with the arithmetic progression on which p doesn't divide the level of our newforms and thus $\mathbf{g}_\Psi$ contains p-stabilizations.

Proposition 5.4.3. *Let $\mathfrak{m}$ be an ideal coprime to p, and let ψ be a finite-order Hecke character of conductor $\mathfrak{m}$ or $\mathfrak{m}\mathfrak{p}$ such that the $\mathfrak{p}$-part of the finite-type of ψ is $\omega_{\mathfrak{p}}^{a-1}$. Let Ψ be the usual family of Hecke characters $\Psi : \mathbb{I}_K/K^\times \to \mathcal{I}$ such that $\psi_m = P_m \circ \Psi = \psi\alpha^{m-1}$, and let Ψ^ρ be the family above such that $P_m \circ \Psi^\rho = \psi_m^\rho$ for $m \equiv a \pmod{p-1}$. Then there is an associated $\mathcal{I}$-adic newform*

$$\mathbf{g}_{\Psi^\rho} = \sum_{\mathfrak{a}:(\mathfrak{a},\overline{\mathfrak{m}\mathfrak{p}})=1} \Psi^\rho(\mathfrak{a}) p^{(m-1)\mathrm{ord}_{\overline{\mathfrak{p}}(\mathfrak{a})}} q^{N(\mathfrak{a})} \in \mathbf{S}^{\mathrm{ord}}(dN(\overline{\mathfrak{m}}), \chi_K \chi_\psi^{-1} \omega^{2a-1}; \mathcal{I})$$

such that $P_m(\mathbf{g}_\Psi) = (g^\rho_{\psi_m})^\sharp = (g^\flat_{\psi_m})^\rho$ for $m \equiv a \pmod{p-1}$ and $P_m(\mathbf{g}_\Psi) = (g_{\psi_m})^\rho \otimes \omega^{a-m}$ for $m \not\equiv a \pmod{p-1}$. If we view ψ_m as a classical (complex-valued) Hecke character, in the $m \equiv a \pmod{p-1}$ case the roots of the Hecke polynomial are $\alpha_{g^\rho} = \overline{\psi}_{m,\mathbb{C}}(\mathfrak{p})$ and $\beta_{g^\rho} = \overline{\psi}_{m,\mathbb{C}}(\overline{\mathfrak{p}})$.

We remark that this is the best we could hope for - the characters of the conjugated forms $(g_{\psi_m})^\rho$ vary as ω^m, but specializations of any $\mathcal{I}$-adic modular form need to vary as ω^{-m}, so a twist is necessary to correct this outside of a single arithmetic progression.

5.5. **Shifts of $\mathcal{I}$-adic forms.** Finally, we take two general operations that one can do with classical modular forms and describe explicitly how we apply these to $\mathcal{I}$-adic forms. The first is replacing a modular form $g(z)$ by $g(Mz/L)$ for M, L natural numbers; if g lies in the space $S_m(N,\chi)$ then one checks $g(Mz/L)$ lies in the space $S_m(\Gamma_0(MN, L), \chi)$. The effect on q-expansions is easy to understand: if $g(z) = \sum a_n q^n$ then $g(Mz/L) = \sum a_n q^{Mn/L}$. We take the notation that $g|_{M/L}$ denotes the modular form $g|_{M/L}(z) = g(Mz/L)$. If $\mathbf{g} = \sum A_n q^n$ is a $\mathcal{I}$-adic modular form, it's clear that to have this effect on specializations we'd just want to use $\sum A_n q^{Mn/L}$.

Lemma 5.5.1. *Suppose M, L are positive integers that are coprime to each other. Let $\mathbf{g} = \sum A_n q^n$ be a $\mathcal{I}$-adic form in $\mathbf{S}(N, \chi; \mathcal{I})$. Then if we define*

$$\mathbf{g}|_{M/L} = \sum A_n q^{Mn/L},$$

we have $\mathbf{g}|_{M/L} \in \mathbf{S}(\Gamma_0(NM, L), \chi; \mathcal{I})$ and moreover $P_m(\mathbf{g}|_{M/L}) = P_m(\mathbf{g})|_{M/L}$.

The second operation is taking a modular form $g(z)$ and replacing it with the multiple $f(z)g(z)$, where f is another fixed modular form. We want to take a $\mathcal{I}$-adic form $\mathbf{g}$ and "shift" it by a constant modular form f. We can certainly multiply together the two formal q-series to get a new series such that applying P_m gives us $f \cdot P_m(\mathbf{g})$, a modular form of weight $m + k$.

This is almost what we need; we just need to modify the series so that the specializations have the correct weight. For Λ-adic forms, we can do this by applying the automorphism $\sigma_{-k} : \Lambda \to \Lambda$ specified by $\sigma_{-k}(1+X) = (1+X)(1+p)^{-k}$, as it's clear that $P_m \circ \sigma_{-k} = P_{m-k}$. To do this in the $\mathcal{I}$-adic setting we need to extend σ_{-k} to an automorphism of $\mathcal{I}$; for the $\mathcal{I}$ we construct in Section 5.2 we can define σ_{-k} explicitly as an automorphism taking γ_π to $\gamma_\pi(1+\pi)^{-k}$ (which satisfies $P_m \circ \sigma_{-k} = P_{m-k}$ for our canonical extensions of P_m to $\mathcal{I}$).

Proposition 5.5.2. *Suppose that we have an extension of σ_{-k} to $\mathcal{I}$ as discussed above. Let $\mathbf{g} = \sum A_n q^n \in \mathbf{S}(\Gamma, \chi; \mathcal{I})$ be a $\mathcal{I}$-adic form, and $f = \sum b_m q^m \in M_k(\Gamma_f, \chi_f; \mathcal{O}_F)$ be a fixed p-adic modular form. Then the product of formal q-series*

$$f \cdot \mathbf{g} = \left(\sum_m b_m q^m \right) \left(\sum_m \sigma_{-k}(A_n) q^n \right)$$

is a $\mathcal{I}$-adic form in $\mathbf{S}(\Gamma \cap \Gamma_f, \chi\chi_f; \mathcal{I})$ with specializations $P_m(f \cdot \mathbf{g}) = f \cdot P_{m-k}(\mathbf{g})$.

6. $\mathcal{I}$-adic Petersson inner products

Now that we've set up the basic framework of $\mathcal{I}$-adic modular forms (for the types of extensions $\mathcal{I}/\Lambda$ discussed), in this section we'll recall results of Hida that allow us to define an element in $\mathcal{I}$ that interpolates the Petersson inner products of specializations of two $\mathcal{I}$-adic cusp forms. This will be our main tool from Hida theory that will let us take the identities we get from Ichino's formula and turn them into an identity of p-adic analytic functions.

6.1. **Hida's linear functional.** Naively, one might want a pairing on the space of $\mathcal{I}$-adic cusp forms that interpolates the Petersson inner products, but this is much to hope for. Instead, if we're given an ordinary $\mathcal{I}$-adic newform $\mathbf{h}$ of level N_0, we can define a linear functional $\ell_{\mathbf{h}} : \mathbf{S}^{\mathrm{ord}}(N_0p, \chi; \mathcal{I}) \to \mathcal{I}$ such that $\ell_{\mathbf{h}}(\mathbf{g})$ interpolates the Petersson inner products of $\mathbf{h}_P$ and $\mathbf{g}_P$. Hida constructs such forms in Section 7 of [Hid88], and we give a construction along the same lines. It's built from the results quoted in Section 4.5, in particular the existence of an idempotent $1_{\mathbf{h}}$ and the congruence number $H_{\mathbf{h}}$.

Proposition 6.1.1. *Let $\mathbf{h} \in \mathbf{S}^{\mathrm{ord}}(N_0p, \chi; \mathcal{I})$ be a $\mathcal{I}$-adic newform of level N_0. Then for any r we can define a linear functional $\ell_{\mathbf{h}} : \mathbf{S}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I}) \to \mathcal{I}$ by the formula*

$$\ell_{\mathbf{h}}(\mathbf{g}) = a(1, 1_{\mathbf{h}}\mathbf{g})H_{\mathbf{h}}$$

where $H_{\mathbf{h}}$ is the congruence number associated to $\mathbf{h}$ and $1_{\mathbf{h}}$ is the idempotent of the Hecke algebra associated to $\mathbf{h}$ (both as defined in the previous section).

Proof. The assignment $\mathbf{g} \mapsto a(1, 1_{\mathbf{h}}\mathbf{g})$ is evidently linear. The only issue is that by definition, $1_{\mathbf{h}}$ is only an element of $\mathbb{T}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I}) \otimes Q(\mathcal{I})$, so we only get a linear functional $\mathbf{S}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I}) \to Q(\mathcal{I})$. However, by definition $H_{\mathbf{h}}$ is exactly what's needed to fix this; it annihilates the congruence module

$$\mathcal{C}_0(\mathbf{h}; \mathcal{I}) = \frac{\mathbb{T}(\mathbf{h})_{\mathcal{I}} \oplus \mathbb{T}(\mathbf{h})_{\mathcal{I}}^{\perp}}{\mathbb{T}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I})};$$

since $1_{\mathbf{h}}$ lies in $\mathbb{T}(\mathbf{h})_{\mathcal{I}}$ by construction, $H_{\mathbf{h}} \cdot 1_{\mathbf{h}}$ is trivial in this quotient, i.e. $H_{\mathbf{h}} \cdot 1_{\mathbf{h}}$ lies in $\mathbb{T}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I})$. So $H_{\mathbf{h}}1_{\mathbf{h}}\mathbf{g} \in \mathbf{S}^{\mathrm{ord}}(N_0p^r, \chi; \mathcal{I})$ and so its first Fourier coefficient is actually in in $\mathcal{I}$. ☐

We remark that if $r < s$ then the linear functional for level N_0p^s restricts to the linear functional for N_0p^r; this follows because we can check that the idempotent $1_{\mathbf{h}}$ for level N_0p^s projects to $1_{\mathbf{h}}$ for level N_0p^r. We can also extend this to a linear functional $\mathbf{S}(N_0p^r, \chi; \mathcal{I}) \to \mathcal{I}$ by precomposing it with the ordinary projector e; on this larger space it's thus given by

$$\ell_{\mathbf{h}}(\mathbf{g}) = a(1, e1_{\mathbf{h}}\mathbf{g})H_{\mathbf{h}}.$$

For simplicity, we'll abuse notation and let $1_{\mathbf{h}} \in \mathbb{T}(N_0p^r, \chi; \mathcal{I})$ denote $e1_{\mathbf{h}}$, since this just amounts to taking the image of $1_{\mathbf{h}} \in \mathbb{T}^{\mathrm{ord}}$ under the inclusion $\mathbb{T}^{\mathrm{ord}} \to \mathbb{T}$ coming from multiplication by e. If $P \in \mathcal{X}(\mathcal{I}, \mathcal{O}_F)_{\mathrm{alg}}$ then we know we have

$$\ell_{\mathbf{h}}(\mathbf{g})_P = a(1, 1_{\mathbf{h}_P}\mathbf{g}_P)H_{\mathbf{h},P}.$$

and the right-hand side is computed in the appropriate space of classical cusp forms (with F coefficients).

6.2. **Realizing the functional as a Petersson inner product.** To write the values $\ell_{\mathbf{h}}(\mathbf{g})_P$ explicitly, we need to compute $a(1, 1_hg)H_h$ for the classical modular forms $h = \mathbf{h}_P$ and $g = \mathbf{g}_P$. We won't say anything about H_h for now, and instead focus the term $a(1, 1_hg)$. In this section we deal with the case where g is has the same level as the newform h, and discuss how to work with g of higher level in the next section. Since $\mathbf{h}$ is an ordinary $\mathcal{I}$-adic newform, h will always be either an ordinary newform (of level N_0p) or a p-stabilization of an ordinary newform (of level N_0); we'll need to deal with those two cases separately. The form g, meanwhile, can be any cusp form of level N_0p.

Our first observation is that the projection 1_h is characterized by the properties that $1_hh = h$, and $1_hh' = 0$ if h' is an eigenform of any Hecke operator t and has a different eigenvalue from h. The latter property follows from noting that if t is such an operator, and $th' = \lambda'h'$ with $\lambda_h(t) \neq \lambda'$, then we can compute

$$\lambda_h(t)1_hh' = t \cdot (1_hh') = 1_h \cdot (th') = 1_h \cdot (\lambda'h') = \lambda'1_hh'$$

which forces $1_hh' = 0$.

If we assume $h \in S_m(N_0p, \chi; F)$ actually has coefficients in a number field $F_0 \subseteq F$, and restrict to elements $g \in S_k(N_0p, \chi; F_0)$ then we can compute $a(1, 1_h g) \in F_0$ by computing this in $\mathbb{C}$ (where we can pass to an element $1_h \in \mathbb{T}_{\mathbb{C}}(S_k(N_0p, \chi))$ characterized by the same property as before). Since $g \mapsto a(1, 1_h g)$ is then a linear functional on $S_k(N_0p, \chi)$, by duality for the Petersson inner product there exists a modular form $h' \in S_k(N_0p, \chi)$ such that $\langle g, h' \rangle = a(1, 1_h g)$ for all g. In fact, it's sufficient to find h' such that $g \mapsto \langle g, h' \rangle$ is a scalar multiple of $g \mapsto a(1, 1_h g)$, and then comparing values at h gives us $a(1, 1_h g) = \langle g, h' \rangle / \langle h, h' \rangle$. It will turn out that h' is either equal to or closely related to h (and its exact form will come out from our analysis). We start by checking:

Lemma 6.2.1. *Let $h \in S_m(N_0p, \chi)$ be either a p-ordinary newform or the p-stabilization of an ordinary newform. Then the unique form $h' \in S_m(N_0p, \chi)$ such that $\langle g, h' \rangle = a(1, 1_h g)$ lies in the prime-to-N_0p Hecke eigenspace $U(h) \subseteq S_m(N_0p, \chi)$ of h.*

Proof. We know $S_k(N_0p, \chi)$ decomposes as an orthogonal direct sum of prime-to-N_0p Hecke eigenspaces, and in fact this is a decomposition of spaces $U(f)$ as g' ranges over all newforms of character χ and level dividing N_0p. Moreover, we know 1_h annihilates these subspaces for every $f \neq h$, and thus $a(1, 1_h -)$ is trivial on all of them. The only way that $\langle -, h' \rangle$ can be trivial on all of them is if h' has no component in each of these spaces $U(f)$, i.e. $h' \in U(h)$. □

We now need to split up into cases based on whether h is a newform or a p-stabilization of one. If $h \in S_m(N_0p, \chi)$ is actually a newform of level N_0p, he multiplicity one theorem tells us that $U(h)$ is one-dimensional, so h' must be a scalar multiple of h itself. Thus we get

Corollary 6.2.2. *Let $h \in S_m(N_0p, \chi)$ be a p-ordinary newform. Then for any $g \in S_m(N_0p, \chi)$ we have*

$$a(1, 1_h g) = \frac{\langle g, h \rangle}{\langle h, h \rangle}.$$

So, for the rest of this section we work with the more involved case of the p-stabilization of a newform. For convenience we change our notation a bit for this case, and now let h denote the original p-ordinary newform in $S_m(N_0, \chi)$. We know that $U(h) \subseteq S_m(N_0p, \chi)$ is two-dimensional, spanned by $h(z)$ and $h(pz)$. Moreover, the operator $T(p)$ of level N_0p has two eigenforms in this space with different eigenvalues; more specifically we take the notation that

$$x^2 - a_p(h)x + p^{m-1}\chi(p) = (x - \alpha_h)(x - \beta_h)$$

with $|\alpha_h|_p = 1$ and $|\beta_h|_p < 1$. By Lemma 4.4.2 the two eigenforms are

$$h^\sharp(z) = h(z) - \beta_h h(pz) \qquad\qquad h^\flat(z) = h(z) - \alpha_h h(z),$$

with eigenvalues α_h and β_h, respectively. Thus $h^\sharp$ spans the one-dimensional ordinary subspace of $U(h)$, and $h^\sharp$ is what actually arises as the specialization of a $\mathcal{I}$-adic newform.

So, we want to compute the linear functional $a(1, 1_{h^\sharp} -)$ on $S_m(N_0p, \chi)$. We can pin this down exactly because we know it's zero on the orthogonal complement of the two-dimensional space $U(h)$, and we also know its behavior on $U(h)$ because $1_{h^\sharp} h^\sharp = h^\sharp$ (by definition) and $1_h h^\flat = 0$ (since $h^\flat$ has a $T(p)$-eigenvalue different from $h^\sharp$). So $a(1, 1_{h^\sharp} -)$ is (up to a scalar) equal to the form $\langle -, h' \rangle$ for some $h' \in U(h) = \mathbb{C}h(z) \oplus \mathbb{C}h(pz)$ that's orthogonal to $h^\flat$. Finding the appropriate linear combination h' is still a bit delicate, though; to do this we'll need to understand how to compute the Petersson inner products $\langle h(z), h(pz) \rangle$, $\langle h(pz), h(z) \rangle$, and $\langle h(pz), h(pz) \rangle$ in terms of $\langle h(z), h(z) \rangle = \langle h, h \rangle$. The starting point is computing the last product; this can be done in great generality by change-of-variables as follows.

Lemma 6.2.3. *Let f, g be two weight-k cusp forms (of any level). Then for any prime p and any integer i, we have*

$$\langle f(pz+i), g(pz+i) \rangle = p^{-k} \langle f(z), g(z) \rangle.$$

Proof. We can pick some common congruence subgroup $\Gamma(M)$ for which all of $f(z)$, $f(pz+i)$, $g(z)$, and $g(pz+i)$ are modular. We have

$$\langle f(pz+i), g(pz+i) \rangle = \frac{1}{[\overline{\Gamma} : \overline{\Gamma}(M)]} \int_{D(M)} f(pz+i) g(pz+i) \operatorname{Im}(z)^k d\mu(z)$$

$$= p^{-k}\frac{1}{[\overline{\Gamma} : \overline{\Gamma}(M)]}\int_{D(M)} f(pz+i)g(pz+i)\,\mathrm{Im}(pz+i)^k d\mu(z),$$

where $d\mu$ is the standard hyperbolic measure on $\mathbb{H}$ and $D(M)$ is a fundamental domain for $\Gamma(M)$. Using change-of-variables on the Möbius transformation $\sigma : z \mapsto pz+i$, under which $d\mu$ is invariant, gives

$$\langle f(pz+i), g(pz+i)\rangle = p^{-k}\frac{1}{[\overline{\Gamma} : \overline{\Gamma}(M)]}\int_{\sigma D(M)} f(z)g(z)\,\mathrm{Im}(z)^k d\mu(z).$$

The set $\sigma D(M)$ is a fundamental domain for the group $\sigma\overline{\Gamma}(M)\sigma^{-1}$, and we can compute that this conjugate still lies in $\overline{\Gamma}$ and has index $[\overline{\Gamma} : \overline{\Gamma}(M)]$. Thus the right hand side of our equation is $p^{-k}\langle f(z), g(z)\rangle$ computed on this new congruence subgroup. □

Returning to our situation where h is an eigenform of level N_0 with $p \nmid N_0$, we can use the result above plus the Hecke relation for $T(p)h$ to work out what $\langle h(z), h(pz)\rangle$ and $\langle h(pz), h(z)\rangle$ are.

Lemma 6.2.4. *Let $h \in S_m(N_0, \chi)$ be a $T(p)$-eigenform with eigenvalue a_p, for a prime $p \nmid N_0$. Then we have*

$$\langle h(z), h(pz)\rangle = \frac{a_p}{p^{m-1}(p+1)}\langle h, h\rangle \qquad \langle h(pz), h(z)\rangle = \frac{\overline{\chi}(p)a_p}{p^{m-1}(p+1)}\langle h, h\rangle.$$

Proof. To compute these, we start by noting that f being an eigenform tells us

$$a_p h(z) = \chi(p)p^{m-1}h(pz) + \frac{1}{p}\sum_{i=0}^{p-1} h\left(\frac{z+i}{p}\right).$$

Applying $\langle -, h(z)\rangle$ to both sides and using linearity in the first coordinate we get

$$a_p\langle h, h\rangle = \chi(p)p^{m-1}\langle h(pz), h(z)\rangle + \frac{1}{p}\sum_{i=0}^{p-1}\left\langle h\left(\frac{z+i}{p}\right), h(z)\right\rangle.$$

The previous lemma plus invariance of h under $z \mapsto z+i$ gives that $\langle h(z), h(pz)\rangle = p^{-m}\langle h(\frac{z+i}{p}), h(z)\rangle$ and thus

$$a_p\langle h, h\rangle = \chi(p)p^{m-1}\langle h(pz), h(z)\rangle + p^m\langle h(z), h(pz)\rangle.$$

An identical computation with $\langle h(z), -\rangle$ (using antilinearity in the second coordinate) tells us

$$\overline{\chi}(p)a_p\langle h, h\rangle = \overline{\chi}(p)p^{m-1}\langle h(z), h(pz)\rangle + p^m\langle h(pz), h(z)\rangle.$$

Here we use the relation $\overline{a}_p = \overline{\chi}(p)a_p$, which follows from the identity $T(p)^* = \chi(p)T(p)$. We can then take these two equations and solve for $\langle h(z), h(pz)\rangle$ and $\langle h(pz), h(z)\rangle$ to obtain the lemma. □

We can then complete our computation of $a(1, 1_h g)$ in the case when we're working with a p-stabilization of a newform.

Proposition 6.2.5. *Suppose $p \nmid N_0$ and $h \in S_m(N_0, \chi)$ is a p-ordinary newform, and let $\alpha = \alpha_h$ and $\beta = \beta_h$ be the roots of $x^2 - a_p(h)x + p^{m-1}\chi(p)$ with $|\alpha|_p = 1$ and $|\beta|_p < 1$, as before. Then the linear functional $a(1, 1_{f^\sharp}-)$ is given by*

$$a(1, 1_{h^\sharp}g) = \frac{\langle g, h^\natural\rangle}{\langle h^\sharp, h^\natural\rangle} \qquad h^\natural = h(z) - p\beta h(pz)$$

Proof. By the above, we know we can write $a(1, 1_{h^\sharp}-) = \langle -, h'\rangle/\langle h^\sharp, h'\rangle$ for any h' in a unique one-dimensional subspace of this. Assuming the subspace isn't $\mathbb{C}h(pz)$ (which will be justified when we find a different subspace that works), we can take $h' = h(z) - Ch(pz)$; since $\langle h^\sharp, h'\rangle/\langle h^\sharp, h'\rangle = 1 = a(1, 1_h-)$ trivially, we just need to choose C such that we have $\langle h^\flat, h'\rangle = 0 = a(1, 1_{h^\sharp}h^\flat)$.

Thus, we want to solve the equation

$$\langle h^\flat, h'\rangle = \langle h(z) - \alpha h(pz), h(z) - Ch(pz)\rangle = 0$$

for C. Expanding this by linearity we get

$$0 = \langle h(z), h(z)\rangle - \alpha\langle h(pz), h(z)\rangle - \overline{C}\langle h(z), h(pz)\rangle + \alpha\overline{C}\langle h(pz), h(pz)\rangle.$$

Substituting in the results of the previous two lemmas, we get

$$0 = \langle h, h\rangle - \alpha\frac{\overline{\chi}(p)a_p}{p^{m-1}(p+1)}\langle h, h\rangle - \overline{C}\frac{a_p}{p^{m-1}(p+1)}\langle h, h\rangle + \alpha\overline{C}p^{-m}\langle h, h\rangle.$$

Rearranging and simplifying we ultimately get $\overline{C} = \overline{\chi}(p)\alpha p$. By the Ramanujan conjecture (which is elementary in the CM case we're interested in), we have $\overline{\chi}(p)\alpha = \overline{\beta}$, so we get that $C = p\beta$ is a solution to our equation, and thus $h^\natural = h(z) - p\beta h(pz)$ satisfies $\langle h^\flat, h^\natural\rangle = 0$. □

We end this section with a few remarks. First of all, in Lemma 4.4.2 we define $h^\sharp$ and $h^\flat$ for any modular form h that's an eigenform of the prime-to-p Hecke operator $T(p)$, so it is reasonable to also define $h^\natural = h(z) - p\beta h(pz)$ for any such h. In later sections we will use this notation in the case where $h(z) = h_0(Lz)$ for h_0 a newform and L an integer prime to p. Such a form remains at prime-to-p level and an eigenform for $T(p)$; note that in this case $h_0^\sharp(Lz)$ and $h^\sharp(z)$ agree and likewise for $\flat$ and $\natural$, so there is no chance for notational confusion.

Also, we recall that Hida makes a similar computation in Proposition 4.5 of [Hid85], and concludes that he can take $h^\natural = h_\rho^\sharp | w_{N_0 p}$, where w_N is the matrix

$$w_N = \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}$$

and if $h = \sum a_n q^n$ we let $h_\rho = \sum \overline{a}_n q^n$. Comparing our results (and using the Atkin-Lehner relation that $h_\rho | w_{N_0}$ is a multiple of h) we find that if h is a p-ordinary newform of level N_0 then $h^\natural$ and $h_\rho^\sharp | w_{N_0 p}$ are scalar multiples of each other. (Quoting Hida's result and doing this would be a quicker way to get our formula for $h^\natural$, but the computations and techniques above will be important later on)

6.3. **Getting Petersson inner products for higher levels.** To summarize the computations above, for a $\mathcal{I}$-adic newform $\mathbf{h} \in \mathbf{S}^{\mathrm{ord}}(N_0 p, \chi; \mathcal{I})$ of level N_0 and an arbitrary $\mathcal{I}$-adic form $\mathbf{g} \in \mathbf{S}(N_0 p, \chi; \mathcal{I})$, the element $\ell_{\mathbf{h}}(\mathbf{g}) \in \mathcal{I}$ has specialization at $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_F)$ given by

$$a(1, 1_h g) = \frac{\langle g(z), h(z)\rangle}{\langle h(z), h(z)\rangle} H_{\mathbf{h},P} \qquad \text{or} \qquad a(1, 1_{h^\sharp} g) = \frac{\langle g(z), h^\natural(z)\rangle}{\langle h^\sharp(z), h^\natural(z)\rangle} H_{\mathbf{h},P}$$

in the cases where $\mathbf{h}_P = h$ is a newform or $\mathbf{h}_P = h^\sharp$ is a p-stabilization of a newform, respectively (and setting $g = \mathbf{g}_P$). In this section we want to expand this construction to allow $\mathbf{g}$ to be a $\mathcal{I}$-adic form in a space $\mathbf{S}(M_0 p^r, \chi; \mathcal{I})$ of higher level, which we'll need for our applications.

First of all, we recall that we've actually defined extensions of our linear functional $\ell_{\mathbf{h}}$ to spaces $\mathbf{S}(N_0 p^r, \chi; \mathcal{I})$ where we allow higher p-power level. So we just need to compute $\ell_{\mathbf{h}}(\mathbf{g})$ and thus $\ell_h(g)$ on higher-level forms g. Fortunately for us there's a trick we can use involving the $T(p)$ operator, as in Sections 3 and 4 of [Hid85]. On one hand $T(p)$ is known to take $S_m(M_0 p^r, \chi)$ into $S_m(M_0 p^{r-1}, \chi)$ for $r \geq 2$. On the other, since h or $h^\sharp$ (as appropriate) is an eigenform for $T(p)$ with eigenvalue α_h (the root of the Hecke polynomial which is a p-adic unit under our embedding into $\overline{\mathbb{Q}}_p$) we can see $1_h T(p) = 1_h \alpha_h$ or $1_{h^\sharp} T(p) = 1_{h^\sharp}\alpha_h$.

Analyzing our functional in these two different ways, we find that if $g \in S_m(M_0 p^r, \chi)$ then we have

$$a(1, 1_h g) = \alpha_h^{-(r-1)} a(1, 1_h T(p)^{r-1} g) = \alpha_h^{-(r-1)} \frac{\langle T(p)^{r-1} g(z), h(z)\rangle}{\langle h(z), h(z)\rangle}$$

or

$$a(1, 1_{h^\sharp} g) = \alpha_h^{-(r-1)} a(1, 1_{h^\sharp} T(p)^{r-1} g) = \alpha_h^{-(r-1)} \frac{\langle T(p)^{r-1} g(z), h^\natural(z)\rangle}{\langle h^\sharp(z), h^\natural(z)\rangle}.$$

Finally, we note that if we view the Petersson inner products here as being over $\Gamma_0(N_0 p^r)$, we can rewrite it in a way that avoids the $T(p)$ operator, using the following lemma.

Lemma 6.3.1. *Suppose f, f' are two elements of $S_k(N_0 p^r, \chi)$ for $r \geq 1$. Then we have*

$$\langle T(p) f, f'\rangle = p^k \langle f(z), f'(pz)\rangle.$$

Proof. We can write out the definition of the Hecke operator and get

$$\langle T(p) f, f'\rangle = \left\langle \sum_{i=0}^{p-1} p^{k/2-1} f\bigg|_k \begin{bmatrix} 1 & i \\ 0 & p \end{bmatrix}, f'\right\rangle = p^{k/2-1} \sum_{i=0}^{p-1} \left\langle f\bigg|_k \begin{bmatrix} 1 & i \\ 0 & p \end{bmatrix}, f'\right\rangle.$$

Now, slash operators don't preserve individual spaces like $S_k(N_0p^r,\chi)$, but they can be viewed as operators on the infinite-dimensional space of all weight-k modular forms. There, it's straightforward to check that if α is a matrix, the slash operator $[\alpha]_k$ has an adjoint given by $[\beta]_k$ where $\beta = \det(A)A^{-1}$ is the adjugate matrix. Thus:

$$\langle T(p)f, f'\rangle = p^{k/2-1}\sum_{i=0}^{p-1}\left\langle f, f'\bigg|_k \begin{bmatrix} p & -i \\ 0 & 1 \end{bmatrix}\right\rangle$$

$$= p^{k/2-1}\sum_{i=0}^{p-1}\langle f(z), p^{k/2}f'(pz-i))\rangle = p^k\langle f(z), f'(pz)\rangle. \quad \square$$

Therefore, we conclude:

Proposition 6.3.2. *Let* $\mathbf{h}$ *be a* $\mathcal{I}$*-adic newform in* $\mathbf{S}(N_0p,\chi;\mathcal{I})$ *with weight-*m *specialization* h_m *or* $h_m^\sharp$ *(as appropriate), and* $\mathbf{g}$ *any* $\mathcal{I}$*-adic form in* $\mathbf{S}(N_0p^r,\chi;\mathcal{I})$ *with weight-*m *specialization* g_m*. Then* $P_m(\ell_{\mathbf{h}}(\mathbf{g}))$ *is equal to*

$$\frac{p^{m(r-1)}}{\alpha_h^{r-1}}\frac{\langle g(z), h(p^{r-1}z)\rangle}{\langle h(z), h(z)\rangle}H_{\mathbf{h},P} \qquad or \qquad \frac{p^{m(r-1)}}{\alpha_h^{r-1}}\frac{\langle g(z), h^\natural(p^{r-1}z)\rangle}{\langle h^\sharp(z), h^\natural(z)\rangle}H_{\mathbf{h},P}$$

as appropriate.

Next we deal with raising the prime-to-p part of the level. In this case $\ell_{\mathbf{h}}$ isn't already defined on $\mathbf{S}(M_0p^r,\chi;\mathcal{I})$, so we have to do some sort of construction to extend it. Here we use some *trace operators*. If $\Gamma_0(M_0p^r, L_0)$ contains $\Gamma_0(N_0p^r)$, and we want to start with a modular form on the former group and obtain one on the latter, we can define a linear function

$$\mathrm{tr}: S_k(\Gamma_0(M_0p^r,L_0),\chi) \to S_k(N_0p^r,\chi) \qquad \mathrm{tr}(f) = \sum_\gamma \chi^{-1}(\gamma)f|[\gamma]_k$$

where γ runs over a set of coset representatives of $\Gamma_0(M_0p^r,L_0)\backslash\Gamma_0(N_0p^r)$. This has the property that if $f \in S_k(\Gamma_0(M_0p^r,L_0),\chi)$ and $f' \in S_k(N_0p^r,\chi)$ we have

$$\langle \mathrm{tr}(f), f'\rangle = \sum_\gamma \chi^{-1}(\gamma)\langle f|[\gamma]_k, f'\rangle = \sum_\gamma \langle f, \chi(\gamma)f'|[\gamma^{-1}]_k\rangle = \sum_\gamma \langle f, f'\rangle.$$

Thus the resulting sum is just $[\Gamma_0(M_0p,L_0):\Gamma_0(N_0p)]$ times $\langle f, f'\rangle$, so we have

$$\langle \mathrm{tr}(f), f'\rangle = \langle f, f\rangle\frac{M_0L_0}{N_0}\prod_{q|M_0L_0, q\nmid N_0}(1+q^{-1}).$$

It's straightforward that this linear transformation on complex vector spaces preserves K_0-rationality, so would pass to a transformation

$$\mathrm{tr}: S_k(\Gamma_0(M_0p^r,L_0),\chi;K) \to S_k(N_0p^r,\chi;K).$$

Furthermore, in this case it preserves integrality in K. Since we aren't changing the p-power of our level, we can choose our coset representatives to satisfy $\gamma \equiv 1 \pmod p$ and then the action of these γ's matches up with the action of $\mathrm{SL}_2(\mathbb{Z}/M_0p\mathbb{Z})$ discussed in Section 1.IV of [Hid88], which preserve the space $\overline{S}(M_0p;\mathcal{O}_F)$. Thus, if χ is any character of $(\mathbb{Z}/N_0\mathbb{Z})^\times$, we have a trace operator

$$\mathrm{tr}: \overline{S}(\Gamma_0(M_0,L_0);\mathcal{O}_F) \to \overline{S}(M_0;\mathcal{O}_F) \qquad \mathrm{tr}(f) = \sum_\gamma \chi^{-1}(\gamma)f|[\gamma]_k$$

Finally we can extend this to a $\mathcal{I}$-linear operator

$$\mathrm{tr}: \mathbf{S}(\Gamma_0(M_0p^r,L_0),\chi;\mathcal{I}) \to \mathbf{S}(N_0p^r,\chi;\mathcal{I})$$

via our measure-theoretic perspective, defining $\mu_{\mathrm{tr}(\mathbf{g})}(x \mapsto f(x)) = \mathrm{tr}(\mu_{\mathbf{g}}(x \mapsto f(x)))$. We conclude:

Proposition 6.3.3. *Let* $\mathbf{h}$ *be a* $\mathcal{I}$*-adic newform of level* N_0 *and character* χ*, and let* h_m *be the newform associated to* $P_m(\mathbf{h})$ *(so either* $h_m = P_m(\mathbf{h})$ *is of level* N_0p*, or* h_m *is of level* N_0 *and* $P_m(\mathbf{h}) = h_m^\sharp$*) and* α_h

the $T(p)$-eigenvalue of $P_m(\mathbf{h})$. If $\mathbf{g} \in \mathbf{S}(\Gamma_0(M_0p^r, L_0), \chi; \mathcal{I})$ is any $\mathcal{I}$-adic form (for $r \geq 1$ and M_0 a multiple of N_0), and we let $g_m = P_m(\mathbf{g})$, then the specialization of the element $\ell_{\mathbf{h}}(\mathrm{tr}(\mathbf{g})) \in \mathcal{I}$ at P_m is given by

$$C\frac{\langle g_m(z), h_m(p^{r-1}z)\rangle}{\langle h_m(z), h_m(z)\rangle}H_{\mathbf{h},P} \qquad \textit{or} \qquad C\frac{\langle g_m(z), h_m^{\natural}(p^{r-1}z)\rangle}{\langle h_m^{\sharp}(z), h_m^{\natural}(z)\rangle}H_{\mathbf{h},P}$$

in the cases where $P_m(\mathbf{f})$ is h_m or $h_m^{\sharp}$, respectively, where the constant is

$$C = \frac{p^{m(r-1)}}{\alpha_h^{r-1}}\frac{M_0L_0}{N_0}\prod_{q|M_0L_0, q\nmid N_0}(1+q^{-1}).$$

6.4. **Euler factors from p-stabilizations.** From the previous section, we've seen that our element $\ell_{\mathbf{h}}(\mathbf{g}) \in \mathcal{I}$ interpolating Petersson inner products will have specializations of the form

$$\frac{\langle g, h^{\natural}\rangle}{\langle h^{\sharp}, h^{\natural}\rangle}$$

when we specialize at points P where $\mathbf{h}_P$ is the p-stabilization of a newform h. Similarly, g may also be a p-stabilization of a newform (or related to one). On the other hand, our automorphic formulas usually only involve the newforms themselves. So we'd like to relate the ratio above to one not involving p-stabilizations.

More specifically, to fit the setup and notation we'll need later on, we'll take the following setup. We will take f, g, h to be modular forms of weights k, $m-k$, and m, respectively, such that the central characters satisfy $\chi_f\chi_g = \chi_h$. We assume the forms g and h are of prime-to-p level, and are p-ordinary eigenforms of the prime-to-p Hecke operator $T(p)$. On the other hand, f is not assumed to be either p-ordinary or to have prime-to-p level; but we do require that one of the following conditions two holds for it:

- f is of prime-to-p level, and an eigenform for the prime-to-p Hecke operator $T(p)$.
- f has level exactly divisible by p^r for some $r > 1$, f is an eigenform for the p-level Hecke operator $T(p)$, and f is new at level p^r.

The reason we specify "eigenforms of $T(p)$" rather than "newforms" in our setup is because we will need to allow forms like $h_0(Lz)$ where h_0 is an actual newform and L is a prime-to-p integer; such forms do indeed remain eigenforms for the appropriate $T(p)$. Also, for our purposes f being "new at level p^r" means that f is perpendicular to $\psi(z)$ and $\psi(pz)$ for any ψ of level N_0p^{r-1}. Thus if f_0 is a newform of level N_0p^r and L is coprime to p, $f_0(Lz)$ is still new at p^r.

Given this setup, we will need to compute an explicit constant $(*)$ such that we have an equality

$$\frac{\langle f(z)g^{\sharp}(z), h^{\natural}(p^{r-1}z)\rangle}{\langle h^{\sharp}(z), h^{\natural}(z)\rangle} = (*)\frac{\langle f(z)g(z), h(p^{r_0}z)\rangle}{\langle h(z), h(z)\rangle}$$

where p^{r_0} is the exact power of p dividing the level of f and $r = \max\{1, r_0\}$. The constant $(*)$ will turn out to be the Euler factor at p which needs to be removed from the p-adic L-function we're constructing. It will be described in terms of the roots of the Hecke polynomial at p for f, g, and h. Such roots will always include $\alpha_g, \beta_g, \alpha_h, \beta_h$ in the notation of Lemma 4.4.2. For f, we will have α_f, β_f determined similarly if $r_0 = 0$ (though neither will necessarily be a p-adic unit), or just a_f if $r_0 \geq 1$ (which will satisfy $a_f = 0$ if $r_0 \geq 2$ in the cases we're considering).

We break the computation of $(*)$ up into three parts: one relating the denominators on both sides, the second relating the numerators if p does not divide the level of f, and the last relating the numerators when p does divide the level of f. This computation is related to the one carried out in Section 4 of [DR14].

6.4.1. *Relating $\langle h^{\sharp}, h^{\natural}\rangle$ to $\langle h, h\rangle$.* We start by working with the denominator of the expression, since here the computation is an extension of what we've already done in the previous section. Expanding out the definition of $h^{\sharp}$ and $h^{\natural}$ tells us that $\langle h^{\sharp}, h^{\natural}\rangle$ equals

$$\langle h(z), h(z)\rangle - \beta_h\langle h(pz), h(z)\rangle - p\overline{\beta}_h\langle h(z), h(pz)\rangle + p\beta_h\overline{\beta}_h\langle h(pz), h(pz)\rangle.$$

We can use Lemmas 6.2.3 and 6.2.4 to express each inner product in terms of $\langle h, h\rangle = \langle h(z), h(z)\rangle$ to obtain

$$\langle h^{\sharp}, h^{\natural}\rangle = \langle h, h\rangle\left(1 - \frac{\overline{\chi}(p)\beta_h a_p(f)}{p^{k-1}(1+p)} - \frac{p\overline{\beta}_h a_p(f)}{p^{k-1}(1+p)} + \frac{p\beta_h\overline{\beta}_h}{p^k}\right).$$

Now, we can get rid of the complex conjugates via the Ramanujan conjecture: we know $\overline{\beta}_h = \overline{\chi}(p)\alpha_h$ and also $|\beta_h|^2 = p^{k-1}$ (so the rightmost term is identically 1). Thus the term in parentheses simplifies to

$$2 - \frac{\overline{\chi}(p)(\beta_h + p\alpha_h)a_p(f)}{p^{k-1}(1+p)}.$$

Using that $p^{k-1} = \overline{\chi}(p)\alpha_h\beta_h$, that $a_p(f) = \alpha_h + \beta_h$, and collecting everything with a common denominator gives

$$\frac{2(1+p)\alpha_h\beta_h - (\beta_h + p\alpha_h)(\alpha_h + \beta_h)}{\alpha_h\beta_h(1+p)} = \frac{\alpha_h\beta_h + p\alpha_h\beta_h - \beta_h^2 - p\alpha_h^2}{\alpha_h\beta_h(1+p)}$$

Cancelling and factoring gives

$$\frac{1 + p - \beta_h/\alpha_h - p\alpha_h/\beta_h}{1+p} = \frac{(1-\beta_h/\alpha_h)(1-p\alpha_h/\beta_h)}{(1+p)}.$$

For our purposes later it's helpful to rearrange this as

Proposition 6.4.1. *With our setup as above (with h a newform of prime-to-p level), we have*

$$\langle h^\sharp, h^\natural\rangle = \langle h, h\rangle \cdot \frac{(-\alpha_h/\beta_h)(1-\beta_h/\alpha_h)(1-p^{-1}\beta_h/\alpha_h)}{(1+p^{-1})}.$$

6.4.2. *Relating $\langle fg^\sharp, h^\natural\rangle$ to $\langle fg, h\rangle$ when the level of f is prime to p.* To simplify notation in this case, we set $\langle fg, h\rangle = \langle f(z)g(z), h(z)\rangle$, $\langle fg_p, h\rangle = \langle f(z)g(pz), h(z)\rangle$, $\langle fg_p, h_p\rangle = \langle f(z)g(pz), h(pz)\rangle$, and so on; with this convention our goal is to express

$$\langle fg^\sharp, h^\natural\rangle = \langle fg, h\rangle - \beta_g\langle fg_p, h\rangle - p\overline{\beta}_h\langle fg, h_p\rangle + p\beta_g\overline{\beta}_h\langle fg_p, h_p\rangle$$

as a constant times $\langle fg, h\rangle$.

To express $\langle fg_p, h_p\rangle$ in terms of $\langle fg, h\rangle$ we will obtain two equations that are linear in the quantities $\langle fg_p, h_p\rangle$, $\langle f_pg, h\rangle$, and $\langle fg, h\rangle$ and solve. The first equation comes from the Hecke eigenvalue relation for f:

Lemma 6.4.2. *In our setup, we have*

$$(\alpha_f + \beta_f)\langle fg, h\rangle = \alpha_f\beta_f\langle f_pg, h\rangle + p^m\langle fg_p, h_p\rangle.$$

Proof. This follows from starting with the equation $a_f f(z) = T(p)f(z)$; expanding out the definition of the Hecke operator we can write this as

$$(\alpha_f + \beta_f)f(z) = \alpha_f\beta_f f(pz) + \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{z+i}{p}\right).$$

We then apply the linear functional $\langle -g, h\rangle$ to both sides of this. Since we can compute $\langle f(\frac{z+i}{p})g(z), h(z)\rangle = p^m\langle f(z)g(pz), h(pz)\rangle$ by substituting $z \mapsto pz - i$ and applying Lemma 6.2.3, we obtain the desired identity. □

To get a second equation we will use the fact that the Petersson inner product is invariant under applying a slash operator to both sides simultaneously; in particular we will apply an Atkin-Lehner operator defined by a matrix

$$\omega_p = \begin{bmatrix} p & 1 \\ Npc & pd \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ Nc & pd \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} = \gamma_p\delta_p$$

where N is the common level of f, g, h (prime to p by assumption) and c, d chosen so this has determinant p (which forces $pd \equiv 1 \pmod N$); we follow the conventions of [Asa76]). Then:

Lemma 6.4.3. *In our setup, we have $\langle f_pg, h\rangle = \overline{\chi}_f(p)p^{m-k}\langle fg_p, h_p\rangle$.*

Proof. By invariance of the Petersson inner product we have $\langle f_pg, h\rangle = \langle f_p|[\omega_p]_k g|[\omega_p]_{m-k}, h|[\omega_p]_m\rangle$ so we just need to determine what these translates are. This is straightforward; h is invariant under $\gamma_p \in \Gamma_1(N)$ so we get $h|[\omega_p]_m = h|[\delta_p]_m = p^{m/2}h(pz)$ and similarly for g. Meanwhile writing $f_p = p^{-k/2}f|[\delta_p]_k$ and computing

$$\delta_p\gamma_p\delta_p = \begin{bmatrix} ap & b \\ Nc & d \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$$

we find $f_p|[\omega_p]_k = p^{-k/2}\overline{\chi}_f(p)f(z)$ as the first matrix is in $\Gamma_0(N)$ so transforms f by $\chi_f(d) = \overline{\chi}_f(p)$ and the second fixes any modular form. ☐

Combining the equations in the two lemmas allows us to conclude

$$\langle fg_p, h_p\rangle = \frac{\alpha_f + \beta_f}{p^{m-1}(1+p)}\langle fg, h\rangle.$$

We can then carry out the process of the first lemma for the eigenform equations for g and h, and the process of the second lemma for $\langle fg_p, h\rangle$ and $\langle fg, h_p\rangle$ and obtain

$$\langle fg_p, h\rangle = \overline{\chi}_g(p)\frac{\alpha_g + \beta_g}{p^{m-k-1}(1+p)}\langle fg, h\rangle, \qquad \langle fg, h_p\rangle = \chi_h(p)\frac{\overline{\alpha}_h + \overline{\beta}_h}{p^{m-1}(1+p)}\langle fg, h\rangle.$$

We now can substitute these terms to our expansion of $\langle fg^\sharp, h^\natural\rangle$ above and obtain

$$\langle fg, h\rangle\left(1 - \frac{\beta_g\overline{\chi}_g(p)(\alpha_g+\beta_g)}{p^{m-k-1}(1+p)} - \frac{p\overline{\beta}_h\chi_h(p)(\overline{\alpha}_h+\overline{\beta}_h)}{p^{m-1}(1+p)} + \frac{p\beta_g\overline{\beta}_h(\alpha_f+\beta_f)}{p^{m-1}(1+p)}\right).$$

The term in parenthesis needs to be manipulated and factored. This tedious but elementary computation is omitted here, but is carried out in detail in Section 3.4.2 of the author's thesis [Col15].

Proposition 6.4.4. *Suppose we're in the above setup (in particular with the level of f coprime to p). Then we have*

$$\langle f(z)g^\sharp(z), h^\natural(z)\rangle = \langle f(z)g(z), h(z)\rangle\frac{(-\alpha_h/\beta_h)}{1+p^{-1}}\left(1 - \frac{\beta_g\alpha_f}{\alpha_h}\right)\left(1 - \frac{\beta_g\beta_f}{\alpha_h}\right).$$

6.4.3. *Relating $\langle fg^\sharp, h^\natural\rangle$ to $\langle fg, h\rangle$ when p divides the level of f.* In this case f has level divisible by some power p^r with $r \geq 1$, and our goal is to relate the quantity $\langle f(z)g^\sharp(z), h^\natural(p^{r-1}z)\rangle$ to the quantity $\langle f(z)g(z), h(p^rz)\rangle$ which appears in our automorphic theory. Here we follow the same approach of writing

$$\langle fg^\sharp, h^\natural\rangle = \langle fg, h\rangle - \beta_g\langle fg_p, h\rangle - p\overline{\beta}_h\langle fg, h_p\rangle + p\beta_g\overline{\beta}_h\langle fg_p, h_p\rangle,$$

though here h and h_p denote $h(p^{r-1}z)$ and $h(p^rz)$ respectively. In this situation several of the terms will be zero, with the underlying reason being that f is new at p while g and h are not:

Lemma 6.4.5. *Suppose f is a modular form of weight k and level M_0p^r which is new at p^r, and g_0, h_0 are any modular forms of weight $m-k$ and m, respectively, which are modular for M_0p^{r-1}. Then $\langle f(z)g_0(z), h_0(z)\rangle = 0$.*

We remark that it's straightforward to check (using trace operators as in the proof here) that condition of "being new at p^r" does not depend on the prime-to-p level. Thus it doesn't matter if the M_0 in the statement of the lemma is larger than the original prime-to-p level of f.

Proof. Similar to what we discussed in Section 6.3, we can construct a trace operator from modular forms of level M_0p^r to M_0p^{r-1}, taking a form ψ to the sum of all translates $\chi^{-1}(\gamma)\psi|[\gamma]$ for γ running over coset representatives of $\Gamma_0(M_0p^r)\backslash\Gamma_0(M_0p^{r-1})$. Furthermore, we have that if ψ is modular of level M_0p^{r-1} then $\langle \mathrm{tr}(f), \psi\rangle$ is a scalar multiple of $\langle f, \psi\rangle$.

Since f being new at p^r means it's perpendicular to every ψ of level M_0p^{r-1}, we can thus conclude $\langle \mathrm{tr}(f), \psi\rangle = 0$ for every such ψ. Since $\mathrm{tr}(f)$ lies in this space of modular forms M_0p^{r-1} we conclude that $\mathrm{tr}(f) = 0$.

Thus we obviously have $0 = \langle \mathrm{tr}(f)g_0, h_0\rangle$. But we can also carry out a similar manipulation for this Petersson inner product:

$$\langle \mathrm{tr}(f)g_0, h_0\rangle = \sum_\gamma \langle \chi_f^{-1}(\gamma)f|[\gamma]_kg_0, h_0\rangle = \sum_\gamma \langle \chi_f^{-1}(\gamma)fg_0|[\gamma^{-1}]_{m-k}, h_0|[\gamma^{-1}]_m\rangle,$$

and using the transformation laws for $\gamma \in \Gamma_0(M_0p^{r-1})$ on h_0, g_0 we conclude that $\langle \mathrm{tr}(f)g_0, h_0\rangle$ is a scalar multiple of $\langle fg_0, h_0\rangle$, and thus that $\langle fg_0, h_0\rangle = 0$. ☐

This lemma directly implies $\langle fg, h\rangle = \langle f(z)g(z), h(p^{r-1}z)\rangle$ vanishes. A further argument allows it to handle $\langle fg_p, h_p\rangle$ as well.

Lemma 6.4.6. *We have $\langle fg_p, h_p\rangle = 0$.*

Proof. By Lemma 6.3.1, we have $\langle f(z)g(pz), h(p^r z)\rangle = p^m\langle T(p)(f(z)g(pz)), h(p^{r-1}z)\rangle$. If we write if $f = \sum a_n q^n$ and $g = \sum b_n q^n$ then we have

$$T(p)(f(z)g(pz)) = T(p)\left(\sum_{n=0}^{\infty}\Big(\sum_{i+pj=n} a_i b_j\Big)q^n\right) = \sum_{n=0}^{\infty}\Big(\sum_{i+pj=pn} a_i b_j\Big)q^n = \sum_{n=0}^{\infty}\Big(\sum_{i'+j=n} a_{pi'} b_j\Big)q^n,$$

which is $T(p)(f(z))\cdot g(z) = a_f f(z)g(z)$. Thus $\langle fg_p, h_p\rangle = a_f p^k\langle fg, h\rangle = 0$. □

At this point we've reduced our equation to

$$\langle fg^\sharp, h^\natural\rangle = -\beta_g\langle fg_p, h\rangle - p\overline{\beta}_h\langle fg, h_p\rangle.$$

If $r > 1$ then we can in fact also apply Lemma 6.4.5 to $\langle fg_p, h_p\rangle$ and conclude:

Proposition 6.4.7. *Suppose that f is new of level p^r with $r \geq 2$. Then we have*

$$\langle f(z)g^\sharp(z), h^\natural(p^{r-1}z)\rangle = -p\overline{\beta}_h\langle f(z)g(z), h(p^r z)\rangle.$$

If $r = 1$ then $\langle fg_p, h\rangle$ will generally not be zero. Instead we can relate it to $\langle fg, h_p\rangle$ by using the Atkin-Lehner operator ω_p as in Lemma 6.4.3.

Lemma 6.4.8. *In the case $r = 1$, we have $\langle f_p g, h\rangle = -pa_f\overline{\chi}_h(p)\langle fg_p, h_p\rangle$.*

Proof. As in Lemma 6.4.3 we may write $\langle fg_p, h\rangle = \langle f|[\omega_p]_k g_p|[\omega_p]_{m-k}, h|[\omega_p]_m\rangle$, and the arguments of that lemma give $g_p|[\omega_p]_{m-k} = \overline{\chi}_g(p)p^{-(m-k)/2}g$ and $h|[\omega_p]_m = p^{m/2}h_p$. Meanwhile, one can compute $f|[\omega_p]_k = -p^{1-k/2}\overline{a}_p f$ arguing with Atkin-Lehner operators as in [Asa76] or Section 4.6 of [Miy06]; in particular, the arguments of Theorem 1 and Lemma 3 of [Asa76] go through in our case (specifically because f is an eigenform of $T(p)$ with eigenvalue a_f, p exactly divides the level of f, and the character χ_f has trivial p-part due to the equation $\chi_f\chi_g = \chi_h$). Combining these equations and using $\overline{a}_p = \overline{\chi}_f(p)a_f$ and $\overline{\chi}_f(p)\overline{\chi}_g(p) = \overline{\chi}_h(p)$ gives the lemma. □

Substituting, we have

$$\langle fg^\sharp, h^\natural\rangle = (pa_f\beta_g\overline{\chi}_h(p) - p\overline{\beta}_h)\langle fg, h_p\rangle.$$

Using the identity $\overline{\chi}_h(p) = \overline{\beta}_h/\alpha_h$ gives:

Proposition 6.4.9. *Suppose that f is new of level p^r with $r = 1$. Then we have*

$$\langle f(z)g^\sharp(z), h^\natural(p^{r-1}z)\rangle = -p\overline{\beta}_h\left(1 - \frac{\beta_g a_f}{\alpha_h}\right)\langle f(z)g(z), h(p^r z)\rangle.$$

We remark that we have carried out numerical computations to verify the formulas obtained in in Propositions 6.4.1, 6.4.4, 6.4.7, and 6.4.9. This is described in Section 5.2 of our paper [Col18].

6.5. **The module of congruences for a CM family.** Finally, the last bit of Λ-adic theory we'll need to use is an explicit computation of the congruence number $H_{\mathbf{g}}$ in the case $\mathbf{g} = \mathbf{g}_\Psi$ is a $\mathcal{I}$-adic CM newform as constructed in the previous section. As we've mentioned, this computation is essentially a consequence of the main conjecture of Iwasawa theory for imaginary quadratic fields, which was proven by Rubin [Rub91]. The result is that $H_{\mathbf{g}}$ is essentially a p-adic L-function associated to an anticyclotomic family of Hecke characters closely related to Ψ; the connection between the main conjecture and the computation of $H_{\mathbf{g}}$ is studied by Hida and Tilouine in [HT93] and [HT94].

A number of constructions have been given for these *p-adic Hecke L-functions*, for instance by Manin-Vishik [VM74], by Katz [Kat76] and [Kat78], and by Coates-Wiles [CW78]; see also expositions in Chapter II.4 of [dS87] and Chapter 9 of [Hid13]. We'll give the necessary setup to state the special case of these results that we'll actually use.

The main thing we need to do is to define the appropriate periods for algebraicity results. We actually define a pair $(\Omega_\infty, \Omega_p)$ of a complex and a p-adic period. The idea is to fix an auxiliary prime-to-p conductor c, and take A_0 to be the elliptic curve defined by $A_0(\mathbb{C}) = \mathbb{C}/\mathcal{O}_c$, which we can show is defined over a number field. We then pick a nowhere-vanishing differential ω defined over this number field. Then, working over $\mathbb{C}$ we have a standard differential ω_∞ coming from the standard coordinate on $\mathbb{C}$ passed to $\mathbb{C}/\mathcal{O}_c$. Working over $\mathcal{O}_{\mathbb{C}_p}$ we can get an isomorphism between $\widehat{\mathbb{G}}_m$ and the formal completion $\widehat{\mathcal{A}}_0$ of a good integral model

$\mathcal{A}_0$ of A_0, well-defined up to p-adic units; from this we get a differential ω_p induced by the standard period dt/t on $\widehat{\mathbb{G}}_m$. We can then define our periods:

Definition 6.5.1. Given a fixed differential ω defined over $\mathcal{O}_c$ as above, we define two periods $\Omega_\infty \in \mathbb{C}$ and $\Omega_p \in \mathbb{C}_p$ by

$$\omega = \Omega_\infty \omega_\infty \qquad \omega = \Omega_p \omega_p.$$

The pair $(\Omega_\infty, \Omega_p)$ is evidently well-defined up to algebraic scalars. We usually take the convention that we choose our scalar so that Ω_p is a p-adic unit; thus $(\Omega_\infty, \Omega_p)$ is well-defined up to p-adic units of $\overline{\mathbb{Q}}$ (via the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ that we've fixed since the beginning). Also, we note that while we started with a conductor c (and defined our elliptic curve A_0 as the one having CM by the order $\mathcal{O}_c$ of $\mathcal{O}_K$), the dependence on c is entirely illusory: since there's a prime-to-p isogeny between the resulting curves for any two such choices of c, we can choose the same pair of periods $(\Omega_\infty, \Omega_p)$ to arise from both cases.

With our periods constructed, we can state the interpolation property that characterizes the p-adic Hecke L-function, and quote the existence result of this L-function. In full generality, it can be defined to interpolate special values of Hecke L-functions where we can vary the character over all things in a certain range of conductors; we will be interested in a specialization that gives us something defined on a domain similar to Λ, which interpolates the special values in a particular anticyclotomic family of characters. In particular we'll consider a family $\Phi : \mathbb{I}_K \to \mathcal{I}$ constructed along the lines of what we did in Section 5.2, in particular satisfying $P_m \circ \Phi = \varphi_m = \varphi\alpha^{m-1}(\alpha^c)^{-(m-1)}$. More specifically, we'll ask that the finite-order character φ here has prime-to-p conductor $\mathfrak{c}$, and that the p-part of its finite-type is $\omega_{\mathfrak{p}}^a \omega_{\overline{\mathfrak{p}}}^{-a}$ for a residue class a mod $p-1$. Thus φ_m will be unramified at p (and have conductor exactly $\mathfrak{c}$) for $m \equiv a \pmod{p-1}$.

So, the interpolation property we want is at points P_m for $m \equiv a \pmod{p-1}$. There is a slight additional wrinkle: even though our family Φ of Hecke characters are defined over $\mathcal{I}$, the p-adic L-function will need to be defined over a larger ring; to deal with the periods we need to extend scalars to $\mathcal{O}_F^{\mathrm{ur}}$, the integer ring of the maximal unramified extension F^{ur}/F. So we define $\mathcal{I}^{\mathrm{ur}} = \mathcal{I} \otimes_{\mathcal{O}_F} \mathcal{O}_F^{\mathrm{ur}} = \mathcal{O}_F^{\mathrm{ur}}[[\Gamma']]$. On this larger ring, our distinguished specializations $P_m : \mathcal{I} \to \mathcal{O}_F$ have a unique $\mathcal{O}_F^{\mathrm{ur}}$-linear extension to $\mathcal{I}^{\mathrm{ur}} \to \mathcal{O}_F^{\mathrm{ur}}$ that we also denote P_m, characterized in the usual way (by mapping the distinguished topological generator γ_π to $(1+\pi)^m$). Then:

Theorem 6.5.2. *Suppose $\Phi = \varphi\mathcal{A}\alpha^{-1}(\mathcal{A}^c)^{-1}\alpha^c$ is an anticyclotomic family as described above, so that for $m \equiv a \pmod{p-1}$ we have that φ_m has infinity-type $(m-1, -m+1)$ and conductor $\mathfrak{c}$ coprime to p. Then, for $m \geq 3$ satisfying $m \equiv a \pmod{p-1}$ we can define $L_{\mathrm{alg}}(\varphi_m, 1) \in \overline{\mathbb{Q}}$ and $L_p(\varphi_m, 1) \in \mathbb{C}_p$ satisfying*

$$\frac{L_p(\varphi_m, 1)}{\Omega_p^{2m-2}(1 - \varphi_m^{-1}(\overline{\mathfrak{p}})/p)(1 - \varphi_m(\mathfrak{p}))} = L_{\mathrm{alg}}(\varphi_m, 1) = \frac{(m-1)!\pi^{m-2}}{2^{m-2}\sqrt{d}^{m-2}} \frac{L(\varphi_m, 1)}{\Omega_\infty^{2m-2}},$$

(with these equalities really being under our embeddings ι_∞ and ι_p) and there exists an element $\mathcal{L}_p(\Phi) \in \mathcal{I}^{ur}$ such that specializing at P_m for such m gives $L_p(\varphi_m, 1)$.

The theorem holds (with the same interpolation formula) for families of the form $\varphi\mathcal{A}^c(\alpha^c)^{-1}\mathcal{A}^{-1}\alpha$ as well.

Next, we connect this back to the problem of finding the congruence number associated to a $\mathcal{I}$-adic family $\mathbf{g}_\Psi$ where Ψ is a family with $P_m \circ \Psi = \psi_m = \psi\alpha^{m-1}$ of infinity-type $(m-1, 0)$ (where here α has infinity-type $(1, 0)$ and finite-type $\omega_{\mathfrak{p}}^{-1}$). As we've said, the Iwasawa main conjecture for the imaginary quadratic field K is enough to compute this. By a general argument in deformation theory (see Section 3.2 of [Hid96], for instance), the congruence module $\mathcal{C}(\mathbf{f}, \mathcal{I})$ can be identified with (the Pontryagin dual of) a Selmer group for the adjoint Galois representation $\mathrm{ad}(\mathbf{f})$. In the case where $\mathbf{f} = \mathbf{g}_\Psi$ is a CM family, this adjoint Galois representation decomposes as a direct sum of $\mathrm{Ind}(\Psi/\Psi^c)$ and χ_K, and thus the Selmer group breaks up similarly, and finally the Selmer group of $\mathrm{Ind}(\Psi/\Psi^c)$ is related to $\mathcal{L}_p(\Psi/\Psi^c)$ by the main conjecture.

The precise relation between $H_{\mathbf{g}_\Psi}$ to $\mathcal{L}_p(\Psi/\Psi^c)$ is worked out [HT93] and [HT94], and we quote a version of their result.

Theorem 6.5.3. *Suppose that our family is such that for $m \equiv a \pmod{p-1}$, the character ψ_m has conductor $\mathfrak{c}$ which is a nonempty product of inert primes. Then the CM forms g_{ψ_m} have residually irreducible p-adic*

Galois representations, and the element

$$\left(\prod_{q|N(\mathfrak{c})}(1+q^{-1})(1-q^{-2})\right)h_K\mathcal{L}_p(\Psi/\Psi^c)\in\mathcal{I}^{ur}$$

is equal (up to a unit in $\mathcal{I}^{ur}$) to the congruence number $H_{\mathbf{g}_\Psi}\in\mathcal{I}$. Here h_K is the class number of K.

Similarly, the congruence number $H_{\mathbf{g}_{\Psi^\rho}}$ (for the conjugate Λ-adic family of Proposition 5.4.3) is

$$\left(\prod_{q|N(\mathfrak{c})}(1+q^{-1})(1-q^{-2})\right)h_K\mathcal{L}_p(\Psi^c/\Psi).$$

We remark that the hypothesis about residually irreducible Galois representations is automatically satisfied if Ψ is ramified at any inert prime, since then even the induced local representation at that prime is residually irreducible (since it corresponds to a supercuspidal). Also, along this chain of reasoning we implicitly use the fact that the local ring of the Hecke algebra through which $\lambda_{\mathbf{g}_\Psi}$ factors is Gorenstein (as needed in Theorem 4.5.6 to conclude the existence of $H_{\mathbf{g}_\Psi}$ in the first place); this Gorenstein condition follows from modularity lifting work, e.g. from [Wil95], Corollary 2 to Theorem 2.1.

Finally, it is useful to link the values $h_K L_{\mathrm{alg}}(\psi_m/\psi_m^c,1)$ that arises from this congruence number computation back to $L(\mathrm{ad}(g_{\psi_m}),1)$ and then to the related Petersson inner product $\langle g_{\psi_m},g_{\psi_m}\rangle$ - we can roughly think about it as the "algebraic part" of these quantities. The relation between the adjoint L-value and the Petersson inner product is stated in Theorem 3.1.1. We then want to specialize to the case of $f=g_\psi$ and relate $L_N(\mathrm{ad}\,g_\psi,1)$ to $L(\psi/\psi^c,1)$, the L-value we're interpolating. On the level of automorphic L-functions we have an identity

$$L(\mathrm{ad}\,g_\psi,1)=L(\psi/\psi^c,1)L(\chi_K,1)$$

coming from a corresponding decomposition of Weil-Deligne groups. To get an identity involving $L_N(\mathrm{ad}\,g_\psi,1)$ we need to work out the Euler factors of $L(\psi/\psi^c,1)L(\chi_K,1)$ at places dividing N. For our purposes we have:

Proposition 6.5.4. *Let $K=\mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field (with $-d$ an odd fundamental discriminant) and ψ a Hecke character for K which has conductor $\mathfrak{c}$ divisible only by inert primes of K. Then we have*

$$L(\psi/\psi^c,1)L(\chi_K,1)=L_N(\mathrm{ad}\,g_\psi,1)\prod_{q|d}(1-q^{-1})^{-1}\prod_{q|N(\mathfrak{c})}(1-q^{-2})^{-1}(1+q^{-1})^{-1}$$

Combining this, Theorem 3.1.1, and the analytic class number formula $h_K=(w_K\sqrt{d}/2\pi)L(\chi_K,1)$ lets us conclude

Corollary 6.5.5. *Let $K=\mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field (with $-d$ an odd fundamental discriminant) and ψ a Hecke character of infinity-type $(m-1,0)$ for K which has conductor $\mathfrak{c}$ divisible only by inert primes of K. Then we have*

$$\left(\prod_{q|N(\mathfrak{c})}(1+q^{-1})(1-q^{-2})\right)h_K L_{\mathrm{alg}}(\psi/\psi^c,1)=\frac{2^m\pi^{2m-1}w_K}{3\sqrt{d}^{m-3}}\frac{\langle g_\psi,g_\psi\rangle}{\Omega_\infty^{2m-2}}\prod_{q|dN(\mathfrak{c})}(1+q^{-1}).$$

6.6. **Summarizing our results.** We now restate the constructions of this section in a way that mirrors the development of p-adic L-functions in Section 2: passing from a Petersson inner product $\langle fg,h\rangle$ to an algebraic value $\langle fg,h\rangle_{\mathrm{alg}}$ and then to a p-adic value $\langle fh,h\rangle_p$, and realizing these p-adic values as specializations of a p-adic function $\langle f\mathbf{g},\mathbf{h}\rangle\in\mathcal{I}^{\mathrm{ur}}$. We work with the following setup:

- f is a fixed newform of weight k, level $N_f p^{r_0}$ and character χ_f, with N_f prime to p.
- $\mathbf{g}$ is a $\mathcal{I}$-adic family of newforms, such that for a particular residue class $m-k\equiv a \pmod{p-1}$ we have g_{m-k} is of prime-to-p level N_g and character χ_g.
- $\mathbf{h}$ is a $\mathcal{I}$-adic family of newforms, such that for $m\equiv a \pmod{p-1}$ (the same a as above) we have h_m is of prime-to-p level N_g and character $\chi_h=\chi_f\chi_g$.

- We want to interpolate the Petersson inner products of the form
$$\langle f|_{M_f} \cdot g_{m-k}|_{M_g}, h_m|_{M_h p^{r_0}}\rangle = \langle f(M_f z) \cdot g_{m-k}(M_g z), h_m(M_h p^{r_0} z)\rangle$$
for $m \equiv a \pmod{p-1}$, for some prime-to-p integers M_f, M_g, M_h.

For convenience we'll sometimes suppress the constants $M_f, M_g, M_h p^{r_0}$ in the above notation and simply write $\langle f g_{m-k}, h_m\rangle$ for the inner product there. This inner product can be computed on $\Gamma_0(N_{fgh}p^r)$, where
$$N_{fgh} = \mathrm{lcm}(N_f M_f, N_g M_g, N_h M_g)$$
is the common prime-to-p level of the three modular forms involved and $r = \max\{r, 1\}$.

To define $\langle f g_{m-k}, h_m\rangle_{\mathrm{alg}}$, we want to use $a(1, 1_{h_m} f g_{m-k})$, which we know is an algebraic number that's equal to the ratio of $\langle f g_{m-k}, h_m\rangle$ by $\langle h_m, h_m\rangle = \langle h_m(z), h_m(z)\rangle$; this will be an algebraic number corresponding to the orthogonal projection of $f g_{m-k}$ onto h_m in the space of cusp forms. We also need to multiply this by the algebraic part of the congruence number H_{g_m}. Since we've only discussed congruence numbers for CM forms, we restrict to that situation:

- The $\mathcal{I}$-adic form $\mathbf{h}$ is actually a CM form $\mathbf{g}_\Psi$, where Ψ is a family of Hecke characters for $K = \mathbb{Q}(\sqrt{-d})$ such that ψ_m has prime-to-p conductor $\mathbf{c}$ (so $N_h = dN(\mathfrak{c})$), and also such that the Galois representations associated to g_{ψ_m} are residually irreducible.

This covers both the "usual" families $\mathbf{g}_\Psi$ of Section 5.3 and the "conjugate" families $\mathbf{g}_{\Psi^\rho}$ of Section 5.4. Of course, if one wants to have some other $\mathcal{I}$-adic newform $\mathbf{h}$ and works out the congruence ideal $H_\mathbf{h}$, it's easy to modify what follows for that situation.

We then define
$$\begin{aligned}\langle f g_{m-k}, h_m\rangle_{\mathrm{alg}} = {} & \frac{\langle f|_{M_f} \cdot g_{m-k}|_{M_g}, h_m|_{M_h p^{r_0}}\rangle}{\langle h_m, h_m\rangle} \\ & \cdot h_K L_{\mathrm{alg}}\big(\psi_m/\psi_m^c, 1\big) \left(\prod_{q | N(\mathfrak{c})} (1+q^{-1})(1-q^{-2})\right) \\ & \cdot \frac{p^{m(r-1)}}{\alpha_{h_m}^{r-1}} \frac{N_{fgh} M_h^{m+1}}{N_h} \left(\prod_{q | N_{fgh}, q\nmid N_h} (1+q^{-1})\right)\end{aligned}$$
and
$$\begin{aligned}\langle f g_{m-k}, h_m\rangle_p = {} & \frac{\langle f|_{M_f} \cdot g^\sharp_{m-k}|_{M_g}, h^\natural|_{M_h p^{r_0 - 1}}\rangle}{\langle h^\sharp_m, h^\natural_m\rangle} \\ & \cdot h_K L_p\big(\psi_m/\psi_m^c, 1\big) \left(\prod_{q | N(\mathfrak{c})} (1+q^{-1})(1-q^{-2})\right) \\ & \cdot \frac{p^{m(r-1)}}{\alpha_{h_m}^{r-1}} \frac{N_{fgh} M_h^{m+1}}{N_h} \left(\prod_{q | N_{fgh}, q\nmid N_h} (1+q^{-1})\right)\end{aligned}$$
This is set up so that if we define
$$\langle f\mathbf{g}, \mathbf{h}\rangle = \ell_\mathbf{h}(\mathrm{tr}(f|_{M_f/M_h} \cdot \mathbf{g}|_{M_g/M_h})) \in \mathcal{I}^{\mathrm{ur}}$$
(using the shifting and scaling constructions of Section 5.5 and the trace operator $\mathrm{tr} : \mathbf{S}(\Gamma_0(N_{fgh}p^r, M_h), \chi; \mathcal{I}) \to \mathbf{S}(N_h p^r, \chi; \mathcal{I})$ defined in Section 6.3, and taking the congruence number $H_\mathbf{h}$ to actually be the element of $\mathcal{I}^{\mathrm{ur}}$ written in Theorem 6.5.3), then we have $P_m(\langle f\mathbf{g}, \mathbf{h}\rangle) = \langle f g_{m-k}, h_m\rangle_p$. We can see this by applying Proposition 6.3.3 and unravelling the definitions; we get
$$\begin{aligned}P_m\Big(\ell_\mathbf{h}(\mathrm{tr}(f|_{M_f/M_h} \cdot \mathbf{g}|_{M_f/M_h}))\Big) = {} & \frac{\langle f|_{M_f/M_h} \cdot g^\sharp_{m-k}|_{M_g/M_h}, h^\natural_m|_{p^{r-1}}\rangle}{\langle h^\sharp_m, h^\natural_m\rangle} \cdot H_{\mathbf{h}, P_m} \\ & \cdot \frac{p^{m(r-1)}}{\alpha_{h_m}^{r-1}} \frac{N_{fgh} M_h}{N_h} \prod_{q | N_{fgh}, q\nmid N_h} (1+q^{-1})\end{aligned}$$

Making the change of variables $z \mapsto M_h z$ (by Lemma 6.2.3) we further see that this is equal to

$$\frac{\langle f|_{M_f} \cdot g^{\sharp}_{m-k}|_{M_g}, h^{\natural}_m|_{M_h p^{r-1}}\rangle}{\langle h^{\sharp}_m, h^{\natural}_m\rangle} \cdot H_{\mathbf{h},P_m} \cdot \frac{p^{m(r-1)}}{\alpha^{r-1}_{h_m}} \frac{N_{fgh} M_h^{m+1}}{N_h} \prod_{q \mid N_{fgh}, q \nmid N_h} (1+q^{-1}).$$

Since $H_{\mathbf{h}}$ is realized by $h_k \mathcal{L}_p(\Psi(\Psi^c)^{-1})$ by Theorem 6.5.3, we conclude that this is exactly the value of $\langle f g_{m-k}, h_m\rangle_p$:

Theorem 6.6.1. *Suppose that we have f, $\mathbf{g}$, and $\mathbf{h} = \mathbf{g}_\Psi$ satisfying the conditions listed above for a residue class a. Then the element $\langle f\mathbf{g}, \mathbf{h}\rangle \in \mathcal{I}$ defined above satisfies $P_m(\langle f\mathbf{g}, \mathbf{h}\rangle) = \langle f g_{m-k}, h_m\rangle_p$ for all $m \equiv a \pmod{p-1}$.*

To properly *use* this theorem, we'll want formulas more directly relating our initial complex-analytic value $\langle f g_{m-k}, h_m\rangle$ to the algebraic value $\langle f g_{m-k}, h_m\rangle_{\mathrm{alg}}$ and then the p-adic value $\langle f g_{m-k}, h_m\rangle_p$. We do this as follows. (Remember that our notation suppresses the shifts $|_{M_f}$ and so on).

Proposition 6.6.2. *Suppose that we have f, $\mathbf{g}$, and $\mathbf{h} = \mathbf{g}_\Psi$ satisfying the conditions listed above for a residue class a. Then we have*

$$\langle f g_{m-k}, h_m\rangle_{\mathrm{alg}} = \frac{\langle f g_{m-k}, h_m\rangle}{\Omega_\infty^{2m-2}} \cdot \frac{2^m w_K \pi^{2m-1}}{3\sqrt{d}^{m-3}} \frac{p^{m(r-1)}}{\alpha^{r-1}_{h_m}} \frac{N_{fgh} M_h^{m+1}}{N_h} \prod_{q \mid N_{fgh}} (1+q^{-1})$$

and also

$$\langle f g_{m-k}, h_m\rangle_p = e_p(f g_{m-k}, h_m) \Omega_p^{2m-2} \langle f g_{m-k}, h_m\rangle_{\mathrm{alg}}$$

where $e_p(f g_{m-k}, h_m)$ is a removed "Euler factor" given by

- $e_p(f g_{m-k}, h_m) = (1 - \frac{\beta_g \alpha_f}{\alpha_h})(1 - \frac{\beta_g \beta_f}{\alpha_h})$ *if* $r_0 = 0$,
- $e_p(f g_{m-k}, h_m) = p\overline{\alpha}_h (1+p^{-1})(1 - \frac{\beta_g a_f}{\alpha_h})$ *if* $r_0 = 1$,
- $e_p(f g_{m-k}, h_m) = p\overline{\alpha}_h (1+p^{-1})$ *if* $r_0 \geq 2$.

Here r_0 is the exact power of p dividing the level of f.

Proof. The equation for $\langle f g_{m-k}, h_m\rangle_{\mathrm{alg}}$ follows from taking the definition and plugging in Corollary 6.5.5 and noting that $N_h = dN(\mathfrak{c})$ so the $\prod_{q \mid N_{fgh}, q\nmid N_h}$ and $\prod_{q \mid dN(\mathfrak{c})}$ combine to a single product.

The equation for $\langle f g_{m-k}, h_m\rangle_p$ follows from comparing the definitions and noting we have

$$\langle f g_{m-k}, h_m\rangle_p = \frac{\langle f g^\sharp, h^\natural\rangle}{\langle f g, h\rangle} \frac{\langle h, h\rangle}{\langle h^\sharp, h^\natural\rangle} \frac{L_p(\psi_m/\psi^c_m, 1)}{L_{\mathrm{alg}}(\psi_m/\psi^c_m, 1)} \langle f g_{m-k}, h_m\rangle_p.$$

The ratio L_p/L_{alg} is

$$\Omega_p^{2m-2} \left(1 - \frac{\psi^c_m}{\psi_m}(\overline{\mathfrak{p}}) p^{-1}\right)\left(1 - \frac{\psi_m}{\psi^c_m}(\mathfrak{p})\right) = \Omega_p^{2m-2}\left(1 - \frac{\beta_h}{\alpha_h} p^{-1}\right)\left(1 - \frac{\beta_h}{\alpha_h}\right)$$

by definition, which multiplies together with the ratio

$$\frac{\langle h, h\rangle}{\langle h^\sharp, h^\natural\rangle} = \frac{(1+p^{-1})}{(-\alpha_h/\beta_h)(1-\beta_h/\alpha_h)(1-p^{-1}\beta_h/\alpha_h)}$$

given by Proposition 6.4.1 to give $\Omega_p^{2m-2}(1+p^{-1})$. The specified equations for $e_p(f g_{m-k}, h_m)$ then follow from the computations of $\langle f g^\sharp, h^\natural\rangle / \langle f g, h\rangle$ in Propositions 6.4.4, 6.4.9, and 6.4.7, respectively. □

7. Constructing the p-adic L-function

We now proceed with the main result of this paper, the construction of the anticyclotomic p-adic L-function $\mathcal{L}_p(f, \Xi^{-1})$, by combining our explicit version of Ichino's formula from Theorem 3.4.2 with the Λ-adic modular form theory we've developed in the previous sections.

Our version of Ichino's formula relates $|\langle f(z) g_\varphi(z), g_\psi(c^2 N z)\rangle|^2$ to the product of $L(f, \varphi\psi^{-1}, 0)$ and $L(f, \psi^{-1}\varphi^{-1} N^{m-k-1}, 0)$. With this in hand, our strategy for constructing $\mathcal{L}_p(f, \Xi^{-1})$ is as follows: let the characters φ, ψ vary in $\mathcal{I}$-adic families Φ, Ψ such that $\Xi = \Psi\Phi N^{-m+k+1}$ is the family we're aiming for, while $\Phi\Psi^{-1}$ is the constant family for some character η. Rearranging, we get a family of equations relating

$L(f, \xi_m^{-1}, 0)$ to a product of things we can p-adically interpolate: various explicit constants and terms with mild dependence on m (which can either be p-adically interpolated or folded into the definition of the algebraic part of the L-value), the reciprocal of a single L-value $L(f, \eta^{-1}, 0)$, and finally a product of Petersson inner products. Carrying this out will give us our p-adic L-function.

7.1. **Precise setup for the calculation.** To carry this out, we first of all need to specify exactly what our input data will be, and our hypotheses on it. To start off, we fix the following things (where the hypotheses listed are fundamental to the method we're using):

- A holomorphic newform f of some weight k, level $N = N_0 p^{r_0}$, and character χ_f.
- An imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ (where $-d$ is the fundamental discriminant).
- An odd prime p which splits in K.
- A Hecke character $\underline{\xi}_a$ of weight $(a-1, -a+k+1)$ for some integer a satisfying $a-1 \equiv k \pmod 2$, with central character $\chi_\xi = \underline{\xi}_{a,\mathrm{fin}}|_{\mathbb{Z}}$ equal to χ_f. This determines a $\mathcal{I}$-adic family $\Xi = \underline{\xi}_a \alpha^{-a}(\alpha^c)^a \mathcal{A}(\mathcal{A}^c)^{-1}$ as in Lemma 5.2.1.

In addition we impose the following hypotheses on these objects. For the most part these assumptions come from our version of Ichino's formula, Theorem 3.4.2. In principle one could eliminate most or all of these by working out more cases of the local integrals arising in Ichino's formula - so these hypotheses should not be fundamental roadblocks to carrying out the arguments of this paper.

- The character χ_f of the newform f is assumed to be trivial. Note that this forces the weight k and thus the integer a to be even.
- The fundamental discriminant $-d$ is odd (i.e. d is squarefree and satisfies $d \equiv 3 \pmod 4$), and d is assumed coprime to N.
- The Hecke character $\underline{\xi}_a$ has conductor (c) for c some $c \in \mathbb{Z}$ coprime to p, d, and N.

Given this setup, we want to define families of characters Φ and Ψ which give us $\mathcal{I}$-adic CM forms $\mathbb{g}_\Phi$ and $\mathbb{g}_\Psi$ so that we can carry out the computation outlined in the previous section. More precisely, we want to construct finite-order characters φ, ψ and take the associated families $\Phi = \varphi\alpha^{-1}\mathcal{A}$ and $\Psi = \psi\alpha^{-1}\mathcal{A}$ with specializations $\varphi_{m-k} = \varphi\alpha^{m-k-1}$ and $\psi_m = \psi\alpha^{m-1}$. If we look at the right-hand side of Ichino's formula, the L-values involved are

$$L(f, \varphi\psi^{-1}\alpha^{-k}, 0)L(f, \psi^{-1}\varphi^{-1}\alpha^{-(2m-k-2)}N^{m-k-1}, 0).$$

So we want to define ψ and φ such that the product $\psi^{-1}\varphi^{-1}\alpha^{2m-2}N^{m-k-1}$ equals ξ_m^{-1} for $m \equiv a \pmod{p-1}$, and such that $\varphi\psi^{-1}\alpha^{-k}$ is a "well-behaved" auxiliary character.

For applications, it will be important to be able to have flexibility in what we're allowed to choose as $\varphi\psi^{-1}$, and also the Hida theory we want to apply demands that our CM newform $\mathbb{g}_\Psi$ has residually irreducible Galois representations. We can deal with both of these issues by introducing a character ν mod ℓ for any prime ℓ; if we modify ψ by ν and φ by ν^{-1}, then the product $\psi\varphi$ (which corresponds to the family Ξ we've already fixed) doesn't change while $\psi\varphi^{-1}$ is modified by ν^2. Accordingly, we add in the following auxiliary data to our setup:

- An auxiliary prime ℓ not dividing $2pcdN$ that is inert in K, and an auxiliary character ν of $(\mathcal{O}_K/\ell^{c_\ell}\mathcal{O}_K)^\times$ of conductor ℓ^{c_ℓ} and which is trivial on $(\mathbb{Z}/\ell^{c_\ell}\mathbb{Z})^\times$.

Also, since the field $K = \mathbb{Q}(\sqrt{-3})$ has some extra roots of unity, in that case we need to add some additional hypotheses.

- If $K = \mathbb{Q}(\sqrt{-3})$ we also require ν is trivial on $\mathcal{O}_K^\times$ (this is automatic in all other cases), and that $a - 1 \equiv 0 \pmod 6$.

We can then explicitly construct a Hecke character satisfying the properties we want.

Lemma 7.1.1. *Suppose we have the data and hypotheses specified above. Then we can define finite-order Hecke characters φ and ψ of K (giving rise to families Φ, Ψ and Λ-adic CM forms $\mathbb{g}_\Phi, \mathbb{g}_\Psi$) with the following properties*

(1) *The conductor of φ is either $c\ell^{c_\ell}$ or $\mathfrak{p}c\ell^{c_\ell}$, and for $m \equiv a - 1 \pmod{p-1}$ the character φ_{m-k} has conductor ℓ^{c_ℓ} and finite-type $\xi_{a,fin}\nu^{-1}$.*

(2) *The conductor of* ψ *is either* ℓ^{c_ℓ} *or* $\mathfrak{p}\ell^{c_\ell}$*, and for* $m \equiv a-1 \pmod{p-1}$ *the character* ψ_m *has conductor* ℓ^{c_ℓ} *and finite-type* ν*.*
(3) *The product* $\varphi_{m-k}^{-1}\psi_m = \varphi^{-1}\psi\alpha^k$ *is independent of* m *and has conductor* $c\ell^{c_\ell}$ *and finite-type* $\xi_{a,fin}^{-1}\nu^2$*.*
(4) *For* $m \equiv a \pmod{p-1}$*, the product*

$$\varphi_{m-k}\psi_m N^{-(m-k-1)} = \varphi\psi\alpha^{2m-k-2}N^{-(m-k-1)}$$

is equal to ξ_m*.*

Proof. We want to define φ and ψ to have finite-type parts

$$\varphi_{\rm fin}(x) = \omega_{\mathfrak{p}}(x)^{a-k-1}\xi_{a,{\rm fin}}(x)\nu^{-1}(x) \qquad\qquad \psi_{\rm fin}(x) = \omega_{\mathfrak{p}}(x)^{a-1}\nu(x).$$

For these to define finite-order Hecke characters we just need to check that the right-hand sides of these characters are trivial for x in the unit group $\mathcal{O}_K^\times$. This follows from the following the following three facts:

- We have $\nu(x) = 1$ because we've specified ν is trivial on units (as an explicit hypothesis for $\mathbb{Q}(\sqrt{-3})$, and as a consequence of it being trivial on $\mathbb{Z}$ in all other cases).
- We have $a-1 \equiv 0 \pmod{|\mathcal{O}_K|^\times}$ and thus $\omega_{\mathfrak{p}}(x)^{a-1} = 1$ (as an explicit hypothesis for $\mathbb{Q}(\sqrt{-3})$, and as a consequence of $a-1 \equiv k \equiv 0 \pmod 2$ in the other cases). Thus $\psi_{\rm fin}(x) = 1$ for all $x \in \mathcal{O}_K^\times$.
- We have $\xi_{a,{\rm fin}}(x) = x^{-(2a-k-2)}$ (because ξ_a is a Hecke character of weight $(a-1, -a+k+1)$) and $\omega_{\mathfrak{p}}(x)^{a-1} = x^{a-1}$ (by construction of the Teichmüller character and because x must be a root of unity of order prime to p), so $\varphi_{\rm fin}(x) = x^{-(a-1)} = 1$ because $a-1 \equiv 0 \pmod{|\mathcal{O}_K|^\times}$.

So $\varphi_{\rm fin}, \psi_{\rm fin}$ give us characters on $K^\times/\mathcal{O}_K^\times$; embedding this into the group of ideals we can extend these (in $|\operatorname{Cl}_K|$ different ways) to finite-order Hecke characters φ, ψ. In fact, to make (4) true we extend φ arbitrarily and then define ψ by $\psi = \xi_a\varphi^{-1}\alpha^{-2a+k+2}N^{m-k-1}$ (which we can check has the correct finite-type and infinity-type).

So we've constructed ψ and φ in such a way that (1) and (2) are true, (3) is immediate, and (4) is true at least for $m = a$. To prove (4) for general $m \equiv a \pmod{p-1}$ we remember ξ_m is defined as $\xi_a\alpha^{m-a}(\alpha^c)^{a-m}$ and use the fact that $\alpha\alpha^c$ is a twist of N by a Teichmüller character $\omega^{-1} \circ N_{\mathbb{Q}}^K$ and thus $(\alpha\alpha^c)^{p-1} = N^{p-1}$ (see Section 5.4). □

So, for the rest of this section we fix the characters ψ, φ and the associated families Ψ, Φ as constructed in this lemma. We also take the notation that $\eta = \varphi^{-1}\psi\alpha^k$ is the constant Hecke character (of infinity-type $(k, 0)$). Then, for every positive weight $m \equiv a \pmod{p-1}$, Theorem 3.4.2 tells us we have an identity

$$\langle fg_{\varphi,m-k}, g_{\psi,m}|_{c^2N}\rangle\langle f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m}|_{c^2N}\rangle = \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2}2^{4m-2}d\ell^{2c_\ell}(c^2N)^{m+1}} \cdot \prod_{q|cNd\ell}(1+q^{-1})^{-2} \cdot (*)_{f,\ell} \cdot L(f, \eta^{-1}, 0)L(f, \xi_m^{-1}, 0)$$

where we set

$$(*)_{f,\ell} = \left(\sum_{i=0}^{c_\ell}\left(\frac{\alpha}{\ell^{(k-1)/2}}\right)^{2i-c_\ell} - \frac{1}{\ell}\sum_{i=i}^{c_\ell-1}\left(\frac{\alpha}{\ell^{(k-1)/2}}\right)^{2i-c_\ell}\right).$$

with $\alpha_{f,\ell}$ one of the roots of the Hecke polynomial for f at ℓ.

7.2. **The equation for algebraic and p-adic values.** To use the above family of equations to produce a p-adic L-function, we first need to manipulate it to give equations relating the "algebraic parts" of Petersson inner products (as defined in Section 6.6) to those of L-values (as defined in section 2.3), and then equations of the corresponding "p-adic parts".

Following the notation of Section 6.6, we suppress the shift by c^2N and write $\langle fg_{\varphi,m-k}, g_{\psi,m}\rangle$ in place of $\langle fg_{\varphi,m-k}, g_{\psi,m}|_{c^2N}\rangle$. Then using Proposition 6.6.2 we write

$$\langle fg_{\varphi,m-k}, g_{\psi,m}\rangle_{\rm alg}\langle f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m}\rangle_{\rm alg} = C_1\frac{\langle fg_{\varphi,m-k}, g_{\psi,m}\rangle}{\Omega_\infty^{2m-2}}\frac{\langle f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m}\rangle}{\Omega_\infty^{2m-2}}$$

for

$$C_1 = \frac{2^{2m}w_K^2\pi^{4m-2}}{3^2d^{m-3}}c^{4m+8}N_0^{2m+4}\frac{p^{2m(r-1)}}{\psi_m(\overline{\mathfrak{p}})^{2(r-1)}}\prod_{q|cN_0d\ell}(1+q^{-1})^2,$$

noting that the original formula gives us powers of $\alpha_{g_{\psi,m}} = \psi_m(\overline{\mathfrak{p}})$ and $\alpha_{g^\rho_{\psi,m}} = \overline{\beta}_{g_{\psi,m}} = \overline{\psi}_m(\mathfrak{p}) = \psi_m(\overline{\mathfrak{p}})$ in the denominator. Similarly, using our definitions in Section 2.3 we have

$$L_{\mathrm{alg}}(f, \xi_m^{-1}, 0) L_{\mathrm{alg}}(f, \eta^{-1}, 0) = C_2 \frac{L(f, \xi_m^{-1}, 0)}{\Omega_\infty^{4m-4-2k}} \frac{L(f, \eta^{-1}, 0)}{\Omega_\infty^{2k}}$$

where

$$C_2 = w_K^2 (m-2)!(m-k-1)!(k-1)! \frac{c^{2m+6} \ell^{2c_\ell(k+4)} \pi^{2m-4}}{d^{m-2} 2^{2m-4}}.$$

Taking our explicit Ichino's formula and manipulating it to get the constants C_1 and C_2 on each side, we obtain

$$\langle f g_{\varphi,m-k}, g_{\psi,m} \rangle_{\mathrm{alg}} \langle f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m} \rangle_{\mathrm{alg}} = C_3 \cdot L_{\mathrm{alg}}(f, \xi_m^{-1}, 0) L_{\mathrm{alg}}(f, \eta^{-1}, 0)$$

for

$$C_3 = \frac{2^2 N_0^{m+3}}{\ell^{2c_\ell(k+5)}} (*)_{f,\ell} \cdot \frac{p^{(m-1)(r-1)}}{\psi_m(\overline{\mathfrak{p}})^{2(r-1)}} \left((1+p^{-1})^2 p^{(m+1)} \right)^{r-r_0-1};$$

note that $1 + r_0 - r$ is 0 if p does not divide the level of f and -1 if p does divide the level of f, so the parenthesized term on the right is 1 in the former case and is $(1+p^{-1})^{-2} p^{-(m+1)}$ in the latter case.

Next we pass to an equation of p-adic values (except for $L_{\mathrm{alg}}(f, \eta^{-1}, 0)$ which we will be treating as a constant). Again, going back to from Proposition 6.6.2 and Section 2.3 we see that there are removed Euler factors on each side - $e_p(f g_{\varphi,m-k}, g_{\psi,m}) e_p(f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m})$ on the left and $e_p(f, \xi^{-1})$ on the right.

To compare these, we start by considering the case $r_0 = 0$. Here we have

$$e_p(f g_{\varphi,m-k}, g_{\psi,m}) = \left(1 - \frac{\varphi_{m-k}(\mathfrak{p}) \alpha_f}{\psi_m(\overline{\mathfrak{p}})}\right) \left(1 - \frac{\varphi_{m-k}(\mathfrak{p}) \beta_f}{\psi_m(\overline{\mathfrak{p}})}\right)$$

from our computations of α and β for a CM form in Section 5.3. We can also check that $\xi_m(\overline{\mathfrak{p}}) = \varphi_{m-k}(\overline{\mathfrak{p}}) \psi_m(\overline{\mathfrak{p}}) N(\mathfrak{p})^{-(m-k-1)} = \psi_m(\overline{\mathfrak{p}}) / \varphi_{m-k}(\mathfrak{p})$, and therefore conclude

$$e_p(f g_{\varphi,m-k}, g_{\psi,m}) = (1 - \xi_m^{-1}(\overline{\mathfrak{p}}) \alpha_f)(1 - \xi_m^{-1}(\overline{\mathfrak{p}}) \beta_f) = (1 - a_p \xi^{-1}(\overline{\mathfrak{p}}) + \xi^{-2}(\overline{\mathfrak{p}}) p^{k-1});$$

this matches up with (the square root of) $e_p(f, \xi^{-1})$.

We can similarly analyze $e_p(f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m})$, recalling from Section 5.4 that $\alpha_{h^\rho} = \overline{\beta}_h$ and $\beta_{h^\rho} = \overline{\alpha}_h$; we get

$$e_p(f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m}) = \left(1 - \frac{\overline{\alpha}_g \overline{\beta}_f}{\overline{\beta}_h}\right) \left(1 - \frac{\overline{\alpha}_g \overline{\alpha}_f}{\overline{\beta}_h}\right).$$

By comparing the equalities $\alpha_f \beta_f = \chi_f(p) p^{k-1}$ (by definition) and $\alpha_f \overline{\alpha}_f = p^{k-1}$ (by the Ramanujan conjecture) we know $\overline{\alpha}_f = \overline{\chi}_f(p) \beta_f$, and we get similar identities for all of the other conjugates; and since $\chi_f \chi_g = \chi_h$ the character values we have cancel and we conclude

$$e_p(f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m}) = \left(1 - \frac{\beta_g \alpha_f}{\alpha_h}\right) \left(1 - \frac{\beta_g \beta_f}{\alpha_h}\right).$$

So for $r_0 = 0$ we get an equality

$$e_p(f g_{\varphi,m-k}, g_{\psi,m}) e_p(f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m}) = e_p(f, \xi^{-1}).$$

For $r_0 \geq 2$, we directly calculate

$$e_p(f g_{\varphi,m-k}, g_{\psi,m}) e_p(f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m}) = p^2 (1+p^{-1})^2 \psi_m(\mathfrak{p})^2,$$

and for $r_0 = 1$ similar arguments give

$$e_p(f g_{\varphi,m-k}, g_{\psi,m}) e_p(f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m}) = p^2 (1+p^{-1})^2 \psi_m(\mathfrak{p})^2 (1 - \xi_m^{-1}(\overline{\mathfrak{p}}) a_f)^2.$$

On the other hand, we've defined $e_p(f, \xi^{-1})$ on a case-by-case basis to consist of exactly the Euler factors that appear here, times an additional factor of $(p^{k/2} \xi_m(\overline{\mathfrak{p}})^{-1})^r$. Further manipulations of our formula result in

Proposition 7.2.1. *We have*

$$\langle f g_{\varphi,m-k}, g_{\psi,m}\rangle_p \langle f^\rho g^\rho_{\varphi,m-k}, g^\rho_{\psi,m}\rangle_p = C_4 \cdot L_p^*(f,\eta^{-1},0) L_p(f,\xi_m^{-1},0)$$

for

$$C_4 = \eta(\overline{\mathfrak{p}})^{-r} (*)_{f,\ell} \frac{2^2 N_0^{m+3}}{\ell^{2c_\ell(k+5)}},$$

where we recall $L_p^*(f,\eta^{-1},0)$ is the value $\Omega_p^{2k} L_{\mathrm{alg}}(f,\eta^{-1},0)$ without an Euler factor removed.

7.3. **The p-adic L-function.** Finally, we can use our final equation of the previous section to finish our construction of $\mathcal{L}(f,\Xi^{-1})$ in terms of the element

$$\langle f\mathbf{g}_\Phi, \mathbf{g}_\Psi\rangle = \ell_{\mathbf{g}_\Psi}(\mathrm{tr}(f|_{1/c^2N} \cdot \mathbf{g}_\Phi|_{1/c^2N})) \in \mathcal{I}^{\mathrm{ur}}$$

and the conjugate element $\langle f^\rho \mathbf{g}_{\Phi^\rho}, \mathbf{g}_{\Psi^\rho}\rangle \in \mathcal{I}^{\mathrm{ur}}$, as constructed in Chapters 5 and 6.

In our constant C_4, there's only one term that depends on m, the parameter that varies in our p-adic family, the factor of N_0^{m+3}. This is straightforward to interpolate explicitly (see e.g. Section 7.1 of [Hid93]), so we can find an element $\mathcal{C} \in \Lambda^\times$ satisfying

$$P_m(\mathcal{C}) = \eta(\overline{\mathfrak{p}})^r \frac{2^2 \ell^{2c_\ell(k+5)}}{N_0^{m+3}}.$$

We can then conclude our main theorem:

Theorem 7.3.1. *We have the following identity in $\mathcal{I}^{\mathrm{ur}}$:*

$$\langle f\mathbf{g}_\Phi, \mathbf{g}_\Psi\rangle \langle f^\rho \mathbf{g}_{\Phi^\rho}, \mathbf{g}_{\Psi^\rho}\rangle = (*)_{f,\ell} \cdot L_p^*(f,\eta^{-1},0) \cdot \mathcal{C}^{-1} \cdot \mathcal{L}(f,\Xi^{-1}).$$

Proof. For the infinite set of $m > k$ satisfying $m \equiv a \pmod{m-1}$, the specializations of both sides of the equation are equal by Proposition 7.2.1; thus the elements of $\mathcal{I}^{\mathrm{ur}}$ themselves are equal by Lemma 2.2.2. □

We can then rearrange to get an explicit expression for $\mathcal{L}(f,\Xi^{-1})$ (as stated in Theorem 1.0.2), assuming the constants don't make the equation identically zero. If the constants are in fact units mod p, then our rearrangement genuinely gives an expression occurring in $\mathcal{I}^{\mathrm{ur}}$.

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