# NUMERICAL COMPUTATION OF PETERSSON INNER PRODUCTS AND $q$-EXPANSIONS

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ABSTRACT. In this paper we discuss the problem of numerically computing Petersson inner products of modular forms, given their $q$-expansion at $\infty$. A formula of Nelson [Nel15] reduces this to obtaining $q$-expansions at all cusps, and we describe two algorithms based on linear interpolation for numerically obtaining such expansions. We apply our methods to numerically verify constants arising in an explicit version of Ichino's triple-product formula relating $\langle fg, h \rangle$ to the central value of $L(f \times g \times \bar{h}, s)$, for three modular forms $f, g, h$ of compatible weights and characters.

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## 1. INTRODUCTION

The Petersson inner product on the space of holomorphic cusp forms $S_k(N, \chi)$ of a given weight, level, and character is a standard part of the theory of modular forms, defined by (up to a normalizing factor)

$$\langle f, g \rangle = \int_{\mathbb{H} \backslash \Gamma} f(x + iy) \overline{g(x + iy)} y^k \frac{dx \, dy}{y^2}.$$

Specific values of this (and related integrals) arise often in the arithmetic theory of newforms, their corresponding automorphic representations, and associated geometric objects such as elliptic curves; in particular

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special values of $L$-functions are often realized as such integrals. Thus it is of interest to numerically compute such quantities.

We discuss how to compute $\langle f, g \rangle$ given just the $q$-expansions of these forms at $\infty$, and give some example applications of our method. Actually, the problem we really consider is that of finding $q$-expansions of $f$ and $g$ at all cusps, at which point we use a formula of Nelson [Nel15] which gives the Petersson inner product as a sum over all cusps $s$:

$$\langle f, g \rangle = \frac{4}{\mathrm{vol}(\mathbb{H}\backslash\Gamma)} \sum_s \frac{h_{s,0}}{h_s} \sum_{n=1}^{\infty} \frac{a_{n,s}\overline{b}_{n,s}}{n^{k-1}} \sum_{m=1}^{\infty} \left(\frac{x}{8\pi}\right)^{k-1} \left(xK_{k-1}(x) - K_{k-2}(x)\right) \qquad x = 4\pi m \sqrt{\frac{n}{h_s}}$$

(this formula explained in more detail in Theorem 4.2).

The computation of $q$-expansions of modular forms at cusps other than $\infty$ (given the $q$-expansion at infinity) is a surprisingly subtle problem, and the main result of this paper is to give an algorithm that can numerically compute these $q$-expansions for use in Nelson's formula. Recalling that the $q$-expansion of $f$ at any cusp can be viewed as the $q$-expansion of $f|[\alpha]_k$ at $\infty$ for some matrix $\alpha$, our approach is to calculate various values of $f|[\alpha]_k$ (using the original $q$-expansion of $f$), and then linearly interpolate these in a way that gives us a good numerical approximation of the expansion at $\infty$. One version of our algorithm (assuming absolutely nothing about $f$ beyond it being a modular form that we know the $q$-expansion for) is Algorithm 2.3, which directly interpolates the coefficients of the $q$-expansion. A second version is given in Algorithm 2.6, which assumes that $f$ is an eigenform away from bad primes and has the advantage that the computation does not grow even as the number of coefficients we want does.

While we only discuss cusp forms and Petersson inner products in this paper, we remark that this approach should be easily modified to other situations. Nelson's formula can be applied to general integrals of automorphic functions on quotients of the upper half-plane. Certainly any other sort of integral constructed from modular forms could be handled this way, and our interpolation approach could be modified to handle other classes of functions that can be described reasonably in terms of a Fourier expansion (e.g. Maass forms).

**Our motivation, and comparison with other approaches.** Our specific motivation for studying this comes from the situation where we have *three* newforms $f, g, h$ such that the product $fg$ has the same weight and character as $h$. A general formula of Ichino [Ich08] gives a relation between $|\langle fg, h \rangle|^2$ and the central value of a triple-product $L$-function which we may write as

$$|\langle fg, h \rangle|^2 = C \cdot L(f \times g \times \overline{h}, m - 1) \cdot \prod_{\text{bad primes } p} I_p^{**},$$

where the constant $C$ and the local constants at bad primes $I_p^{**}$ are things that can be in principle evaluated from the setup of the problem, but in practice the computations are quite subtle. In [Col16] we establish a completely explicit formula in some cases, and use it to construct $p$-adic $L$-functions.

In a context like this it is important to know that the algebraic part of our constants are precisely correct, because we ultimately want to study $p$-integrality and congruences modulo $p$ for our $p$-adic $L$-function. Hence, we wish to numerically compute the ratio of $|\langle fg, h \rangle|^2$ and $L(f \times g \times \overline{h}, m - 1)$ in many cases and verify this agrees with the constants we obtain in our formula. Numerical agreement in a representative sample of examples provides a very convincing argument that the constants are indeed correct, because errors in the theoretical calculations generally result in things like the constants containing extraneous powers of 2 or incorrect Euler-like factors such as $(1 + 1/p)$.

To implement this calculation, there is a well-known algorithm of Dokchitser [Dok04] that we can use to compute the $L$-value. However, we were not able to find in the literature a satisfactory method for computing Petersson inner products for our purposes. Ideally, we would like our algorithm to have the following characteristics:

- Works directly with the $q$-expansions of our modular forms at infinity, since this is how our modular forms are given.
- Avoids computing with full spaces of cusp forms as much as possible; in examples we want to test $f, g, h$ may all be of reasonably large levels that are coprime to each other, so any space $S_k(N, \chi)$ containing both $fg$ and $h$ may be of large enough dimension to make it impractical to work with.

The most commonly-suggested method, perhaps, is to us the connection with adjoint $L$-functions - for a newform $f$, there is an explicit formula relating between $\langle f, f \rangle$ and $L(\mathrm{ad}\, f, 1)$. However, using this for something like $\langle fg, h \rangle$ requires decomposing $fg$ in terms of an eigenbasis, which ultimately would involve computing a full space of cusp forms that is potentially very large. Also, we will see in Section 4.2 that it is a nontrivial task just to implement the formula relating $\langle f, f \rangle$ and $L(\mathrm{ad}\, f, 1)$ for newforms of arbitrary level! Another approach is given in [Coh13], but this is based on numerical integration from the values of the function itself, which isn't ideal for modular forms given as $q$-expansions.

The most promising approach seemed to be to use Nelson's formula, which expresses the Petersson inner product as a straightforward infinite sum (involving some $K$-Bessel functions) over the $q$-expansions. Of course, this requires a method to get the $q$-expansions at other cusps, and once again there are an assortment of results in the literature but none that were satisfactory for our purposes. Asai [Asa76] uses Atkin-Lehner operators to give a full expression of expansions at all cusps for modular forms of squarefree level, but there are not any results nearly as nice for the general case. Some partial results are given in the thesis of Delaunay [Del02], and a formula and algorithm for expansions at cusps of width one was given in the recent thesis of Chen [Che16]. The only general algorithm we are aware of is in Section 3.6.8 of the book [EC11], but this involves computations with a full space of modular forms (actually, of even higher level than what one starts with) so would be impractical for the applications we have in mind.

**Overview of this paper.** In Section 2 we present the core results of this paper: setting up the problem of determining $q$-expansions at all cusps, and then presenting our algorithms for numerically computing these expansions. Section 2.3 presents our first algorithm, which solves for the coefficients of $f|[\alpha]_k = \sum b_n q^n$ by truncation of the sum and direct interpolation of the coefficients $b_n$. Our second algorithm, in Section 2.4, applies to the case that $f$ is an eigenform and instead interpolates $f|[\alpha]_k$ as a linear combination of a basis for the eigenspaces of $f$ and its twists. The theoretical result guaranteeing that $f|[\alpha]_k$ arises as such a linear combination is the following:

**Theorem 1.1.** *Let $f \in S_k(N, \chi)$ be an eigenform of the Hecke operators $T_p$ for $p \nmid N$ (i.e. an oldform associated to a newform $f_0 \in S_k(N_0, \chi)$ for some $N_0 | N$). Then $f|[\alpha]_k$ (its expansion at another cusp, normalized to have integer exponents in its $q$-expansion) is a linear combination of twists $(f_0 \otimes \mu)(mz)$ that lie in $S_k(\Gamma_1(Nh))$.*

This is stated later on as Theorem 2.4, which is proven in Section 3.1. In Section 2.5 we discuss how to narrow down the space $S_k(\Gamma_1(Nh))$ in which $f|[\alpha]_k$ may live, and thus the list of twists potentially needed. We remark that determining all of the twists of the appropriate level requires knowing the minimal-level twist of $f_0$. Finding this minimal level twist is the only place our current algorithm may require working with a full space of cusp forms $S_k(N, \chi)$; we discuss this and potential ways to avoid it in Section 4.3.

We combine our $q$-expansion algorithms with Nelson's formula in Section 4 to describe an algorithm for numerically computing Petersson inner products. This is followed with some examples of computing self-Petersson inner products $\langle f, f \rangle$ for newforms $f$, and comparing with the known formula for $\langle f, f \rangle$ in terms of $L(\mathrm{ad}\, f, 1)$, plus some computations of ratios of Petersson inner products such as $\langle f(pz), f(z) \rangle / \langle f(z), f(z) \rangle$ which are relevant in the study of $p$-adic $L$-functions. In Section 5 we describe how to best implement our methods to compute products $\langle fg, h \rangle$, and then describe several computations we have made to verify formulas proven in [Col16].

## 2. Approaches to numerical computation of $q$-expansions at cusps

2.1. **Precise setup of the problem.** Before describing our methods for computing the $q$-expansion of a modular form at all cusps, we want to be precise about how we're formulating the problem and about what spaces all of the relevant modular forms live in. Throughout we will let $f \in M_k(N, \chi)$ be a modular form of weight $k$ on $\Gamma_0(N)$ with character $\chi$. Our goal is to start with the $q$-expansion

$$f(z) = \sum a_n e^{2\pi i n z} = \sum a_n q^n$$

of $f$ at infinity and, from that, compute the $q$-expansions of the translates

$$f|[\alpha]_k(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) \qquad \alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for all choices of $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Of course since we know how $f$ transforms under $\Gamma_0(N)$ this reduces to looking at finitely many matrices representing the cosets of $\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})$.

The problem can be further condensed by passing from a matrix $\alpha$ as above to the corresponding *cusp* in $\mathbb{P}^1(\mathbb{Q})$, which we take to be the image of $\infty$ under the action of $\alpha$ by a Möbius transformation: $\alpha\infty = a/c$. If two matrices $\alpha, \beta$ correspond to the same cusp, we will explicitly describe how the $q$-expansions differ at the end of this section. So we really just need to understand $f|[\alpha]_k$ for one matrix $\alpha$ corresponding to each cusp. An explicit description of the cusps can be given as in Proposition 1.43 of [Shi94]; all we'll really need is that each non-$\infty$ cusp can be represented as $a/c$ for $c$ a proper divisor of $N$ and $(a, c) = 1$.

So now we consider a cusp $a/c$ of this form, and fix a choice of matrix

$$\alpha_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We know $f|[\alpha_1]_k$ is a modular form for the group $\alpha_1^{-1}\Gamma_0(N)\alpha_1$ with character induced by $\chi$ under conjugation, which is a congruence subgroup containing $\Gamma(N)$. However, it does not contain $\Gamma_1(N)$ and thus the $q$-expansion of $f|[\alpha_1]_k$ may involve fractional powers. To avoid this we replace $f|[\alpha_1]_k(z)$ by some $f|[\alpha_1]_k(hz)$ which is a modular form in some $M_k(\Gamma_1(N'))$, ideally with $h$ as small as possible. We can equivalently write $f|[\alpha_1]_k(hz)$ as (a scalar multiple of) $f|[\alpha_h]_k$ for

$$\alpha_h = \alpha_1 \cdot \tau_h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ah & b \\ ch & d \end{bmatrix}.$$

**Lemma 2.1.** *Fix $N$, $\chi$, and $a/c$ as above. Let $h|(N/c)$ be an integer satisfying both*

- *$N$ divides $c^2 h$.*
- *$\chi$ is trivial on the subgroup $(1 + ch\mathbb{Z})/N\mathbb{Z}$ of $(\mathbb{Z}/N\mathbb{Z})^\times$.*

*Then for any $f \in M_k(N, \chi)$, we have $f|[\alpha_h]_k \in M_k(\Gamma_1(Nh))$.*

Note that the smallest $h$ satisfying the first condition is exactly the width of the cusp $a/c$ for $\Gamma_0(N)$, and the smallest $h$ satisfying both is the width of $a/c$ for $\ker \chi \leq \Gamma_0(N)$. So this $h$ is indeed the smallest integer such that $[\alpha_h]_k$ takes $M_k(N, \chi)$ into any $M_k(\Gamma_1(N'))$.

*Proof.* The first step is showing that $\Gamma_1(Nh)$ is a subgroup of the group $\alpha_h^{-1}\Gamma_0(N)\alpha_h$ for which $f|[\alpha_h]_k$ is modular; equivalently, we have to show that if $\gamma \in \Gamma_1(Nh)$ then $\alpha_h \gamma \alpha_h^{-1} \in \Gamma_0(N)$. If we write

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod{Nh},$$

then an explicit calculation (using $c^2 h \equiv 0 \pmod{N}$) shows that

$$\alpha_h \gamma \alpha_h^{-1} \equiv \begin{bmatrix} 1 - Bach & ** \\ 0 & 1 + Bach \end{bmatrix} \pmod{N}.$$

This means $f|[\alpha_h]_k$ lies in $M_k(\Gamma_1(N), \chi')$ for $\chi'$ the character given by $\chi'(\gamma) = \chi(\alpha_h \gamma \alpha_h^{-1})$. Since we don't want a character on $\Gamma_1(N)$ we need to insist that this is trivial, i.e. that we've chosen $h$ large enough so the elements $1 \pm Bach$ on the diagonal are actually in the kernel of $\chi$. $\qquad\square$

The main goal of this paper is to present practical methods for determining the $q$-expansion $f|[\alpha_h] = \sum b_n q^n$ for any cusp $a/c$, working from the original $q$-expansion $f = \sum a_n q^n$. In some cases there is a satisfactory theoretical way to find $f|[\alpha_h]$ using Atkin-Lehner operators (we will discuss this, as well as a more general refinement of the above Lemma, in Section 2.5). But if the level $N$ is divisible by large powers of a prime, then the exact determination of $f|[\alpha_h]$ is a delicate problem in local representation theory. So instead we will look for a way to *numerically* compute the coefficients $b_n$.

*Expansions for other matrices at the same cusp.* When expanding at a cusp $a/c$ we'll usually work with a fixed matrix $\alpha_1$ as above, but in some cases we'll need to consider other matrices too. Suppose

$$\beta_1 = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

is any other matrix that takes $\infty$ to the cusp $a/c$ of $\Gamma_0(N)$. Cusps can be described as double cosets in $\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$ where $\Gamma_\infty$ is the stabilizer of the cusp infinity in $\mathrm{SL}_2(\mathbb{Z})$, i.e. $\Gamma_\infty = \{\pm\delta_x : x \in \mathbb{Z}\}$ where we write

$$\delta_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

So, if $\alpha_1$ and $\beta_1$ represent the same cusp, there is $\gamma \in \Gamma_0(N)$ and $x \in \mathbb{Z}$ with $\beta_1 = \gamma\alpha_1(\pm\delta_x)$. If we set $\beta_h = \beta_1\tau_h$ we then get

$$f|[\beta_h] = \chi(\gamma)\big(f|[\alpha_h]_k\big)|[\tau_h^{-1}\delta_x\tau_h]_k.$$

A computation gives that $\tau_h^{-1}\delta_x\tau_h = \delta_{x/h}$, so $f|[\beta_h]_k$ is equal to $f|[\alpha_h]_k$ with the slash operator $[\delta_{x/h}]_k$ applied and times a constant. It's straightforward to check that $\delta_{x/h}$ normalizes $\Gamma_1(Nh)$ so $f|[\beta_h]_k$ still lies in $M_k(\Gamma_1(Nh))$. Also, $\delta_{x/h}$ acts in a predictable way on the $q$-expansion, which we summarize in the following proposition.

**Proposition 2.2.** *Suppose*

$$\beta_1 = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \qquad\qquad \beta_1' = \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}$$

*are two matrices taking $\infty$ to the same cusp $a/c$ for $\Gamma_0(N)$, with width $h$ as in the above proposition. If $f|[\beta_h]_k$ has q-expansion $\sum b_n q^n$, then we have*

$$f|[\beta_h']_k = \chi\big(a'd''h - b'c'' - a'c''x\big)\sum b_n \exp(2\pi inx/h)q^n$$

*where $x$ is an integer chosen such that $c''d' - c'd''h + c'c''x \equiv 0 \pmod{N}$.*

*Proof.* The claim that $\beta_1, \beta_1'$ take $\infty$ to the same cusp $a/c$ means that they are in the same double coset in $\Gamma_0(N)\backslash\mathrm{SL}_2(Z)/\Gamma_\infty$, i.e. that there's $\gamma \in \Gamma_0(N)$ and $\delta_x \in \Gamma_\infty$ (where we WLOG move the factor of $\pm I$ to the matrix in $\Gamma_0(N)$) such that $\beta_1' = \gamma\beta_1\delta_x$. Right-multiplying by $\tau_h$ and rearranging we get

$$\gamma = \beta_h'\delta_{x/h}^{-1}\beta_h^{-1}.$$

Computing out the product on the right-hand side we find the bottom-left entry is $c'd'' - c''d + c'c''x$, so our assumption that $\gamma \in \Gamma_0(N)$ forces $x$ to satisfy the specified congruence. Since $\beta_h' = \gamma\beta_h\delta_{x/h}$, we can compute the $q$-series of $f|[\beta_h']$ by applying these three matrices - $\gamma$ transforms $f$ via $\chi$ applied to its lower-right entry, $\beta_h$ gives the expansion $\sum b_n q^n$, and $\delta_{x/h}$ replaces $q$ by

$$\exp(2\pi in(z + x/h)) = q \cdot \exp(2\pi inx/h). \qquad\qquad \square$$

*Expansions of $f(mz)$ in terms of expansions of $f(z)$.* If one has a modular form $f(z)$ and applies a degeneracy map to it to obtain a modular form $f(mz)$ for some positive integer $m$, the expansion of $f(mz)$ at any cusp can be obtained from the expansions of $f(z)$ at a possibly different cusp. We will describe explicitly how to do this here; note that this reduces the problem of finding expansions of eigenforms just to the case of newforms.

It is helpful to consider the case where $m = p$ is as prime, which divides up into two situations: the case where $p \nmid c$ (the denominator of our cusp) and the case $p|c$. In the former case we can choose our matrix $\alpha_1$ to have $d|p$, at which point we write

$$f(z)\Big|_k \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}\Big|_k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = f(z)\Big|_k \begin{bmatrix} ap & b \\ c & d/p \end{bmatrix}\Big|_k \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$$

In the latter case we instead have

$$f(z)\Big|_k \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}\Big|_k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = f(z)\Big|_k \begin{bmatrix} a & bp \\ c/p & d \end{bmatrix}\Big|_k \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}.$$

If $m$ is composite we can iterate this procedure one prime at a time to get that $f(mz)|[\alpha]_k$ is equal to $(f|[\alpha'])(m'z)$ for some matrix $\alpha' \in \mathrm{SL}_2(\mathbb{Z})$ and some rational number $m'$.

To give the general case explicitly, suppose $f \in M_k(N, \chi)$ is a modular form, $m$ is an integer, and we want to consider the expansion of $f(mz) \in M_k(Nm, \chi)$ at a cusp $a/c$ of $\Gamma_0(Nm)$. As usual we assume $c|N$ and $(a, c) = 1$, and fix a matrix

$$\alpha_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

taking $\infty$ to $a/c$. Let $m_1 = (c, m)$ and $m_2 = m/m_1$; note that this implies $c/m_1$ and $m_2$ are coprime, and therefore we may find an integer $y$ such that $d - (c/m_1)y$ is divisible by $m_2$. Then we have

$$f(mz)|[\alpha_1]_k = m^{-k/2} \cdot f(z)\bigg|_k \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}\bigg|_k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = m^{-k/2} \cdot f(z)\bigg|_k \begin{bmatrix} am_2 & bm \\ c/m_1 & d \end{bmatrix}\bigg|_k \begin{bmatrix} m_1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we can further expand

$$\begin{bmatrix} am_2 & bm \\ c/m_1 & d \end{bmatrix} = \begin{bmatrix} am_2 & bm - yam_2 \\ c/m_1 & d - yc/m_1 \end{bmatrix}\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} am_2 & bm_1 - ya \\ c/m_1 & \frac{d-yc/m_1}{m_2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix},$$

at which point our initial expression is written in terms of an expansion of $f$ at the cusp $(am_2)/(c/m_1)$.

## 2.2. Attempt via Fourier analysis.

As above, suppose we have $f = \sum a_n q^n \in M_k(N, \chi)$ and that we want to compute the coefficients in the expansion $f|[\alpha_h]_k = \sum b_n q^n$ at a cusp $a/c$. A first approach one might try is to simply use Fourier inversion to obtain a formula for each $b_n$. We describe this computation, and why it does not turn out to give us a practical algorithm.

We can single out the Fourier coefficient $b_m$ by integrating $f|[\alpha_h]_k(x+iy) \cdot \exp(-2\pi im(x+iy))$ from $x = 0$ to $x = 1$ (for a fixed value of $y$):

$$b_m = \int_0^1 \left(\sum b_n q^n\right) \exp(-2\pi im(x+iy))dx = \exp(2\pi my)\int_0^1 f|[\alpha_h]_k(x+iy)\exp(-2\pi imx)dx$$

$$= \exp(2\pi my)\int_0^1 h^{k/2}(ch(x+iy)+d)^{-k}f\left(\frac{ah(x+iy)+b}{ch(x+iy)+d}\right)\exp(-2\pi imx)dx.$$

Since $f(z) = \sum a_n q^n$ we can simply substitute this in and rearrange to get

$$b_m = h^{k/2}\exp(2\pi my)\sum_{n=0}^{\infty} a_n \int_0^1 \frac{1}{(chx+d+ichy)^k}\exp\left(2\pi i\left(n\frac{ahx+b+iahy}{chx+d+ichy} - mx\right)\right)dx.$$

This gives a series converging to $b_m$. However, it does not seem to be practical to compute $b_m$ this way - the series can take quite a while to converge, and without a very efficient method for computing the integrals (for all values of both $m$ and $n$ up to whatever cutoffs we need) the computation will be very slow.

## 2.3. Approach 1: Least squares for the $q$-series.

Another approach to determining the Fourier coefficients of $f|[\alpha_h]_k = \sum b_n q^n$ is to treat the $b_n$'s as variables to be filled in by interpolating from the known values that the function takes. As stated this has infinitely many variables, but truncating we can approximate it as $\sum_{n=0}^{K} b_n q^n$. We can evaluate $f|[\alpha_h]_k(z)$ at many points, and try to find the coefficients $b_0, \ldots, b_K$ that best fit the data.

If we choose points $z_1, \ldots, z_M$ on the upper half-plane, and let $q_j = \exp(2\pi iz_j)$, then after computing each $q_j$ and its powers plus each value $f|[\alpha_h](z_j)$ (from the original $q$-expansion of $f$), the problem is to choose the vector of values $b_0, \ldots, b_K$ that offers the best solution to the matrix equation

$$\begin{bmatrix} 1 & q_1 & q_1^2 & \cdots & q_1^K \\ 1 & q_2 & q_2^2 & \cdots & q_2^K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & q_M & q_M^2 & \cdots & q_M^K \end{bmatrix}\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_K \end{bmatrix} = \begin{bmatrix} f|[\alpha_h](z_1) \\ f|[\alpha_h](z_2) \\ \vdots \\ f|[\alpha_h](z_M) \end{bmatrix}$$

where $q_l = \exp(2\pi iz_l)$.

If we interpret "best solution" as asking for the smallest Euclidean distance between the two sides as elements of $\mathbb{R}^M$, then this is just a standard problem in linear algebra, and the least-squares solution to the equation $Ax = b$ is the actual solution to $(A^*A)x = A^*b$ where $A^*$ is the conjugate transpose of $A$. It's then straightforward to implement this as an algorithm: given $M$, $K$, and the points $z_1, \ldots, z_M$ we can compute the matrix of powers of $q$ and the vector of values of $f|[\alpha_h]$ as floating-point complex numbers, and then perform solve the floating-point linear system $(A^*A)x = A^*b$.

The next question is how to best choose $M$, $K$, and the points $z_j$. The number $K$ of coefficients to look for and the imaginary parts of the $z_j$ are closely related to the accuracy we want from the calculation. Specifically, since we've chosen as $K$ as our cutoff, then $\sum_{j=0}^{K} b_j q^j$ will differ from the actual value of $f|[\alpha_h](z)$ by the tail $\sum_{j=K=1}^{\infty} b_j q^j$, which is on the order of $|q^K| = \exp(-2\pi K \operatorname{Im}(z))$, so every part of our computation will have an error of around this size. Also, when determining the accuracy of the coefficient $b_j$, it's actually the product $b_j q^j$ which can be expected to have error of size $\exp(-2\pi K \operatorname{Im}(z))$, so the error of $b_j$ will be about the order of $\exp(-2\pi(K - j) \operatorname{Im}(z))$.

So, what we can do is specify a number $K_0$ of coefficients we definitely want, an absolute error $10^{-E}$ for our calculations, and an exponential decay rate $e^{-C_0}$ such that we'd like the error of $b_j e^{-C_0}$ to be on the order of $10^{-E}$. (This is a reasonable requirement, because for our applications we'll be computing sums where $b_j$ is multiplied by some exponentially-decreasing factor). For the actual computation we need to aim for an exponential decay rate $C$ and a number of coefficients $K$ such that $e^{-KC} \approx 10^{-E}$ and thus we truncate our sum at around the correct place, so start with $K = K_0$ and $C = C_0$ and either increase $K$ or decrease $C$ to get $KC \approx \log(10)E$.

To be able to compute the coefficients with decay rate $e^{-C}$, we sample at points $z_j$ where $|q_j| \approx e^{-C}$, i.e. $\operatorname{Im}(z_l) \approx C/2\pi$. Moreover, when computing the values of $f|[\alpha_h](z_j)$ the factor of automorphy $(chz_j + d)$ affects location of the translated point $\alpha_h z_j$ and thus the speed of convergence of the sum, so to optimize this we prefer to choose points $z_j$ with $\operatorname{Re}(z_j) \approx -d/ch$ to minimize this.

In our implementations, we chose points with $\operatorname{Im}(z_l) = C/2\pi$, and with $\operatorname{Re}(z_l)$ chosen randomly in an interval of length 1 centered at $-d/ch$. Fixing the imaginary part leaves the magnitude of all of our computations equal. Since we're working directly with powers of $\exp(2\pi iz)$ that are periodic under $z \mapsto z+1$ there's no reason to work outside of an interval of length 1, but the interpolation seems somewhat sensitive to working in any smaller range. The number of points sampled $M$ needs to be at least as large as $K$ for our interpolation problem to be solvable in principle, and the larger $M$ is the more accurate the computation is likely to be; we settled on $M = 2K$ as a workable choice.

**Algorithm 2.3** (Least-squares for $q$-expansion). Suppose we have a modular form $f = \sum a_n q^n \in M_k(N, \chi)$ and we want to compute its expansion $f|[\alpha_h] = \sum b_n q^n$ at a cusp given by a matrix $\alpha_h$ in our notation above. Suppose further that we've fixed constants $E$, $K_0$, and $C_0$ such that for $n \le K_0$ we would like to compute the coefficient $b_n$ to with an error of approximately $10^{-E} e^{nC_0}$. We proceed as follows:

- Either increase $K = K_0$ or decrease $C = C_0$ so that $KC \approx \log(10)E$, and work with interpolating the truncation $\sum_{n=0}^{K_0} b_n q^n$ of the expansion for $f|[\alpha_h]$.
- Choose $M$ (we used $2K_0$) and pick $M$ points $z_1, \ldots, z_M$ with $\operatorname{Im}(z_j) = C/2\pi$ and $\operatorname{Re}(z_l)$ is picked randomly in the interval of length 1 centered around $-d/ch$ (for $c, d, h$ the parameters from the matrix $\alpha_h$).
- Numerically compute the values $f|[\alpha_h](z_l) = h^{k/2}(chz_l + d)^{-k} f(\alpha_h z_l)$ using the $q$-expansion for $f$, truncating when we've reached an accuracy a bit past $10^{-E}$, and fill these into a vector $b$.
- Numerically compute the values $q_j^n = \exp(2\pi inz_j)$ and fill these into a matrix $A$.
- Numerically find the least squares solution to $Ax = b$ as the exact solution to $(A^*A)x = A^*b$. The solution vector $x$ is our numerical approximation to the coefficients $b_0, b_1, \ldots, b_K$.

Given the nature of the least-squares approximation, it seems very unlikely to be able to establish rigorous error bounds for this algorithm (even if the points were picked deterministically rather than randomly). Nonetheless it seems to work well in practice, and testing with various examples it returns values for the coefficients with accuracy close to what we hope.

For example, consider the unique newform

$$f = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 - 8q^8 + \cdots \in S_4(\Gamma_0(6));$$

because this has squarefree level the results of Asai [Asa76] tell us that its expansion at any cusp should be a multiple of itself. Sure enough, if we run the algorithm above with $E = 15$ and $C = 1$, we need to compute $K = 35$ coefficients and thus sample at 70 points. An example run of this for the cusp $1/3$ and the matrix

$$\alpha_1 = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$$

and $h = 2$ required using around 270 coefficients of $f$ for the slowest-converging sum, and returns that $f|[\alpha_2]_k$ is approximately

$(1.0000000000000147 + .0000000000000235i)q + (-1.9999999999999052 + .0000000000000885i)q^2$

$\quad + (-2.9999999999996767 - .0000000000002597i)q^3 + (3.9999999999998517 + .0000000000000770i)q^4$

$\quad + (5.9999999999967810 + .0000000000018893i)q^5 + (6.0000000000018602 - .0000000000051318i)q^6 + \cdots,$

which is an approximation of $f$ itself with errors on the scale we wanted.

Expansions at cusps for non-squarefree levels can get more complicated and seem less well-understood theoretically. For instance, one can take the newform

$$f = q - 3q^2 + q^4 - 15q^5 - 25q^7 + 21q^8 + 45q^{10} + \cdots \in S_4(\Gamma_0(27))$$

and looks at the cusp $1/3$ where we take the same matrix $\alpha_1$ as above but this time with width $h = 3$. If we want $E = 15$ and $C = 1$ once again we find we need to take $K = 35$ and sample at 70 points. This time a sample run-through used approximately 410 coefficients of $f$ for its slowest-converging sum, and returns that $f|[\alpha_3]_k$ is approximately

$(.9396926207858713 - .3420201433255586i)q + (2.2981333293573119 - 1.9283628290595167i)q^2$

$\quad + (-.0000000000004964 - .0000000000003253i)q^3 + (-.1736481776683433 + .9848077530113447i)q^4$

$\quad + (2.6047226650051819 + 14.7721162951836733i)q^5 + (-.0000000000019237 - .0000000000090777i)q^6 + \cdots$

Here the coefficients are much less readily recognizable, but one can identify the first coefficient as being the inverse of the usual primitive 18th root of unity $\zeta_{18}$. Similarly the other coefficients appear to also be related to 18th roots of unity times the corresponding coefficient of the original modular form $f$, and our computations suggest

$$f|[\alpha_3]_k = \zeta_{18}^{-1}q + 3\zeta_{18}^{-2}q^2 + 0q^3 + \zeta_{18}^5 q^4 + 15\zeta_{18}^4 q^5 + 0q^6 + \cdots.$$

In the next section we will approach this problem from a different angle and make it somewhat more clear where these coefficients are coming from.

2.4. **Approach 2: Least squares for an eigenbasis.** A downside to the least-squares algorithm applied to $q$-expansions is that if we need many coefficients of our modular form (which will happen when we compute Petersson inner products using Nelson's formula), the algorithm gets quite slow: to obtain $M$ coefficients we need to compute values at $2M$ points and then numerically solve a least-squares problem for a $2M \times M$ matrix. But modular forms are determined by only a finite number of coefficients, so in principle we should be able to make this computation independent of the number of coefficients we want.

One way to accomplish this is to simply compute a basis of the space $M_k(\Gamma_1(Nh))$ containing $f|[\alpha_h]_k$, and then perform a least-squares computation to find a best approximation of $f|[\alpha_h]_k$ as a linear combination of this basis by evaluating at a collection of points in the upper half-plane. If our basis consists of $d$ modular forms, then evaluating at $2d$ points should give us a good numerical approximation of the coefficients of the linear combination from which we can recover numerical approximations for any number of coefficients we want. The downside of this naive approach is that the dimension $d$ of $M_k(\Gamma_1(Nh))$ grows linearly in terms of the weight $k$ and quadratically in terms of the level $Nh$, and for even fairly small levels and weights $d$ may end up much larger than the number of coefficients we want to obtain.

So if $f$ is an arbitrary modular form in $M_k(N, \chi)$ then it seems unlikely that a least-squares approach attempting to realize $f$ as a linear combination of other modular forms would be efficient. However, for most of the examples we care about $f$ is far from arbitrary: the modular forms $f$ of most interest are eigenforms. In this case we could hope that $f|[\alpha_h]_k$ is a linear combination of a comparatively small number of basis elements. Indeed this is true; the following theorem will be proven in Section 3.1. (We restrict to cuspidal

eigenforms at this point, because our interest is in modular forms in the old subspace corresponding to a particular newform, but the argument should extend to Eisenstein series as well).

**Theorem 2.4.** *Let $f \in S_k(N, \chi)$ be an eigenform of the Hecke operators $T_p$ for $p \nmid N$ (i.e. an oldform associated to a newform $f_0 \in S_k(N_0, \chi)$ for some $N_0 | N$). Then $f|[\alpha_h]_k$ is a linear combination of twists $(f_0 \otimes \mu)(mz)$ that lie in $S_k(\Gamma_1(Nh))$.*

Here $f_0 \otimes \mu$ denotes the newform that is a twist of $f_0$ by a Dirichlet character $\mu$, so $f_0 \otimes \mu$ may differ from the "naive twist" $f_{0,\mu} = \sum \mu(n) a_n q^n$ which may not be a newform itself (but is an oldform associated to the newform $f_0 \otimes \mu$).

This result gives us a reasonably small subspace of $S_k(\Gamma_1(Nh))$ to look for $f|[\alpha_h]_k$ in, making the computation much more reasonable than working with a full basis. We just need to figure out which forms $(f_0 \otimes \mu)(mz)$ are actually modular for $\Gamma_1(N)$. The first step of doing this is to locate a twist $g$ of $f_0$ which is twist-minimal (i.e. $g$ is not itself a twist of any lower-level newforms) - this is clearly a finite computation, which we make some remarks on in Section 4.3. Once we have $g$ we can determine the level and $p$-th Fourier coefficient of any twist $g \otimes \mu$ of it via a prime-by-prime analysis, either working classically (as in Section 3 of [AL78] and in [Asa76]) or adelically (where it's clear what happens if the local component of the representation is principal series or special, but more complicated if it's supercuspidal; see the discussion in Sections 2 and 4 of [LW12] and Section 2 of [Hum15]). The results of this analysis are summarized in the following lemma.

**Lemma 2.5.** *Let $g = \sum b_n q^n$ be a twist-minimal newform of level $N_g$ and character $\chi_g$, and let $N_{g,\chi}$ be the conductor of $\chi_g$. Fix a prime $p$ and let $p^{r_g}$ be the exact power of $p$ dividing $N_g$, $p^{r_{g,\chi}}$ the exact power dividing $N_{g,\chi}$, and $\nu$ a Dirichlet character of prime-power conductor $p^u$.*

- *If we don't have $r_g = r_{g,\chi} > 0$, then $g \otimes \nu$ has level $\mathrm{lcm}(N_g, p^{2u})$ and equals the naive twist $g_\nu$.*
- *If $r_g = r_{g,\chi} > 0$ and $u \neq r_{g,\chi}$ then $g \otimes \nu$ has level $\mathrm{lcm}(N_g, p^{u+r_{g,\chi}}, p^{2u})$ and equals $g_\nu$.*
- *If $r_g = r_{g,\chi} > 0$ and $u = r_{g,\chi}$, but the $p$-part of the conductor of $\chi_g \nu$ is $p^{r'} > 1$, then $g \otimes \nu$ has level $\mathrm{lcm}(N_g, p^{u+r'})$ and equals $g_\nu$.*
- *If $r_g = r_{g,\chi} > 0$, $u = r_{g,\chi}$, and $\chi_g \nu$ is unramified at $p$, then $g \otimes \nu$ has level $N_g$ and does not equal the naive twist $g_\nu$; instead it has a coefficient of $(\chi_g \nu)(p) \bar{b}_p$ for $q^p$ and thus can be explicitly written as*

$$(g \otimes \nu) = \sum_{(n,p)=1} \nu(n) b_n q^n + \sum_{n=p^i n'} (\chi_g \nu)(p)^i \nu(n') \bar{b}_p^i b_{n'} q^n.$$

*Proof.* The first case corresponds to the local representation of $g$ at $p$ either being unramified, special of level $p$, or supercuspidal. In all three cases it's clear that the twisted local representation will result in $g \otimes \nu$ having a trivial $p$-th Fourier coefficient so $g \otimes \nu = g_\nu$. In the first two cases one can explicitly compute the conductor of the twisted local representation to be $p^{2u}$, and for the supercuspidal case we know that the conductor will be bounded above by $\max(p^{2u}, p^{r_g})$ with equality if $2u > r_g$ via Section 3 of [AL78], and equality if $2u \leq r_g$ by our assumption of twist-minimality.

The remaining type of twist-minimal local representations are principal series $\pi(\chi_1, \chi_2)$ where one of the two characters $\chi_i$ is ramified; the final three possibilities cover subcases of this situation. In any case we know $g \otimes \nu$ has local representation $\pi(\chi_1 \nu_p, \chi_2 \nu_p)$ where $\nu_p$ is the local character associated to the adelic lift of $\nu$. Here it is clear how to analyze the conductor of this principal series representation (since $\chi_1$ is unramified and $\chi_1 \chi_2$ is the $p$-part of the adelic lift of $\chi_g$, the conductor of $\chi_1 \nu_p$ is $p^u$ and the conductor of $\chi_2 \nu_p$ equals the conductor of $\chi_g \nu$). In the case where $\chi_2 \nu_p$ is unramified, its value at $p$ will give rise to the coefficient of $q^p$ in $g \otimes \nu$ which is killed off in the naive twist $g_\nu$, and using the relations between the characters lets us compute this coefficient to be $(\chi_g \nu)(p) \bar{b}_p$. $\square$

With this analysis it's easy to come up with a list of twists $g \otimes \mu$ of level at most $Nh$ and moreover find the exact level of each $g \otimes \mu$ so we can determine exactly which oldforms $(g \otimes \mu)(mz)$ are of level $Nh$ as well. This gives us a finite list $g_1, \ldots, g_M$ of modular forms of which we know $f|[\alpha_h]_k$ is a linear combination of, and we can proceed with a computation similar to the one of the previous section: we sample at some collection of more than $M$ points, compute the values of $g_l$ and $f|[\alpha_h]_k$ at each point, and use least-squares approximation to find the best fit for the list of coefficients in the relation $f|[\alpha_h]_k = \sum c_l g_l$.

Once again it seems very difficult to establish any sort of rigorous bounds on the error in this computation, but in practice it works quite well and heuristically one expects that the error in the computation will be near the same order of magnitude as where we truncated our sums. More specifically, if w e normalize all of our values $f|[\alpha_h]_k(z_j)$ and $g_l(z_j)$ by dividing by $q_j = \exp(2\pi i z_j)$ and then numerically compute our values $f|[\alpha_h]_k(z_j)/q_j$ and $g_l(z_j)/q_j$ to within an error of $10^{-E}$, then we expect the numerical values of $c_l$ will be such that the product $c_l \cdot (g_l(z_j)/q_j)$ is accurate to about $10^{-E}$ as well. For the $g_l$'s that are actually newforms, the coefficient of $q$ is 1 so $g_l(z_j)/q_j \approx 1$, and thus these $c_l$'s themselves should be accurate to about $10^{-E}$. For $g_l$'s of the form $(g_0 \otimes \mu)(mz)$ for $m > 1$, the value of $g_l(z_j)/q_j$ is significantly smaller (approximately $\exp(-2\pi(m-1)\operatorname{Im}(z_j))$) so the error in $c_l$ might be larger, but we can compensate for this by making our original computation more accurate (as described in the algorithm below).

The last thing to decide is what points $z_j$ we want to sample. In this case we have quite a bit of flexibility, and we are free to pick points $z_j$ to try to minimize the number of terms needed to be used when computing the values of our modular forms from the $q$-expansion of $f$ and its twists. Roughly speaking this amounts to trying to simultaneously minimize both $|\exp(2\pi i z)|$ and $|\exp(2\pi i(\alpha_h \cdot z))|$, i.e. to simultaneously maximize $\operatorname{Im}(z)$ and $\operatorname{Im}\left(\frac{ahz+b}{chz+d}\right) = \frac{h\operatorname{Im}(z)}{|chz+d|^2}$. Comparing these we can compute that the best choice for $z$ has $\operatorname{Im}(z) = \sqrt{h}/\sqrt{2}c$ and $\operatorname{Re}(z) = -d/c$; expanding this a bit since we need multiple points we can calculate that if we choose $z$ in the rectangle

$$\operatorname{Im}(z) \in \left[\frac{1}{2c\sqrt{h}}, \frac{1}{c\sqrt{h}}\right] \qquad \operatorname{Re}(z) \in \left[\frac{-d-\sqrt{h/2}}{ch}, \frac{-d+\sqrt{h/2}}{ch}\right]$$

then $|\exp(2\pi i z)|$ and $|\exp(2\pi i \alpha_h z)|$ are both bounded above by $\exp(-\pi/c\sqrt{h})$.

**Algorithm 2.6** (Least-squares for twists of an eigenform). Suppose we have $f = \sum a_n q^n \in M_k(N, \chi)$ an eigenform for all prime-to-$N$ Hecke operators, and we want to compute its expansion $f|[\alpha_h] = \sum b_n q^n$ at a cusp given by a matrix $\alpha_h$ in our notation above. Suppose further that we've fixed constants $E_0$, $K$, and $C$ such that for $n \leq K$ we would like to compute the coefficient $b_n$ to with an error of approximately $10^{-E_0}e^{nC}$. We proceed as follows:

- Determine the newform $f_0$ associated to $f$ and a twist-minimal newform $g_0$ that's a twist of $f_0$.
- For Dirichlet characters $\mu$ of modulus $N$, determine the level of the twist $g_0 \otimes \mu$; create a list $g_1, \ldots, g_L$ of all forms $(g_0 \otimes \mu)(mz)$ that have level $Nh$.
- Pick $M$ random points $z_1, \ldots, z_M$ (we use $M = 2L$) with $1/2c\sqrt{h} \leq \operatorname{Im}(z) \leq 1/c\sqrt{h}$ and $(-d - \sqrt{h/2})/ch \leq \operatorname{Re}(z) \leq (-d + \sqrt{h/2})/ch$.
- Set our truncation point for sums to be when the tail is size $10^{-E}$ where $E = E_0 + \frac{m_0 - 1}{\log 10}(2\pi\frac{\sqrt{h}}{c} - C)$ (or $E = E_0$, if $2\pi\frac{\sqrt{h}}{c} < C$) where $m_0$ is the largest integer $\leq K$ such that we have a modular form $(g_0 \otimes \mu)(m_0 z)$ on our list.
- Numerically compute the values $f|[\alpha_h](z_j)$ using the $q$-expansion for $f$ to accuracy $10^{-E}$, and fill these into a vector $b$.
- Numerically compute the values $g_l(z_j)$ to an accuracy of $10^{-E}$, using the $q$-expansions for the twists as described in Lemma 2.5, and fill these into a matrix $A$.
- Numerically find the least squares solution to $Ax = b$, which approximates the values of $c_1, \ldots, c_L$ in our linear combination. Use these values plus the $q$-expansions of the $g_l$ to provide a numerical approximation for the $q$-expansion of $f|[\alpha_h] = \sum c_l g_l$.

The change of the truncation point to $10^{-E}$ is to guarantee that we've computed everything out far enough so that even the coefficient of the (small) values of $(g_0 \otimes \mu)(m_0 z)$ can be computed with as much accuracy as we want. In principle this could go quite far beyond the original accuracy $10^{-E_0}$ we were interested in, and if this becomes an issue the choice of points $z_j$ could be adjusted instead. However for most practical purposes the change is not a serious problem, and the number of terms needed to be computed in the sums usually stays far below the number needed for the algorithm in the previous section.

For an example, we return to the modular form

$$f = q - 3q^2 + q^4 - 15q^5 - 25q^7 + 21q^8 + 45q^{10} + \cdots \in S_4(\Gamma_0(27))$$

considered in the previous section, and look at the expansion $f|[\alpha_3]_k$ at the cusp $1/3$ (with the matrix $\alpha_1$ considered there). Now we know that this translate must be a linear combination of twists lying in $S_k(\Gamma_1(81))$. One can check directly that $f$ is twist-minimal and the list of possible basis elements are

$$f(z), f(3z), (f \otimes \mu_1)(z), (f \otimes \mu_1^2)(z), (f \otimes \mu_1^3)(z), (f \otimes \mu_1^3)(3z), (f \otimes \mu_1^4)(z), (f \otimes \mu_1^5)(z),$$

where we fix $\mu_1$ to be the Dirichlet character modulo 9 defined on the multiplicative generator 2 of $(\mathbb{Z}/9\mathbb{Z})^{\times}$ by $\mu_1(2) = \zeta_6$. Then a numerical computation finds that $f|[\alpha_3]_k(z)$ is approximately

$$(.469846310392954 - .171010071662834i)(f \otimes \mu_1)(z) + (.469846310392954 + .171010071662834i)(f \otimes \mu_1^2)(z)$$

$$+ (.469846310392954 - .171010071662834i)(f \otimes \mu_1^4)(z) + (-.469846310392954 - .171010071662834i)(f \otimes \mu_1^5)(z)$$

(omitting the factors where the numerically-calculated coefficients are very close to zero). Numerically summing up this linear combination of Fourier expansions, one gets a numerical $q$-series that (up to our expected error) agrees with the one computed by our other algorithm in the previous section. What's more interesting is to try to identify the complex numbers appearing as coefficients here: they all seem to be approximating $\frac{1}{2}$ times an 18th root of unity, and suggest that

$$f|[\alpha_3]_k(z) = \frac{\zeta_{18}^{-1}}{2}(f \otimes \mu_1)(z) + \frac{\zeta_{18}}{2}(f \otimes \mu_1^2)(z) + \frac{\zeta_{18}^{-1}}{2}(f \otimes \mu_1^4)(z) + \frac{-\zeta_{18}}{2}(f \otimes \mu_1^5)(z).$$

Combining these $q$-series one can work out explicitly that if $f(z) = \sum a_n q^n$ then $f|[\alpha_3]_k(z) = -\sum \zeta_{18}^{8n} a_n q(n)$. So for this example, the expansion is (up to a scalar) an additive twist of the original $q$-expansion of $f$. Other modular forms have other behavior; for instance if we consider the newform and matrix

$$f = q - q^2 - 7q^3 - 7q^4 + 7q^6 + 6q^7 + 15q^8 + 22q^9 + \cdots \in S_4(\Gamma_0(25)) \qquad \alpha_1 = \begin{bmatrix} 1 & -1 \\ 5 & -4 \end{bmatrix},$$

the numerically-calculated expansion of $f|[\alpha_1]_k(z)$ includes nonzero coefficients for all four twists of $f = \sum a_n q^n$ by Dirichlet characters modulo 5 and can be expressed as $f|[\alpha_1]_k = \sum \xi(n) q^n$ where $\xi$ is periodic modulo 5 and satisfies

$$\xi(1) \approx -.809016994374947 + 1.11351636441161i \approx \frac{\cos(6\pi/5)}{\cos(7\pi/10)} \zeta_{20}^7$$

$$\xi(2) \approx .309016994374947 - .100405707943114i \approx \frac{\cos(2\pi/5)}{\cos(\pi/10)} \zeta_{20}^{19}$$

$$\xi(3) \approx .309016994374947 + .100405707943114i \approx \frac{\cos(2\pi/5)}{\cos(\pi/10)} \zeta_{20}$$

$$\xi(4) \approx -.809016994374947 - 1.11351636441161i \approx \frac{\cos(6\pi/5)}{\cos(7\pi/10)} \zeta_{20}^{13}$$

So we have numerically identified the expansion of $f(z)$ at the cusp $1/5$ as being a "twist" of $f$ by a periodic function $\xi$ with coefficients that are are algebraic numbers in $\mathbb{Q}(\zeta_{20})$. We do not pursue a theoretical understanding of how or why these specific coefficients arise; for our purposes we just need the numerical values.

2.5. **When can the eigenspace be narrowed down?** In this section we refine Lemma 2.1, to narrow down the space in which we can be guaranteed $f|[\alpha_h]_k$ lives (and accordingly prune the list of potential twists considered in Algorithm 2.6). For some cusps the result is close to optimal already, but for others it fails quite badly - most notably the cusp 0 (which always has width $N$), with the matrix

$$\alpha_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

If $f$ is a newform, Lemma 2.1 and Theorem 2.4 can only tell us that $f|[\alpha_N]_k$ is a linear combination of twists of $f$ (and their images under degeneracy maps) which are modular of level $N^2$ for some character. But $|[\alpha_N]_k$ is just the Atkin-Lehner operator $W_N$, and the theory of newforms tells us that $f|[\alpha_N]_k$ is a scalar multiple of one particular newform $f_\rho$ of level $N$. So in this case our result is quite far from sharp, and running Algorithm 2.6 naively may take quite some time due to including a great many unneeded basis elements.

However, the situation is not quite as simple as it seems - while it is true that for the particular matrix $\alpha_1$ above that $f|[\alpha_N]_k$ is always a scalar multiple of a single newform, this will fail for other choices of matrices taking $\infty$ to the cusp 0 (even other ones with $a = 0$ and $c = 1$). The key point is actually that the lower-right entry $d$ is zero; this makes the behavior of the lower-left and lower-right entries of the conjugate $\alpha_h \gamma \alpha_h^{-1}$ sensible and allows us to conclude $f|[\alpha_h]_k$ transforms reasonably under $\gamma$. So a refinement of Lemma 2.1 can only reasonably hold if we are careful to choose our matrix $\alpha_1$ carefully. In the case of a general cusp $a/c$, we'd like the product $cd$ to be as close to divisible by $N$ as possible - since $c$ and $d$ must be coprime, in particular $d$ should be divisible by the prime-to-$c$ part of $N$.

So, fix a modular form $f \in M_k(N, \chi)$, and a cusp $a/c$ with associated width $h$ for $f$ (so $h$ is determined from $N$, $c$, and $\chi$ as in Lemma 2.1). Factor $N$ as $c_0 \cdot d_0$, where $c_0$ and $c$ have the same prime divisors and $d_0$ is coprime to $c_0$. Then choose a matrix

$$\alpha_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

with $a$ and $c$ as in our cusp ($\alpha_1$ takes $\infty$ to $a/c$) and with $d$ divisible by $d_0$ (which we can do because $d_0$ is coprime to $c$). Factor the width $h$ as $h_c \cdot h_d$ where $h_c|c_0$ and $h_d|d_0$, and also let $\chi_c$ and $\chi_d$ denote the restrictions of $\chi$ to $\mathbb{Z}/c_0\mathbb{Z}$ and $\mathbb{Z}/d_0\mathbb{Z}$, respectively.

**Proposition 2.7.** *In the above setup, for any matrix*

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_0(Nh_c)$$

*satisfying* $(A - D)c \equiv 0 \pmod{c_0}$ *we have* $(f|[\alpha_h]_k)|[\gamma]_k = (\chi_c \chi_d^{-1})(D) f|[\alpha_h]_k$.

*Proof.* We have $f|[\alpha_h]_k|[\gamma]_k = f|[\alpha_h \gamma \alpha_h^{-1}]_k|[\alpha_h]_k$, so we want to show that $f$ transforms under $\alpha_h \gamma \alpha_h^{-1}$ by the scalar $(\chi_c \chi_d^{-1})(D)$. An explicit computation gives

$$\alpha_h \gamma \alpha_h^{-1} = \begin{bmatrix} * & * \\ -Bc^2 h + (A - D)cd + Cd^2/h & Bach + Dad - Abc - Cbd/h \end{bmatrix}.$$

By construction of $d$ and assumption that $C \equiv 0 \pmod{Nh_c}$ and $(A - D)c \equiv 0 \pmod{c_0}$ we can conclude that each of the three terms in the lower-left entry are divisible by $N$, and thus $\alpha \gamma \alpha_h^{-1} \in \Gamma_0(N)$. Thus $f|[\alpha_h \gamma \alpha_h^{-1}]$ equals $\chi(Bach + Dad - Abc - Cbd/h) \cdot f$.

So we just need to simplify

$$\chi(Bach + Dad - Abc - Cbd/h) = \chi(D + (D - A)bc + Bach),$$

where we can remove $Cbd/h \equiv 0 \pmod{N}$ immediately. To further work with this we split $\chi$ as our product $\chi_c \chi_d$. Since $\chi_c$ is defined modulo $c_0$ and $(D - A)bc \equiv 0 \pmod{c_0}$ we have

$$\chi_c(D + (D - A)bc + Bach) = \chi_c(D + Bach) = \chi_c(D)\chi_c(1 + D'Bach) = \chi_c(D)$$

with the last equality because $h$ is defined so that $\chi$ (and thus $\chi_c$) is trivial on $1 + ch\mathbb{Z}$. Similarly since $\chi_d$ is defined modulo $d_0$ and we have

$$\chi_d(D + (D - A)bc + Bach) = \chi_d(D + (D - A)(-1)) = \chi_d(A) = \chi_d^{-1}(D),$$

using that working modulo $d_0$ we have $ch \equiv 0$, $bc \equiv -(ad - bc) = -1$, and $AD \equiv AD - BC = 1$. $\square$

So by properly choosing the lower-right entry $d$ in our matrix $\alpha_1$, we can guarantee that $f|[\alpha_h]_k$ is actually modular of level $Nh_c$ rather than just $Nh$, and moreover get at least some control of the character. In the case that $c$ and $N/c$ are coprime (i.e. $c = c_0$ in our notation above), this proposition fully determines the character and states that $f|[\alpha_h]_k$ lies in $M_k(Nh_c, \chi_c \chi_d^{-1})$, but in the general case where there are primes dividing both $c$ and $N/c$ we can only give a transformation rule for matrices $\gamma$ such that $A \equiv D \pmod{c_0/c}$, i.e. for some intermediate congruence group $\Gamma_H(Nh_c)$. This allows $f|[\alpha_h]_k$ to be a linear combination of forms with characters that agree with $\chi_c \chi_d^{-1}$ on $H$ - and based on numerical examples in such cases this seems to be the best one could hope for.

We can restate the result of our computation as follows, which we can view as a strengthened version of Lemma 2.1 but which only applies if the matrix $\alpha_1$ has a "correctly-chosen" lower-right entry.

**Proposition 2.8.** *Fix a modular form $f \in M_k(N, \chi)$ and a cusp $a/c$ with width $h$ (for $f$). Choose a matrix $\alpha_1 \in \mathrm{SL}_2(\mathbb{Z})$ taking $\infty$ to $a/c$ with bottom-right entry $d$ divisible by the prime-to-$c$ part of $N$. Then we have*

$$f|[\alpha_h]_k \in \bigoplus_{\chi'} M_k(Nh_c, \chi')$$

*where $\chi'$ runs over all characters of $(\mathbb{Z}/N\mathbb{Z})^{\times}$ which agree with $\chi_c \chi_d^{-1}$ on the subgroup $H \leq (\mathbb{Z}/N\mathbb{Z})^{\times}$ which is the kernel of the map $(\mathbb{Z}/N\mathbb{Z})^{\times} \to (\mathbb{Z}/\frac{c_0}{c}\mathbb{Z})^{\times}$ given by $a + N\mathbb{Z} \mapsto a^2 + \frac{c_0}{c}\mathbb{Z}$.*

Suppose $p$ is a prime with exact power $p^m$ dividing $N$, and let $\chi_p$ denote the $p$-component of $\chi$ (a character of $(\mathbb{Z}/p^m\mathbb{Z})^{\times}$). If $p \nmid c$ (i.e. $p|d$) the restriction on $\chi'$ above requires that $\chi'_p = \chi_p^{-1}$. For $p|c$ we need to consider the exact power $p^{m'}$ dividing $c_0/c$, and the restriction is that $\chi'_p$ agrees with $\chi_p$ on the multiplicative subgroup $1+p^{m'}\mathbb{Z}$ (and that $\chi'_p$ and $\chi_p$ have the same sign, but this is determined by the parity of $k$ anyway).

To apply this in Algorithm 2.6, we need to consider which twists ($f \otimes \mu$) will lie in the space considered above. Since the character of the twist is $\chi\mu^2$ we see that for $p|d$ we need $\chi_p \mu_p^2 = \chi_p^{-1}$, i.e. that $\mu_p = \chi_p^{-1}$ up to a quadratic character. For $p|c$ we need that $\chi_p \mu_p^2 = \chi_p$ on $1+p^{m'}\mathbb{Z}$, i.e. that $\mu_p^2$ is trivial modulo $p^{m'}$. In the case when $p$ is odd, if $m' = 0$ this requires $\mu_p$ to be either trivial or the unique quadratic character, while if $m' \geq 1$ then $\mu_p$ must have conductor at most $p^{m'}$. For $p = 2$, if $m' = 0, 1$ then $\mu_p$ may be trivial or any of the four quadratic characters, while if $m' \geq 2$ then $\mu_p$ may be any character of conductor at most $2^{m'+1}$.

So Proposition 2.8 represents a significant restriction of potential twists appearing in Algorithm 2.6 compared to our original result from Lemma 2.1. However, it still does not recover the full strength of what newform theory tells us for the cusp 0 (where we're applying the Atkin-Lehner involution $W_N$) or the full strength of Asai's result [Asa76] covering the case when $N$ is squarefree, because our result cannot see distinguish differences by quadratic characters. However, we remark that combining Proposition 2.8 with the restriction to level $Nh_c$ often rules out incorrect twists; for instance in the squarefree case we know $f|[\alpha_h]_k$ will still have $N$ and this rules out most incorrect twists because they would have a higher level.

We suspect that one could analyze how Hecke operators interact with our slash operators $|[\alpha_h]_k$ (similarly to what is done in Chapter 4.6 of [Miy06], or in [Asa76]) and further narrow down the list of twists needed to be considered in Algorithm 2.6; we do not attempt to carry this out here.

## 3. Theoretical results on transferring modular forms to other cusps

**3.1. Transformations of eigenforms to other cusps.** In this section we prove Theorem 2.4, that if $f \in S_k(N, \chi)$ is an eigenform of all Hecke operators $p \nmid N$, then the translate to another cusp $f|[\alpha_h]_k \in S_k(\Gamma_1(Nh))$ arises as a linear combination of twists of $f$ (and their images under degeneracy maps). To begin our analysis we split $[\alpha_h]_k$ into its two parts

$$S_k(N, \chi) \xrightarrow{\;[\alpha_1]_k\;} S_k(\Gamma_1(N, h)) \xrightarrow{\;[\tau_h]_k\;} S_k(\Gamma_1(Nh)) \ .$$

To study this we recall the general definition of Hecke operators on these spaces. We can consider congruence subgroups of the form

$$\Gamma_H(N, n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \equiv 0 \pmod{N}, \ b \equiv 0 \pmod{n}, \ a + n\mathbb{Z}, d + n\mathbb{Z} \in H \right\} \subseteq \mathrm{SL}_2(\mathbb{Z})$$

for $n|N$ and $H$ a subgroup of $(\mathbb{Z}/n\mathbb{Z})^{\times}$. For such a subgroup $\Gamma = \Gamma_H(N, n)$ we set

$$\Delta = \Delta_H(N, n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \equiv 0 \pmod{N}, \ b \equiv 0 \pmod{n}, \ a + n\mathbb{Z} \in H, \ ad - bc > 0 \right\} \subseteq M_2(\mathbb{Z}).$$

For $m > 0$ we take the subset $\Delta_m = \{\beta \in \Delta : \det(\beta) = m\}$. Then, for $\chi$ a character of $H \leq (\mathbb{Z}/N\mathbb{Z})^{\times}$ that we view as a character of $\Delta$ by acting on the upper-left entry $a$, we define the Hecke operator $T_m$ on $M_k(\Gamma, \chi)$ by taking a decomposition of $\Delta_m$ in terms of left cosets of $\Gamma$:

$$\Delta_m = \coprod_i \Gamma \beta_i, \qquad T_m f = m^{k/2-1} \sum_i \chi(\beta_i) f|[\beta_i]_k.$$

The theory of Hecke operators is worked out in this generality in Chapter 3 of [Shi94]. In particular, Proposition 3.36 gives an explicit formula for $T_m$ that lets us conclude that passing to a larger congruence subgroup preserves Hecke operators prime to the level.

**Proposition 3.1.** *Suppose we have two subgroups of the above form satisfying $\Gamma_{H'}(N', n') \leq \Gamma_H(N, n)$ (which implies $N|N'$, $n|n'$, and the pullback of $H$ to $(\mathbb{Z}/N'\mathbb{Z})^\times$ contains $H'$); for any character $\chi$ of $H$ (and its corresponding restriction $\chi'$ to $H'$) we have an inclusion*

$$M_k(\Gamma_H(N, n), \chi) \subseteq M_k(\Gamma_{H'}(N', n'), \chi').$$

*If $m$ is an integer prime to $N'$, then the Hecke operators $T_m$ on these two spaces are compatible with the inclusion map.*

With this setup it's easy to check that our map $[\tau_h]_k$ is compatible with Hecke operators $T_m$ for $(m, N) = 1$.

**Lemma 3.2.** *Fix an integer $N$ and a divisor $h$ of it, and consider the map*

$$[\tau_h]_k : S_k(\Gamma_1(N, h)) \to S_k(\Gamma_1(Nh)).$$

*Then if $(m, N) = 1$ the Hecke operators $T_m$ on each space are compatible with $[\tau_h]_k$: we have $T_m(f|[\tau_h]_k) = (T_m f)|[\tau_h]_k$ for all $f$.*

*Proof.* One can check that conjugation by $\tau_h$ takes $\Gamma_1(N, h)$ to $\Gamma_H(Nh)$ for $H \leq (\mathbb{Z}/hN\mathbb{Z})^\times$ the kernel of the projection to $(\mathbb{Z}/N\mathbb{Z})^\times$, and also $\tau_h^{-1}\Delta_1(N, h)_m \tau_h = \Delta_H(Nh)_m$. From this we can see that $[\tau_h]_k$ maps from $S_k(\Gamma_1(N, h))$ to $S_k(\Gamma_H(Nh))$ and preserves $T_m$, and we can include into $S_k(\Gamma_1(Nh))$. $\square$

On the other hand, the interaction of $[\alpha_1]_k$ with Hecke operators seems less well-known. In trying to analyze this we run into the problem that $f|[\alpha_1]_k$ is invariant under the subgroup $\alpha_1^{-1}\Gamma_0(N)\alpha_1$ which is hard to identify and may not be one of the types of subgroups we've already studied. We can always find a congruence subgroup inside of it that is (what we've proven is that $\Gamma_1(N, h)$ is contained in $\alpha_1^{-1}\Gamma_0(N)\alpha_1$), but there isn't a direct link between the Hecke operators involved. However, we can see that $[\alpha_1]_k$ is compatible with some of the Hecke operators as follows.

**Proposition 3.3.** *For a matrix $\alpha_1 \in \mathrm{SL}_2(\mathbb{Z})$ and the associated integer $h$ as above, the operator $[\alpha_1]_k : S_k(N, \chi) \to S_k(\Gamma_1(N, h))$ is compatible with Hecke operators $T_m$ for $m \equiv 1 \pmod N$.*

*Proof.* Consider the following diagram of spaces of modular forms:

$$
\begin{array}{ccc}
S_k(N, \chi) & \xrightarrow{\;[\alpha_1]_k\;} & S_k(\Gamma_1(N, h)) \\
\downarrow & & \downarrow \\
S_k(\Gamma(N)) & \xrightarrow{\;[\alpha_1]_k\;} & S_k(\Gamma(N))
\end{array}
\quad ;
$$

we've already established that $[\alpha_1]_k$ defines a map between the top two spaces, and it clearly also defines one between the bottom two spaces because $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$. The diagram evidently commutes because the operator $[\alpha_1]_k$ is defined independently of the ambient space it's used on. To prove that $T_m$ is compatible with the top map $[\alpha_1]_k$, it's sufficient to prove it's compatible with the bottom one and use compatibility of the vertical inclusions.

So we want to prove that for $m \equiv 1 \pmod N$, the endomorphisms $T_m$ and $[\alpha_1]_k$ on $S_k(\Gamma(N))$ commute. For this, note that by definition $\Delta(N)_m$ is all matrices

$$\delta = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix} \pmod N \qquad AD - BC = m;$$

if $m \equiv 1 \pmod N$ then this forces $D \equiv 1 \pmod N$ and thus $\delta \equiv I \pmod N$. Then conjugating such a $\delta$ by $\alpha_1$ gives another matrix congruent to $I$ modulo $N$, and we conclude that conjugation by $\alpha_1$ is an automorphism of $\Delta(N)_m$. Thus if $\Delta(N)_m = \coprod \Gamma(N)\beta_i$ is a coset decomposition, conjugating gives that $\Delta(N)_m = \coprod \Gamma(N)\alpha_1^{-1}\beta_i\alpha_1$ is also one, and $T_m$ can be written in terms of either, and thus

$$T_m(f|[\alpha_1]_k) = m^{k/2-1}\sum_i f|[\alpha_1]_k[\alpha_1^{-1}\beta_i\alpha_1]_k = m^{k/2-1}\sum_i f|[\beta_i]_k[\alpha_1]_k = (T_m f)|[\alpha_1]_k. \qquad \square$$

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Putting things together we have:

**Theorem 3.4.** *The operator* $[\alpha_h]_k : S_k(N, \chi) \to S_k(\Gamma_1(Nh))$ *is compatible with the Hecke operators* $T_m$ *defined on both spaces for* $m \equiv 1 \pmod{N}$. *Thus, if* $f_0 = \sum a_n q^n \in S_k(N_0, \chi)$ *is a newform of level* $N_0 | N$ *and* $f \in S_k(N, \chi)$ *is anything lying in the corresponding prime-to-$N$ eigenspace, then* $f | [\alpha_h]_k$ *satisfies* $T_m(f | [\alpha_h]_k) = a_m f | [\alpha_h]_k$ *for* $m \equiv 1 \pmod{N}$.

So if we start with $f$ an eigenform of the prime-to-$N$ Hecke algebra on $S_k(N, \chi)$ (associated to a newform $f_0 = \sum a_n q^n$ but perhaps itself an oldform), then $f | [\alpha_h]_k$ is a "partial" eigenform lying in the subspace

$$\{g \in S_k(\Gamma_1(Nh)) : T_m(g) = \lambda_m g, m \equiv 1 \pmod{N}\}.$$

This subspace breaks up as a direct sum of prime-to-$N$ eigenspaces, each of which is associated to some newform $g_{0,i}$. The next theorem lets us pin down these $g_{0,i}$'s as being twists of $f$.

**Theorem 3.5.** *Suppose* $f_0 = \sum a_n q^n$ *is a newform, and* $g_0 = \sum b_n q^n$ *is another newform such that* $a_m = b_m$ *for* $m \equiv 1 \pmod{N}$. *Then* $g_0$ *is a twist* $f_0 \otimes \mu$ *for some Dirichlet character* $\mu$ *modulo* $N$.

The idea is essentially to define $\mu(m + N\mathbb{Z}) = b_m/a_m$ and check that this is independent of the representative of $m$ and defines a Dirichlet character. If we have plenty of coefficients where $a_m \neq 0$ then this makes sense and the argument goes through easily (claims 2 and 3 below are the main idea); making the argument go through for forms where we may have many $a_m$'s equal to zero just requires a little more care.

*Proof.* Define a subset $H \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$ consisting of all residue classes $c + N\mathbb{Z}$ such that there exists an infinite set $\{l_i\}$ of representatives of $c + N\mathbb{Z}$ with the $l_i$ pairwise coprime and satisfying $a_{l_i} \neq 0$. Note that for any given integer $L$, all but finitely many elements of $\{l_i\}$ will be coprime to $L$. (In most cases we'd expect to be able to take infinitely many primes $p \equiv c \pmod{N}$ with $a_p \neq 0$ as such a set).

*Claim 1: $H$ is a subgroup.* Suppose we have two residue classes $c + N\mathbb{Z}$ and $c' + N\mathbb{Z}$ satisfying our condition, with infinite sets $\{l_i\}$ and $\{l'_j\}$. Then $\{l_i l'_j : (l_i, l'_j) = 1\}$ is a set of representatives of $cc' + N\mathbb{Z}$ with $a_{l_i l'_j} = a_{l_i} a_{l'_j} \neq 0$ for all of its elements, and there exists an infinite subset of it that's pairwise coprime (for any finite subset that's pairwise coprime, we only have finitely many $l_i$'s and $l'_j$'s involved, and this only throws away a finite list of possible things to add, so a maximal such subset must be infinite).

*Claim 2: We have a well-defined function* $\mu : H \to \mathbb{C}$ *given by setting* $\mu(c + N\mathbb{Z}) = b_l/a_l$ *for any* $l \in c + N\mathbb{Z}$ *with* $a_l \neq 0$. By assumption there exist plenty of such $l$'s, so we need to check that if $l, l' \in c + N\mathbb{Z}$ satisfy $a_l, a_{l'} \neq 0$ then $b_l/a_l = b_{l'}/a_{l'}$. By claim (1) we know $H$ is a subgroup so $(c + N\mathbb{Z})^{-1}$ satisfies our assumption, and thus we can pick $l'' \in (c + N\mathbb{Z})^{-1}$ which is coprime to both $l$ and $l'$. Then $ll'' \equiv 1 \pmod{N}$ so we have

$$b_l b_{l''} = b_{ll''} = a_{ll''} = a_l a_{l''} \neq 0$$

giving $b_l/a_l = a_{l''}/b_{l''}$. An identical computation says $b_{l'}/a_{l'} = a_{l''}/b_{l''}$ too.

*Claim 3: $\mu$ is a multiplicative character on $H$.* For two cosets $c + N\mathbb{Z}$ and $c' + N\mathbb{Z}$ in $H$, by assumption we can pick representatives $l \in c + N\mathbb{Z}$ and $l' \in c' + N\mathbb{Z}$ with $l, l'$ coprime and $a_l, a_{l'} \neq 0$. Then we have

$$\mu(cc') = \frac{a_{ll'}}{b_{ll'}} = \frac{a_l a_{l'}}{b_l b_{l'}} = \mu(c)\mu(c').$$

*Claim 4: $\mu$ extends to a Dirichlet character on $(\mathbb{Z}/N\mathbb{Z})^\times$.* Since we have a character $H \to \mathbb{C}^\times$ on a subgroup $H$ of an abelian group $(\mathbb{Z}/N\mathbb{Z})^\times$, it's a general fact that we can extend it to a character of the full group.

*Claim 5: For every prime $p$ lying in a residue class $c + N\mathbb{Z}$ in $H$, we have $b_p = \mu(p)a_p$.* If $a_p \neq 0$ then this is immediate from definition of $\mu(p)$. If $a_p = 0$ then picking some $l \in (c + N\mathbb{Z})^{-1}$ with $p \nmid l$ and $a_l \neq 0$ gives $b_p b_l = b_{pl} = a_{pl} = a_p a_l$ which forces $b_p = 0 = \mu(p)a_p$.

*Claim 6: For all but finitely many primes $p$ lying in a residue class $c + N\mathbb{Z}$ not in $H$, we have $b_p = a_p = 0$.* The set of $p \in c + N\mathbb{Z}$ with $a_p \neq 0$ is certainly finite, since otherwise $c + N\mathbb{Z}$ would be in our set $H$ by definition. The set of $p$ with $b_p \neq 0$ but $a_p = 0$ can't be as large as the order of $c + N\mathbb{Z}$ in $(\mathbb{Z}/N\mathbb{Z})^\times$, since if we had distinct primes $p_1, \cdots, p_f$ with $b_{p_i} \neq 0$ and $p_1 \cdots p_f \equiv 1 \pmod{N}$ then we'd have

$$0 = a_{p_1} \cdots a_{p_f} = a_{p_1 \cdots p_f} = b_{p_1 \cdots p_f} = b_{p_1} \cdots b_{p_f} \neq 0.$$

*Claim 7: The newform $f_0 \otimes \mu$ equals $g_0$.* By strong multiplicity one, it's sufficient to check that these newforms have the same coefficients for all but finitely many primes $p$. For primes $p \nmid N$, $f_0 \otimes \mu$ and $g_0$ have

$p$-th coefficients $\mu(p)a_p$ and $b_p$, respectively, and combining claims 5 and 6 we've verified that all but finitely many of these are equal. $\qquad\square$

Combining Theorem 3.4 and Theorem 3.5 establishes Theorem 2.4.

## 4. Computing the self-Petersson inner product and comparing to the adjoint $L$-function

4.1. **Computing the Petersson inner product numerically.** In this section we describe how to numerically compute the Petersson inner product of two modular forms $f, g \in S_k(N, \chi)$, given $q$-expansions of both at $\infty$. This is done by applying a formula of Nelson [Nel15] that expresses the Petersson inner product in terms of the Fourier expansions of $f$ and $g$ at all cusps, combined with our methods for computing these Fourier expansions. To start, we state the definition of the Petersson inner product we'll be working with:

**Definition 4.1.** Let $f, g$ be two cusp forms of level $k$ (or even one cusp form and one modular form) for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. If $\Gamma$ is any congruence subgroup for which both are modular, we define their *normalized Petersson inner product* as

$$\langle f, g \rangle = \frac{1}{\mathrm{vol}(\mathbb{H}\backslash\Gamma)} \int_{\mathbb{H}\backslash\Gamma} f(x+iy)\overline{g(x+iy)}y^k \frac{dx\ dy}{y^2}.$$

Here $y^{-2}dx\ dy$ is the standard volume measure on the upper half-plane. Our normalization is by $\mathrm{vol}(\mathbb{H}\backslash\Gamma) = \frac{\pi}{3}[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$, and allows the definition to be independent of the choice of congruence subgroup $\Gamma$ that we view both forms as modular with respect to. Notation in the literature varies, with some places defining $\langle f, g \rangle$ without this normalizing factor, and others simply using the index $[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$ rather than the volume of $\mathbb{H}\backslash\Gamma$.

Nelson's formula (Theorem 5.6 of [Nel15]) applies to quite general integrals on modular curves, and our methods for computing Fourier coefficients at all cusps could be applied to many situations. For the purposes of this paper we are interested in Petersson inner products, so we specialize the formula to that case (see Example 5.7 of Nelson's paper):

**Theorem 4.2** (Nelson). *Suppose $f = \sum_n a_n q^n$ and $g = \sum b_n q^n$ are two cusp forms in $S_k(N, \chi)$. Then we have*

$$\langle f, g \rangle = \frac{4}{\mathrm{vol}(\mathbb{H}\backslash\Gamma)} \sum_s \frac{h_{s,0}}{h_s} \sum_{n=1}^{\infty} \frac{a_{n,s}\overline{b}_{n,s}}{n^{k-1}} \sum_{m=1}^{\infty} \left(\frac{x}{8\pi}\right)^{k-1} \left(xK_{k-1}(x) - K_{k-2}(x)\right) \qquad x = 4\pi m\sqrt{\frac{n}{h_s}},$$

*where $K_v$ is a $K$-Bessel function, $s$ runs over all cusps of $\Gamma_0(N)$, $h_{s,0}$ is the width of that cusp for $\Gamma_0(N)$, $h_s$ is the width for that cusp for $f$ as described in Lemma 2.1, and we choose a single matrix $\alpha_1$ taking $\infty$ to $s$ and write $f|[\alpha_{h_s}]_k = \sum a_{n,s} q^n$ and $g|[\alpha_{h_s}]_k = \sum b_{n,s} q^n$.*

So to apply this formula we just need to compute the Fourier expansions of $f$ and $g$ at each cusp, via our methods from Section 2. Since Bessel functions decay exponentially in their arguments, this matches up well with our algorithms returning Fourier coefficients with accuracy up to an exponentially decaying factor, and thus makes it so that each term of the sum over $n$ has an absolute error on the order of whatever magnitude we want to specify. We can implement this as follows:

**Algorithm 4.3** (Petersson inner product of two modular forms of level $N$). Let $f, g \in S_k(N, \chi)$ be two cusp forms of the same level and character. Then we can compute their Petersson inner product to an approximate accuracy of $10^{-E}$ as follows:

- List all of the cusps $s$ of $\Gamma_0(N)$, the widths $h_{s,0}$ for $\Gamma_0(N)$, and their widths $h_s$ of Lemma 2.1.
- Iterate over cusps $s$, and for each do the following:
  - Iterate over $n$ and compute the inner sum over $m$ that involves Bessel functions (we'll denote this sum $S_{s,n}$); each sum over $m$ can be truncated when the terms get some safe factor smaller than $10^{-E}$. Record these sums $S_{s,n}$ for each $n$, until we reach some $n_s$ where $S_{s,n}$ is a safe factor smaller than $10^{-E}$.
  - Use one of our previous algorithms to compute the Fourier expansions of $f$ and $g$ at the cusp $s$, with absolute accuracy $10^{-E}$, relative decay $C$ chosen so that $e^{-Cn} \geq S_n$ for all $n$, and number of terms desired equal to the number of terms $n_s$ we found in the previous step. (Of course for the cusp $\infty$ we can skip this and use the Fourier expansion directly).

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- Compute the products $a_{n,s}\bar{b}_{n,s}/n^{k-1} \cdot S_{s,n}$ and sum them up from $n = 1$ to $n_s$. This is the contribution of the cusp $s$ to our formula for the Petersson inner product.
- Add up the contributions for all cusps $s$, and normalize by the constant at the front of the formula.

For the case we're ultimately interested in, we'll work with three modular forms natively of different levels; there we want to compute Fourier expansions for each at their native level to avoid any redundant computation. This is discussed in Section 5.1. For modular forms natively of the *same* level, the main case of interest is when $f = g$ are the same newform; in this case the self-Petersson inner product $\langle f, f \rangle$ is related to an adjoint $L$-value.

In fact the standard way to compute $\langle f, f \rangle$ is by way of computing this $L$-value instead, and we cannot claim our algorithm will be a better way. Instead, we can use the relation of $\langle f, f \rangle$ with the special value $L(\mathrm{ad}\, f, 1)$ to provide some numerical verification that Algorithm 4.3, serving as an introduction to the sort of comparisons we'll be making in Section 5.2.

### 4.2. Comparing with adjoint $L$-values.
It is well-known that if $f$ is a newform, its self-Petersson product $\langle f, f \rangle$ is related to a value of the adjoint $L$-function associated to $f$ (or of its shift, the symmetric square $L$-function for $f$). This is found in papers of Shimura and Hida (see [Shi76], Section 5 of [Hid81], and Section 10 of [Hid86]), following ideas going back to Petersson; if one considers the automorphic adjoint $L$-function $L(\mathrm{ad}\, f, s)$ defined correctly at all factors and uses the normalization of the Petersson inner product we do, the identity can be written as

$$L(\mathrm{ad}\, f, 1) = \frac{\pi^2}{6} \frac{(4\pi)^k}{(k-1)!} \langle f, f \rangle \prod_p (*)_p$$

where $(*)_p$ is an explicit factor for primes $p$ dividing the level of $N$ which we will describe soon. In this section we'll recall how to numerically compute the adjoint $L$-value, and then show several examples where we numerically compare both sides of this formula and see that our method for computing $\langle f, f \rangle$ returns the correct results.

An efficient algorithm for computing values of $L$-functions has been given by Dokchitser [Dok04], which is implemented in SageMath [Dev16]. This algorithm relies on the functional equation for the $L$-function in question, and thus requires knowledge of various parameters for the functional equation in addition to the coefficients (or equivalently, the Euler factors) of the $L$-function itself. In the case of $L(\mathrm{ad}\, f, s)$ some of the parameters are easy: the weight is 1 (i.e. the functional equation relates $s$ and $1 - s$), the gamma factor is $\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)$, and the sign $\varepsilon$ is always $+1$. The analytic conductor $N_{\mathrm{ad}}$ of the functional equation, however, is more subtle to determine.

The determination of the analytic conductor $N_{\mathrm{ad}} = \prod_p N_{\mathrm{ad},p}$ is a local problem that needs to be solved at each bad prime $p$, as is the determination of the correct Euler factor $L_p(\mathrm{ad}\, f, s)$ and the correction factor $(*)_p$ in our formula above. This breaks into a case-by-case analysis based on the local representation $\pi_{f,p}$ of the automorphic representation associated with $f$. Actually, since $L(\mathrm{ad}\, f, s)$ is invariant under replacing $f$ by a twist, the first step is to replace $f$ by a twist $g$ which is twist-minimal and proceed with analyzing the newform $g$ which has level $N_g$ and character $\chi_g$ of conductor $N_{\chi,g}$. Let $p^r$ be the exact power of $p$ dividing $N_g$, and $p^{r_\chi}$ the exact power of $p$ dividing $N_{g,\chi}$. Then we have:

- If $r = 0$ (i.e. $p \nmid N_g$, even if we have $p|N$) our $L$-function is unramified: $N_{\mathrm{ad},p} = 1$ and the "good" Euler factor is $L_p(\mathrm{ad}\, f, s) = L_p(\mathrm{ad}\, g, s) = (1 - \frac{\alpha_p}{\beta_p}p^{-s})^{-1}(1 - \frac{\beta_p}{\alpha_p}p^{-s})^{-1}(1 - p^{-s})^{-1}$ where $\alpha_p, \beta_p$ arise from $(X^2 - a_p X + \chi(p)p^{k-1}) = (X - \alpha_p)(X - \beta_p)$.
  - If $p \nmid N$ (i.e. $f$ is twist-minimal at $p$) then $p$ is a good prime so $(*)_p$ doesn't need to be defined, but if $p|N$ ($f$ is not twist-minimal at $p$) we have $(*)_p = (1 + 1/p)L_p(\mathrm{ad}\, f, 1)$.
- If $r = 1$ but $r_\chi = 0$, the local representation of $g$ at $p$ is an unramified special representation and we have $N_{\mathrm{ad},p} = p^2$, $L_p(\mathrm{ad}\, f, s) = (1 - \frac{1}{p}p^{-s})^{-1}$.
  - If $p\|N$ ($f$ is twist-minimal at $p$) then $(*)_p = (1 + \frac{1}{p})$, while if $p^2|N$ ($f$ is not twist-minimal at $p$) then $(*)_p = (1 + \frac{1}{p})(1 - \frac{1}{p^2})^{-1}$.
- If $r = r_\chi \geq 1$, the local representation of $g$ at $p$ is a half-ramified principal series and we have $N_{\mathrm{ad},p} = p^{2r}$ and $L_p(\mathrm{ad}\, f, s) = (1 - p^{-s})^{-1}$.

– If $p^r \| N$ ($f$ is twist-minimal at $p$) then $(*)_p = (1 + \frac{1}{p})$, while if $p^{r+1} | N$ ($f$ is not twist-minimal at $p$) then $(*)_p = (1 + \frac{1}{p})(1 - \frac{1}{p})^{-1}$.

- If $r \geq 2$ and $r > r_\chi$, $\pi_{g,p}$ is supercuspidal. At this point it becomes harder to give a clean description of all of our quantities, but $N_{\mathrm{ad},p} = p^e$ for some $e \leq 2r$, and the $L$-function splits up into two cases:
  - If $\pi_{g,p} \cong \eta \otimes \pi_{g,p}$ for $\eta$ the unramified quadratic character of $\mathbb{Q}_p^\times$, then $L(\mathrm{ad}\, f, s) = (1 + p^{-s})^{-1}$ and $(*)_p = 1$.
  - If $\pi_{g,p} \not\cong \eta \otimes \pi_{g,p}$, then $L(\mathrm{ad}\, f, s) = 1$ and $(*)_p = (1 + 1/p)$.

In the supercuspidal case we have not described how to full determine $N_{\mathrm{ad},p}$ nor how to determine if $\pi_{g,p} \cong \eta \otimes \pi_{g,p}$, though in principle this can be done by the algorithm of Loeffler-Weinstein [LW12] which explicitly determines $\pi_{g,p}$. In the case of central trivial character, Nelson-Pitale-Saha [NPS14] give a finer characterization of the conductor in Proposition 2.5. In any case we remark that Dokchitser's algorithm gives a way to numerically check the functional equation for any guesses of $N_{\mathrm{ad},p}$ and $L(\mathrm{ad}\, f, s)$, so one can always recover the correct values that way.

We give some examples of the resulting computations and comparisons. For $f_1 = \Delta$, the $\Delta$-function of weight 12 and level 1 (which has no bad places), we compute

$$\frac{L(\mathrm{ad}\, f_1, 1)}{\langle f_1, f_1 \rangle} \approx \frac{0.6317929457\ldots}{9.8869793538\ldots \cdot 10^{-7}} \approx 639015.136088\ldots \approx \frac{\pi^2}{6} \frac{(4\pi)^{12}}{11!}.$$

For $f_2 = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 + \cdots$ the unique newform of weight 6 and level 3 with trivial character, the local representation at 3 is special and we get

$$\frac{L(\mathrm{ad}\, f_2, 1)}{\langle f_2, f_2 \rangle} \approx \frac{0.9879391307\ldots}{0.00001372666446\ldots} \approx 71972.2648922\ldots \approx \frac{\pi^2}{6} \frac{(4\pi)^6}{5!} \left(1 + \frac{1}{3}\right).$$

On the other hand, the twist $f_2' = q + 6q^2 + 4q^4 - 6q^5 + \cdots$ of weight 6 and level 9 has the same $L$-value but the Petersson inner product differs

$$\frac{L(\mathrm{ad}\, f_2', 1)}{\langle f_2', f_2' \rangle} \approx \frac{0.9879391307\ldots}{0.00001220147952\ldots} \approx 80968.7980038\ldots \approx \frac{\pi^2}{6} \frac{(4\pi)^6}{5!} \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{9}\right)^{-1}.$$

The example $f_3 = q - 4q^3 - 2q^5 + \cdots$ of weight 4, level 8, and trivial character has a supercuspidal local component at $p = 2$. By Proposition 2.5 of [NPS14] we know $N_{\mathrm{ad},2} = 16$, $L_2(\mathrm{ad}\, f, 1) = 1$, and $(*)_2 = 1 + \frac{1}{2}$. Here the computation gives

$$\frac{L(\mathrm{ad}\, f_3, 1)}{\langle f_3, f_3 \rangle} \approx \frac{0.8047560912\ldots}{0.0000784759013\ldots} \approx 10254.8180648\ldots \approx \frac{\pi^2}{6} \frac{(4\pi)^4}{3!} \left(1 + \frac{1}{2}\right).$$

For the newform $f_4 = q + 6\sqrt{10}q^2 + 232q^4 - 96\sqrt{10}q^5 + \cdots$ of weight 8, level 9, and trivial character we can find (either by a computation via Loeffler-Weinstein's algorithm, or by trial and error with the $L$-function parameters) find that $\pi_{f,3}$ is isomorphic to its twist by $\eta$ and we have $N_{\mathrm{ad},3} = 9$, so $L_3(\mathrm{ad}\, f, 1) = (1 + 1/3)^{-1}$ and $(*)_3 = 1$ and sure enough

$$\frac{L(\mathrm{ad}\, f_4, 1)}{\langle f_4, f_4 \rangle} \approx \frac{1.6698026860\ldots}{8.2275074570\ldots \cdot 10^{-6}} \approx 202953.652096\ldots \approx \frac{\pi^2}{6} \frac{(4\pi)^8}{7!}.$$

### 4.3. Comments on computing minimal twists.
Thus far, all of our algorithms have appeared to achieve our goal of avoiding ever working with full spaces of modular forms of a given weight, level, and character. Instead, if we are given the $q$-expansion of a modular form $f$, we have at worst needed to work with a collection of twists of it. However, there is a bit of a caveat to this: to correctly find all of the twists of $f$ and their levels, we need to start with a *minimal* twist of $f$.

In practice, for most cases we work with $f$ will either be twist-minimal in the first place, or we will have specifically picked it out as a twist of a lower-level form. But in general a minimal twist needs to be searched for. We do this by a brute-force search of lower-level modular forms, and Loeffler-Weinstein [LW12] have a more sophisticated algorithm. Both of these approaches involve computing full spaces of modular forms, however, and it would be desirable to have an algorithm that doesn't.

One approach we could consider taking would be to start with $f$ of some level $N$, take its naive twists $f_\mu$, and then check numerically if $f_\mu$ is actually of some smaller level; since $f_\mu$ is automatically modular under

some $\Gamma_0(N')$, to check modularity under any $\Gamma_0(M)$ we'd just need to check whether it transforms correctly under

$$\begin{bmatrix} 1 & 0 \\ M & 1 \end{bmatrix}.$$

It would be straightforward to check it the transformation rule appears to hold numerically for a handful of points. This would not provide a *proof* that $f_\mu$ is modular of our lower level, but in the spirit of the numerical computations in this paper it would be a strong justification.

The hole in this strategy is that it only checks modularity of the naive twist $f_\mu$ but we know in some cases the true twist $f \otimes \mu$ will have extra Fourier coefficients at bad primes that were "twisted away" in $f$. To deal with all cases, we would need a way to recover the lost coefficients of $f$ at bad primes, either theoretically or numerically. We are not sure if there is a known way to do this, and in any case have not pursued it since the brute-force approach is sufficient for the cases we want to handle.

4.4. **Computing a ratio of Petersson inner products.** One feature of our method for computing Fourier expansions, and thus Petersson inner products, is that it doesn't require the modular forms involved to be newforms. Even with the method described in 2.4, we can take $f$ to be any oldform associated to a newform $f_0$ and work with $f$ directly, only needing to use $f_0$ itself to determine a basis for the space $f|[\alpha_h]_k$ lies in. This is useful for our purposes of numerically verifying computations made in [Col16], as some of these calculations involve taking a newform $h$ and relating $\langle h, h \rangle$ to $\langle h', h'' \rangle$ where $h', h''$ are particular oldforms associated to $h$. We give a few examples of computations verifying such calculations here, illustrating a simpler version of the more complex comparisons needed to be made in [Col16].

For instance, in Section 6.2 of [Col16] we calculate the formula

$$\frac{\langle h(pz), h(z) \rangle}{\langle h(z), h(z) \rangle} = \frac{a_p}{p^{m-1}(p+1)}$$

when $h(z) = \sum a_n q^n$ is a weight-$m$ eigenform the prime-to-$p$ Hecke operator $T(p)$. We can then numerically check this in the case $h = \Delta$ is the $\Delta$-function and $p = 11$ (so $a_p = 534612$), where we get

$$\frac{\langle \Delta(11z), \Delta(z) \rangle}{\langle \Delta(z), \Delta(z) \rangle} \approx \frac{1.5438373630\ldots \cdot 10^{-13}}{9.8869793538\ldots \cdot 10^{-7}} \approx 1.5614853715\ldots \cdot 10^{-7} \approx \frac{534612}{11^{11} \cdot 12}.$$

This formula was used as an intermediate in [Col16] for computations with $p$-stabilizations of a $p$-ordinary form $h$ (one where $a_p$ is not divisible by $p$). If we let $\alpha_p$ and $\beta_p$ be the roots of the Hecke polynomial for $a_p$ such that $\alpha_p$ is a $p$-adic unit for a given embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\beta_p$ is not, then one can define the $p$-stabilization as $h^\sharp(z) = h(z) - \beta_p h(pz)$ and also $h^\flat = h(z) - \alpha_p h(pz)$. We then calculated that $h^\sharp(z) = h(z) - p\beta_p h(pz)$ is orthogonal to $h^\flat$ under the Petersson inner product, which allowed us to realize "projection onto $h^\sharp$" as a scalar multiple of the functional $\langle -, h^\sharp \rangle$, and proved the following formula

$$\frac{\langle h^\sharp, h^\sharp \rangle}{\langle h, h \rangle} = \frac{(-\alpha/\beta)(1 - \beta/\alpha)(1 - p^{-1}\beta/\alpha)}{(1 + p^{-1})}.$$

This ratio of arises when determining removed Euler factors in the $p$-adic $L$-functions we were working with.

In our example of $h = \Delta$ and $p = 11$ (the smallest prime for which $\Delta$ is $p$-ordinary), we take $\alpha, \beta$ to be the roots $(a_{11} \pm \sqrt{a_{11} - 4 \cdot 11^{11}})/2$ and we can numerically compute that $\langle \Delta^\flat, \Delta^\sharp \rangle \approx 0$ and moreover that

$$\frac{\langle \Delta^\sharp, \Delta^\sharp \rangle}{\langle \Delta, \Delta \rangle} \approx \frac{1.4821834825\ldots \cdot 10^{-6} + 7.1394620388\ldots \cdot 10^{-7}i}{9.8869793538\ldots \cdot 10^{-7}} \approx 1.4991267095\ldots - 0.7221075096\ldots i$$

which does indeed agree with the expected ratio above.

## 5. Inner products involving three eigenforms

5.1. **Working with eigenforms of different levels.** In this section we give examples of computations involving Petersson inner products of the form $\langle fg, h \rangle$ where $f, g, h$ are three modular forms of levels $k$, $m - k$, and $m$ for $0 < k < m$; thus the product $fg$ is of weight $m$ and it makes sense to pair it with $h$. We will generally also assume they satisfy $\chi_f \chi_g = \chi_h$ since otherwise the inner product is trivially zero.

Once again, our general setup will be that we are given the $q$-expansions of $f$, $g$, and $h$ at infinity. Since the $q$-expansion of $fg$ is just the product of the $q$-expansions of $f$ and $g$, we can apply Theorem 4.2, which we can write out explicitly as follows.

**Theorem 5.1.** *Suppose $k, m$ are integers satisfying $0 < k < m$, and $f = \sum_n a_n q^n \in S_k(N_f, \chi_f)$, $g = \sum b_n q^n \in S_{m-k}(N_g, \chi_g)$, and $h = \sum c_n q^n \in S_m(N_h, \chi_h)$ are three cusp forms. Set $N = \mathrm{lcm}(N_f, N_g, N_h)$. Then we have*

$$\langle fg, h \rangle = \frac{4}{\mathrm{vol}(\mathbb{H}\backslash\Gamma)} \sum_s \frac{h_{s,0}}{h_s} \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n-1} a_{i,s} b_{n-i,s} \right) \frac{\bar{c}_{n,s}}{n^{k-1}} \sum_{m=1}^{\infty} \left( \frac{x}{8\pi} \right)^{k-1} \left( x K_{k-1}(x) - K_{k-2}(x) \right)$$

*setting $x = 4\pi m \sqrt{\frac{n}{h_s}}$. Again $K_v$ is a K-Bessel function, $s$ runs over all cusps of $\Gamma_0(N)$, $h_{s,0}$ is the width of that cusp for $\Gamma_0(N)$, and $h_s$ is a common width such that if we fix a matrix $\alpha_1$ taking $\infty$ to $s$ then $f|[\alpha_{h_s}]_k = \sum a_{n,s} q^n$, $g|[\alpha_{h_s}]_{m-k} = \sum b_{n,s} q^n$, and $h|[\alpha_{h_s}]_m = \sum c_{n,s} q^n$ all have integer exponents of $q$.*

Thus we can numerically compute $\langle fg, h \rangle$ by numerically computing the $q$-expansions of $f$, $g$, and $h$ at each cusp of $\Gamma_0(N)$ and applying this formula just like in Algorithm 4.3. In practice we will want to implement this slightly differently, because usually $N_f$, $N_g$, and $N_h$ will be distinct so we only need to compute expansions for $f$ at the cusps of the congruence subgroup it's naturally defined over. Doing this requires modifying our algorithm to first look over all cusps of $\Gamma_0(N)$ and note which is the most accuracy we need from each expansion of $f, g, h$, then compute each of these expansions for the "natural" cusps, and finally use them to get the appropriate expansions for each cusp of $\Gamma_0(N)$ (remembering that expansions at different representatives of a single cusp will differ as explained in Proposition 2.2).

A first application of the above methods is to verify the computations in Section 6.4 of [Col16], where we compare a Petersson inner product $\langle fg, h \rangle$ compared with a $p$-stabilized version $\langle fg^\sharp, h^\natural \rangle$; these give rise to the removed Euler factors at $p$ for the $p$-adic $L$-functions we construct. The setup is similar to what was described above in Section 4.4; we do not go into detail beyond saying that we have run a variety of numerical examples that agree with our computed formulas (which also agree with the conjectured form for removed Euler factors in general).

### 5.2. Numerically verifying an explicit Ichino formula.

We now turn to our main application, of offering various numerical verifications of an explicit form of Ichino's triple-product formula needed in [Col16]. Ichino [Ich08] proved a general result about automorphic representations on $\mathrm{GL}_2$, which can be applied to the case of three holomorphic newforms $f, g, h$ (of compatible weights and characters, as discussed previously) to obtain a formula relating $|\langle fg, h \rangle|^2$ to the central value of the triple-product $L$-function $L(f \times g \times \bar{h}, s)$. It is clear in principle that the formula will give us an explicit constant (a certain power of $\pi$ times an algebraic number) relating these two quantities. However, determining the algebraic part of the constant may involve many delicate calculations, and our goal is to provide a computational verification of the resulting formula.

Specifically, in Theorem 3.1.2 of [Col16] we establish the following explicit version of Ichino's formula. We remark that if $f, g, h$ are newforms such that one of them is new at a prime $p$ and the other two are old there, then $\langle fg, h \rangle$ is automatically zero; the factors $M_f, M_g, M_h$ are introduced to avoid this.

**Theorem 5.2.** *Fix integers $m > k > 0$, and let $f \in S_k(N_f, \chi_f)$, $g \in S_{m-k}(N_g, \chi_g)$, and $h \in S_m(N_h, \chi_h)$ be classical newforms such that the characters satisfy $\chi_f \chi_g = \chi_h$. Take $N_{fgh} = \mathrm{lcm}(N_f, N_g, N_h)$ and choose positive integers $M_f, M_g, M_h$ such that the three numbers $M_f N_f$, $M_g N_g$, $M_h N_h$ divide $N_{fgh}$ and moreover none of the three is divisible by a larger power of any prime $p$ than both of the others. Then we have*

$$|\langle f_{M_f} g_{M_g}, h_{M_h} \rangle|^2 = \frac{3^2 (m-2)! (k-1)! (m-k-1)!}{\pi^{2m+2} 2^{4m-2} M_f^k M_g^{m-k} M_h^m} L(f \times g \times \bar{h}, m-1) \prod_{p | N_{fgh}} I_p^{**},$$

*where $f_{M_f}(z)$ denotes $f(M_f z)$, and the constants $I_p^{**}$ are values of (slightly re-normalized) "Ichino local integrals".*

The bulk of the difficulty in making this completely explicit is in determining the constants $I_p^{**}$ at the bad primes. Before starting on what is known about this we first want to check the formula for newforms of

level 1 to verify that the other part of the constant is correct (especially the power of 2 in the denominator). In the case where $f, g, h$ are all of level 1 the formula reduces to

$$|\langle fg, h \rangle|^2 = \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2}2^{4m-2}} L(f \times g \times \overline{h}, m-1).$$

The simplest case to test is when $f = g$ is the $\Delta$-function of weight 12 and $h$ is a newform of weight 24 (there are two conjugate such newforms, but for explicitness pick the one with $540 - 12\sqrt{144169}$ as the coefficient of $q^2$). We can compute $\langle fg, h \rangle$ by our usual algorithm, and $L(f \times g \times \overline{h}, m-1)$ via Dokchitser's algorithm [Dok04]. Since all of our forms are of level 1 the conductor of this $L$-function is 1 and all of the Euler factors are the naive triple-product ones. (The other parameters for Dokchitser's algorithm follow from the analytic theory of such $L$-functions and doesn't depend on the levels: the weight is $2m - 2$, the local constant is 1, and the gamma factors are 0, 1, $-k+1$, $-k+2$, $-(m-k)+1$, $-(m-k)+2$, $-m+2$, and $-m+3$). Running this we get:

$$\frac{|\langle fg, h \rangle|^2}{L(f \times g \times \overline{h}, 23)} \approx \frac{1.2769689139\ldots \cdot 10^{-16}}{1.1302460925\ldots} \approx 1.1298149335\ldots \cdot 10^{-16} \approx \frac{3^2 \cdot 22! \cdot 11! \cdot 11!}{\pi^{50}2^{94}}.$$

With the main constant in the formula verified, we can move on to checking local factors $I_p^{**}$ in various cases. This local factor arises as follows (which we explain in detail in Section 3.2 of [Col16]): first $I_p$ is defined as a local integral of matrix coefficients of newvector of local constituents, then it is normalized by some $L$-factors to a value $I_p^*$ (which is the standard quantity considered in the literature), and we modify it slightly further to get the constant $I_p^{**}$ appearing in our formula. (Specifically, in the process of making Ichino's formula explicit we get $\langle f, f \rangle$ on one side and $L(\mathrm{ad}\, f, 1)$ on the other, and similarly for the other two forms, so $I_p^{**}$ takes into account the factors $(*)_p$ arising form this comparison as detailed in Section 4.2).

*Case of one conductor-$p$ special representation and two unramified representations.* The simplest nontrivial case for our local integrals is when $\pi_{f,p}$, $\pi_{g,p}$, and $\pi_{h,p}$ (the local representations at $p$ for our three newforms $f, g, h$) consist of two unramified representations and one special representation of conductor $p$, in some order. In this case the local integral was calculated by Woodbury in [Woo12] to give

$$I_p^* = \frac{1}{p}\left(1 + \frac{1}{p}\right)^{-1} \qquad I_p^{**} = \frac{1}{p}\left(1 + \frac{1}{p}\right)^{-2}.$$

Also in this case, the local factor of the $L$-function is

$$L_p(f \times g \times \overline{h}, s) = \prod_{i,j=1}^{2}(1 - \alpha_i\beta_j\gamma p^{-s})^{-1}$$

where $\alpha_1, \alpha_2$ and $\beta_1, \beta_2$ are the roots of the Hecke polynomials at $p$ for the two of $f, g, \overline{h}$ that are unramified, and $\gamma$ is the coefficient of $p$ for the one that is special. The local contribution to the conductor of the functional equation is $p^4$.

As a numerical verification, we apply Dokchitser's algorithm to compute $L(f \times g \times \overline{h}, 17)$ and ours to compute $|\langle f(z)g(3z), h(z) \rangle|^2$ where $f$ is the unique newform of weight 6 and level 3, $g$ is the unique newform of weight 12 and level 1 (the $\Delta$-function), and $h$ is the unique newform of weight 18 and level 1. Running this computation gives

$$\frac{|\langle f(z)g(3z), h(z) \rangle|^2}{L(f \times g \times \overline{h}, 17)} \approx \frac{4.7335974505\ldots \cdot 10^{-23}}{1.3684877005\ldots} \approx 3.4589988997\ldots \cdot 10^{-23} \approx \frac{3^2 \cdot 16! \cdot 5! \cdot 11!}{\pi^{38}2^{70}3^{12}} \cdot \frac{1}{3}\left(1 + \frac{1}{3}\right)^{-2}.$$

We remark for this computation (and the ones to follow), the time-intensive part is computing the $L$-value. For a computation that resulted in about 15 decimal points of accuracy in the case above, the $L$-function algorithms built into Sage asked for over 30000 terms of the Dirichlet series, which in turn required finding the coefficients of the three modular forms at all primes up to at least 30000. Using the default modular symbol methods in Sage for working with modular forms, this took several hours on the author's laptop computer - a lengthy computation but not one requiring special resources.

*Case of two conductor-p principal series representations and one unramified representation.* The next case we can consider is when two of our representations are principal series of conductor $p$. We carry out this computation in [Col16], and obtain the following local factors:

$$I_p^* = \frac{1}{p} \qquad\qquad I_p^{**} = \frac{1}{p}\left(1 + \frac{1}{p}\right)^{-2}.$$

Again the conductor is $p^4$, and the local $L$-factor is

$$L_p(f \times g \times \overline{h}, s) = \prod_{i=1}^{2}(1 - \alpha_i\beta\gamma p^{-s})^{-1}(1 - \alpha_i^{-1}\beta^{-1}\gamma^{-1}p^{-s})^{-1}$$

where as before $\alpha_1, \alpha_2$ are the roots of the Hecke polynomial for the one of $f, g, \overline{h}$ unramified at $p$ and $\beta, \gamma$ are the $p$-th coefficients for the other two.

As a test of this particular case, we take $f = g$ to both be the newform $q - 2i\sqrt{11}q^2 + 6i\sqrt{11}q^3 + \cdots$ of weight 6, level 5, and of the unique even character $\chi$ of conductor 5, and again take $h = \Delta$. This gives

$$\frac{|\langle f(z)g(z), h(z)\rangle|^2}{L(f \times g \times \overline{h}, 11)} \approx \frac{1.6015746784\ldots \cdot 10^{-16}}{1.4547492648\ldots} \approx 1.1009283297\ldots \cdot 10^{-16} \approx \frac{3^2 \cdot 10! \cdot 5! \cdot 5!}{\pi^{26}2^{46}} \cdot \frac{1}{5}\left(1 + \frac{1}{5}\right)^{-2}.$$

*Other conductor-p cases.* There are a handful of other cases to consider where all representations are of conductor $\leq p$, and most have been computed in the literature. We do not need these in the specific setup considered in [Col16], but we have carried out numerical computations as a verification of each of them as well.

- Two conductor-$p$ special and one unramified: Here the local conductor is $p^4$ and $L_p(f \times g \times \overline{h}, s) = \prod_{i=1}^{2}(1 - \alpha_i\beta\gamma p^{-s})^{-1}(1 - \alpha_i\beta\gamma p^{-s+1})^{-1}$ where once again $\alpha_1, \alpha_2$ are the roots of the Hecke polynomial for the one of $f, g, \overline{h}$ unramified at $p$ and $\beta, \gamma$ are the $p$-th coefficients for the other two. In this case the local factor was worked out by Woodbury [Woo12] as $I_p^* = \frac{1}{p}$ and $I_p^{**} = \frac{1}{p}(1 + \frac{1}{p})^{-2}$.

- Three conductor-$p$ special: The local conductor is $p^5$ and $L_p(f \times g \times \overline{h}, s) = (1 - \alpha\beta\gamma p^{-s})^{-1}(1 - \alpha\beta\gamma p^{-s+1})^{-2}$ where $\alpha, \beta, \gamma$ are the $p$-th coefficients of $f, g, \overline{h}$. This is the only case where the $\varepsilon$-factor for $L_p(f \times g \times \overline{h}, s)$ is not automatically one, and is instead given by $-\alpha\beta\gamma/p^{m-2}$; here the local factor is also calculated by Woodbury [Woo12] as $I_p^* = (1 - \varepsilon)\frac{1}{p}(1 + \frac{1}{p})$ and $I_p^{**} = (1 - \varepsilon)\frac{1}{p}(1 + \frac{1}{p})^{-2}$.

- Two conductor-$p$ principal series and one conductor-$p$ special: The local conductor is $p^6$ and $L_p(f \times g \times \overline{h}, s) = (1 - \alpha\beta\gamma p^{-s})^{-1}(1 - \alpha^{-1}\beta^{-1}\gamma^{-1}p^{-s})$ where $\alpha, \beta, \gamma$ are the $p$-th coefficients of $f, g, \overline{h}$. In this case the local factors are computed by Humphries [Hum18] giving $I_p^* = \frac{1}{p}(1 + \frac{1}{p})$ $I_p^{**} = \frac{1}{p}(1 + \frac{1}{p})^{-2}$.

- Three conductor-$p$ principal series: The local conductor is $p^6$ and $L_p(f \times g \times \overline{h}, s) = (1 - \alpha\beta\gamma p^{-s})^{-1}(1 - \alpha^{-1}\beta^{-1}\gamma^{-1}p^{-s})^{-2}$ where $\alpha, \beta, \gamma$ are the $p$-th coefficients of $f, g, \overline{h}$. We do not know of a place in the literature explicitly dealing with this case, but it should be a special case of the results of Hsieh [Hsi17]. A numerical test suggests the values should be $I_p^* = \frac{1}{p}(1 + \frac{1}{p})$ and $I_p^{**} = \frac{1}{p}(1 + \frac{1}{p})^{-2}$.

Surprisingly, the Ichino local integrals seem much more uniform across these various cases when expressed in our modified normalization $I_p^{**}$ (intended for working with classical modular forms) than the standard one $I_p^*$ coming from the adelic formulation. We remark that the factors of $(1 + \frac{1}{p})^{-2}$ arise from us normalizing our Petersson inner products by $\mathrm{vol}(\mathbb{H}\backslash\Gamma)$. If we left it unnormalized instead then the local integrals would have an even simpler form.

*Case of one representation of conductor $\geq p^2$ and two unramified representations.* We can also consider the general case when two of our three representations $\pi_1, \pi_2$ are unramified. The case when the third representation is conductor-$p$ special was discussed above, so we're left with the case of conductor $p^c$ for $c \geq 2$.

The overall condition that the product of central character is trivial forces $\pi_3$ to have an unramified central character itself. The most interesting case of such a $\pi_3$ is when it is supercuspidal, but there's also the possibility a principal series (corresponding to two characters with the product unramified) or a special representation (a twist of the conductor-$p$ one by a character with its square unramified). In all three cases,

the local $L$-factor $L_p(f \times g \times \overline{h}, s)$ is trivial, but the local conductor for the $L$-function is $p^{4c}$, making the algorithms for finding the $L$-value quite computationally intensive.

Each of these three cases needs to be analyzed separately, all of them are considered in [Hu17], and using our normalization $I_p^{**}$ all of them have the same form

$$I_p^{**} = \frac{1}{p^c}\left(1 + \frac{1}{p}\right)^{-2}.$$

As before, we can check an assortment of examples for these cases. For instance, as a test of the supercuspidal case we can take $f = q - 12q^3 + 54q^5 + \cdots$ to be the unique newform of weight 6, level 4, and trivial character, $g = \Delta$ the delta-function, and $h = q - 528q^2 - 4284q^3 + \cdots$ the unique newform of weight 18 and level 1. We then find

$$\frac{|\langle f(z)g(4z), h(z)\rangle|^2}{L(f \times g \times \overline{h}, 17)} \approx \frac{4.2746854\ldots \cdot 10^{-25}}{0.6583584\ldots} \approx 6.4929462\ldots \cdot 10^{-25} \approx \frac{3^2 \cdot 16! \cdot 5! \cdot 11!}{\pi^{38}2^{70}4^{12}} \cdot \frac{1}{4}\left(1 + \frac{1}{2}\right)^{-2}.$$

We have carried out similar computations checking the special and principal series cases as well.

*One case with two supercuspidals and one unramified representation.* Finally, the last case we will consider is where one of the local representations $\pi_{f,p}, \pi_{g,p}, \pi_{h,p}$ is unramified, and the other two representations are both isomorphic to a single supercuspidal $\pi$ with trivial central character. In particular we'll consider the case where $\pi \cong \pi \otimes \eta$ where $\eta$ is the unramified quadratic character of $\mathbb{Q}_p^\times$; this is "type 1" in the notation of [NPS14]. Nelson-Pitale-Saha prove that in this case we have

$$I_p^* = I_p^{**} = \frac{1}{p^c}\left(1 + \frac{1}{p}\right)^{-2}\left(\frac{(\alpha^{c/2+1} - \alpha^{-c/2-1}) - p^{-1}(\alpha^{c/2-1} - \alpha^{-c/2+1})}{\alpha - \alpha^{-1}}\right)^2,$$

where $p^c$ is the conductor of $\pi$, we assume $p^c$ is also the conductor of $\pi \times \pi$ (which will be true in our cases of interest), and $\alpha, \alpha^{-1}$ are the Satake parameters of the third unramified representation (so, for example, if $f$ is the one unramified at $p$, then $\alpha p^{(k-1)/2}, \alpha p^{(k-1)/2}$ are the roots of the Hecke polynomial for $f$ at $p$).

In this situation, the local conductor is $p^{2c}$ and the local $L$-factor

$$L(f \times g \times \overline{h}, s) = (1 - \alpha p^{-s})^{-1}(1 + \alpha p^{-s})^{-1}(1 - \alpha^{-1}p^{-s})^{-1}(1 + \alpha^{-1}p^{-s})^{-1}.$$

We can then proceed to numerical examples. Our first example will take $f = g = q + 6\sqrt{10}q^2 + 232q^4 + \cdots$ to be the newform of weight 8, level 9, trivial character, and which isn't a twist of a newform of level 3, and $h = q + 216q^2 - 3348q^3 + \cdots$ is the unique newform of weight 1 and level 16. We then numerically compute

$$\frac{|\langle f(z)g(z), h(z)\rangle|^2}{L(f \times g \times \overline{h}, 15)} \approx \frac{2.1021427352\ldots \cdot 10^{-17}}{7.9702799221\ldots}$$

$$\approx 2.6374766705\ldots \cdot 10^{-18} \approx \frac{3^2 \cdot 14! \cdot 7! \cdot 7!}{\pi^{34}2^{62}} \cdot \frac{1}{3^2}\left(1 + \frac{1}{3}\right)^{-2} \cdot \left(\frac{3348^2}{3^{15}}\right).$$

Here, $c = 2$ so the last term in $I_p^{**}$ becomes $(\frac{\alpha^2 - \alpha^{-2}}{\alpha - \alpha^{-1}})^2 = (\alpha + \alpha^{-1})^2$ (note the second half of the numerator drops out since $c/2 - 1 = 0$), and by definition $\alpha + \alpha^{-1}$ is just $a_p/p^{(m-1)/2}$.

For our second example we take $f = g = q - \frac{3+\sqrt{129}}{2}q^2 + \frac{5+3\sqrt{129}}{2}q^4 + \cdots$ to be a newform of weight 6, level 81, and trivial character (the unique such newform that isn't a twist of a lower-level form), and $h = \Delta$ the delta-function. We then compute

$$\frac{|\langle f(z)g(z), h(z)\rangle|^2}{L(f \times g \times \overline{h}, 11)} \approx \frac{4.2156534297\ldots \cdot 10^{-18}}{0.8058589132\ldots}$$

$$\approx 5.2312549509\ldots \cdot 10^{-18} \approx \frac{3^2 \cdot 10! \cdot 5! \cdot 5!}{\pi^{26}2^{46}} \cdot \frac{1}{3^4}\left(1 + \frac{1}{3}\right)^{-2} \cdot \left(\frac{252^2}{3^{11}} - 1 - \frac{1}{3}\right)^2.$$

Here $c = 4$ so the final term in $I_p^{**}$ is $(\frac{(\alpha^3 - \alpha^{-3}) - p^{-1}(\alpha - \alpha^{-1})}{\alpha - \alpha^{-1}})^2$, giving $\alpha^2 + 1 + \alpha^{-2} - p^{-1} = (\alpha + \alpha^{-1})^2 - 1 - p^{-1}$.

*Other cases.* The results of Nelson-Pitale-Saha, [NPS14] and Hu [Hu17] compute Ichino local integrals $I_p^*$ in more generality than we have discussed, and more recently Hsieh [Hsi17] has computed them in many more situations. We do not claim to have checked any of these beyond what is discussed above, but in principle this could be done by the same sorts of calculations that we have given.

## References

[AL78]    A. O. L. Atkin and Wen Ch'ing Winnie Li, *Twists of newforms and pseudo-eigenvalues of W-operators*, Invent. Math. **48** (1978), no. 3, 221–243. MR 508986 (80a:10040)

[Asa76]   Tetsuya Asai, *On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution*, J. Math. Soc. Japan **28** (1976), no. 1, 48–61. MR 0427235 (55 #270)

[Che16]   Hao Chen, *Computational aspects of modular parametrizations of elliptic curves*, Ph.D. thesis, University of Washington, 2016.

[Coh13]   Henri Cohen, *Haberland's formula and numerical computation of Petersson scalar products*, ANTS X—Proceedings of the Tenth Algorithmic Number Theory Symposium, Open Book Ser., vol. 1, Math. Sci. Publ., Berkeley, CA, 2013, pp. 249–270. MR 3207417

[Col16]   Dan Collins, *Anticyclotomic p-adic L-functions and Ichino's formula*, Preprint: arXiv:1612.06948 [math.NT], Dec 2016.

[Del02]   Christophe Delaunay, *Formes modulaires et invariants de courbes elliptiques définies sur $\mathbb{Q}$*, Ph.D. thesis, L'Université Bordeaux I, 2002.

[Dev16]   The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 7.4)*, 2016, http://www.sagemath.org.

[Dok04]   Tim Dokchitser, *Computing special values of motivic L-functions*, Experiment. Math. **13** (2004), no. 2, 137–149. MR 2068888

[EC11]    Bas Edixhoven and Jean-Marc Couveignes (eds.), *Computational aspects of modular forms and Galois representations*, Annals of Mathematics Studies, vol. 176, Princeton University Press, Princeton, NJ, 2011, How one can compute in polynomial time the value of Ramanujan's tau at a prime. MR 2849700

[Hid81]   Haruzo Hida, *Congruence of cusp forms and special values of their zeta functions*, Invent. Math. **63** (1981), no. 2, 225–261. MR 610538 (82g:10044)

[Hid86]   _____, *Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms*, Invent. Math. **85** (1986), no. 3, 545–613. MR 848685 (87k:11049)

[Hsi17]   Ming-Lun Hsieh, *Hida families and p-adic triple product L-functions*, Preprint: arXiv:1705.02717 [math.NT], 2017.

[Hu17]    Yueke Hu, *Triple product formula and the subconvexity bound of triple product L-function in level aspect*, Amer. J. Math. **139** (2017), no. 1, 215–259. MR 3619914

[Hum15]   Peter Humphries, *Spectral multiplicity for Maass newforms of non-squarefree level*, Preprint: arXiv:1502.06885 [math.NT], 2015.

[Hum18]   _____, *Quantum unique ergodicity in shrinking sets for dihedral Maass newforms*, In preparation, 2018.

[Ich08]   Atsushi Ichino, *Trilinear forms and the central values of triple product L-functions*, Duke Math. J. **145** (2008), no. 2, 281–307. MR 2449948 (2009i:11066)

[LW12]    David Loeffler and Jared Weinstein, *On the computation of local components of a newform*, Math. Comp. **81** (2012), no. 278, 1179–1200. MR 2869056 (2012k:11064)

[Miy06]   Toshitsune Miyake, *Modular forms*, english ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006, Translated from the 1976 Japanese original by Yoshitaka Maeda. MR MR2194815 (2006g:11084)

[Nel15]   Paul D. Nelson, *Evaluating modular forms on Shimura curves*, Math. Comp. **84** (2015), no. 295, 2471–2503. MR 3356036

[NPS14]   Paul D. Nelson, Ameya Pitale, and Abhishek Saha, *Bounds for Rankin-Selberg integrals and quantum unique ergodicity for powerful levels*, J. Amer. Math. Soc. **27** (2014), no. 1, 147–191. MR 3110797

[Shi76]   Goro Shimura, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), no. 6, 783–804. MR 0434962 (55 #7925)

[Shi94]   _____, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, NJ, 1994, Reprint of the 1971 original, Kanô Memorial Lectures, 1. MR 1291394 (95e:11048)

[Woo12]   Michael Woodbury, *Explicit trilinear forms and the triple product L-function*, Preprint: Available at http://www.mi.uni-koeln.de/~woodbury/research/trilinear.pdf, 2012.

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