Anticyclotomic p-adic L-functions and Ichino's formula

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Abstract

We investigate the anticyclotomic p-adic L-function that interpolates the central value of the Rankin-Selberg L-function $L(f \times \chi, s)$, where f is a fixed classical modular form and χ a Hecke character of an imaginary quadratic field varying in an anticyclotomic family. Such p-adic L-functions were first constructed by Bertolini-Darmon-Prasanna using toric integrals. We give an alternative construction using Ichino's triple-product formula, applied to a fixed modular form and a pair of CM forms. Using two CM forms gives us flexibility to obtain precise results in cases that are difficult to analyze from other constructions, such as when the family of Hecke characters leads to residually reducible Galois representations after induction. In particular our results and techniques apply to the family passing through the trivial character, and thus are suited for future arithmetic applications towards elliptic curves.

The first step of the computation is to make Ichino's formula completely explicit in a classical context, resulting in an equation relating a classical Petersson inner product to a product of two Rankin-Selberg L-values. To work out the constant exactly, we need to evaluate certain local integrals at finite places that arise from Ichino's formula. To handle these, we collect various results and techniques from the literature and make some computations ourselves. Once we have an explicit equation, we simultaneously vary the two CM forms in Hida families. By combining some Hida theory with the Main Conjecture of Iwasawa Theory for imaginary quadratic fields (a theorem of Rubin) we show that the Petersson inner products on one side of our equation vary p-adically analytically. From this we conclude that the L-values we're interested in vary p-adically, and moreover obtain a formula relating the p-adic family interpolating Petersson inner products to a product of our p-adic L-function and a fixed L-value.

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To my friends and family.

Contents

Chapter 1

Introduction and main results

1.1 Introduction

L-functions have a central place in modern number theory, because they mediate the conjectural Langlands correspondence between motivic and automorphic objects, and because they encode many deep properties of the objects they correspond to. One way that these properties are encoded is as special values - values of L-functions at certain integers are expected to be related to subtle arithmetic invariants. In some of the better-understood cases, there are precise conjectures stating that an L-value should equal a canonical transcendental factor times an algebraic number coming from the arithmetic of the underlying object; the most famous of these is the conjecture of Birch and Swinnerton-Dyer on the central L-value for elliptic curves. Such conjectures are very deep, and in most cases we do not yet have a good handle on them.

However, in some of these situations we are at least able to use automorphic methods to prove "algebraicity results", that a special value is given by the expected transcendental factor times some algebraic number. We can then define this algebraic number to be the *algebraic part* of the L-value, and attempt to study it further. A fruitful way of doing so is to work with p-adic families of such values, usually for an odd prime p (as the case $p = 2$ introduces many difficulties and isn't normally handled). This was first done by Kubota and Leopoldt for Dirichlet L-functions; for a fixed Dirichlet character they constructed a p-adic analytic function $L_p(s, \chi)$ such that its values at nonpositive integers n interpolate classical Dirichlet L-values at those integers. Studying this p-adic L-function leads to new results - for instance it has its own "special values" at integers outside of the range of interpolation such as $s = 1$, and these lead to a p-adic class number formula based on the value $L_p(1, \chi)$ analogous to the classical one.

The theory of p-adic L-functions was significantly deepened by the work of Iwasawa. He reinterpreted the Kubota-Leopoldt p-adic L-function as an element of the power series ring $\Lambda = \mathbb{Z}_p[\![X]\!]$, and based on his study proposed a deep connection with Galois cohomology that came to be called the Main Conjecture of Iwasawa Theory. This conjecture was ultimately proven by Mazur and Wiles [MW84], providing tools to study p-adic L-functions (and thus classical L-values) by way of working with Galois cohomology, and vice-versa.

The idea of p-adic L-functions, and the corresponding concepts in Iwasawa theory, have been vastly generalized to other sorts of L-functions beyond the Kubota-Leopoldt case (which in the Langlands framework corresponds to the group GL_1 and the field $\mathbb{Q}$). In the GL_1 case, p-adic L-functions for Hecke characters over totally real fields and CM fields, respectively, were constructed by Deligne-Ribet [DR80] and Katz [Kat78] (in both cases, using the theory of p-adic modular forms). The corresponding Main Conjecture in the totally real field case was proven by Wiles [Wil90] and the one in the imaginary quadratic case by Rubin [Rub91], with the general CM case having been studied but still open.

In another direction, one can study p-adic L-functions and Iwasawa theory over groups other than GL_1. The basic case to deal with is for GL_2 over $\mathbb{Q}$, for L-functions corresponding to classical modular forms (and thus to elliptic curves as a special case).

In this setting, when f is a fixed classical newform of weight k, Amice-Vélu [AV75] and Vishik [Vis76] constructed a "cyclotomic" p-adic L-function interpolating the special values $L(f, \chi, m)$ with χ varying over p-power-conductor Dirichlet characters and m in a certain range. The Iwasawa main conjecture in this setting is proven (under some hypotheses) by the work of Kato [Kat04] and Skinner-Urban [SU14].

Another sort of p-adic L-function for GL_2 is an "anticyclotomic" p-adic L-function, interpolating special values of the Rankin-Selberg L-function $L(f, \xi, s) = L(f \times g_\xi, s)$ where ξ is a Hecke character of an imaginary quadratic field and g_ξ is the associated classical CM form. Such a p-adic L-function was first constructed by Bertolini-Darmon-Prasanna [BDP13], and recently a result in the direction of the corresponding Iwasawa main conjecture was proven by Wan [Wan15]. We remark that Bertolini-Darmon-Prasanna also provide a special-value formula for the p-adic L-functions, interpreting its values outside of the range of interpolation as p-adic regulators.

The purpose of this thesis is to give a new method of construction of this type of anticyclotomic p-adic L-function, associated to a fixed newform f and an anticyclotomic p-adic family of Hecke characters Ξ. A naive first approach for constructing such a p-adic L-function $\mathcal{L}(f, \Xi)$ might be to use the Rankin-Selberg unfolding method to write $L(f \times g_\xi, s)$ as a Petersson inner product $\langle fE, g_\xi \rangle$, and then attempt to fit this into a p-adic family of inner products $\langle f\mathbb{E}, \mathrm{g}_\Xi \rangle$ using Hida's theory of Λ-adic modular forms. This approach is sensible, but it can only be pinned down exactly in the case when the residual Galois representations associated to the Λ-adic CM form g_Ξ are residually irreducible. Unfortunately, this excludes the most interesting cases, such as when Ξ passes through the trivial character! To apply to all cases, the original construction of Bertolini-Darmon-Prasanna and subsequent generalizations such as by Hsieh [Hsi14] are instead based on using toric integrals.

However, one would still like to have a construction using Λ-adic interpolations of Petersson inner products such as $\langle f\mathbb{E}, \mathrm{g}_\Xi \rangle$, since these are easy to work with; for instance, the construction makes it readily clear that if f, g are two newforms of the same weight and level satisfying $f \equiv f' \pmod{p^n}$, then we have $\langle f\mathbb{E}, \mathrm{g}_\Xi\rangle \equiv \langle f'\mathbb{E}, \mathrm{g}_\Xi\rangle \pmod{p^n}$ and thus could conclude that the associated p-adic L-functions are congruent modulo p^n. The approach of this thesis is to obtain an interpolation still involving these sorts of Petersson inner products, of the form

$$\langle f\mathrm{g}_\Phi, \mathrm{g}_\Psi\rangle^2 = (*) \cdot L_p(f, \eta) \cdot \mathcal{L}_p(f, \Xi),$$

where Φ, Ψ are two p-adic families such that $\Phi\Psi = \Xi$, and $L_p(f, \eta)$ is a single L-value coming from $\Phi^{-1}\Psi$. The flexibility allowing ourselves two such families is that even if Ξ has residually reducible Galois representations, we can choose Ψ and Φ to have residually irreducible ones, and thus put ourselves in a position where $\langle f\mathrm{g}_\Phi, \mathrm{g}_\Psi\rangle$ can be readily worked with. We will always assume that p is an odd prime, since much of the theory we need is not worked out for the $p = 2$ case.

Our approach to obtaining such an equation is to use Ichino's formula for triple-product L-functions [Ich08]. In Chapter 2, we specialize from Ichino's general automorphic formulation to get an explicit formula of the form

$$|\langle f g_\varphi, g_\psi\rangle|^2 = (*)L(0, f, \varphi\psi^{-1})L(0, f, \varphi^{-1}\psi^{-1}N^{m-k-1})$$

where f is a newform of weight k and g_φ, g_ψ are classical CM newforms of weights $m - k$ and m, respectively. The bulk of the work in getting this explicit formula is evaluating the local integrals that arise in Ichino's general formulation. Various special cases of these have been studied in the literature, and in Section 2.2.6 we give a discussion of techniques that have been used to compute them, and obtain the results we need in our case either by quoting them from elsewhere in the literature (especially from [Hu14], [NPS14], and [Sah15]) or computing them ourselves using similar techniques.

In Chapter 3 we recall the basic theory of p-adic and Λ-adic modular forms that we will need, and then assemble some results of Hida and others that we will apply. Ultimately we show how to construct $\mathcal{I}$-adic families of CM forms $\mathbb{g}_\Psi$ (where $\mathcal{I} \supseteq \Lambda$ is a certain finite extension) and how to construct an element $\langle \mathbb{f}, \mathbb{g}_\Psi \rangle \in \mathcal{I}$ interpolating Petersson inner products $\langle f, g_\psi \rangle$ where f, g_ψ are specializations of the families $\mathbb{f}$ and $\mathbb{g}_\Psi$. Our analysis of the specializations of $\langle \mathbb{f}, \mathbb{g}_\Psi \rangle$ lets us explicitly compute the Euler factors at p that will come out of the expression $\langle f\mathbb{g}_\Phi, \mathbb{g}_\Psi \rangle$ we ultimately want to study, and see that we get the same factors as in [BDP13] (which are the expected ones from the general theory of p-adic L-functions).

Finally, Chapter 4 contains our main calculation: we obtain an infinite family of equations from Ichino's formula where we vary the weights of g_φ and g_ψ together, and then manipulate these to become an infinite family of equations in $\mathbb{C}_p$ involving specializations of $\langle f\mathbb{g}_\Phi, \mathbb{g}_\Psi \rangle$ and $\mathcal{L}_p(f, \Xi)$, which finally lets us conclude an equation between these elements (which lie in an extension of Λ). As a first consequence, we can show that a congruence between modular forms $f \equiv g$ gives a congruence between the p-adic L-functions.

The result stated in this thesis requires the prime p to not divide the level of f, but a slight extension of the methods will allow us to remove that hypothesis. We then anticipate several arithmetic applications of the formula we obtain, for example to the Main Conjecture for this anticyclotomic L-function in certain cases not covered by Wan's results (for instance, for f corresponding to an elliptic curve E such that $E(\mathbb{Q})$ has p-torsion).

We state our main results precisely in Section 1.4, and provide a discussion of our hypotheses there. While we do make several assumptions about the modular form f, the quadratic field K, and the anticyclotomic family Ξ, we emphasize that very few of these are critical roadblocks to the method: for the most part they could be removed by computing more local integrals, which is a tractable problem. Moreover, the methods we use are quite flexible and very general; by changing around our setup we could study other Rankin-Selberg or triple-product p-adic L-functions, and our techniques could be applied just as well to the case of Hilbert modular forms or to automorphic forms on quaternion algebras.

1.2 Notation and conventions

Embeddings. Throughout, we will be fixing a prime number $p > 2$ and then studying p-adic families (of modular forms, L-values, etc.). Thus we will fix once and for all embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

We will also be working with an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, which again we will fix subject to some hypotheses specified later. We will treat K as being a subfield of $\overline{\mathbb{Q}}$, and thus having distinguished embeddings into $\mathbb{C}$ and into $\overline{\mathbb{Q}}_p$ via ι_∞ and ι_p, respectively. The embedding $\iota_p : K \hookrightarrow \overline{\mathbb{Q}}_p$ is a place of K, and thus corresponds to a prime ideal $\mathfrak{p}$ lying over p. We will always be working in the situation where p splits in K; thus we can always take $\mathfrak{p}$ to denote the prime ideal lying over p that corresponds to ι_p, and $\overline{\mathfrak{p}}$ the other one. We let ∞ denote the unique infinite place of K, coming from composing the embedding $\iota_\infty : K \hookrightarrow \mathbb{C}$ with the complex absolute value.

Hecke characters. We will frequently be using Hecke characters associated to the field K, so we want to explicitly describe our conventions for them in all of their guises. If $\mathfrak{m}$ is a nonzero ideal of $\mathcal{O}_K$ and $I^{S(\mathfrak{m})}$ is the group of fractional ideals coprime to $\mathfrak{m}$, a classical Hecke character is a group homomorphism $\varphi : I^{S(\mathfrak{m})} \to \mathbb{C}^\times$ satisfying

$$\varphi(\alpha\mathcal{O}_K) = \varphi_{\text{fin}}(\alpha)\alpha^a\overline{\alpha}^b$$

for all $\alpha \in \mathcal{O}_K$ that are coprime to $\mathfrak{m}$, where $\varphi_{\text{fin}} : (\mathcal{O}_K/\mathfrak{m})^\times \to \mathbb{C}^\times$ is a character (the *finite part* or *finite-type* of φ) and a, b are complex numbers (with the pair (a, b) called the *infinity type*). With this convention, the norm character determined by $N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$ on prime ideals has trivial conductor (i.e. $\mathfrak{m} = \mathcal{O}_K$), trivial finite part, and infinity-type $(1, 1)$. Also, we sometimes view φ as being defined on all fractional ideals of $\mathcal{O}_K$, implicitly setting $\varphi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not coprime to $\mathfrak{m}$. We caution that many places in the literature take the opposite convention of calling $(-a, -b)$ the infinity-type, so one must be careful when comparing different papers. In particular, our convention matches up with that of Bertolini-Darmon-Prasanna [BDP13], but is opposite of Hsieh [Hsi14] and from most of Hida's papers.

Next, we know that (primitive) classical Hecke characters are in bijection with adelic Hecke characters, i.e. continuous homomorphisms $\mathbb{I}_K/K^\times \to \mathbb{C}^\times$. A straightforward way to describe this bijection is by letting $\text{id}(\alpha) = \prod \mathfrak{p}^{v_\mathfrak{p}(\alpha_\mathfrak{p})}$ denote the ideal associated to an idele $\alpha = (\alpha_v)$; then a classical Hecke character $\varphi : I^{S(\mathfrak{m})} \to \mathbb{C}$ corresponds to a continuous character $\varphi_\mathbb{C} : \mathbb{I}_K/K^\times \to \mathbb{C}^\times$ such that $\varphi_\mathbb{C}(\alpha) = \varphi(id(\alpha))$ for every $\alpha \in \mathbb{I}_K^{S(\mathfrak{m}),\infty}$ (the set of ideles that are trivial at infinite places and places in S). Under this correspondence we find that if φ had infinity-type (a, b) then the local factor $\varphi_{\mathbb{C},\infty}$ at the infinite place is given by $\varphi_{\mathbb{C},\infty}(z) = z^{-a}\overline{z}^{-b}$. Thus our convention for the infinity-type for an adelic Hecke character is that it corresponds to the *negatives* of the exponents of z and $\overline{z}$ in the local component at the infinite place. In particular the adelic absolute value $\|\cdot\|_\mathbb{A}$ is a character of infinity type $(-1, -1)$, and it corresponds to the inverse of the norm character.

A classical or adelic Hecke character of K is called *algebraic* if its infinity-type (a, b) consists of integers. Algebraic adelic Hecke characters are in bijection with algebraic p-adic Hecke characters; a p-adic Hecke character is a continuous homomorphism $\psi : \mathbb{I}_K/K^\times \to \overline{\mathbb{Q}}_p^\times$, and ψ is algebraic of weights (a, b) if its local factors $\psi_\mathfrak{p}$ and $\psi_{\overline{\mathfrak{p}}}$ on $K_\mathfrak{p}^\times \cong \mathbb{Q}_p^\times$ and $K_{\overline{\mathfrak{p}}}^\times \cong \mathbb{Q}_p^\times$ are given by $\psi_\mathfrak{p}(x) = x^{-a}$ and $\psi_{\overline{\mathfrak{p}}}(x) = x^{-b}$ on some neighborhoods of the identity in these multiplicative groups of local fields. Then, an algebraic adelic Hecke character $\varphi_{\mathbb{C}}$ of infinity-type (a,b) corresponds to a p-adic Hecke character $\varphi_{\mathbb{Q}_p}$ of weight (a,b) by the formula

$$\varphi_{\mathbb{Q}_p}(\alpha) = (\iota_p \circ \iota_\infty^{-1})\big(\varphi_{\mathbb{C}}(\alpha)\alpha_\infty^a \overline{\alpha}_\infty^b\big)\alpha_{\mathfrak{p}}^{-a}\alpha_{\overline{\mathfrak{p}}}^{-b}$$

for any idele $\alpha = (\alpha_v)$. It is straightforward that this defines a continuous character $\mathbb{I}_K \to \overline{\mathbb{Q}}_p^\times$, and it's trivial on $K^\times$ because if $\alpha \in K^\times$ is treated as a principal idele then $\iota_\infty^{-1}(\alpha_\infty) = \iota_p^{-1}(\alpha_{\mathfrak{p}})$ and $\iota_\infty^{-1}(\overline{\alpha}_\infty) = \iota_p^{-1}(\alpha_{\overline{\mathfrak{p}}})$ via how we set up our embeddings.

Finally, p-adic Hecke characters (algebraic or not) are in bijection with p-adic Galois characters $G_K \to \overline{\mathbb{Q}}_p^\times$ via precomposing with the reciprocity map of class field theory $\mathrm{Art}_K : \mathbb{I}_K \to G_K^{ab}$; under this bijection the algebraic Hecke characters of weight (a,b) correspond to the Hodge-Tate Galois characters of weight (a,b). (For completeness this correspondence requires us to specify a few conventions as well. We can choose to take Art_K to be arithmetically normalized, i.e. so that it takes a uniformizer $\pi_{\mathfrak{q}} \in K_{\mathfrak{q}}^\times \subseteq \mathbb{I}_K$ to an arithmetic Frobenius $\mathrm{Frob}_{\mathfrak{q}} \in G_{K_{\mathfrak{q}}}$; thus our correspondence identifies the norm Hecke character N with the usual cyclotomic Galois character ε. If we take our convention for Hodge-Tate weights to be so that the cyclotomic character has weights 1 rather than -1, we have the stated correspondence between weights of p-adic Hecke characters and of Galois characters. However, since we won't do much explicitly on the Galois side in this thesis, the choices of conventions here aren't especially important).

So, whenever we have an algebraic Hecke character φ we can consider any of the four types of realizations of it discussed above. We will pass between them fairly freely, only being as explicit as we need to be clear and to make precise computations. In particular we'll often abuse notation and allow φ to denote whichever of the four associated Hecke or Galois characters that's most convenient for our purposes at any given time. Also, we'll use the terminology "weights" and "infinity-type" interchangeably for the pair of parameters (a, b).

Modular forms. We'll heavily use the theory of classical modular forms, as developed in e.g. [Miy06] or [Shi94]. In particular we'll primarily be concerned with modular forms for the congruence subgroup $\Gamma_0(N)$; if χ is a Dirichlet character modulo N we let $M_k(N, \chi) = M_k(\Gamma_0(N), \chi)$ be the $\mathbb{C}$-vector space of modular forms that transform under $\Gamma_0(N)$ with weight k and character χ. We let $S_k(N, \chi)$ denote the subspace of cusp forms; on this space we have the Petersson inner product, which we always take to be normalized by the volume of the corresponding modular curve:

$$\langle f, g \rangle = \frac{1}{\mathrm{vol}(\mathbb{H}\backslash\Gamma_0(N))} \int_{\mathbb{H}\backslash\Gamma_0(N)} f(z)\overline{g(z)}\,\mathrm{Im}(z)^k \frac{dx\ dy}{y^2}.$$

Any modular form $f \in M_k(N, \chi)$, as a function on the upper half-plane $\mathbb{H}$ which is invariant under $z \mapsto z + 1$, has a Fourier expansion $f = \sum a_n q^n$ for $q = \exp(2\pi i z)$. We'll work heavily with such expansions, so we take the notation that $a(n, f)$ denotes the coefficient of q^n in the Fourier expansion of f. By using Fourier expansions, for any subring $A \subseteq \mathbb{C}$ we can define $M_k(N, \chi; A)$ as the subspace of $M_k(N, \chi)$ consisting of forms with all of their Fourier coefficients $a(n, f)$ lying in A.

Our use of modular forms will go in in two distinct directions. In Chapter 2, we'll work automorphically, and work with a classical newform f by way of studying its associated automorphic representation π_f for $\mathrm{GL}_2/\mathbb{Q}$. We tend to follow the conventions and notations from standard sources on the automorphic theory for GL_2, such as [JL70] and [Bum97].

In the other direction, in Chapter 3 we work with the theory of p-adic and Λ-adic modular forms, where modular forms with algebraic q-expansions are interpreted as formal power series with coefficients in a p-adic field, and studied in p-adic families. Much of the groundwork of this theory was laid by Hida, and we follow his writings (especially [Hid88] and [Hid93]). However, we tend to rephrase his results in more modern language (treating Λ-adic forms as formal power series, or as measures); this formulation is known to experts and is used in the modern literature, but we are not aware of any comprehensive references written using it.

1.2.1 List of Notation

We end this section by giving a list of what various different symbols and variables are used to denote in this thesis, and where they're defined (if applicable). The following are our usual uses of various symbols:

- p will be a fixed rational prime such that we study things p-adically, ℓ will be a fixed auxiliary prime, and q will be used to denote arbitrary primes. (q is also used to denote a power series variable in early parts of Chapter 3, following the usual convention of writing Fourier expansions of modular forms as "q-series", but this shouldn't cause any confusion).

- f, g, h (and occasionally ψ) will denote modular forms; much of the time $f \in S_k(N, \chi_f)$ will usually be a fixed newform of weight k, level N, and character χ_f.

- When working automorphically, if f is a newform we usually let π_f denote the associated automorphic representation, and $F \in \pi_f$ the distinguished automorphic lift of f as defined in Section 2.2.1.

- In Chapter 2, π_1, π_2, π_3 will be denote three automorphic representations for $\mathrm{GL}_2/\mathbb{Q}$, except in Section 2.3 where instead they will denote three local representations of $\mathrm{GL}_2(\mathbb{Q}_q)$. In either case, we'll always assume the product of the central characters of π_1, π_2, π_3 are trivial.

- $K = \mathbb{Q}(\sqrt{-d})$ will be a fixed imaginary quadratic field of discriminant $-d$, for an integer $d > 0$. We'll always assume the prime p splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$. General ideals of K will be denoted $\mathfrak{a}$; things like $\mathfrak{c}$ and $\mathfrak{m}$ are usually ideals that are (parts of) conductors of Hecke characters.

- $\varphi, \psi, \xi, \eta, \alpha, \beta$ will usually be used to denote various Hecke characters of K; χ will be used to denote Dirichlet characters. In particular, χ_f is the character of the modular form f, χ_K is the quadratic character associated to the field K, and χ_φ is the restriction to $\mathbb{Z}$ of the finite-type part of a Hecke character φ.

- F will denote a finite extension of $\mathbb{Q}_p$, and $\mathcal{O}_F$ its integer ring. $\Lambda = \Lambda_F = \mathcal{O}_F$ will be the power series ring $\mathcal{O}_F[\![X]\!]$, which for our purposes is more naturally viewed as the (isomorphic) completed group ring $\mathcal{O}_F[\![\Gamma]\!]$ for $\Gamma = (1+p)^{\mathbb{Z}_p} \subseteq \mathbb{Z}_p^\times$. $\mathcal{I}$ will denote a finite flat extension of Λ, for our purposes of the form described in Definition 3.2.8.

- Φ, Ψ, Ξ will denote certain Λ-adic (or $\mathcal{I}$-adic) families of Hecke characters we'll construct in Section 3.5.1.

- $\mathbb{f}, \mathbb{g}$ will denote Λ-adic (or $\mathcal{I}$-adic) modular forms, as defined in Section 3.2.

We also give a list of where we properly define or construct a number of objects that will be referred to at various points.

- In Section 1.3 we define a collection of Hecke characters $\Sigma_{cc}(\mathfrak{N}, c)$ on which the BDP L-function will be defined, and a subset $\Sigma_{cc}^{(2)}(\mathfrak{N}, c)$ where it will interpolate classical L-values.

- In Section 1.3 we define the p-adic L-values $L_p^*(f, \xi^{-1})$ and $L_p^{BDP}(f, \xi^{-1})$ (and the associated algebraic parts), and an element $\mathcal{L}_p^{BDP}(f, \Xi)$ interpolating them for a family of Hecke characters Ξ. We also define intermediate constants $C(f, \xi, c)$ and $w(f, \xi)$ used in the definition of these L-values.

- In Theorem 2.2.3 we define $L^H(\mathrm{ad}\,\psi, 1)$, a modified version of the adjoint L-value $L(\mathrm{ad}\,\psi, 1)$ associated to a modular form ψ.

- Local integrals I_v^{**} are defined in Section 2.2.6 and computed in Section 2.3.

- Quite a bit of notation related to local representations is introduced in Section 2.3 and only used there; we don't recall that here.

- g_φ will be the classical CM newform associated to a Hecke character φ of appropriate infinity-type, as defined in Section 2.4.

- Spaces of p-adic modular forms $\overline{M}(N, \chi; \mathcal{O}_F)$ and $\overline{S}(N, \chi; \mathcal{O}_F)$, as well as their associated actions of Λ and their Hecke algebras, are defined in Section 3.1.

- Spaces of $\mathcal{I}$-adic modular forms $\mathbb{M}(N, \chi; \mathcal{I})$ and $\mathbb{S}(N, \chi; \mathcal{I})$ are defined in Section 3.2, as are the sets $\mathcal{X}(\mathcal{I}; \mathcal{O}_F)$ and $\mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_F)$ of "points" at which we can specialize them.

- If f is a p-ordinary newform of level prime to p, the associated eigenforms $f^\sharp$ and $f^\flat$ of level pN are defined in Lemma 3.3.3. A third element called $f^\natural$ is defined in Proposition 3.4.6.

- The element $H_{\mathbb{f}}$ that we call a "congruence number" of a $\mathcal{I}$-adic newform $\mathbb{f}$ (i.e. a generator of the annihilator of the module of congruences for $\mathbb{f}$) is defined in Section 3.3.2.

- $\mathcal{I}$-adic families $\mathbb{g}_\Psi$ and $f\mathbb{g}_\Phi^*$ involving CM newforms are defined in Section 3.5.

- The Katz p-adic L-function $\mathcal{L}_p^{Katz}(\Phi)$ and the associated values $L_p^{Katz}(\varphi_m, 1)$ are defined in Section 3.6.

- Chapter 4 amounts to a long calculation applying the material from before it; the precise hypotheses and constructions that are used there are given in

Sections 4.1.3 and 4.3. (In particular, the elements $\langle f \mathrm{g}_\Phi^\dagger, \mathrm{g}_\Psi \rangle$ and $\langle f \mathrm{g}_\Phi^\ddagger, \mathrm{g}_\Psi \rangle$ of $\mathcal{I}$ are defined in the latter section).

1.3 The BDP p-adic L-function

The focus of this thesis is on studying the p-adic L-function constructed by Bertolini, Darmon, and Prasanna in [BDP13]. Given a classical modular form f and a Hecke character ξ associated to an imaginary quadratic field, f determines an automorphic representation π_f for $\mathrm{GL}_2/\mathbb{Q}$ and ξ determines an automorphic representation for GL_1/K which can be induced to π_ξ for $\mathrm{GL}_2/\mathbb{Q}$; we are then interested in studying the central value $L(\pi_f \times \pi_\xi, 1/2)$ of the $\mathrm{GL}_2 \times \mathrm{GL}_2$ L-function corresponding to this pair. The BDP p-adic L-function interpolates these special values when f is fixed and ξ varies with weights large relative to the weight of f. We will briefly recall their setup and results, specialized somewhat to the setting we'll ultimately study in this thesis.

To set up the construction of the BDP L-function precisely, let us fix a classical newform $f = \sum a_n q^n \in S_k(N, \chi_f)$. We then fix a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-d})$ with fundamental discriminant $-d$, where $d > 3$ and every prime dividing N splits in K. (The assumption that d is odd is imposed in [BDP13] to avoid complicated local calculations at the prime 2. The assumption that primes dividing N split in K implies that $\mathcal{O}_K$ has a cyclic ideal $\mathfrak{N}$ of norm N and thus is a special case of the "Heegner hypothesis" made there. The assumption $d > 3$ is added in this thesis to avoid some annoyances that occur when $\mathcal{O}_K^\times$ is larger than $\{\pm 1\}$).

The setup in [BDP13] is phrased in terms of the classical Rankin-Selberg L-function associated to f and the theta function of ξ, which we can write explicitly as

$$L(f, \xi, s) = \sum_{n=1}^{\infty} a_n \left(\sum_{N(\mathfrak{a})=n} \xi(\mathfrak{a}) \right) n^{-s}.$$

In fact they study the special value $L(f, \xi^{-1}, 0)$ for certain characters ξ; first of all, one can check that if ξ has infinity-type (ℓ_1, ℓ_2) then $L(f, \xi^{-1}, s)$ is *critical* at the point $s = 0$ in the sense of Deligne iff it fits into one of the following three categories:

(1) $1 \leq \ell_1, \ell_2 \leq k - 1$.

(2) $\ell_1 \geq k$ and $\ell_2 \leq 0$.

(2') $\ell_1 \leq 0$ and $\ell_2 \geq k$.

Thus if we let Σ denote the set of all critical characters, it's naturally partitioned as $\Sigma = \Sigma^{(1)} \sqcup \Sigma^{(2)} \sqcup \Sigma^{(2')}$ via these three conditions. Further, we call a critical character *central critical* if it satisfies $\ell_1 + \ell_2 = k$ and $\chi_\xi = \chi_f$; this guarantees that $L(f, \xi^{-1}, s)$ is self-dual and has $s = 0$ as the center of its functional equation. We let Σ_{cc} denote the set of central critical characters; the partition of Σ restricts to a partition $\Sigma_{cc} = \Sigma_{cc}^{(1)} \sqcup \Sigma_{cc}^{(2)} \sqcup \Sigma_{cc}^{(2')}$.

The BDP L-function will then be defined on characters in $\Sigma_{cc}^{(2)}$. More specifically, it will be defined on any given families of characters with the same ramification (and will be normalized in a way that depends on the conductor). There are a few different ways one can formalize this; the point of view in [BDP13] is to realize the L-function as a continuous function on a set of characters viewed as a function space. Another approach (which in principle gives a stronger statement, but in practice falls out of essentially the same ideas) is to parametrize a class of characters as a rigid analytic space and show that the BDP L-function is analytic on that domain. We will describe the setup of [BDP13] that uses the former perspective, and then describe what the corresponding result is in the latter one.

In addition to fixing an ideal $\mathfrak{N}$ with $\mathcal{O}_K/\mathfrak{N}\mathcal{O}_K \cong \mathbb{Z}/N\mathbb{Z}$, we fix an auxiliary integer c, which for simplicity we assume is odd and coprime to both N and d. We'll then consider characters which have prime-to-N conductor equal to c. We can formalize this by letting $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$ be the order of index c in $\mathcal{O}_K$, and then setting U_c to be the compact open subgroup $\prod_q (U_c \otimes \mathbb{Z}_q^x)$ of $\widehat{\mathcal{O}}_K^\times = \prod_{\mathfrak{q}} \mathcal{O}_{\mathfrak{q}}^\times \subseteq \mathbb{I}_K$. If N_χ is the conductor of χ_f and $\mathfrak{N}_\chi$ is the unique ideal dividing $\mathfrak{N}$ of norm N_ε, we can define a character $\widehat{\chi}_f : U_c \to \mathbb{C}^\times$ by

$$U_c \twoheadrightarrow \left(\frac{U_c}{(\mathfrak{N}_\chi \cap U_c) U_c} \right)^\times \cong \left(\frac{\widehat{\mathcal{O}}_K}{\mathfrak{N}_\chi \widehat{\mathcal{O}}_K} \right)^\times \cong (\mathbb{Z}/N_\varepsilon \mathbb{Z})^\times \to \mathbb{C}^\times$$

with the final map given by χ_f^{-1}. With this setup, we define $\Sigma_{cc}^{(i)}(\mathfrak{N}, c)$ (denoted by $\Sigma_{cc}^{(i)}(\mathfrak{N})$ in [BDP13]) to be the subset of $\Sigma_{cc}^{(i)}$ consisting of character ξ with $\xi|_{U_c} = \widehat{\chi}_f$ and with c dividing the conductor of ξ. (Such a ξ has its behavior entirely determined except at primes $q|c$, where we've at least specified that its restriction to $\mathbb{Z}_q^\times$ is trivial and that its conductor exactly divides c).

Given this setup, if ξ is a character in $\Sigma_{cc}^{(2)}(\mathfrak{N}, c)$, the paper of Bertolini-Darmon-Prasanna proceeds to prove the following result:

Theorem 1.3.1 (Bertolini-Darmon-Prasanna)**.** *Given f, $K = \mathbb{Q}(\sqrt{-d})$, $\mathfrak{N}$, and c as above, if $\xi \in \Sigma_{cc}^{(2)}(\mathfrak{N}, c)$ has infinity-type $(k+j, -j)$, then the quantity*

$$L_{\text{alg}}^{BDP}(f, \xi^{-1}, 0) = \frac{C(f, \xi, c)}{w(f, \xi)} \frac{L(f, \xi^{-1}, 0)}{\Omega_\infty^{2(k+2j)}}$$

(where the constants $C(f, \xi, c)$, $w(f, \xi)$, and Ω_∞ will be described below) is algebraic. In fact, it lies in the number field F_0 generated by the Fourier coefficients of f, the values of ξ, and a field $\widetilde{H}_c \supseteq K$ that depends only on K and c.

We outline their argument, describing the constants along the way as they come in. The first (and most involved) step is using methods of Waldspurger that relate period integrals to L-values to conclude

$$C(f, \xi, c) \cdot L(f, \xi^{-1}, 0) = |S(f, \xi, c)|^2$$

where $S(f, \xi, c)$ is a certain period integral (that actually is a finite sum). The constant

$C(f,\xi,c)$ that appears in this computation is (under our hypotheses) given by

$$C(f,\xi,c) = \Gamma(j+1)\Gamma(k+j)\left(\frac{2\pi}{c\sqrt{d}}\right)^{k+2j-1}\prod_{q|c}\left(\frac{q-\chi_K(q)}{q-1}\right).$$

Next, the constant $w(f,\xi)$ is the number such that $\overline{S(f,\xi,c)} = w(f,\xi)S(f,\xi,c)$, thus allowing us to write

$$\frac{C(f,\xi,c)}{w(f,\xi)}\cdot L(f,\xi^{-1},0) = S(f,\xi,c)^2.$$

This modification is important because the function $S(f,\xi,c)$ will be something that can be related to a p-adic function, and thus so is its square, but complex conjugation can't be dealt with p-adically! The constant $w(f,\xi)$ can be worked out explicitly as

$$w(f,\xi) = \frac{w_f(-N)^{k/2+j}\xi(\mathfrak{b})N(\mathfrak{b})^j}{\chi_f(N(\mathfrak{b}))b_N^{k+2j}},$$

where w_f is the Atkin-Lehner eigenvalue for f and $\mathfrak{b}\subseteq\mathcal{O}_c$ and $b_N\in\mathcal{O}_c$ are an ideal and an element, respectively, such that $\mathfrak{b}$ is coprime to Nc and $\mathfrak{b}\mathfrak{N} = (b_N)$. The resulting quantity can be checked to be independent of the choice of $\mathfrak{b}$ and b_N (or this can be inferred from the computation that $w(f,\xi) = \overline{S(f,\xi,c)}/S(f,\xi,c)$!).

Finally, the quantity $S(f,\xi,c)$ itself is generally transcendental, and we need to understand it in terms of a more concrete transcendental period. It turns out that the correct period is related to the period Ω_∞ associated to the Katz p-adic L-function; we defer the construction to Section 3.6 where that p-adic L-function is discussed. In fact, the complex period Ω_∞ naturally comes along with a p-adic period $\Omega_p\in\mathbb{C}_p$, and both are well-defined up to simultaneously scaling them by an algebraic p-adic unit. Carrying out this computation lets Bertolini, Darmon, and Prasanna conclude that

$$L_{\text{alg}}^{BDP}(f,\xi^{-1},0) = \frac{C(f,\xi,c)}{w(f,\xi)}\frac{L(f,\xi^{-1},0)}{\Omega_\infty^{2(k+2j)}}$$

is indeed algebraic.

Since $L^{BDP}_{\text{alg}}(f,\xi^{-1},0)\in\mathbb{C}$ lies in the embedded copy of $\overline{\mathbb{Q}}$, we can then pass it to an element of $\overline{\mathbb{Q}}_p$ via the pair of embeddings $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C},\overline{\mathbb{Q}}_p$ we've always been fixing. To define the "p-adic version" of this L-value, we multiply by $\Omega_p^{2(k+2j)}$; while the pair (Ω_∞,Ω_p) was only well-defined up to p-adic units, the quantity $\Omega_p^{2(k+2j)}L^{BDP}_{\text{alg}}(f,\xi^{-1},0)$ is completely independent of that choice of scalar. We also want to multiply by an Euler factor, in particular

$$(1-\xi^{-1}(\overline{\mathfrak{p}})a_p+\xi^{-2}(\overline{\mathfrak{p}})\chi_f(p)p^{k-1})^2=(1-\alpha_p\xi^{-1}(\overline{\mathfrak{p}}))^2(1-\beta_p\xi^{-1}(\overline{\mathfrak{p}}))^2,$$

the inverse square of the Euler factor $L_{\overline{\mathbb{F}}_p}(f,\xi^{-1},0)$. Thus we define the p-adic L-value $L_p^{BDP}(f,\xi^{-1})\in\mathbb{C}_p$ as

$$L_p^{BDP}(f,\xi^{-1})=(1-\xi^{-1}(\overline{\mathfrak{p}})a_p+\xi^{-2}(\overline{\mathfrak{p}})\chi_f(p)p^{k-1})^2\Omega_p^{2(k+2j)}\cdot\iota_p\iota_\infty^{-1}\left(L^{BDP}_{\text{alg}}(f,\xi^{-1},0)\right).$$

Then, the p-adic interpolation result proven in [BDP13] is of the following form:

Theorem 1.3.2. *Given f, $K=\mathbb{Q}(\sqrt{-d})$, $\mathfrak{N}$, and c as above, the map $\Sigma^{(2)}_{cc}(\mathfrak{N},c)\to\mathbb{C}_p$ defined by $\xi\mapsto L_p^{BDP}(f,\xi^{-1})$ is uniformly continuous (relative to a certain function space topology on the domain) and thus extends to the closure of $\Sigma^{(2)}_{cc}(\mathfrak{N},c)$ (which includes other functions such as the characters in $\Sigma^{(1)}_{cc}(\mathfrak{N},c)$).*

As mentioned above, we will want a stronger statement, that the function $\xi\mapsto L_p^{BDP}(f,\xi^{-1})$ is actually analytic for certain p-adic analytic families of characters ξ. While [BDP13] does not formulate such a statement, one can be recovered from their proof of the above theorem. Alternatively we can quote results of Hsieh [Hsi14], who constructs such L-functions in more generality.

The natural setting for discussing p-adic analyticity of such an L-function is rigid geometry. However, in our setting we can get away with working entirely algebraically, only using the geometry for motivation. Suppose we fix a character $\xi_0 \in \Sigma_{cc}(\mathfrak{N}, c)$ of infinity-type $(a-1, -a+k+1)$; in Section 3.5 we'll discuss how to put ξ_0 in a family of characters Ξ. In particular, Ξ will be a continuous homomorphism $\mathbb{I}_K^\times \to \mathcal{I}^\times$ where $\mathcal{I}$ is a certain finite extension of the power series ring $\Lambda = \mathcal{O}_F[\![X]\!]$ (for F the completion of the number field F_0). If we define a specialization map $P_m : \Lambda = \mathcal{O}_F[\![X]\!] \to \mathcal{O}_F$ by $X+1 \mapsto (1+p)^m$ and extend this to $\mathcal{I}$ in a certain way that will be made precise later, the family Ξ will be constructed so that $P_m \circ \Xi = \xi_0\beta^{m-a}$ for a character β of infinity-type $(1, -1)$ and conductor $\mathfrak{p}\overline{\mathfrak{p}}$. In particular, β^{p-1} is unramified, so if $m \equiv a \pmod{p-1}$ then ξ_m has infinity-type $(m-1, -m+k-1)$ and the same conductor and finite part as ξ_0; so such ξ_m's are in $\Sigma_{cc}(\mathfrak{N}, c)$. (We remark that it is entirely a convention we're making to have the specialization at P_m be of infinity-type $(m-1, -m+k+1)$ rather than $(m, -m+k)$ or anything else; we could shift this by simply multiplying Ξ by a fixed power of β. The convention we take is the one that will ultimately be the most convenient for us, but all of our families and p-adic L-functions can always be shifted as desired.)

Geometrically, Λ is the coordinate ring of the rigid analytic p-adic open unit disc, and the specialization maps $P : \mathcal{O}_F[\![X]\!] \to \mathcal{O}_F$ are the F-valued points of that disc. The finite extension $\mathcal{I}/\Lambda$ then defines a rigid analytic space that's a finite cover of the disc. Thus Ξ can be thought of as a collection of Hecke characters $P \circ \Xi$ by this parametrized by this space, which is "analytic" and is such that at the distinguished points P_m we get a classical Hecke character of a weight related to m. In view of this, we'd like our p-adic L-function to be a function on this rigid analytic space (i.e. an element of $\mathcal{I}$, or of some related ring) that assigns a point P to the p-adic L-value of $P \circ \Xi$. We can make sense of this by looking at the points P_m for $m \geq k$ with $m \equiv a \pmod{p-1}$; these are dense in the open disc, and the characters $P_m \circ \Xi$ are classical Hecke characters $\xi_m \in \Sigma_{cc}^{(2)}(\mathfrak{N}, c)$ for which we've defined $L_p^{BDP}(f, \xi_m^{-1})$. Thus, we have the following formulation of the existence of the p-adic L-function that is suited to our purposes:

Theorem 1.3.3. *Fix a Hecke character $\xi_m \in \Sigma_{cc}(\mathfrak{N}, c)$ of infinity-type $(a-1, a+k-1)$, and put it into a family Ξ as above. Then there exists $\mathcal{L}_p^{BDP}(f, \Xi) \in \Lambda^{\mathrm{ur}}$ that satisfies $P_m(\mathcal{L}_p^{BDP}(f, \Xi)) = L_p^{BDP}(f, \xi_m^{-1})$ for all $m \geq k$ satisfying $m \equiv a \pmod{p-1}$. Here $\Lambda^{\mathrm{ur}} = \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_F^{\mathrm{ur}} = \mathcal{O}_F^{\mathrm{ur}}[\![X]\!]$, where $\mathcal{O}_F^{\mathrm{ur}}$ is the ring of integers of F^{ur}, the completion of the maximal unramified extension of F, and $P_m : \Lambda^{\mathrm{ur}} \to \mathcal{O}_F^{\mathrm{ur}}$ is the unique $\mathcal{O}_F^{\mathrm{ur}}$-linear extension of $P_m : \Lambda \to \mathcal{O}_F$.*

Finally, we remark that if we want to work with a single L-value $L(0, f, \eta^{-1})$, it's more convenient to define algebraic and p-adic parts in a simpler way, without adding in the factors that were necessary for making L_p^{BDP} a p-adic analytic function. In particular we can leave off $w(f, \eta)$ and the Euler factors, and define:

$$L^*_{\mathrm{alg}}(f, \xi^{-1}, 0) = \frac{C(f, \xi, c) L(f, \xi^{-1}, 0)}{\Omega_\infty^{2(k+2j)}} \qquad\qquad L^*_p(f, \xi^{-1}, 0) = L^*_{\mathrm{alg}}(f, \xi^{-1}, 0)\Omega_p^{2(k+2j)}.$$

1.4 Main results

We can now state our main result (which will be proven in Section 4.6), giving an equation relating the BDP p-adic L-function $\mathcal{L}_p^{BDP}(f, \Xi) \in \Lambda^{\mathrm{ur}}$ to a function (perhaps in an extension of Λ^{ur}) that interpolates certain Petersson inner products. This can be viewed as an alternate construction of the BDP L-function as an element of Λ^{ur}, since Hida [Hid88] has already constructed interpolations of the Petersson inner products in the form we need. To be precise, we suppose we have the following setup:

- An odd prime p.
- A newform f of weight k, level N, and trivial character (note this forces k to be even). We assume that our prime p is coprime to the level N.

- An imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ with $d \equiv 3 \pmod 4$ and $d \geq 7$, such that all primes dividing pN split in K. Given this we take $\mathfrak{N}$ to be a cyclic ideal of norm N in K.

- A residue class modulo $p-1$, represented by $a \in \{0, \ldots, p-1\}$, such that $a-1$ is even.

- A weight $\underline{a} \equiv a \pmod{p-1}$ and a character $\xi_{\underline{a}} \in \Sigma_{cc}(\mathfrak{N}, c)$ of weight $(a-1, -a+k+1)$ for some integer c coprime to $2pdN$.

- An auxiliary prime ℓ not dividing $2pcdN$ that is inert in K, and an auxiliary character ν of $(\mathcal{O}_K/\ell^{c_\ell}\mathcal{O}_K)^\times$ of conductor ℓ^{c_ℓ} and which is trivial on $(\mathbb{Z}/\ell^{c_\ell}\mathbb{Z})^\times$.

Given this setup, we let F be p-adic field large enough that it contains all of the algebraic quantities of interest (i.e. the field K, the values of all of our characters, the coefficients of our newform, and the algebraic BDP L-values). We take $\Lambda = \mathcal{O}_F[\![X]\!]$, and we can construct an anticyclotomic family of Hecke characters $\Xi : \mathbb{I}_K \to \Lambda^\times$ (which we'll do precisely in Section 3.5) with specializations ξ_m of weights $(m-1, -m+k+1)$ such that $\xi_{\underline{a}}$ is our given character $\xi_{\underline{a}}$.

From this, in Section 3.5 we'll construct certain explicit $\mathcal{I}$-adic modular forms g_Ψ, $f\mathrm{g}_\Phi^\dagger$, and $f\mathrm{g}_\Phi^\ddagger$ (with specializations of g_Ψ being CM newforms, and specializations of the other two each being products of the f we started with times CM forms that vary, but with one of the forms precomposed with a scaling $z \mapsto Lz$); here $\mathcal{I}$ is a finite extension of $\Lambda = \mathcal{O}_F[\![X]\!]$ of the form $\mathcal{O}_F[\![X^{1/p^e}]\!]$. Then, Hida's theory lets us construct elements $\langle f\mathrm{g}_\Phi^\dagger, \mathrm{g}_\Psi\rangle$ and $\langle f\mathrm{g}_\Phi^\ddagger, \mathrm{g}_\Psi\rangle$ in $\mathcal{I}$ that interpolate the Petersson inner products of specializations of the appropriate $\mathcal{I}$-adic forms. By using Ichino's formula in conjunction with this, we can prove our main result:

Theorem 1.4.1. *We have an equality in an extension $\mathcal{I}^{\mathrm{ur}}[1/p]$ of Λ^{ur}*

$$\langle f\mathrm{g}_\Phi^\dagger, \mathrm{g}_\Psi\rangle\langle f\mathrm{g}_\Phi^\ddagger, \mathrm{g}_\Psi\rangle = \mathcal{C} \cdot L_p^*(f, \eta^{-1}) \cdot \mathcal{L}_p^{BDP}(f, \Xi).$$

Here, $\eta \in \Sigma_{cc}^{(2)}(\mathfrak{N}, c\ell^{c_\ell})$ is a character with weight $(k, 0)$ and finite-type given by the finite-type of $\underline{\xi}_{\underline{a}}$ times our auxiliary character ν. Also, the element $\mathcal{C} \in \mathcal{I}[1/p]$ is a product of the following terms:

- *An element of Λ interpolating the function $m \mapsto (c/2^4)^m$.*
- *An element $\prod_{q|d}(1 - q^{-1})^{-2} \in \mathbb{Q}_p$ that may be divisible by (negative) powers of p.*
- *A p-adic unit in $\overline{\mathbb{Q}}_p$.*
- *A specific Euler factor at our auxiliary prime ℓ, which can be made a p-adic unit by a suitable choice of ℓ.*

In Theorem 4.6.1 we state this with a more explicit description of the element $\mathcal{C}$. The only other difference between this statement and that in Section 4.6 is whether the ℓ-part of η is given by ν or ν^2; but it's easy to pass between these because ν is a character of an odd-order cyclic group

$$\frac{(\mathcal{O}_K/\ell^{c_\ell}\mathcal{O}_K)^\times}{(\mathbb{Z}/\ell^{c_\ell}\mathbb{Z})^\times} \cong Z_{\ell^{c_\ell}}.$$

As a consequence to this theorem we also obtain a congruence result:

Corollary 1.4.2. *Let $f, f' \in S_k(N)$ be newforms such that $f \equiv f' \pmod{p^n}$. Letting η and Ξ be as above, with the prime ℓ chosen so that $(\star\star)_\ell$ is a unit, we have*

$$L_p^*(f, \eta^{-1}) \cdot \mathcal{L}_p^{BDP}(f, \Xi) \equiv L_p^*(f', \eta^{-1}) \cdot \mathcal{L}_p^{BDP}(f', \Xi) \pmod{p^n}.$$

As mentioned in Section 4.6, we expect to be able to refine this to obtain the most natural congruence $\mathcal{L}_p^{BDP}(f, \Xi) \equiv \mathcal{L}_p^{BDP}(f', \Xi)$.

We end this chapter by adding some remarks on the purpose of each of the hypotheses in our initial list of assumptions, and how necessary they are for our techniques. We proceed roughly in order from the least critical to the most. We recall that the assumption that $\xi_{\underline{a}}$ is in $\Sigma_{cc}(\mathfrak{N}, c)$ amounts to specifying its infinity-type, its central character, its behavior at primes dividing N, and its prime-to-N conductor.

- The assumption that p does not divide the level of f is to simplify the calculations in Section 3.4. We are in progress of removing this assumption.

- The assumption that f has trivial central character is only in place to simplify the cases of local integrals we need to consider; we anticipate extending to allow f to have arbitrary character.

- The assumptions that $d \equiv 3 \pmod 4$ and $d \geq 7$ are to simplify our calculations, but should be removable if needed. Allowing other other values of d would give us even fundamental discriminants which would complicate the local integral at the prime 2. Allowing $K = \mathbb{Q}(\sqrt{-3})$ (or $K = \mathbb{Q}(i)$) would just require a bit more bookkeeping with roots of unity.

- The assumption that all primes dividing N split in K can be removed; we primarily keep it to follow the notation of [BDP13] exactly.

- The assumption on the infinity-type of $\xi_{\underline{a}}$ is essentially a normalization issue that can always be achieved by shifting by a power of the norm character. The assumption on the finite-type part of $\xi_{\underline{a}}$ is mostly a matter of convenience; all that's really necessary is a condition on the central character we'll mention below. In particular, we could relax the assumption that the conductor c is coprime to $2pdN$, though this may introduce many complications (especially if we allow $p|c$).

- If $\xi_{\underline{a}}$ has residually reducible Galois representation, we need to modify Ψ somehow to guarantee it has residually irreducible representations, and the easiest way to do this is to add in an inert auxiliary prime ℓ. Depending on our goal, we could choose to do something else, or to add in other auxiliary primes if it would be convenient.

- The assumption that p splits in K (i.e. that $K/\mathbb{Q}$ is p-ordinary) is necessary for forming the anticyclotomic families of characters Ξ we work with.
- The assumption that p is odd is a requirement for much of the Hida theory and other results we cite, so removing it would be impractical.
- The central-character assumption that $\chi_f = \xi_{\underline{a},\mathrm{fin}}|_{\mathbb{Z}}$ and the parity assumption $a - 1 \equiv k \pmod 2$ are absolutely necessary for our method (and indeed for $\xi_{\underline{a}}$ to exist and $L(f \times \xi_{\underline{a}}, 0)$ to be a central critical L-value).

Chapter 2

The automorphic theory

2.1 Ichino's formula

In this chapter we collect the automorphic results and definitions we will need to make use of, and relate them to corresponding notions for classical holomorphic newforms. We will work on the groups GL_2, $\mathrm{GL}_2 \times \mathrm{GL}_2$, and $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ (all over $\mathbb{Q}$, though nearly everything can generalize immediately to other number fields) and their corresponding L-functions. We use [JL70], [Jac72], [Bum97], and [GH11] as references for the foundations of this theory (and in particular the construction of L-functions and related objects for GL_2 and $\mathrm{GL}_2 \times \mathrm{GL}_2$).

The primary tool we'll use in the theory of automorphic representations is a formula proven by Ichino in 2008 [Ich08]. If we're given three automorphic representations π_1, π_2, π_3 (with the product of their central characters trivial), this formula relates global trilinear form on $\pi_1 \times \pi_2 \times \pi_3$ (coming from integration) to a product of local trilinear forms with the scaling constant coming from the L-value $L(\pi_1 \times \pi_2 \times \pi_3, 1/2)$. Our goal is to apply this to a situation where π_1 corresponds to a modular form f we're interested in, and π_2 and π_3 are CM representations we get to choose. We'll verify later in this chapter (Proposition 2.4.2) that in this case $L(\pi_1 \times \pi_2 \times \pi_3, 1/2)$

splits up as a product $L(f, \eta^{-1}, 0)L(f, \xi^{-1}, 0)$ of two L-values of the type considered by Bertolini-Darmon-Prasanna. By applying Ichino's formula and computing all of the local trilinear forms, we will ultimately end up with an explicit formula relating a certain Petersson inner product (coming from the global trilinear form) to this product $L(f, \eta^{-1}, 0)L(f, \xi^{-1}, 0)$; this is what we will use to interpolate such L-values.

For the rest of this section, we give the background and setup needed to state Ichino's result precisely in the automorphic language of [Ich08], so that in later sections we can move on to giving an explicit version of it in the case of holomorphic modular forms.

First of all, we recall that an integral representation for the standard triple-product L-function on GSp_6 was given by Garrett [Gar87] and Piatetski-Shapiro and Rallis [PSR87], and proven to be correct at all places by [Ike99] and [Ram00] (for ramified nonarchimedean places). This means that if π_1, π_2, π_3 are three local representations of $\mathrm{GL}_2(\mathbb{Q}_q)$ (for $q \leq \infty$) which correspond to Weil-Deligne representations ρ_1, ρ_2, ρ_3 via local Langlands, then the L-factor

$$L_q(\pi_1 \times \pi_2 \times \pi_3, s) = L_q(\rho_1 \otimes \rho_2 \otimes \rho_3, s)$$

can be realized by certain zeta integral, and moreover this zeta integral can be used to show that the full L-function $L(\pi_1 \times \pi_2 \times \pi_3, s)$ (initially defined by an Euler product of the local factors) also has an integral representation globally that gives it a meromorphic continuation and shows that it satisfies a functional equation. In particular, this tells us that our L-function is actually defined at the central value $s = 1/2$ we want to study!

The study of trilinear forms, meanwhile, was started by Dipendra Prasad. For a nonarchimedean local field K, we let D_K denote the non-split quaternion algebra, and then:

Theorem 2.1.1 (Prasad)**.** *Suppose that π_1, π_2, π_3 are irreducible smooth representations of $G = \mathrm{GL}_2(K)$ or $G = D_K^\times$ such that the product of their central characters is trivial. Then the space of G-invariant linear forms is at most 1-dimensional. Moreover, if $G = \mathrm{GL}_2(K)$ then exactly one of the following two situations holds:*

1. *There exists a nonzero $\mathrm{GL}_2(K)$-invariant form on $\pi_1 \otimes \pi_2 \otimes \pi_3$.*

2. *All of the π_i's are discrete series, and there exists $D_K^\times$-invariant form on $\pi_1^D \otimes \pi_2^D \otimes \pi_3^D$ (where π_i^D is the Jacquet-Langlands transfer of π_i to a local representation of $D^\times = D_K^\times$)*

When the residue characteristic is not equal to 2, situations (1) and (2) occur exactly when the triple product ε-factor $\varepsilon(\pi_1 \otimes \pi_2 \otimes \pi_3)$ is $+1$ or -1, respectively.

A similar result holds at Archimedean places, and thus by combining local results one can show that if π_1, π_2, π_3 are irreducible cuspidal automorphic representations of $\mathrm{GL}_2(F)$ for a number field F (or more generally of $D^\times(F)$ for a quaternion algebra D) then there is at most a one-dimensional space of $G(\mathbb{A}_F)$-invariant forms on $\pi_1 \otimes \pi_2 \otimes \pi_3$. Motivated by Prasad's work, Jacquet conjectured that given irreducible cuspidal automorphic representations π_1, π_2, π_3 of $\mathrm{GL}_2(F)$, then the central value $L(\pi_1 \times \pi_2 \times \pi_3, 1/2)$ is nonzero iff there exists a nontrivial invariant functional on $\pi_1^D \otimes \pi_2^D \otimes \pi_3^D$ for some quaternion algebra D. This conjecture was proven by Harris and Kudla [HK91] and [HK04], by way of establishing a formula for the integral of a certain theta-series that's closely related to the global integration functional I_D on $\pi_1^D \otimes \pi_2^D \otimes \pi_3^D$.

Theorem 2.1.2 (Harris-Kudla)**.** *Let F be a number field. Let π_1, π_2, π_3 be irreducible cuspidal automorphic representations of $\mathrm{GL}_2(F)$ with the product of their central characters trivial. If D is a quaternion algebra F such that each π_i^D is defined, and S is* *a set of bad places then we have an equation of the form*

$$\int_{(\mathbb{A}_F^\times D^\times(F)\backslash D^\times(\mathbb{A}_F))^2} \theta(f,\varphi) = \zeta_{F,S}(2)^{-2} L_S(\pi_1 \times \pi_2 \times \pi_3, 1/2) \prod_{v \in S} Z_v(0, W_{f_v}^{\psi_v}, F_{\varphi_v}).$$

Then, we have $L(\pi_1 \otimes \pi_2 \otimes \pi_3, 1/2) \neq 0$ iff there exists such a quaternion algebra D and vectors $\varphi_i \in \pi_i^D$ such that

$$I_D(\varphi_1, \varphi_2, \varphi_3) = \int_{\mathbb{A}_F^\times D^\times(F)\backslash D^\times(\mathbb{A}_F)} \varphi_1(x)\varphi_2(x)\varphi_3(x)dx \neq 0.$$

Afterwards, a number of people refined the formula established by Harris-Kudla in various special cases, to allow them to make more precise conclusions about the integral $I_D(\varphi_1, \varphi_2, \varphi_3)$ and the L-value $L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)$. In particular, Watson [Wat02] considered the case of three Maass newforms $\varphi_1, \varphi_2, \varphi_3$, and gave a precise equality between $|\int \varphi_1\varphi_2\varphi_3|^2$ and $L(\pi_1 \times \pi_2 \times \pi_3, 1/2)$. Finally, Ichino [Ich08] gave a definitive general identity encompassing these previous results, in the following form. (For our purposes we take the local representations at infinite places to be Hilbert space representations of $\mathrm{GL}_2(\mathbb{R})$, rather than the corresponding $(\mathfrak{g}, K)$-modules, so that we can integrate over the entire group with them).

Theorem 2.1.3 (Ichino's Formula)**.** *Let π_1, π_2, π_3 be irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_2(F)$ (for a number field F) with the product of their central characters trivial, and let D be a quaternion algebra such that Jacquet-Langlands associates an automorphic representation π_i^D on D to each π_i. Then if we set $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$, we have an equality of $D^\times(\mathbb{A}_F)$-linear functionals $\Pi \otimes \widetilde{\Pi} \to \mathbb{C}$ of the form*

$$\frac{I(\varphi, \widetilde{\varphi})}{\langle \varphi, \widetilde{\varphi} \rangle} = \frac{C}{8} \frac{\zeta_F^*(2)^2 L^*(\pi_1 \times \pi_2 \times \pi_3, 1/2)}{L^*(\mathrm{ad}\,\pi_1, 1) L^*(\mathrm{ad}\,\pi_2, 1) L^*(\mathrm{ad}\,\pi_3, 1)} \prod_v \frac{I_v^*(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v}.$$

The notation used in this theorem is as follows. The left hand side involves global integrals on the quotient set $[D^\times(\mathbb{A}_F)] = \mathbb{A}_F^\times D^\times(F)\backslash D^\times(\mathbb{A}_F)$. In particular, $I = I^D$ is the global integration functional given on simple tensors $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ and $\widetilde{\varphi} = \widetilde{\varphi}_1 \otimes \widetilde{\varphi}_2 \otimes \widetilde{\varphi}_3$ by

$$I(\varphi, \widetilde{\varphi}) = \left(\int_{[D^\times(A_{\mathbb{F}})]} \varphi_1(g)\varphi_2(g)\varphi_3(g) dx_T \right) \left(\int_{[D^\times(A_{\mathbb{F}})]} \widetilde{\varphi}_1(g)\widetilde{\varphi}_2(g)\widetilde{\varphi}_3(g) dx_T \right)$$

(since each φ_i is an automorphic form on $D^\times(\mathbb{A}_F)$, automorphy plus our central character assumption implies the product is trivial on $\mathrm{GL}_2(F)$ and on the center $\mathbb{A}_F^\times$, so these integrals makes sense). The global pairing $\langle \varphi, \widetilde{\varphi} \rangle$ is given on simple tensors by

$$\langle \varphi, \widetilde{\varphi} \rangle = \prod_{i=1}^{3} \left(\int_{[D^\times(A_{\mathbb{F}})]} \varphi_i(g)\widetilde{\varphi}_i(g) dx_T \right).$$

All of the L-functions (and the ζ-value) are written as L^* and ζ^* to denote that these are taken to include their Γ-factors at infinity. The triple-product L-function is the one discussed earlier in the section; $L(\mathrm{ad}\,\pi, s)$ is the trace-zero adjoint L-function associated to π (originally constructed by Gelbart and Jacquet in [GJ78], and closely related to symmetric square L-functions).

The right-hand side involves a product of local functionals I_v^*, each of which is a $D^\times(F_v)$-invariant functional on $\Pi_v^D \otimes \widetilde{\Pi}_v^D$. To set this up we fix an invariant bilinear local pairing $\langle\ ,\ \rangle_{i,v}$ on $\pi_{v,i} \otimes \widetilde{\pi}_{v,i}$ (coming from the fact that these representations are contragredients) for each place v and each $i = 1, 2, 3$, and use this to define a pairing $\langle\ ,\ \rangle_v$ on $\Pi_v^D \otimes \widetilde{\Pi}_v^D$ determined on simple tensors $\varphi_v = \varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}$ and $\widetilde{\varphi}_v = \widetilde{\varphi}_{1,v} \otimes \widetilde{\varphi}_{2,v} \otimes \widetilde{\varphi}_{3,v}$ by

$$\langle \varphi_v, \widetilde{\varphi}_v \rangle_v = \langle \varphi_{1,v}, \widetilde{\varphi}_{1,v} \rangle_{1,v} \langle \varphi_{2,v}, \widetilde{\varphi}_{2,v} \rangle_{2,v} \langle \varphi_{3,v}, \widetilde{\varphi}_{3,v} \rangle_{3,v}.$$

We then define a functional I_v on $\Pi_v^D \otimes \widetilde{\Pi}_v^D$ by

$$I_v(\varphi_v, \widetilde{\varphi}_v) = \int_{F_v^\times \backslash \mathrm{GL}_2(F_v)} \langle \pi(g_v)\varphi_v, \widetilde{\varphi}_v \rangle_v dx_v$$

We then normalize this functional with (the reciprocal of) the local factors of the L-functions that show up in the global equation to get I_v^*:

$$I_v^*(\varphi_v, \widetilde{\varphi}_v) = \frac{L_v(\operatorname{ad}\pi_1, 1)L_v(\operatorname{ad}\pi_2, 1)L_v(\operatorname{ad}\pi_3, 1)}{\zeta_F(2)^2 L_v(\pi_1 \times \pi_2 \times \pi_3, 1/2)} I_v(\varphi_v, \widetilde{\varphi}_v).$$

Finally, we need to say a bit about the measures used in these formulas. We take the global measure dx_T to be the Tamagawa measure for $\mathrm{PGL}_2(\mathbb{A}_F)$. The local Haar measures dx_v on $\mathrm{PGL}_2(F_v)$ may be chosen arbitrarily, but we must assume that all but finitely many of the ones at nonarchimedean places are such that the volume of $\mathrm{PGL}_2(\mathcal{O}_v)$ is 1. The constant C in the formula is the scaling constant such that $dx_T = C \cdot \prod dx_v$.

With this setup, Ichino showed that

$$\frac{I_v^*(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v} = 1$$

whenever v is a place such that all of the π_v's are unramified, φ_v and $\widetilde{\varphi}_v$ are spherical vectors, and $\mathrm{PGL}_2(\mathcal{O}_v)$ has volume 1. Because of this, the product over all places in Ichino's formula is actually a finite product.

2.2 A classical version of Ichino's formula

In this section we want to take Ichino's general formula (Theorem 2.1.3) and adapt it to get a precise formula in a specific situation we're interested in. In particular, we will fix integers $m > k > 0$ and take classical newforms of weights k, $m-k$, and

m, respectively. We set notation that N_f and χ_f denote the level and character of f, and similarly for g and h; let $N_{fgh} = \operatorname{lcm}(N_f, N_g, N_h)$. We ultimately will want to use Ichino's formula to relate the triple product L-value a classical Petersson inner product pairing h with a product of f and g.

The naive idea is simply to take the Petersson inner product $\langle f(z)g(z), h(z)\rangle$. However, this may not quite work if the levels of the newforms don't match up - for instance if the LCM of N_f and N_g is a proper divisor of N_h then certainly $f(z)g(z)$ is an oldform when we consider it as having level N_h and thus pairs trivially with the newform h. In general, we can fix this issue by replacing the newforms with oldforms of higher level associated to them. We will consider a pairing $\langle f(M_f z)g(M_g z), h(M_h z)\rangle$ where M_f, M_g, M_h are integers such that $M_f N_f$, $M_g N_g$, and $M_h N_h$ all divide N_{fgh}, and moreover for every prime q, the largest power of q to divide any of $M_f N_f$, $M_g N_g$, and $M_h N_h$ actually divides (at least) two of those numbers. In practice, we simply accomplish this by considering each prime $q | N_{fgh}$, and if a larger power of q divides (say) of N_f than N_g or N_h then we append an appropriate power of q to one of M_g or M_h to compensate.

2.2.1 The automorphic setup

To apply Ichino's formula, we need to translate this classical setup to an automorphic one. We let π_f, π_g, π_h be the unitary automorphic representations associated to f, g, h. We fix tensor product decompositions

$$\pi_f \cong \bigotimes \pi_{f,v} \qquad \pi_\varphi \cong \bigotimes \pi_{g,v} \qquad \pi_h \cong \bigotimes \pi_{h,v}$$

and also the corresponding tensor product decompositions of the contragredients

$$\widetilde{\pi}_f \cong \bigotimes \widetilde{\pi}_{f,v} \qquad \widetilde{\pi}_\varphi \cong \bigotimes \widetilde{\pi}_{g,v} \qquad \widetilde{\pi}_h \cong \bigotimes \widetilde{\pi}_{h,v},$$

where for the automorphic representations the contragredient is simply the complex conjugate representation. The classical newforms f, g, h correspond to specific vectors in the automorphic representation, namely F, G, H given by

$$F(x) = \big((y^{k/2}f)|[x_\infty]_k\big)(i)\widetilde{\chi}_f(k_0),$$

$$G(x) = \big((y^{(m-k)/2}g)|[x_\infty]_{m-k}\big)(i)\widetilde{\chi}_g(k_0) \qquad H(x) = \big((y^{m/2}h)|[x_\infty]_m\big)(i)\widetilde{\chi}_h(k_0).$$

Here we decompose $x = \gamma x_\infty k_0$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $x_\infty \in \mathrm{GL}_2^+(\mathbb{R})$, and $k_0 \in K_0(N_f)$, and we let $\widetilde{\chi}_f$ be the character of $K_0(N_f)$ given by applying χ_f to the lower-right entry and likewise for $\widetilde{\chi}_g, \widetilde{\chi}_h$.

Of course, since this vector F corresponds to the newform $f(z)$, we need to suitably modify it to get something that will correspond to $f(M_f z)$ instead. To do this we set up some notation; if M is an integer we let

$$\delta_v(M) = \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}_v) \qquad \delta(M) = (\delta_v(M)) \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}).$$

Moreover, if v is a finite place and we've fixed a uniformizer ϖ_v of $\mathbb{Q}_v$ we let $\delta_v^0(M) = \delta_v(\varpi^{v(M)})$, and we let $\delta^0(M) \in \mathrm{GL}_2(\mathbb{A}^{\mathrm{fin}})$ have coordinates $\delta_v^0(M)$. Then, the adelic lift of $f_{M_f}(z) = f(M_f z)$ is given by

$$x \mapsto \big((y^{k/2}f_{M_f})|[x_\infty]_k\big)(i)\widetilde{\chi}_f(k_0) = M_f^{-k}\big((y^{k/2}f)|[\delta_\infty(M_f)x_\infty]_k\big)(i)\widetilde{\chi}_f(k_0)$$

for a decomposition $x = \gamma x_\infty k_0 \in \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2^+(\mathbb{R})K_0(M_f N_f)$.

So, we'd like to find an element y such that $(\pi(y)F)(x) = F(xy)$ gives us this function up to scalars. Thus if $x = \gamma x_\infty k_0$ with $k_0 \in K_0(M_f N_f)$ we want to pick y so that we can write $xy = \gamma' \cdot \delta_\infty(M_f)x_\infty \cdot k_0'$ with k_0' satisfying $\widetilde{\chi}_f(k_0') = \widetilde{\chi}_f(k_0)$; then we'll have

$$F(xy) = \big((y^{k/2}f)|[\delta_\infty(M_f)x_\infty]_k\big)(i)\widetilde{\chi}_f(k_0)$$

as desired. We can accomplish thus by taking $y = \delta^0(M_f^{-1}) \in \mathrm{GL}_2(\mathbb{A}^{\mathrm{fin}})$, where a direct computation shows that at each finite place we get

$$k_{0,v}\delta_v^0(M_f^{-1}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \delta_v^0(M_f^{-1}) = \delta_v(M_f^{-1}) \begin{bmatrix} \varpi^{-v(M_f)}M_f a & M_f b \\ \varpi^{-v(M_f)}c & d \end{bmatrix} = \delta_v(M_f^{-1})k'_{0,v}$$

so if we have $x = \gamma x_\infty k_0 \in \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2^+(\mathbb{R})K_0(M_f N_f)$ we decompose xy as

$$\big(\gamma\delta(M_f^{-1})\big) \cdot \big(\delta_\infty(M_f)x_\infty\big) \cdot k'_0 \in \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2^+(\mathbb{R})K_0(N_f).$$

Then, since k_0 and k'_0 have the same lower-right entries, $\widetilde{\chi}(k_0) = \widetilde{\chi}(k'_0)$, and we recover that

$$x \mapsto F(xy) = \big((y^{k/2}f)|[\delta_\infty(M_f)x_\infty]_k\big)(i)\widetilde{\chi}_f(k'_0)$$

is a scaling of the adelic lift of f_{M_f} that we wanted.

So, we take the input vector

$$\varphi = \delta^0(M_f)F \otimes \delta^0(M_g)G \otimes \delta^0(M_h)\overline{H} = F_{M_f} \otimes G_{M_g} \otimes \overline{H}_{M_h}$$

in $\pi_f \otimes \pi_g \otimes \pi_h$, and $\widetilde{\varphi}$ to be the vector of complex conjugates of these in the contragredients. The condition for the central characters being trivial is equivalent to asking $\chi_f\chi_g = \chi_h$ as an equality of Dirichlet characters; assuming this we have the identity

$$\frac{I(\varphi,\widetilde{\varphi})}{\langle\varphi,\widetilde{\varphi}\rangle} = \frac{C}{8}\frac{\zeta_F^*(2)^2 L^*(\pi_f \times \pi_g \times \widetilde{\pi}_h, 1/2)}{L^*(\mathrm{ad}\,\pi_f,1)L^*(\mathrm{ad}\,\pi_g,1)L^*(\mathrm{ad}\,\widetilde{\pi}_h,1)}\prod_v \frac{I_v^*(\varphi_v,\widetilde{\varphi}_v)}{\langle\varphi_v,\widetilde{\varphi}_v\rangle_v}.$$

which we need to unwind and write everything explicitly in classical terms. We will spend the rest of this section doing this for the various terms in the formula (except for the local integrals at finite places, which will be the focus of the next section).

2.2.2 Explicit choices of measures

To begin, we need to make our choices of measures completely explicit, and compute the constant C in the formula. First of all, we take Haar measures dx_v on the spaces $\mathrm{GL}_2(\mathbb{Q}_v)$, such that if $v = q$ is a finite prime then $\mathrm{GL}_2(\mathbb{Z}_q)$ has volume 1, and if $v = \infty$ then we take the usual Haar measure on $\mathrm{GL}_2(\mathbb{R})$ given in terms of Lebesgue measure on the coordinates of the matrices:

$$x_\infty = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \qquad\qquad dx_\infty = \frac{d\alpha\ d\beta\ d\gamma\ d\delta}{|\det(x_\infty)|^2}.$$

Each $\mathrm{GL}_2(\mathbb{Q}_v)$ has a center $Z_v \cong \mathbb{Q}_v^\times$ consisting of diagonal matrices; we put Haar measure dz_v on this induced from Haar measure on $\mathbb{Q}_v^\times$, again normalizing things so that if $v = p$ is a finite prime then $\mathbb{Z}_q^\times$ has measure 1, and if $v = \infty$ then we take the usual multiplicative Haar measure on $\mathbb{R}^\times$, $d^\times x = dx_{Lebesgue}/|x|$. We then let $d\overline{x}_v$ be the Haar measure on $\mathrm{PGL}_2(\mathbb{Q}_v) = \mathrm{GL}_2(\mathbb{Q}_v)/Z_v$ induced from our given measures; thus it's the unique one that satisfies

$$\int_{\mathrm{PGL}_2(\mathbb{Q}_v)} \left(\int_{Z_v} \varphi(x_v z_v) dz_v \right) d\overline{x}_v = \int_{\mathrm{GL}_2(\mathbb{Q}_v)} \varphi(g_v) dx_v$$

for any integrable function φ on $\mathrm{GL}_2(\mathbb{Q}_v)$. In particular we find that $\mathrm{PGL}_2(\mathbb{Z}_q)$ has measure 1 for each finite prime q.

We then define global measures $dx = \prod dx_v$ on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, $dz = \prod dz_v$ on $Z \cong \mathbb{A}_\mathbb{Q}^\times$, and $d\overline{x} = \prod d\overline{x}_v$ on $\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$, which thus satisfy the same compatibility identity as above:

$$\int_{\mathrm{PGL}_2(\mathbb{A})} \left(\int_{Z} \varphi(xz) dz \right) d\overline{x} = \int_{\mathrm{GL}_2(\mathbb{A})} \varphi(x) dx.$$

On the other hand, $\mathrm{PGL}_2(\mathbb{A})$ comes with a canonical Haar measure, the *Tamagawa measure* $d\overline{x}_T$, that appears in Ichino's formula. We recall that we define a scaling factor C by

$$d\overline{x}_T = C d\overline{x}.$$

This implies that

$$\mathrm{vol}(\mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q}), d\overline{x}_T) = C\,\mathrm{vol}(\mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q}), d\overline{x}),$$

so we can compute C as the ratio of these two volumes. For the Tamagawa measure, the volume is the *Tamagawa number* of PGL_2, which is well-known to be 2 (see e.g. Theorem 3.2.1 of [Wei82]). The volume of $d\overline{x}$ can be computed by working with a fundamental domain for $\mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$; we can explicitly construct one as $D_\infty \cdot \prod_q \mathrm{PGL}_2(\mathbb{Z}_q)$ where D_∞ is a fundamental domain of $\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})$. Thus the volume of $\mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$ under $d\overline{x}$ is the volume of $D_\infty \prod \mathrm{PGL}_2(\mathbb{Z}_q)$. Since $\mathrm{PGL}_2(\mathbb{Z}_q)$ has volume 1 by construction and D_∞ is a fundamental domain, we find

$$\mathrm{vol}(\mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q}), d\overline{x}) = \mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R}), d\overline{x}_\infty).$$

The volume of $\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})$ under $d\overline{x}_\infty$ can then be computed directly by integration, using a fundamental domain derived from the well-known one for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane $\mathbb{H}$, and we find the volume is $\pi^2/3$. Thus:

Lemma 2.2.1. *The constant C such that $d\overline{x}_T = C d\overline{x}$ is given by*

$$C = \frac{2}{\pi^2/3} = \frac{6}{\pi^2}.$$

2.2.3 Global integrals vs. Petersson inner products

Next, we want to rewrite the left-hand side of Ichino's formula in terms of Petersson inner products. This part of the equation consists of global integrals with respect to Tamagawa measure:

$$\frac{I(\varphi,\widetilde{\varphi})}{\langle\varphi,\widetilde{\varphi}\rangle}=\frac{(\int F_{M_f}G_{M_g}\overline{H}_{M_h}dx_T)\cdot(\int\overline{F}_{M_f}\overline{G}_{M_g}H_{M_h}dx_T)}{(\int F_{M_f}\overline{F}_{M_f}dx_T)(\int G_{M_g}\overline{G}_{M_g}dx_T)(\int\overline{H}_{M_h}H_{M_h}dx_T)}.$$

Thus, each of these integrals is of the form $\int\Psi\overline{\Psi}'dx_T$ where Ψ and Ψ' are functions induced from two modular forms ψ,ψ' on some space $S_\kappa(N,\chi)$ by the formulas

$$\Psi(g)=\big((y^{\kappa/2}\psi)|[g_\infty]_k\big)(i)\widetilde{\chi}(k_0)\qquad\text{and}\qquad\Psi'(g)=\big((y^{\kappa/2}\psi')|[g_\infty]_k\big)(i)\widetilde{\chi}(k_0),$$

where ψ,ψ' come from adelic lifts of f_{M_f}, g_{M_g}, and h_{M_h} as discussed earlier. Now, our global integrals are with respect to Haar measure (normalized as Tamagawa measure) on

$$\mathbb{A}_{\mathbb{Q}}^\times\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}).$$

As in the previous section, we can change from $d\overline{x}_T$ to our $d\overline{x}$ and then compute the integrals via a fundamental domain in $\mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}})$, and moreover such a domain can be taken to be of the form $D_\infty K_0(N)$ for D_∞ a fundamental domain of $\Gamma_0(N)\backslash\mathrm{PGL}_2^+(\mathbb{R})$. From this and our definition of Ψ and Ψ' we conclude

$$\int_{\mathbb{A}_{\mathbb{Q}}^\times\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})}\Psi\overline{\Psi}'d\overline{x}_T=C\,\mathrm{vol}(K_0(N))\int_{\Gamma_0(N)\backslash\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^\times}\psi(x)\overline{\psi}(x)y^\kappa dx_\infty.$$

Now, $\mathrm{vol}(K_0(N))$ is $[\overline{\Gamma}(1):\overline{\Gamma}_0(N)]$, and also if we pass from $\Gamma_0(N)\backslash\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^\times$ to $\Gamma_0(N)\backslash\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^\times\mathrm{SO}_2(\mathbb{R})$ we get $\mathbb{H}$ with the standard Poincaré measure. This involves taking a quotient of $\mathrm{SO}_2(\mathbb{R})\backslash\{\pm 1\}$, which has volume π, so:

$$\Psi\overline{\Psi}' d\overline{x}_T = \frac{C\pi}{[\overline{\Gamma}(1):\overline{\Gamma}_0(N)]}\int_{\Gamma_0(N)\backslash\mathbb{H}} \psi(z)\overline{\psi}'(z)y^{\kappa}d\mu.$$

But this latter integral is exactly the non-normalized Petersson inner product of ψ and ψ'; normalization involves scaling by the inverse of $\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H}) = \frac{\pi}{3}[\overline{\Gamma}(1):\overline{\Gamma}_0(N)]$. So we conclude:

Lemma 2.2.2. *Let ψ, ψ' be modular forms as above and Ψ, Ψ' the resulting functions on $\mathrm{PGL}_2(\mathbb{A})$. Then we have*

$$\int_{\mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A})} \Psi(g)\overline{\Psi}'(x)dx_T = C\int_{\mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A})} \Psi(x)\overline{\Psi}'(x)dx$$

$$= C\frac{\pi^2}{3}\langle\psi,\psi'\rangle = 2\langle\psi,\psi'\rangle$$

where $\langle\cdot,\cdot\rangle$ is the Petersson inner product with its canonical normalization.

Now, we note that F_{M_f} is M_f^k times the adelic lift of f_{M_f}, and similarly for G_{M_g} and H_{M_h}; so, the left-hand side of Ichino's formula becomes

$$\frac{(\int F_{M_f}G_{M_g}\overline{H}_{M_h}dx_T)\cdot(\int \overline{F}_{M_f}\overline{G}_{M_g}H_{M_h}dx_T)}{\int F_{M_f}\overline{F}_{M_f}dx_T\int G_{M_g}\overline{G}_{M_g}dx_T\int \overline{H}_{M_h}H_{M_h}dx_T} = \frac{2^2|\langle f_{M_f}g_{M_g},h_{M_h}\rangle|^2}{2^3\langle f_{M_f},f_{M_f}\rangle\langle g_{M_g},g_{M_g}\rangle\langle h_{M_h},h_{M_h}\rangle}.$$

Since we further can see $\langle f_{M_f},f_{M_f}\rangle = M_f^{-k}\langle f,f\rangle$ (and likewise for g and h) by a simple change-of-variables, this becomes

$$\frac{|\langle f_{M_f}g_{M_g},h_{M_h}\rangle|^2}{2M_f^{-k}M_g^{k-m}M_h^{-m}\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle}.$$

Moreover, we computed $C = 6/\pi^2$ in the previous section, and the value of $\zeta^*(2)$ is given by

$$\zeta^*(2) = \pi^{-2/2}\Gamma(2/2)\zeta(2) = \pi^{-1}\frac{\pi^2}{6} = \frac{\pi}{6}.$$

Thus Ichino's formula simplifies to

$$\frac{|\langle f_{M_f} g_{M_g}, h_{M_h}\rangle|^2}{\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle} = \frac{M_f^{-k} M_g^{k-m} M_h^{-m} \cdot L^*(\pi_f \times \pi_g \times \widetilde{\pi}_h, 1/2)}{2^3 \cdot 3 \cdot L^*(\mathrm{ad}\,\pi_f, 1) L^*(\mathrm{ad}\,\pi_g, 1) L^*(\mathrm{ad}\,\widetilde{\pi}_h, 1)} \prod_v \frac{I_v^*(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v\rangle_v}.$$

2.2.4 The adjoint L-values

The next thing we can do to make Ichino's formula more explicit is to eliminate the Petersson inner products in the denominator of the left-hand side and the adjoint L-values on the right-hand side. A formula of Shimura and Hida (see [Shi75], Section 5 of [Hid81], and Section 10 of [Hid86a]) relates $\langle \psi, \psi\rangle$ to a (slightly modified) adjoint L-value for a modular form ψ:

Theorem 2.2.3. *Let $\psi \in S_\kappa(N, \chi)$ be a newform, and let N_χ be the conductor of the Dirichlet character χ (which we take to be primitive). Then we have an equality*

$$L^H(\mathrm{ad}\,\psi, 1) = \frac{\pi^2}{6} \frac{(4\pi)^k}{(k-1)!} \langle \psi, \psi\rangle.$$

Here, $L^H(\mathrm{ad}\,\psi, 1)$ is defined by starting from a shift of the "naive" symmetric square L-function:

$$L_q^{\mathrm{naive}}(\mathrm{ad}\,\psi, s)^{-1} = \left(1 - \overline{\chi}(q)\frac{\alpha_q^2}{q^{k-1}} q^{-s}\right)\left(1 - \overline{\chi}(q)\frac{\alpha_q \beta_q}{q^{k-1}} q^{-s}\right)\left(1 - \overline{\chi}(q)\frac{\beta_q^2}{q^{k-1}} q^{-s}\right).$$

where $L_q(\psi, s)^{-1} = (1 - \alpha_q q^{-s})(1 - \beta_q q^{-s})$, and then setting

$$L^H(\mathrm{ad}\,\psi, 1) = \begin{cases} L_q^{\mathrm{naive}}(\mathrm{ad}\,\psi, 1) & q \nmid N \\ (1 - q^{-2})^{-1}(1 + q^{-1})^{-1} & q \| N, q \nmid N_\chi \\ (1 - q^{-2})^{-1} & q | N, q \nmid (N/N_\chi) \\ (1 + q^{-1})^{-1} & \textit{otherwise} \end{cases}.$$

We remark that $L^{\mathrm{naive}}(\mathrm{ad}\,\psi, 1)$, $L^H(\mathrm{ad}\,\psi, 1)$, and $L(\mathrm{ad}\,\psi, 1)$ only differ at finitely many Euler factors (for some subset of the divisors of N); also by construction $L_q^{\text{naive}}(\operatorname{ad}\psi,1)$ is 1 if either $q|N_\chi$ or if $\alpha_q=\beta_q=0$. We introduce the notation $L^H(\operatorname{ad}\psi,1)$ (where the "H" is for "Hida" since $L_p^H(\operatorname{ad}\psi,1)$ accounts for the factors in Hida's formula) so that we can avoid writing these factors explicitly until the next section when we work in a case-by-case basis anyway. In light of this we can rewrite our equation

$$L(\operatorname{ad}\psi,1)=\frac{\pi^2}{6}\frac{(4\pi)^k}{(k-1)!}\langle\psi,\psi\rangle\frac{L(\operatorname{ad}\psi,1)}{L^H(\operatorname{ad}\psi,1)},$$

knowing that the ratio of L-values on the right just involves finitely many Euler factors that we will deal with when we work out the local integrals in Ichino's formula.

To get from the above formula with $L(\operatorname{ad}\psi,1)$ to one with $L^*(\operatorname{ad}\pi_\psi,1)$, we just use the explicit expression for the archimedean adjoint L-factor:

$$L_\infty(\operatorname{ad}\pi_\psi,s)=2(2\pi)^{-(s+\kappa-1)}\Gamma(s+\kappa-1)\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)$$

and thus

$$L_\infty(\operatorname{ad}\pi_\psi,1)=2(2\pi)^{-\kappa}\Gamma(\kappa)\pi^{-1}\Gamma(1)=\frac{(\kappa-1)!}{2^{\kappa-1}\pi^{\kappa+1}}.$$

Combining this with the above theorem we get

Corollary 2.2.4. *Let $f\in S_\kappa(N,\chi)$ be a newform. Then we have*

$$L^*(\operatorname{ad}\pi_\psi,1)=\frac{2^\kappa\pi}{3}\langle f,f\rangle\frac{L(\operatorname{ad}\psi,1)}{L^H(\operatorname{ad}\psi,1)}$$

We also note that adjoint L-functions are self-dual, so $L^*(\operatorname{ad}\pi_\psi,1)=L^*(\operatorname{ad}\widetilde{\pi}_\psi,1)$. We can now substitute this into Ichino's formula. In particular we find

$$\frac{1}{L^*(\operatorname{ad}\pi_f,1)L^*(\operatorname{ad}\pi_g,1)L^*(\operatorname{ad}\widetilde{\pi}_h,1)}=\frac{1}{\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle}\frac{3^3}{\pi^3 2^k 2^{m-k}2^m}\prod_q\mathcal{E}_q(f,g,h)$$

where

$$\mathcal{E}_q = \frac{L_q^H(\operatorname{ad} f,1)}{L_q(\operatorname{ad} f,1)}\frac{L_q^H(\operatorname{ad} g,1)}{L_q(\operatorname{ad} g,1)}\frac{L_q^H(\operatorname{ad} h,1)}{L_q(\operatorname{ad} h,1)}.$$

Substituting this in and cancelling the product $\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle$ from each side we get

$$|\langle f_{M_f} g_{M_g}, h_{M_h}\rangle|^2 = \frac{3^2}{\pi^3 2^{2m+3}} \frac{L^*(\pi_f\times\pi_g\times\widetilde{\pi}_h,1/2)}{M_f^k M_g^{m-k} M_h^m} \prod_v \frac{I_v^*(\varphi_v,\widetilde{\varphi}_v)}{\langle\varphi_v,\widetilde{\varphi}_v\rangle_v}\mathcal{E}_v,$$

where we take $\mathcal{E}_\infty = 1$.

2.2.5 The triple product L-function

The next thing we can look at is the triple product L-value $L^*(\pi_f\times\pi_g\times\widetilde{\pi}_h,1/2)$ itself. We can start by computing the archimedean factor. Because f,g,h are holomorphic modular forms, we know that the local components of the automorphic representations at ∞ are discrete series of the following form:

$$\pi_{f,\infty}\cong\sigma(k-1,0) \qquad \pi_{g,\infty}\cong\sigma(m-k-1,0) \qquad \pi_{h,\infty}\cong\sigma(m-1,0),$$

and all of these are self-dual. Then the L-factor at ∞ is given by

$$L_\infty(s,\pi_f\otimes\pi_g\otimes\widetilde{\pi}_h) = \Gamma_{\mathbb{C}}(s+m-3/2)\Gamma_{\mathbb{C}}(s+k-1/2)\Gamma_{\mathbb{C}}(s+m-k-1/2)\Gamma_{\mathbb{C}}(s+1/2)$$

and thus

$$L_\infty(1/2,\pi_f\otimes\pi_g\otimes\widetilde{\pi}_h) = \Gamma_{\mathbb{C}}(m-1)\Gamma_{\mathbb{C}}(k)\Gamma_{\mathbb{C}}(k-m)\Gamma_{\mathbb{C}}(1).$$

Recalling $\Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s)$ we get

$$L_\infty(1/2,\pi_f\otimes\pi_g\otimes\widetilde{\pi}_h) = 2^4(2\pi)^{-(m-1)-k-m-1}\Gamma(m-1)\Gamma(k)\Gamma(m-k)\Gamma(1)$$

$$= 2^4(2\pi)^{-2m}(m-2)!(k-1)!(m-k-1)!.$$

Once we have this, we're left with the non-completed L-value $L(\pi_f \times \pi_g \times \widetilde{\pi}_h, 1/2)$. Writing this classically this becomes $L(f \times g \times \overline{h}, m-1)$, since this L-function has its center of symmetry given by

$$\frac{k + (m-k) + m - 2}{2} = m - 1.$$

Putting this back into Ichino's formula gives us:

$$|\langle f_{M_f} g_{M_g}, h_{M_h}\rangle|^2 = \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+3}2^{4m-1}M_f^k M_g^{m-k} M_h^m} L(f \times g \times \overline{h}, m-1) \prod_v \frac{I_v^*(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v\rangle_v} \mathcal{E}_v.$$

2.2.6 The local integrals

Finally, we have to deal with the local integrals at all places v. To do this we first need to know what our input vectors correspond to locally. Unsurprisingly, since we took $F, G, \overline{H}$ to correspond to classical newforms, a result of Casselman [Cas73] tells us that under the isomorphisms $\pi_f \cong \bigotimes \pi_{f,v}$, we have $F \mapsto \bigotimes F_v$ where each F_v is a *newvector* in $\pi_{f,v}$. This means F_v is a nonzero vector in the one-dimensional space invariant under the group

$$K_2(\mathfrak{a}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \in \mathfrak{a}, d \in 1 + \mathfrak{a}, a \in \mathbb{Z}_q^\times, b \in \mathbb{Z}_q \right\}$$

where $\mathfrak{a} = \mathfrak{p}^{c(\pi_{f,v})}$ is the conductor of $\pi_{f,v}$ (so in particular if $\pi_{f,v}$ is unramified then F_v is a spherical vector). We note that we look at newvectors invariant under K_2 rather than K_1 (as in [Cas73]) in accordance with our convention that we extend χ_f to a character $\widetilde{\chi}_f$ of K_0 by applying χ_f to the lower-right entry rather than the upper-left.

So, our input vector $F_{M_f} = \delta^0(M_f^{-1})F$ decomposes under our tensor product as

$\bigotimes \delta_v^0(M_f^{-1})F_v = \bigotimes \delta_v(\varpi_v^{-v(M_f)})F_v$, with each component being either the newvector F_v or a translation of it by one of our diagonal matrices. We let F_{v,M_f} denote the translation $\delta_v(\varpi_v^{-v(M_f)})F_v$ for ease of notation. We also have similar local decompositions for $G, H, \overline{F}, \overline{G}, \overline{H}$.

Then, the unnormalized and normalized local integrals we're considering are explicitly given by

$$\frac{I_v(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v} = \int_{\mathrm{PGL}_2(\mathbb{Q}_v)} \frac{\langle xF_{v,M_f}, \widetilde{F}_{v,M_f} \rangle}{\langle F_{v,M_f}, \widetilde{F}_{v,M_f} \rangle} \frac{\langle xG_{v,M_g}, \widetilde{G}_{v,M_g} \rangle}{\langle G_{v,M_g}, \widetilde{G}_{v,M_g} \rangle} \frac{\langle x\widetilde{H}_{v,M_h}, H_{v,M_h} \rangle}{\langle \widetilde{H}_{v,M_h}, H_{v,M_h} \rangle} dx$$

and

$$\frac{I_v^*(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v} = \frac{L_v(\mathrm{ad}\,\pi_{f,v}, 1) L_v(\mathrm{ad}\,\pi_{g,v}, 1) L_v(\mathrm{ad}\,\widetilde{\pi}_{h,v}, 1)}{\zeta_v(2)^2 L_v(\pi_{f,v} \times \pi_{g,v} \times \widetilde{\pi}_{h,v}, 1/2)} \frac{I_v(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v}.$$

Since this is evidently scaling-invariant in any of the arguments, the fact that newvectors lie in a one-dimensional means that this integral is determined entirely by the representations $\pi_{f,v}, \pi_{g,v}, \widetilde{\pi}_{h,v}$ and the prime factorizations of the numbers M_f, M_g, M_h. So we take the notation

$$I_v^{**} = \frac{I_v^*(\varphi_v, \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v} \mathcal{E}_v$$

to denote our local factor. We use the notation I_v^{**} rather than I_v^* to emphasize that we've inserted a term $\mathcal{E}_v$ that changes some Euler factors from Ichino's automorphic formulation. At a finite place q we have

$$I_q^{**} = \frac{L_q^H(\mathrm{ad}\,f, 1) L_q^H(\mathrm{ad}\,g, 1) L_q^H(\mathrm{ad}\,h, 1)}{\zeta_q(2)^2 L_q(\pi_{f,q} \times \pi_{g,q} \times \widetilde{\pi}_{h,q}, 1/2)} \cdot \frac{I_q(\varphi_q, \widetilde{\varphi}_q)}{\langle \varphi_q, \widetilde{\varphi}_q \rangle_q}.$$

To make Ichino's formula completely explicit, then, we need to compute all of these local integrals. If $v = q$ is a prime not dividing the levels of f, g, or h, then we've taken M_f, M_g, M_h to be coprime to q, we know the local integral is 1 on spherical vectors, and we know that there are no bad Euler factors in $\mathcal{E}_q^{-1}$, so $I_q^{**} = 1$. For the archimedean place $v = \infty$, $\mathcal{E}_\infty = 1$ by construction, and the local integral for the appropriate discrete series representations is computed by Ichino-Ikeda ([II10] Proposition 7.2) and Woodbury ([Woo12] Proposition 4.6) to be $I_\infty^* = 8\pi$, accounting for our normalization for Haar measure being off from theirs by a factor of 4π.

Putting all of this together we get:

Theorem 2.2.5 (Ichino's formula, classical version)**.** *Fix integers $m > k > 0$, and let $f \in S_k(N_f, \chi_f)$, $g \in S_{m-k}(N_g, \chi_g)$, and $h \in S_m(N_h, \chi_h)$ be classical newforms such that the characters satisfy $\chi_f \chi_g = \chi_h$. Take $N_{fgh} = \mathrm{lcm}(N_f, N_g, N_h)$ and choose positive integers M_f, M_g, M_h such that the three numbers $M_f N_f$, $M_g N_g$, $M_h N_h$ divide N_{fgh} and moreover none of the three is divisible by a larger power of any prime q than both of the others. Then we have*

$$|\langle f_{M_f} g_{M_g}, h_{M_h} \rangle|^2 = \frac{3^2 (m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2} 2^{4m-4} M_f^k M_g^{m-k} M_h^m} L(f \times g \times \overline{h}, m-1) \prod_{q | N_{fgh}} I_q^{**},$$

where the I_q^{**}'s are our modified local integrals defined above.

2.3 Some computations of local integrals

To make our classical version of Ichino's formula completely explicit, we need to compute the nonarchimedean local integrals I_q^{**} that will come up for the choices of newforms f, g, h that we want to consider. This computation will break up into many cases based on the local representations associated to our three newforms at each of the primes. Our goal in this section is discuss how to compute these local integrals explicitly, and work out (or recall from the literature) the cases we will need.

For ease of notation, it will be convenient to consider an abstract local setup throughout (until Section 2.3.5, where we take our abstract results and translate them back to our classical setup from the previous section above). So for what follows, we fix a prime q, and fix a uniformizer ϖ_q of $\mathbb{Q}_q^\times$. For concreteness we take the notation that if μ_1, μ_2 are characters of $\mathbb{Q}_q^\times$ then $\pi(\mu_1, \mu_2)$ denotes the representation consisting of smooth functions $f : G \to \mathbb{C}$ satisfying

$$f\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} g\right) = |a/d|^{1/2}\mu_1(a)\mu_2(d)f(g),$$

and in particular if s_1, s_2 are complex numbers we let $\pi(s_1, s_2)$ denote the unramified representation $\pi(|\cdot|^{s_1}, |\cdot|^{s_2})$ (noting that the complex numbers s_1, s_2 are only determined up to $2\pi i/\log q$). We note that the central character of $\pi(\mu_1, \mu_2)$ is $\mu_1\mu_2$, and thus the central character of $\pi(s_1, s_2)$ is $|\cdot|^{s_1+s_2}$.

With this notation, the only reducible principal series are those of the form $\pi(\chi|\cdot|^{\pm 1/2}, \chi|\cdot|^{\mp 1/2})$; we let $\sigma(\chi|\cdot|^{\pm 1/2}, \chi|\cdot|^{\mp 1/2})$ denote the special representation arising from such a principal series. Our representations are normalized so that the contragredient of $\pi(\mu_1, \mu_2)$ is $\pi(\mu_1^{-1}, \mu_2^{-1})$ (so in particular the contragredient of $\pi(s_1, s_2)$ is $\pi(-s_1, -s_2)$) and the contragredient of $\sigma(\chi|\cdot|^{\pm 1/2}, \chi|\cdot|^{\mp 1/2})$ is $\sigma(\chi^{-1}|\cdot|^{\mp 1/2}, \chi^{-1}|\cdot|^{\pm 1/2})$.

Now, we take three smooth representations π_1, π_2, π_3 of $G = \mathrm{GL}_2(\mathbb{Q}_q)$ that can appear as local factors of automorphic representations associated to classical modular forms (or their contragredients). We let ω_i be the central character of π_i, and always assume that the product $\omega_1\omega_2\omega_3$ is trivial. We also let dg denote Haar measure on G, normalized so that $\mathrm{GL}_2(\mathbb{Z}_q)$ has volume 1. We take $\mathfrak{p}^{c_i}$ to be the conductor of π_i, and furthermore we assume WLOG that $c_3 \geq c_2, c_1$.

In this abstract setup, we let $x_i \in \pi_i$ be a newvector for each i, and we consider the local integral

$$I(\pi_1, \pi_2, \pi_3) = \int_{Z\backslash G} \frac{\langle gx_1, x_1\rangle}{\langle x_1, x_1\rangle}\frac{\langle gy_2, y_2\rangle}{\langle y_2, y_2\rangle}\frac{\langle gx_3, x_3\rangle}{\langle x_3, x_3\rangle} dg,$$

where $Z \cong \mathbb{Q}_q^\times$ is the center, and y_2 is the translate $\delta_v(\varpi^{c_3-c_2})x_2$ of our newvector.

(We note that this is evidently independent of the choice of newvectors x_i). We then normalize this by setting

$$I^{**}(\pi_1, \pi_2, \pi_3) = \frac{L^H(\operatorname{ad}\pi_1, 1)L^H(\operatorname{ad}\pi_2, 1)L^H(\operatorname{ad}\pi_3, 1)}{L(\pi_1 \times \pi_2 \times \pi_3, 1/2)\zeta_q(2)^2} I(\pi_1, \pi_2, \pi_3).$$

Here, $L^H(\operatorname{ad}\pi, s)$ is a slight modification of the standard adjoint L-function, designed to match up with the local factors $L_p^H(\operatorname{ad}\psi, 1)$ defined in Theorem 2.2.3: for $\pi = \pi(s_1, s_2)$ an unramified principal series we take

$$L^H(\operatorname{ad}\pi, 1) = L(\operatorname{ad}\pi, 1) = (1 - q^{s_1 - s_2 - 1})^{-1}(1 - q^{-1})^{-1}(1 - q^{s_2 - s_1 - 1})^{-1}.$$

For ramified representations this breaks up into cases (where χ is taken to denote a ramified character below):

$$L^H(\operatorname{ad}\pi, 1) = \begin{cases} (1 - q^{-2})^{-1} & \pi = \pi(|\cdot|^s, \chi) \\ (1 - q^{-2})^{-1}(1 + q^{-1})^{-1} & \pi = \sigma(|\cdot|^{s \pm 1/2}, |\cdot|^{s \pm 1/2}) \\ (1 + q^{-1})^{-1} & \text{otherwise} \end{cases}$$

For convenience in stating results, we let $I^*(\pi_1, \pi_2, \pi_3)$ denote the usual normalized local integrals for Ichino's formulas, i.e. the ones defined as above but with $L(\operatorname{ad}\pi_i, 1)$ in place of $L^H(\operatorname{ad}\pi_i, 1)$.

At the beginning of Section 2.3.5 we'll connect the computations of $I^{**}(\pi_1, \pi_2, \pi_3)$ back to the integrals I_q^{**} in our version of Ichino's formula, checking that the two local integrals match up despite some slight differences in setup, and also that the cases we will consider suffice for the modular forms f, g, h we want to choose.

2.3.1 Some initial results

We now consider how to compute the integral $I(\pi_1, \pi_2, \pi_3)$. The most direct approach is to try to work directly with the three matrix coefficients that appear in the integral. In general this can get very messy very quickly, but in certain special cases it's possible. To start out, the following lemma is convenient for letting us simplify things a bit (for instance, if we're working with an unramified principal series it always lets us assume it's one of the form $\pi(\chi, \chi^{-1})$):

Lemma 2.3.1. *Suppose π_1, π_2, π_3 are any three representations, and χ_1, χ_2, χ_3 are unramified characters satisfying $\chi_1\chi_2\chi_3 = 1$. Let $\chi_i\pi_i = \chi_i \otimes \pi_i$ be the associated twists. Then we have $I(\chi_1\pi_1, \chi_2\pi_2, \chi_3\pi_3) = I(\pi_1, \pi_2, \pi_3)$ and similarly for I^* and I^*∗.*

In particular, we can take χ_i to satisfy $\chi_i(\varpi) = \omega_i(\varpi)^{-1/2}$; replacing π_i with $\chi_i\pi_i$ lets us always assume WLOG that the central characters ω_i satisfy $\omega_i(\varpi) = 1$.

Proof. If π_i is realized as acting on V_i, then $\chi_i\pi_i$ is realized on V_i by the action $(\chi_i\pi_i)(g)v = \chi_i(\det(g))\pi(g)v$, and moreover since this is an unramified twist, if v is a newvector for π_i then it's also a newvector for $\chi_i\pi_i$. Moreover, the π_i-invariant Hermitian pairing $\langle\ ,\ \rangle$ on V_i is also $\chi_i\pi_i$-invariant; thus if we take x_i to be a newvector or a translate of one as appropriate, $I(\chi_1\pi_1, \chi_2\pi_2, \chi_3\pi_3)$ is the integral of

$$\prod_{i=1}^{3} \frac{\langle (\chi_i\pi_i)(g)x_i, x_i\rangle}{\langle x_i, x_i\rangle} = \prod_{i=1}^{3} \chi_i(\det(g))\frac{\langle (\pi_i)(g)x_i, x_i\rangle}{\langle x_i, x_i\rangle} = \prod_{i=1}^{3} \frac{\langle (\pi_i)(g)x_i, x_i\rangle}{\langle x_i, x_i\rangle}$$

(with the last equality by the assumption $\chi_1\chi_2\chi_3 = 1$) and thus equals $I(\pi_1, \pi_2, \pi_3)$. To extend the result to I^* and I^{**}, we note that the adjoint and triple-product L-functions involved in the definitions are invariant under unramified twists. □

The simplest case is when π_1, π_2, π_3 have conductors $c_i \leq 1$ (i.e. all of the original modular forms have squarefree level at q). This is worked out explicitly by Woodbury [Woo12] in the case of trivial central characters (and thus for arbitrary unramified central characters via the above lemma), and is implicit in the computations of Watson [Wat02]. We have:

Proposition 2.3.2. *Suppose π_1, π_2 are unramified principal series and π_3 is the special representation associated to an unramified character. Then $I^*(\pi_1, \pi_2, \pi_3) = q^{-1}(1+q^{-1})^{-1}$ and thus*

$$I^{**}(\pi_1, \pi_2, \pi_3) = \frac{1}{q}(1+q^{-1})^{-2}.$$

Proposition 2.3.3. *Suppose π_1 is an unramified principal series and π_2, π_3 are special representations associated to unramified characters. Then $I^*(\pi_1, \pi_2, \pi_3) = q^{-1}$ and thus*

$$I^{**}(\pi_1, \pi_2, \pi_3) = \frac{1}{q}(1+q^{-1})^{-2}.$$

Proposition 2.3.4. *Suppose π_1, π_2, π_3 are special representations associated to unramified characters. Then we have $I^*(\pi_1, \pi_2, \pi_3) = (1-\varepsilon)q^{-1}(1+q^{-1})$ for $\varepsilon = \varepsilon(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)$, and thus*

$$I^{**}(\pi_1, \pi_2, \pi_3) = (1-\varepsilon)q^{-1}(1+q^{-1})^{-2}.$$

Another case where $I(\pi_1, \pi_2, \pi_3)$ can be computed in a fairly uniform way is when π_3 has a much larger conductor than π_1 or π_2. This is carried out by Hu [Hu14], giving the following result:

Proposition 2.3.5. *Suppose the representations π_1, π_2, π_3 are such that the conductors satisfy $c_3 \geq 2\max\{c_1, c_2, 1\}$. Then we have*

$$I(\pi_1, \pi_2, \pi_3) = \frac{A_1 A_2}{(1+q^{-1})q^{c_3}}$$

where A_i is determined by π_i in the following way:

- *If* $\pi_i = \pi(\chi_1, \chi_2)$ *is an unramified principal series then*

$$A_i = \frac{(1 - (\chi_1/\chi_2)(\varpi)q^{-1})(1 - (\chi_2/\chi_1)(\varpi)q^{-1})}{(1 + q^{-1})}.$$

- *If* π_i *is a special representation associated to an unramified character, then* $A_i = (1 + q^{-1})$.

- *If* $\pi_i = \pi(\chi_1, \chi_2)$ *is a principal series with one of* χ_1 *and* χ_2 *ramified and the other unramified, then* $A_i = 1$.

- *In other cases (i.e. if* π_i *is a principal series associated to two ramified characters, a special representation associated to a ramified character, or a supercuspidal representation) then* $A_i = (1 - q^{-1})^{-1}$.

We will need to apply the above calculations of Woodbury and Hu in cases when we have two unramified representations and a third ramified one, all with unramified central character (and by our first lemma, we can thus assume WLOG that they all have trivial central character). In these cases we find we get a uniform result:

Corollary 2.3.6. *Suppose that we have* π_1, π_2, π_3 *with* π_1, π_2 *unramified and* π_3 *ramified (and as always the product of their central characters trivial, so* π_3 *has unramified central character). Then we have*

$$I^{**}(\pi_1, \pi_2, \pi_3) = q^{-c_3}(1 + q^{-1})^{-2}.$$

The remaining cases we'll have to deal with are when one of the representations is unramified, while the other two are the same ramified representation. Some such computations are carried out by Nelson, Pitale, and Saha [NPS14]; we will quote some of their results and finally make one additional computation for the last situation we need. To make this computation, we'll outline how they reduce to evaluating Rankin-

Selberg integrals, and then discuss a few approaches one could take to making this latter computation.

So, for the rest of this section, we will focus on the situation where at least one of our three local representations is an unramified principal series, and that the other two have the same conductor. By Lemma 2.3.1 we can assume that $\pi_1 = \pi(s, -s)$ is an unramified principal series with trivial central character, and π_2, π_3 are ramified representations of the same conductor (and in fact in all of the applications we need, we'll actually have $\pi_2 \cong \pi_3$). The assumption on the conductors means that in our local integral, all three of the input vectors will be newvectors.

2.3.2 Reduction to Rankin-Selberg integrals

The Rankin-Selberg integral we will want to work with is described in terms of Whittaker models for the representations π_2, π_3 and the usual induced model for the unramified representation π_1. For the Whittaker models, we recall that if $\psi : (\mathbb{Q}_q, +) \to \mathbb{C}$ is an additive character (which we will always take to have conductor $\mathbb{Z}_q$), then we let $\mathcal{W}(G, \psi)$ denote the space of Whittaker functions for ψ, i.e. smooth functions $W : G \to \mathbb{C}$ satisfying

$$W\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) = \psi(x)W(g)$$

for all $g \in G$ and $x \in \mathbb{Q}_q$. This is made into a representation of G via right translation, and then the local representation theory of GL_2 says that any irreducible infinite-dimensional admissible representation π of G is generic, i.e. has a unique *Whittaker model* $W(\pi, \psi) \subseteq \mathcal{W}(G, \psi)$ that's a subrepresentation isomorphic to π itself. If π is unitary, we can explicitly realize the pairing on its Whittaker model $W(\pi, \psi)$ by

$$\langle W_1, W_2 \rangle = \int_{\mathbb{Q}_q^\times} W_1\left(\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\right) \overline{W_2\left(\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\right)} d^\times y.$$

Since we know $\pi \cong W(\pi, \psi)$ has a one-dimensional subspace of newvectors, we can pick out a *normalized Whittaker newvector* W^ψ to be the unique one in this space satisfying $\langle W^\psi, W^\psi \rangle = 1$ and such that $W^\psi(1)$ is a positive real number.

So, suppose π_1, π_2, π_3 are representations that can arise as local constituents of holomorphic newforms, with $\pi_1 = \pi(s, -s)$ an unramified principal series realized in its induced model (with s purely imaginary) and π_2, π_3 realized by their Whittaker models $W(\pi_2, \psi)$ and $W(\pi_3, \overline{\psi})$. We let $W_2^\psi, W_3^{\overline{\psi}}$ be normalized Whittaker newvectors in $W(\pi_2, \psi)$ and $W(\pi_3, \overline{\psi})$, respectively. We also let f_s° be the spherical vector for $\pi(s, -s)$ defined via the Iwasawa decomposition by

$$f_s^\circ\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} k\right) = \left|\frac{a}{d}\right|^{s+1/2}$$

for $k \in K$; we note that this is normalized both in the sense that $f_s^\circ(1) = 1$ and that $\langle f_s^\circ, f_s^\circ \rangle = 1$ under the usual inner product. Then, we define a local Rankin-Selberg integral

$$J(\pi_2, \pi_3; s) = \int_{NZ\backslash G} f_s^\circ(g) W_2^\psi(g) W_3^{\overline{\psi}}(g)\, dg$$

where $N \cong (\mathbb{Q}_q, +)$ is the unipotent radical of the standard upper-triangular Borel in G, and we define a normalized version

$$J^*(\pi_2, \pi_3; s) = \frac{\zeta_q(1+2s)}{L(\pi_2 \times \pi_3, 1/2 + s)} J(\pi_2, \pi_3; s).$$

Thus our integral is the same as the one in [MV10], and can be checked to be invariant of the choice of character ψ. We also remark that, compared to [NPS14], our conventions are such that their "s" equals our "$s + 1/2$", and they use $\overline{W_2^\psi}$ in place of $W_2^{\overline{\psi}}$ (but those are equal under their assumption that the representation has trivial central character).

Now, our ultimate goal is to evaluate the integrals

$$I^{**}(\pi_2, \pi_3; s) = I^{**}\big(\pi(s, -s), \pi_2, \pi_3\big).$$

With the definition above, can state the following result, which is a key lemma from [MV10].

Proposition 2.3.7 (Michel-Venkatesh, [MV10] Lemma 3.4.2)**.** *If π_1, π_2, π_3 are tempered smooth representations of G (e.g. local constituents of automorphic representations of holomorphic newforms), with $\pi_1 \cong \pi(s, -s)$ spherical and such that the central characters satisfy $\omega_1\omega_2\omega_3 = 1$, then we have*

$$I(\pi_1, \pi_2, \pi_3) = I(\pi_2, \pi_3; s) = (1 - q^{-1})^{-1} |J(\pi_2, \pi_3; s)|^2.$$

We can also compute $\overline{J(\pi_2, \pi_3; s)} = J(\widetilde{\pi}_2, \widetilde{\pi}_3; -s)$, and thus write out the result of this proposition as

$$I(\pi_2, \pi_3; s) = (1 - q^{-1})^{-1} J(\pi_2, \pi_3; s) J(\widetilde{\pi}_2, \widetilde{\pi}_3; -s).$$

Some simple manipulations with local L-factors then let us conclude:

Corollary 2.3.8. *If π_1, π_2, π_3 are tempered smooth representations of G with $\pi_1 \cong \pi(s, -s)$ spherical and π_2, π_3 having central characters inverse to each other, then*

$$I^{**}(\pi_2, \pi_3; s) = (1 + q^{-1})^2 L^H(\operatorname{ad} \pi_2, 1) L^H(\operatorname{ad} \pi_3, 1) J^*(\pi_2, \pi_3; s) J^*(\widetilde{\pi}_2, \widetilde{\pi}_3; -s)$$

Proof. If we start with our equality above and multiply by the factors in the definition of $I^{**}(\pi_2, \pi_3; s)$ we find that this quantity is equal to

$$\frac{L^H(\operatorname{ad} \pi_1, 1) L^H(\operatorname{ad} \pi_2, 1) L^H(\operatorname{ad} \pi_3, 1)}{L(\pi_1 \times \pi_2 \times \pi_3, 1/2) \zeta_q(2)^2} (1 - q^{-1})^{-1} J(\pi_2, \pi_3; s) J(\widetilde{\pi}_2, \widetilde{\pi}_3; -s).$$

Now, since $\pi_1 = \pi(s, -s)$ we can show that we have a formal factorization of L-functions

$$L(\pi_1 \times \pi_2 \times \pi_3, 1/2) = L(\pi_2 \times \pi_3, 1/2 + s)L(\widetilde{\pi}_2 \times \widetilde{\pi}_3, 1/2 - s),$$

and using this and the definitions of J^* from J we find

$$\frac{I^*(\pi_2, \pi_3; s)}{J^*(\pi_2, \pi_3; s)J^*(\widetilde{\pi}_2, \widetilde{\pi}_3; -s)} = \frac{L^H(\operatorname{ad}\pi_1, 1)L^H(\operatorname{ad}\pi_2, 1)L^H(\operatorname{ad}\pi_3, 1)}{(1 - q^{-1})\zeta_q(2s + 1)\zeta_q(-2s + 1)\zeta_q(2)^2}.$$

So we just need to simplify the fraction on the right. The last two adjoint L-factors will be left alone, since we can't say much about them without knowing more about π_2 or π_3. The remaining adjoint factor is given by

$$L^H(\operatorname{ad}\pi(s, -s), 1)^{-1} = L(\operatorname{ad}\pi(s, -s), 1)^{-1} = (1 - q^{-2s-1})(1 - q^{-1})(1 - q^{2s-1}).$$

So we get

$$L^H(\operatorname{ad}\pi_1, 1)L^H(\operatorname{ad}\pi_2, 1)\frac{(1 - q^{-2s-1})(1 - q^{2s-1})(1 - q^{-2})^2}{(1 - q^{-1})(1 - q^{-2s-1})(1 - q^{-1})(1 - q^{2s-1})}.$$

Cancelling terms, we're left with the two adjoint L-factors in the front and a factor of $(1 + q^{-1})^2$, as desired. □

So, we're reduced to computing the local integral $J(\pi_1, \pi_2; s)$ on a case-by-case basis for various choices of π_1 and π_2. Before proceeding, we recall the notation of [NPS14] and [Sah15] which we will use in the next few sections. We take the notation for matrices

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad a(y) = \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} \qquad z(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \qquad n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

for $y, t \in \mathbb{Q}_q^\times$ and $x \in \mathbb{Q}_q$, and accordingly we set $A = \{a(y) : y \in \mathbb{Q}_q^\times\}$, $Z = \{z(t) : t \in \mathbb{Q}_q^\times\}$ and $N = \{n(x) : x \in \mathbb{Q}_q\}$ and note that this agrees with our previous definitions of Z and N. With this notation, the usual upper-triangular Borel subgroup is $B = ZNA$. The normalized Haar measures on $\mathbb{Q}_q$ and $\mathbb{Q}_q^\times$ (giving $\mathbb{Z}_q$ and $\mathbb{Z}_q^\times$ volumes 1, respectively) pass to Haar measures on Z, N, and A.

To work out the integral defining $J(\pi_2, \pi_3; s)$ we need to choose a decomposition of G that will work well for our purposes. One such decomposition is the following, which is a generalization of the Iwasawa decomposition to Iwahori subgroups: we have

$$K = \coprod_{i=0}^{n} (B \cap K)\gamma_i K_2(\varpi^n) \qquad \gamma_i = \begin{bmatrix} 1 & 0 \\ \varpi^i & 1 \end{bmatrix}$$

and thus $G = \coprod_{i=0}^{n} B\gamma_i K$. Note in particular in the extreme cases of $i = 0$ and $i = n$ we have $B\gamma_0 K_2(\varpi^n) = BwK_2(\mathfrak{p}^n)$ and $B\gamma_n K_2(\varpi^n) = BK_2(\varpi^n)$. This decomposition is discussed in Section 2.1 of [Sch02] and in Appendix A of [Hu13]. For a function g invariant by $K_2(\varpi^n)$ on the right, it leads to us being able to write an integral over G as

$$\int_G f(g)dg = \sum_{0 \le i \le n} v_i \int_B f(b\gamma_i)db \qquad v_i = \begin{cases} (1+q^{-1})^{-1} & i = 0 \\ (1-q^{-1})(1+q^{-1})^{-1}q^{-i} & 0 < i < n \\ (1+q^{-1})^{-1}q^{-n} & i = n \end{cases},$$

where db is the usual Haar measure on the Borel $B = ZNA$, given by $|y|^{-1}d^\times z d^\times a dn$ on this decomposition. We have a similar expression for integrals over $Z\backslash G$ or $ZN\backslash G$. Thus, if π_2 and π_3 have conductor ϖ^n (so their Whittaker model is right $K_2(\varpi^n)$-invariant) we can write

$$J(\pi_2, \pi_3; s) = \sum_{0 \le i \le n} v_i \int_{\mathbb{Q}_q^\times} |y|^{s-1/2} W_2^{\psi}(a(y)\gamma_i) W_3^{\overline{\psi}}(a(y)\gamma_i) dy,$$

using that $f_s^\circ(a(y)\gamma_i) = |y|^{s+1/2}$ by definition. So, if we can come up with an explicit enough expression for these values of the Whittaker function, we can compute this integral directly via this decomposition.

The arguments of [NPS14] instead the Bruhat decomposition $G = B \sqcup BwN$ to compute $J(\pi_2, \pi_3; s)$. Here Haar measure is expressed as

$$dg = \frac{\zeta_q(2)}{\zeta_q(1)} |y|^{-1} d^\times u d^\times y dx' dx$$

for $g = n(x')a(y)z(u)wn(x)$ in the big Bruhat cell BwN (note the small Bruhat cell B has measure zero). In this setup we can similarly work out that

$$J(\pi_2, \pi_3; s) = \frac{\zeta_q(2)}{\zeta_q(1)} \int_{\mathbb{Q}_q} \int_{\mathbb{Q}_q^\times} \frac{W_2^\psi\big(a(y)wn(x)\big) W_3^{\overline{\psi}}\big(a(y)wn(x)\big)}{\max\{1, |x|\}^{2s+1}} |y|^{s-1/2} d^\times y dx.$$

2.3.3 Some strategies for the computation

To proceed with computing $J(\pi_2, \pi_3; s)$, the obvious first strategy is to try to get our hands dirty and directly compute the values of the Whittaker newvectors we need. This isn't too bad in some simple cases, but can quickly become very complicated for highly ramified representations. To deal with these latter situations, Nelson-Pitale-Saha [NPS14] and Saha [Sah15] develop techniques using local functional equations that can also be used, which we discuss briefly (though we won't need to use them to make any new computations for the cases considered in this thesis).

Explicit evaluation of the Whittaker functions. To start off, we consider how one might try to get an explicit formula for the Whittaker newvector, hopefully in a clean enough form that we could then work out the integral defining $J(\pi_2, \pi_3; s)$. One way we can try to compute the Whittaker newvector W^π for a given representation π is to work in another model of π; if we can describe the newvector in that model explicitly, and then have an explicit formula for passing from the given model to the Whittaker model, we can use this to describe the Whittaker newvector.

This is doable in the case when π is a principal series, and we work with its usual induced model; these arguments could be modified to work with special representations as well. So, suppose $\pi = \pi(\chi_1, \chi_2)$ is a principal series in its usual induced model, where χ_i has conductor n_i and thus π has conductor $n = n_1 + n_2$.

To compute the newvector in the induced model, we use the decomposition $G = \coprod_{i=0}^{n} B\gamma_i K_2(\varpi^n)$; if $f : G \to \mathbb{C}$ is any function in the induced model we know how it transforms on the left by B, and if it's a newvector we also know it's invariant on the right by $K_2(\varpi^n)$. Thus the newvector f must be determined fully by the values $f(\gamma_i)$, by the formula

$$f\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \gamma_i k\right) = \chi_1(a)\chi_2(d)|ad^{-1}|^{1/2} f(\gamma_i).$$

Moreover, if $f(\gamma_i) \neq 0$ then this formula forces to have $\chi_1(a)\chi_2(d)|ad^{-1}|^{1/2} = 1$ for

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in B \cap \gamma_i^{-1} K_2(\varpi^n)\gamma_i.$$

We can compute that this occurs exactly for such matrices with $a, d \in \mathbb{Z}_q^\times$ satisfying $a \equiv 1 \pmod{\varpi^i}$, $d \equiv 1 \pmod{\varpi^{n-i}}$, and $b \in \mathbb{Z}_q$ satisfying $b \equiv (1-a)\varpi^{-i} \pmod{\varpi^{n-i}}$. Thus we're asking that $\chi_1(a) = \chi_2(d) = 1$ for all $a \equiv 1 \pmod{\varpi^i}$ and $d \equiv 1 \pmod{\varpi^{n-i}}$. Since χ_1 has conductor n_1 and χ_2 has conductor n_2 this can happen only if $i \geq n_1$ and $n - i \geq n_2$; since $n = n_1 + n_2$ this forces $i = n_1$. Thus, our computations establish that (up to scalars) the only possible $K_2(\varpi^n)$-invariant vector $f \in \pi_1(\chi_1, \chi_2)$ is given by

$$f\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \gamma_i k\right) = \begin{cases} \chi_1(a)\chi_2(d)|ad^{-1}|^{1/2} & i = n_1 \\ 0 & i \neq n_1 \end{cases}.$$

So we've accomplished the first step, of describing the newvector explicitly in our model. The second step is to describe how to pass it to the Whittaker model. For an irreducible principal series representation, we can describe the isomorphism between the induced model and the Whittaker model via the formula $h \mapsto \int_{\mathbb{Q}_q} \psi(-x)h(wn(x)g)dx$. Thus the Whittaker newvector $W^\psi \in W(\pi, \psi)$ is determined by the induced model newvector f by

$$W^\psi(g) = \int_{\mathbb{Q}_q} \psi(-x)f(wn(x)g)dx.$$

Using our decomposition above, we want to figure out how to compute this integral for an arbitrary element g of each coset $B\gamma_i K_2$. Since we know how Whittaker functions transform under Z and under N, it's sufficient to look at an arbitrary element

$$g = a(y)\gamma_i k \in A\gamma_i K_2(\varpi^n) \subseteq B\gamma_i K.$$

We can compute

$$wn(x)g = \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} wn(x/y)\gamma_i k$$

and therefore (applying the transformation rules for f)

$$W^\psi(g) = \chi_2(y)|y|^{-1/2} \int_{\mathbb{Q}_q} \psi(-x)f(wn(x/y)\gamma_i)dx.$$

So we need to figure out how to evaluate f on vectors of the form $wn(z)\gamma_i$, and thus in particular which double coset $B\gamma_j K_2(\varpi^n)$ such a vector lies in. This breaks up into cases; if $z \in \mathbb{Z}_q$ then we have

$$wn(z)\gamma_i = \begin{bmatrix} 0 & 1 \\ -1 & -z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varpi^i & 1 \end{bmatrix} = \begin{bmatrix} \varpi^i & 1 \\ -1 - z\varpi^i & -z \end{bmatrix} \in K$$

while if $z \notin \mathbb{Z}_q$ then we can write

$$wn(z)\gamma_i = \begin{bmatrix} -z^{-1} & 1 \\ 0 & -z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varpi^i + z^{-1} & 1 \end{bmatrix} \in B \cdot K$$

At this point, the computation starts to depend heavily on the values n_1 and n_2, so we stop with the general argument here. In the next section we'll finish the computation in for $n_1 = 1$ and $n_2 = 0$, the one case that we will need for our purposes (and in particular the case where $\chi_1|_U$ is quadratic, but the argument should generalize).

Studying $J^*(\pi_2, \pi_3; s)$ as a Laurent polynomial. If a direct computation as outlined in the previous section isn't possible, there are several other techniques we can try. One was developed by Nelson-Pitale-Saha to compute their integrals in the case of $\pi_2 = \pi_3$; we will briefly summarize it (since we'll quote the results they obtain but not actually apply their method). The key point is that the integral representation for the $\mathrm{GL}_2 \times \mathrm{GL}_2$ L-function tells us that $J^*(\pi_2, \pi_3; s)$ is a Laurent polynomial $\sum T_m p^{ms}$ in the variable q^{-s}. This reduces the problem to computing the finitely many nonzero coefficients; moreover, the $\mathrm{GL}_2 \times \mathrm{GL}_2$ functional equation gives a relationship between these coefficients of the form $T_{-m+N} = q^{m-N/2} T_m$, where q^N is the conductor of $\pi \times \pi$. Nelson-Pitale-Saha carry out an analysis of the Whittaker newvector that lets them get enough information to compute these coefficients in this case. (This analysis seems to depend crucially on the fact that they're using the newvector itself, and not a translate by $\delta(\varpi^c)$, and also on the fact that $\pi_2 = \pi_3$).

Fourier expansion of Whittaker functions. Finally, we outline a method of sketched in [NPS14] and expanded upon in [Sah15], which uses Fourier analysis and the GL_2 local functional equation to evaluate values of Whittaker newforms in a way that (in principle) would allow us to evaluate $J(\pi_2, \pi_3; s)$ for any choice of π_2, π_3. We follow the argument there, using the Bruhat decomposition and the corresponding expression of $J(\pi_2, \pi_3; s)$ as an integral. We take the notation that $U = \mathbb{Z}_q^\times$ for convenience, and characters μ of U are canonically extended to $\mathbb{Q}_q^\times$ by $\mu(\varpi) = 1$.

In particular, if π is a fixed representation and W^ψ is its Whittaker newvector (for some character ψ), we want to be able to evaluate the function $W^\psi(a(y)wn(x))$. If we fix x and an exponent t, then $u \mapsto W^\psi(a(u\varpi^t)wn(x))$ gives us a function $U \to \mathbb{C}$, which is locally constant by definition of the Whittaker model. Since it's locally constant on a compact domain, it's a finite sum of indicator functions and thus the Fourier inversion formula for the group U applies to it. Since U is compact, its dual $\widehat{U}$ is discrete, and thus Fourier inversion produces an equality as in the following definition.

Definition 2.3.9. Fix an element $x \in \mathbb{Q}_q^\times$ and an integer $t \in \mathbb{Z}$. Then we define constants $C_{x,t}^\psi(\mu) \in \mathbb{C}$ for each $\mu \in \widehat{U}$ characterized by the identity

$$W^\psi(a(u\varpi^t)wn(x)) = \sum_{\mu \in \widehat{U}} C_{x,t}^\psi(\mu)\mu(u)$$

for all $u \in U$. Moreover these coefficients may be expressed as

$$C_{x,t}^\psi(\mu) = \int_U W^\psi(a(u\varpi^t)wn(x))\mu^{-1}(u)du.$$

The key observation used in Saha's argument is that the GL_2 functional equation for π can give us an identity relating the coefficients $C_{x,t}^\psi(\mu)$ to other quantities that are more directly computable, such as Gauss sums. The statement of the functional equation we want is the following:

Theorem 2.3.10 (Jacquet-Langlands)**.** *For $W' \in \mathcal{W}(\pi, \psi)$, a character μ of $\mathbb{Q}_q^\times$, and a complex number $s \in \mathbb{C}$, define the local zeta integral*

$$Z(W', s, \mu) = \int_{\mathbb{Q}_q^\times} W'(a(y))\mu(y)|y|^{s-1/2} d^\times y.$$

Then, we have an identity

$$\frac{Z(W', s, \mu^{-1})}{L(s, \pi\mu^{-1})} q^{(1/2-s)c(\pi\mu)} \varepsilon(1/2, \pi\mu^{-1}, \psi) = \frac{Z(w \cdot W', 1-s, \mu\omega_\pi^{-1})}{L(1-s, \pi\mu\omega_\pi^{-1})}$$

where both sides are polynomials in $q^{\pm s}$.

By applying this to suitably chosen W', we find that one side of the functional equation gives us a polynomial with the elements $C_{x,t}^{\psi}(\mu)$ appearing in its coefficients, and the other side gives us a polynomial with only simpler things showing up! In particular we want to fix x and μ, and apply the functional equation to the vector $W' = wn(x) \cdot W^\psi$ obtained from applying the group action to our newvector W^ψ. We can then compute what both zeta integrals are:

Proposition 2.3.11. *Fix an element $x \in \mathbb{Q}_q$, a character μ of U (extended to $\mathbb{Q}_q^\times$ by our convention), and a number $s \in \mathbb{C}$. If we take $W' = wn(x) \cdot W^\psi$ then we have*

$$Z(W', s, \mu^{-1}) = \sum_{t=-\infty}^{\infty} q^{-ts} q^{t/2} C_{x,t}^{\psi}(\mu).$$

Proposition 2.3.12. *Fix an element $x \in \mathbb{Q}_q$, a character μ of U (extended to $\mathbb{Q}_q^\times$ by our convention), and a number $s \in \mathbb{C}$. If we take $W' = wn(x) \cdot W^\psi$ then we have*

$$Z(w \cdot W', 1-s, \mu\omega_\pi^{-1}) = \sum_{t=-\infty}^{\infty} q^{-st} \cdot q^{t/2} \omega_\pi(-\varpi^t) W^\psi(a(\varpi^{-t})) G(x\varpi^{-t}, \mu, \psi)$$

where $G(z,\mu,\psi)$ is the Gauss sum

$$G(z,\mu,\psi) = \int_U \psi(zu)\mu(u)d^\times u.$$

Now, Gauss sums can be explicitly evaluated (giving either epsilon factors or 0, at least when $\mu \neq 1$), as can the values $W^\psi(a(y))$ (as done in [Sch02]). Thus, plugging the above two computations back into the functional equation gives us an identity of polynomials where the coefficients on the LHS involve the values $C^\psi_{x,t}(\mu)$ we want to compute, and all of the other quantities are things we understand. Thus the equation will let us work out what the coefficients $C^\psi_{x,t}(\mu)$ are, though the computation can get quite involved:

Theorem 2.3.13. *Fix an element $x \in \mathbb{Q}_q$ and a character μ of U. Then we have the following identity between polynomials in the functions $q^{\pm s}$:*

$$\varepsilon(1/2, \pi\mu^{-1}, \psi)L(s, \pi\mu^{-1})^{-1} \sum_{t=-\infty}^{\infty} (q^{-s})^t \cdot q^{t/2} C^\psi_{x,t-c(\pi\mu^{-1})}(\mu)$$

$$= L(1-s, \pi\mu\omega_\pi^{-1})^{-1} \sum_{t=-\infty}^{\infty} (q^{-s})^t \cdot q^{t/2}\omega_\pi(-\varpi^t)W^\psi(a(\varpi^{-t}))G(x\varpi^{-t}, \mu, \psi).$$

Finally, if we take the two representations π_1, π_2 we're interested in and study this decomposition for the associated Whittaker newvectors W_1^ψ and W_2^ψ (with coefficients $C^\psi_{1,x,t}(\mu)$ and $C^{\overline{\psi}}_{2,x,t}(\mu)$, respectively), we find that we can write

$$J(\pi_1, \pi_2; s) = \frac{\zeta_q(2)}{\zeta_q(1)} \sum_{t=-\infty}^{\infty} \int_{\mathbb{Q}_q} \sum_{\mu \in \widehat{U}} q^{-ts} q^{t/2} \frac{C^\psi_{1,x,t}(\mu) C^{\overline{\psi}}_{2,x,t}(\mu^{-1})}{\max\{1, |x|\}^{2s+1}} dx.$$

Again, this is something that evidently could be computed explicitly, though the computations tend to get messy very quickly.

2.3.4 Results in our cases

We now record the computations of the integrals $J(\pi_2, \pi_3; s)$ and thus of $I^{**}(\pi_2, \pi_3; s)$ that we will need.

Results from Nelson-Pitale-Saha In [NPS14], Nelson-Pitale-Saha compute the local integrals $I^*(\pi, \pi; s)$ in the case when π has trivial central character. We quote the cases that are of interest to us, translating them to our modified I^{**}.

Proposition 2.3.14. *Suppose π_1, π_2, π_3 are three local representations such that $\pi_1 = \pi(s, -s)$ is an unramified principal series having trivial central character (and with $s \in i\mathbb{R}$), and $\pi_2 \cong \pi_3 \cong \pi(\chi, \chi^{-1})$ is a ramified principal series with χ having conductor $n \geq 1$. Then we have*

$$I^{**}(\pi_2, \pi_3; s) = q^{-n}(1 + q^{-1})^{-2} \cdot (*)$$

for

$$(*) = \left(\frac{(\alpha^{n+1} - \alpha^{-n-1}) - 2q^{-1/2}(\alpha^n - \alpha^n) + q^{-1}(\alpha^{n-1} - \alpha^{-n+1})}{\alpha - \alpha^{-1}} \right)^2$$

where we set $\alpha = q^s$.

Proposition 2.3.15. *Suppose π_1, π_2, π_3 are three local representations such that $\pi_1 = \pi(s, -s)$ is an unramified principal series having trivial central character (and with $s \in i\mathbb{R}$), and $\pi_2 \cong \pi_3$ supercuspidal with trivial central character, such that $\pi \cong \pi \otimes \eta$ where η is the nontrivial unramified quadratic character of $\mathbb{Q}_p^\times$. Then*

$$I^{**}(\pi_2, \pi_3; s) = q^{-n}(1 + q^{-1})^{-2} \cdot (*)$$

for

$$(*) = \left(\frac{(\alpha^{n+1} - \alpha^{-n-1}) - q^{-1}(\alpha^{n-1} - \alpha^{-n+1})}{\alpha - \alpha^{-1}} \right)^2$$

where we set $\alpha = q^s$.

The remaining case For our purposes, there is one remaining case to consider: when π_2, π_3 are both principal series representations corresponding to one unramified character and one character with quadratic ramification, for q an odd prime. In particular, since we've been assuming WLOG that our representations satisfy $\omega_2(\varpi) = \omega_3(\varpi) = 1$ we will have

$$\pi_2 = \pi(\mu, \mu^{-1}\chi) \qquad\qquad \pi_3 = \pi(\nu, \nu^{-1}\chi)$$

where μ, ν are unramified characters and χ is the unique quadratic character of conductor 1 (i.e. coming from the index-two subgroup of $\mathbb{Z}_q^\times/U_1 \cong Z_{q-1}$).

We start by computing the Whittaker newvector W_2^ψ. For convenience we work with the induced model $\pi(\mu^{-1}\chi, \mu)$; from the setup in the previous section, we have $n_1 = 1$ and $n_2 = 0$ and we know that the newvector f is given by

$$f\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \gamma_i k\right) = \begin{cases} \chi(a)\mu(d/a)|ad^{-1}|^{1/2} & i = 1 \\ 0 & i = 0 \end{cases}.$$

We further computed that the corresponding Whittaker newvector is determined by the values

$$W^\psi(a(y)\gamma_i) = \mu(y)|y|^{-1/2} \int_{\mathbb{Q}_q} \psi(-x) f(wn(x/y)\gamma_i) dx.$$

To evaluate this integral, recall that for $z \in \mathbb{Z}_q$ we computed

$$wn(z)\gamma_i = \begin{bmatrix} 0 & 1 \\ -1 & -z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varpi^i & 1 \end{bmatrix} = \begin{bmatrix} \varpi^i & 1 \\ -1 - z\varpi^i & -z \end{bmatrix} \in K,$$

and we can then see that this lies in $B\gamma_0 K$ unless $i = 0$ and $z \in -1 + \varpi\mathbb{Z}_q$, and in that case the resulting matrix lies in K_2 so $f(wn(z)\gamma_i) = 1$. If $z \notin \mathbb{Z}_q$ we computed

$$wn(z)\gamma_i = \begin{bmatrix} -z^{-1} & 1 \\ 0 & -z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varpi^i + z^{-1} & 1 \end{bmatrix} \in B \cdot K,$$

and we find this decomposition lies in $B\gamma_0 K_2$ if $i = 0$ and in $B\gamma_1 K_2$ if $i = 1$. In fact, in the $i = 1$ case the second matrix is in K_2 already, so $f(wn(z)\gamma_i) = \chi(-z^{-1})\mu(\varpi z^2)|z|^{-1}$. Combining these facts (and using that χ is quadratic) we conclude

$$f(wn(z)\gamma_i) = \begin{cases} 1 & i = 0, z \in -1 + \varpi\mathbb{Z}_q \\ \chi(-z)\mu(z^2)|z|^{-1} & i = 1, z \notin \mathbb{Z}_q \\ 0 & \text{otherwise} \end{cases}.$$

We can then go back to the integral $\int \psi(-x) f(wn(x/y)\gamma_i)dx$ we needed to evaluate to compute $W^\psi(g)$. If $i = 0$ we know that the integrand is nonzero only when $x/y \in -1+\varpi\mathbb{Z}_q$, and the integral becomes the integral of $\psi(-x)$ over $x \in -y+y\varpi\mathbb{Z}_q = y+\varpi^{v+1}\mathbb{Z}_q$ for $v = v(y)$. Taking the substitution $x' = -x-y$ we conclude the integral is

$$\psi(y) \int_{\varpi^{v+1}\mathbb{Z}_q} \psi(x')dx' = \psi(y) \begin{cases} q^{-v-1} & v+1 \geq 0 \\ 0 & v+1 < 0 \end{cases}.$$

Noting that $|y| = q^{-v}$ by definition, we conclude that we have

$$W^\psi(a(y)\gamma_0) = \begin{cases} \mu(y)|y|^{1/2}\psi(y)q^{-1} & v(y) \geq -1 \\ 0 & v(y) < -1 \end{cases}.$$

Similarly, for $i = 1$ our computations tell us that $f(wn(x/y)\gamma_1)$ is nonzero exactly when $x/y \notin \mathbb{Z}_q$, i.e. $v(x) < v = v(y)$. For x satisfying $v(x) = i < v$ we have

$$f(wn(x/y)\gamma_1) = \chi(-x/y)\mu(\varpi)^{2i-2v}q^{i-v}$$

and thus

$$\int \psi(-x)f(wn(x/y)\gamma_1)dx = \sum_{i=-\infty}^{v-1} \chi(y)\mu(\varpi)^{2i-2v}q^{i-v}\int_{\varpi^i\mathbb{Z}_q^\times} \psi(-x)\chi(-x)dx.$$

Now, this latter integral is zero except for the case $i = -1$, when it gives an ε-factor, namely $q^{1/2}\varepsilon(1/2,\chi,\psi)$. Thus we find $\int \psi(-x)f(wn(x/y)\gamma_1)dx$ is zero unless $v(y) \geq 0$, and in that case the integral is $\chi(y)\mu(y)^{-2}\mu(\varpi)^{-2}|y|q^{-1/2}\varepsilon(1/2,\chi,\psi)$. Thus we get

$$W^\psi(a(y)\gamma_1) = \begin{cases} \chi(y)\mu(y)^{-1}\mu(\varpi)^{-2}|y|^{1/2}q^{-1/2}\varepsilon(1/2,\chi,\psi) & v(y) \geq 0 \\ 0 & v(y) < 0 \end{cases}.$$

Finally, we remember that we need to normalize W^ψ so that $\langle W^\psi, W^\psi\rangle = 1$ and $W^\psi(1) > 0$. Noting that $W^\psi(a(y)\gamma_1) = W^\psi(a(y))$ we compute

$$\langle W^\psi, W^\psi\rangle = \int |W^\psi(a(y))|^2 = \int_{v(y)\geq 0} |y|q^{-1}d^\times y = \int_{v(y)\geq 0} q^{-1}dy = q^{-1}.$$

So we need to multiply by $q^{1/2}$ to normalize the absolute value; also we note that $W^\psi(1) = W^\psi(a(1)\gamma_1) = \mu(\varpi)^{-2}q^{-1/2}\varepsilon(1/2,\chi,\psi)$ so we can multiply by $\mu(\varpi)^2$ and $\varepsilon(1/2,\chi,\overline{\psi})$ to guarantee that this is positive. We conclude that the normalized Whittaker newvector is given by:

$$W_2^\psi(a(y)\gamma_i) = \begin{cases} \chi(y)\mu^{-1}(y)|y|^{1/2} & v(y) \geq 0, i = 1 \\ \mu(\varpi)^2\mu(y)|y|^{1/2}q^{-1/2}\psi(y)\varepsilon(1/2,\chi,\overline{\psi}) & v(y) \geq -1, i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Applying the same computation to $\pi_3 = \pi(\nu^{-1}\chi,\nu)$ we get that the normalized Whittaker newvector $W_3^{\overline{\psi}}$ is given by a similar formula:

$$W_3^{\overline{\psi}}(a(y)\gamma_i) = \begin{cases} \chi(y)\nu^{-1}(y)|y|^{1/2} & v(y) \geq 0, i = 1 \\ \nu(\varpi)^2\nu(y)|y|^{1/2}q^{-1/2}\overline{\psi}(y)\varepsilon(1/2,\chi,\psi) & v(y) \geq -1, i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

We're interested in integrating the product of these two quantities, and fortunately we find that when we multiply them together quite a few things cancel:

$$W_2^{\psi}(a(y)\gamma_i)W_3^{\overline{\psi}}(a(y)\gamma_i) = \begin{cases} (\mu\nu)^{-1}(y)|y| & v(y) \geq 0, i = 1 \\ (\mu\nu)(\varpi)^2(\mu\nu)(y)|y|q^{-1} & v(y) \geq -1, i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Now, the ultimate quantity we want to compute is

$$J(\pi_2, \pi_3; s) = (1 + q^{-1}) \sum_{i=0}^{1} q^{-i} \int_{\mathbb{Q}_q^\times} |y|^{s-1/2} W_2^{\psi}(a(y)\gamma_i) W_3^{\overline{\psi}}(a(y)\gamma_i) d^\times y.$$

Since s is ultimately the parameter associated to an unramified character $\xi(y) = |y|^s$ which comes from our third principal series, it's convenient to write $|y|^{s+1/2} = \xi(y)|y|^{1/2}$ for notational purposes. Thus the $i = 1$ term is

$$q^{-1} \int_{v(y)\geq 0} (\xi\mu^{-1}\nu^{-1})(y)|y|^{1/2} d^\times y = q^{-1} \sum_{i=0}^{\infty} (\xi\mu^{-1}\nu^{-1})(\varpi^i) q^{-i/2},$$

which is a geometric series summing to $q^{-1}(1 - (\xi\mu^{-1}\nu^{-1})(\varpi)q^{-1/2})^{-1}$. Similarly, the $i = 0$ term is

$$q^{-1}(\mu\nu)(\varpi)^2 \int_{v(y)\geq -1} (\xi\mu\nu)(y)|y|^{1/2} d^\times y = q^{-1}(\mu\nu)(\varpi)^2 \sum_{i=-1}^{\infty} (\xi\mu\nu)(\varpi^i) q^{-i/2}.$$

which sums to

$$q^{-1}\frac{(\mu\nu)(\varpi)^2\cdot(\xi\mu\nu)(\varpi^{-1})q^{1/2}}{1-(\xi\mu\nu)(\varpi)q^{-1/2}}.$$

So, we conclude

$$J(\pi_2,\pi_3;s)=(1+q^{-1})^{-1}q^{-1}+\left(\frac{1}{1-(\xi\mu^{-1}\nu^{-1})(\varpi)q^{-1/2}}+\frac{(\xi^{-1}\mu\nu)(\varpi)q^{1/2}}{1-(\xi\mu\nu)(\varpi)q^{-1/2}}\right).$$

Collecting terms we find we get

$$J(\pi_2,\pi_3;s)=(1+q^{-1})^{-1}q^{-1}\frac{(\xi^{-1}\mu\nu)(\varpi)q^{1/2}\cdot(1-\xi^2(\varpi)q^{-1})}{(1-(\xi\mu^{-1}\nu^{-1})(\varpi)q^{-1/2})(1-(\xi\mu\nu)(\varpi)q^{-1/2})}.$$

Next, we recall that we get $J^*(\pi_2,\pi_3;s)$ by multiplying this quantity by $\zeta_q(1+2s)/L(\pi_2\times\pi_3,1/2+s)$. But

$$\zeta_q(1+2s)=(1-q^{-1-2s})^{-1}=(1-\xi^2(\varpi)q^{-1})^{-1}$$

cancels a term on the top of our expression above, and similarly

$$L(\pi_2\times\pi_3,1/2+s)=(1-(\mu\nu)(\varpi)q^{-1/2-s})^{-1}(1-(\mu^{-1}\nu^{-1})(\varpi)q^{-1/2-s})^{-1}$$

cancels the bottom. So we conclude

$$J(\pi_2,\pi_3;s)=(1+q^{-1})^{-1}q^{-1/2}(\xi^{-1}\mu\nu)(\varpi).$$

Finally, we recall that we get back the ultimate local integral we want as

$$I^{**}(\pi_2,\pi_3;s)=(1+q^{-1})^2L^H(\mathrm{ad}\,\pi_2,1)L^H(\mathrm{ad}\,\pi_3,1)J^*(\pi_2,\pi_3;s)J^*(\widetilde{\pi}_2,\widetilde{\pi}_3;-s).$$

Since $\widetilde{\pi}_2 = \pi(\mu^{-1}, \mu\chi)$ and $\widetilde{\pi}_3 = \pi(\nu^{-1}, \nu\chi)$ our computation above gives us

$$J(\widetilde{\pi}_2, \widetilde{\pi}_3; -s) = (1 + q^{-1})^{-1} q^{-1/2} (\xi\mu^{-1}\nu^{-1})(\varpi).$$

Thus we conclude

$$J(\pi_2, \pi_3; s) J(\widetilde{\pi}_2, \widetilde{\pi}_3; -s) = (1 + q^{-1})^{-2} q^{-1}$$

and therefore

$$I^{**}(\pi_2, \pi_3; s) = q^{-1} L^H(\operatorname{ad}\pi_2, 1) L^H(\operatorname{ad}\pi_3, 1).$$

By definition, these remaining local factors are $(1 - q^{-2})^{-1}$ and thus we conclude:

Proposition 2.3.16. *Suppose π_1, π_2, π_3 are three local representations (with trivial product of central characters) such that π_1 is an unramified principal series, and π_2, π_3 are both principal series of the form $\pi(\chi_1, \chi_2)$ with χ_1 unramified and $\chi_2|_{\mathbb{Z}_q^\times}$ the quadratic character of rank 1. Then we have*

$$I^{**}(\pi_1, \pi_2, \pi_3) = (1 - q^{-2})^{-2} q^{-1}.$$

2.3.5 Interpretation for classical newforms

Finally, we want to apply the computations of local integrals in this section back to the terms I_q^{**} arising in Theorem 2.2.5. This more or less amounts to considering various cases for these local representations, and rephrasing the results of the previous section in terms of f, g, h themselves. We consider this problem in our case of interest in the next section. For now, we give a few comments to justify why our I_q^{**}'s can be computed as integrals $I^{**}(\pi_1, \pi_2, \pi_3)$ (as defined in this section) - the definitions are almost the same but there are a few minor differences. First of all, we note that the integrals defining I_q^{**} are symmetric in the arguments. Thus we can rearrange the factors $\pi_{f,q}, \pi_{g,q}, \widetilde{\pi}_{h,q}$ and the associated inputs $\delta_q^0(M_f^{-1})F_q, \delta_q^0(M_g^{-1})G_q, \delta_q^0(M_h^{-1})\widetilde{H}_q$, ordering them so that π_3 has the largest conductor, and π_2 is the only one with a (possibly) nontrivial multiple of $\delta_q^0(M^{-1})$.

At this point I_q^{**} and $I^{**}(\pi_1, \pi_2, \pi_3)$ are defined by very similar integrals, so we just need to check that the few differences in the definitions don't actually change anything. One is that our original integrals I_q^{**} involved the bilinear contragredient pairings $\langle gx_i, \widetilde{x}_i\rangle / \langle x_i, \widetilde{x}_i\rangle$ on $\pi_i \times \widetilde{\pi}_i$ while the integrals $I^{**}(\pi_1, \pi_2, \pi_3)$ involve the unitary pairings $\langle gx_i, gx_i\rangle / \langle x_i, x_i\rangle$ on $\pi_i \times \pi_i$. But these pairings are closely related to each other; one can construct a contragredient pairing by suitably manipulating a unitary pairing, and the fact that x_i and $\widetilde{x}_i$ are newvectors (or translates of them) makes these ratios equal for any g.

The other potential difference between the adjoint L-factors involved in each definition; we need to verify that if $\psi \in S_\kappa(N_\psi, \chi_\psi)$ is a newform then $L_q^H(\mathrm{ad}\,\psi, 1)$ (as defined in Theorem 2.2.3) is equal to $L^H(\pi_{\psi,q}, 1) = L^H(\widetilde{\pi}_{\psi,q}, 1)$ (as defined earlier in this section). But this follows from the well-known classification of these $\pi_{\psi,q}$'s. In particular, in the case where $q \nmid N_\psi$ we have $L_q(f,s)^{-1} = (1 - \alpha_q q^{-s})(1 - \beta_q q^{-s})$ then we know $|\alpha_q| = |\beta_q| = q^{\kappa - 1}$ by the Ramanujan conjecture and that $\pi_{\psi,q} = \pi(\chi_1, \chi_2)$ is a tempered unramified principal series with $\chi_1(\varpi) = \alpha_q$ and $\chi_2(\varpi) = \beta_q$ and thus $L_q^H(\mathrm{ad}\,\psi, 1) = L_q^{\mathrm{naive}}(\mathrm{ad}\,\psi, 1)$ and $L(\psi_{\psi,q}, 1)$ are both equal to

$$L_q^{\mathrm{naive}}(\mathrm{ad}\,\psi, s) = \left(1 - \overline{\chi}(q)\frac{\alpha_q^2}{q^{k-1}}q^{-s}\right)^{-1}\left(1 - \overline{\chi}(q)\frac{\alpha_q\beta_q}{q^{k-1}}q^{-s}\right)^{-1}\left(1 - \overline{\chi}(q)\frac{\beta_q^2}{q^{k-1}}q^{-s}\right).$$

In case when $q | N_\psi$ we have $L_q(\psi, s) = (1 - a_q q^{-s})^{-1}$ and we split up into several cases. Let N_χ denote the conductor of the character χ_ψ and let v_q be the power of q dividing an integer.

- If $v_q(N_\psi) = 1$ and $q \nmid N_\chi$ then $\pi_{\psi,q}$ is a special representation $\sigma(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$

associated to an unramified character with $\chi(\varpi) = a_q/q^{k-1}$; then we have

$$L_q^H(\operatorname{ad}\psi, 1) = L^H(\pi_{\psi,q}) = (1-q^{-2})^{-1}(1+q^{-1})^{-1}.$$

- If $v_q(N_\psi) = v_q(N_\chi) \geq 1$ then $\pi_{\psi,q}$ is a principal series representation $\pi(\chi_1, \chi_2)$ with χ_1 unramified and satisfying $\chi_1(\varpi) = a_q/q^{k-1}$ and χ_2 ramified and satisfying $\chi_2 = \chi_{\psi,q}\chi_1^{-1}$. Then we have

$$L_q^H(\operatorname{ad}\psi, 1) = L^H(\pi_{\psi,q}) = (1-q^{-2})^{-1}.$$

- In all other cases, $a_q = 0$, and $\pi_{\psi,q}$ is either supercuspidal, special associated to a ramified character, or principal series associated to two ramified characters; then in both cases we have

$$L_q^H(\operatorname{ad}\psi, 1) = L^H(\pi_{\psi,q}) = (1+q^{-1})^{-1}.$$

So this establishes that the local integrals computed in the previous section are exactly the ones we want for Ichino's formula.

2.4 Applying Ichino's formula to CM forms

Now that we have an explicit version of Ichino's formula for classical newforms, we recall that we want to apply it to a triple consisting of one fixed (arbitrary) newform, and two CM newforms we will get to choose. In this section we collect some results about CM forms and the associated L-functions which we will need, and then specialize Ichino's formula to our case of interest.

To begin, we recall that if ψ is a Hecke character of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, the associated *CM modular form* g_ψ associated to it (if it exists) is characterized by insisting that the L-function $L(g_\psi, s)$ equals the Hecke L-function $L(\psi, s)$. Looking at the Dirichlet series for both, we find that g_ψ must be defined by the Fourier expansion

$$g_\psi(z) = \sum_{\mathfrak{a}\subseteq\mathcal{O}_K} \psi(\mathfrak{a}) e^{2\pi i N(\mathfrak{a}) z} = \sum_{n=1}^{\infty} \left(\sum_{\mathfrak{a}:N(\mathfrak{a})=n} \psi(\mathfrak{a}) \right) q^{2\pi i n z},$$

where we take the convention that $\psi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not coprime to the conductor. It turns out that if ψ has infinity-type $(m, 0)$ for $m \geq 0$ then this definition does indeed give us a newform (see e.g. Section 4.8 of [Miy06]).

Theorem 2.4.1. *Let ψ be an algebraic Hecke character of infinity-type $(m, 0)$ (for an integer $m \geq 0$) for an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$. Then the function g_ψ defined above is a newform of weight $m+1$, level $d \cdot N(\mathfrak{m}_\psi)$, and character $\chi_K \cdot \chi_\psi$.*

2.4.1 Factorization of L-functions

One of the reasons that CM forms are useful is that L-functions associating to them often factor as products of simpler L-functions, ultimately as a consequence of the fact that the Weil-Deligne representation corresponding to g_ψ is $\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \psi$ (where we've abused notation to let ψ denote both a Hecke character and its associated Galois character restricted to the Weil-Deligne group). If we have a module-theoretic construction with such induced representations that decomposes, then we get a corresponding formal factorization of L-functions. For instance, if we have two Hecke characters φ, ψ then we have an decomposition of representations

$$(\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \varphi) \otimes (\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \psi) \cong \left(\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \psi\varphi\right) \oplus \left(\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \psi\varphi^c\right)$$

Using this decomposition (and tensoring with the Weil-Deligne representation associated to f) we get a formal factorization of a triple-product L-function involving two

CM forms.

Proposition 2.4.2. *Let f be a modular form of weight k and g_φ, g_ψ two CM forms of weights $m-k$ and m, respectively, corresponding to Hecke characters φ, ψ of weights $(m-k-1, 0)$ and $(m-1, 0)$, respectively, for the same imaginary quadratic field K. Then we have*

$$L(s, f \times g_\varphi \times \overline{g}_\psi) = L(s-m+1, f, \varphi\psi^{-1})L(s-m+1, f, \psi^{-1}\varphi^{-1}N^{m-k-1}, \chi_\varphi)$$

where N is the norm character and χ_φ is the restriction to $\mathbb{Z}$ of the finite part of φ.

Another place where we get such a factorization is for adjoint L-functions; the trace-free adjoint of our induction decomposes as

$$\mathrm{ad}(\mathrm{Ind}_{W_K}^{W_\mathbb{Q}} \psi) \cong \mathrm{Ind}_{W_K}^{W_\mathbb{Q}}(\psi\psi^c) \oplus \chi_K$$

where χ_K is the one-dimensional Weil-Deligne representation corresponding to the nontrivial Galois character on the group $\mathrm{Gal}(K/\mathbb{Q})$ of order 2 (which corresponds to the Dirichlet character χ_K via class field theory, as the notation suggests). The adjoint L-function of g_ψ thus factors as the Hecke L-function $L(\psi\psi^c, s)$ times the Dirichlet L-function $L(\chi_K, s)$. However, since we primarily deal with the naive adjoint L-function, we will work directly with that and compute the corresponding factorization (and the extra Euler factors that come up) explicitly.

Proposition 2.4.3. *Let ψ be a Hecke character of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ of weight $(m-1, 0)$ and conductor $\mathfrak{m}_\psi$ with norm $N_\psi = N(\mathfrak{m}_\psi)$. Then we have*

$$L^{\mathrm{naive}}(\mathrm{ad}\, g_\psi, s) = L(\psi^2(N \circ \overline{\chi}_\psi), s+m-1)L(\chi_K, s) \prod_{q | N_\psi} L_q(\chi_K, s)^{-1} \prod_{q | d, q \nmid N_\psi} (1 - q^{-s}).$$

Proof. Recall that the Euler factor at a prime q for $L(g_\psi, s) = L(\psi, s)$ are given by

$$L_q(\psi, s)^{-1} = \begin{cases} (1-\psi(\mathfrak{q})q^{-s})(1-\psi(\overline{\mathfrak{q}})q^{-s}) & q = \mathfrak{q}\overline{\mathfrak{q}} \text{ split} \\ (1-\sqrt{\psi(\mathfrak{q})}q^{-s})(1+\sqrt{\psi(\mathfrak{q})}q^{-s}) & q = \mathfrak{q} \text{ inert} \\ (1-\psi(\mathfrak{q})q^{-s}) & q = \mathfrak{q}^2 \text{ ramified} \end{cases}.$$

Since g_ψ has character $\chi_\psi \chi_K$, and χ_K is 1, -1, and 0 in the three cases above, we find that by definition we have

$$L_q^{\text{naive}}(\operatorname{ad} g_\psi, s)^{-1} = (1-\overline{\chi}_\psi(q)\tfrac{\psi(\mathfrak{q})^2}{q^{m-1}}q^{-s})(1-\overline{\chi}_\psi(q)\tfrac{\psi(\mathfrak{q})\psi(\overline{\mathfrak{q}})}{q^{m-1}}q^{-s})(1-\overline{\chi}_\psi(q)\tfrac{\psi(\overline{\mathfrak{q}})^2}{q^{m-1}}q^{-s})$$

in the split case,

$$L_q^{\text{naive}}(\operatorname{ad} g_\psi, s)^{-1} = (1+\overline{\chi}_\psi(q)\tfrac{\psi(\mathfrak{q})}{q^{m-1}}q^{-s})(1-\overline{\chi}_\psi(q)\tfrac{\psi(\mathfrak{q})}{q^{m-1}}q^{-s})(1+\overline{\chi}_\psi(q)\tfrac{\psi(\mathfrak{q})}{q^{m-1}}q^{-s})$$

in the inert case, and $L_q(\psi, s) = 1$ in the ramified case.

In the split case we see that the first and third term give us the L-factor for $L(\psi^2(N \circ \overline{\chi}_\psi), s+m-1)$. For the remaining term, recall that if $q \nmid N(\mathfrak{m}_\psi)$ then the infinity-type for ψ tells us

$$\psi(\mathfrak{q})\psi(\overline{\mathfrak{q}}) = \psi\big((q)\big) = \chi_\psi(q)q^{m-1},$$

and if $q|N_\psi$ then this product is 0 because at least one of $\psi(\mathfrak{q})$ or $\psi(\overline{\mathfrak{q}})$ is zero. So for split q we conclude

$$L_q^{\text{naive}}(\operatorname{ad} g_\psi, s) = \begin{cases} L_q(\psi^2(N \circ \overline{\chi}_\psi), s+m-1)(1-q^{-s})^{-1} & q \nmid N_\psi \\ L_q(\psi^2(N \circ \overline{\chi}_\psi), s+m-1) & q|N_\psi \end{cases}.$$

In the inert case, the first two terms multiply to be

$$(1-\overline{\chi}_\psi(q)^2\psi(\mathfrak{q})^2 q^{-2(s-m+1)}) = L_q(\psi^2(\chi_\psi\circ N), s+m-1)^{-1}.$$

For the last term, again we know $\psi(\mathfrak{q})=0$ if $q|N_\psi$ and $\psi(\mathfrak{q})=\psi((q))=\chi_\psi(q)q^{m-1}$ otherwise, so we get 1 or $(1+q^{-s})$ respectively; thus for inert q

$$L_q^{\text{naive}}(\operatorname{ad} g_\psi, s) = \begin{cases} L_q(\psi^2(N\circ\overline{\chi}_\psi), s+m-1)(1+q^{-s})^{-1} & q\nmid N_\psi \\ L_q(\psi^2(N\circ\overline{\chi}_\psi), s+m-1) & q|N_\psi \end{cases}.$$

Putting this together, we find that in general we have

$$L_q^{\text{naive}}(\operatorname{ad} g_\psi, s) = \begin{cases} L_q(\psi^2(N\circ\overline{\chi}_\psi), s+m-1)L_q(\chi_K, s) & q\nmid dN_\psi \\ L_q(\psi^2(N\circ\overline{\chi}_\psi), s+m-1) & q|N_\psi, q\nmid d \\ 1 & q|d \end{cases}.$$

Since $L_q(\chi_K, 1)=1$ when $q|d$ we can conclude

$$L^{\text{naive}}(\operatorname{ad} g_\psi, s) = L(\psi^2(N\circ\overline{\chi}_\psi), s+m-1)L(\chi_K, s)$$

$$\cdot\left(\prod_{q|N_\psi} L_q(\chi_K, s)^{-1}\right)\left(\prod_{q|d} L_q(\psi^2(N\circ\overline{\chi}_\psi), s+m-1)^{-1}\right).$$

To simplify this, we note that if q is ramified we have

$$L_q(\psi^2(N\circ\overline{\chi}_\psi), s+m-1)^{-1} = (1-\psi(\mathfrak{q})^2\overline{\chi}_\psi(q)q^{-s-m+1});$$

if $q|N_\psi$ then this is 1, while if $q\nmid N_\psi$ then $\psi(\mathfrak{q})^2\overline{\chi}_\psi(q) = q^{m-1}$ and we get $(1-q^{-s})$. $\square$

We can then translate this over to an equality for the value $L^H(\operatorname{ad} g_\psi, 1)$ defined in Theorem 2.2.3.

Corollary 2.4.4. *Let ψ be a Hecke character of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ (with the fundamental discriminant d odd) of weight $(m-1,0)$ and conductor $\mathfrak{m}_\psi$ with norm $N_\psi = N(\mathfrak{m}_\psi)$. Then we have*

$$L^H(\operatorname{ad} g_\psi, 1) = L(\psi^2(N \circ \overline{\chi}_\psi), m)L(\chi_K, 1) \cdot \prod_{q|N_\psi} L_q(\chi_K, 1)^{-1} \prod_{q|dN_\psi} (1+q^{-1})^{-1} \prod_{\substack{q|N_\psi \\ q=\mathfrak{q}\overline{\mathfrak{q}}\nmid\mathfrak{m}_\psi}} (1-q^{-1})^{-1}.$$

Proof. First of all, evaluating the formula of the previous proposition at $s = 1$ gives

$$L^{\text{naive}}(\operatorname{ad} g_\psi, 1) = L(\psi^2(N \circ \overline{\chi}_\psi), m)L(\chi_K, 1) \prod_{q|N_\psi} L_q(\chi_K, 1)^{-1} \prod_{q|d, q\nmid N_\psi} (1-q^{-1}).$$

Next, recall that the level of g_ψ is $N_\psi d$, and let N_χ denote the level of the character $\overline{\chi}_\psi \chi_K$ of g_ψ. Then by definition we have

$$L^H(\operatorname{ad} g_\psi, 1) = L^{\text{naive}}(\operatorname{ad} g_\psi, 1) \cdot \prod_{q|dN_\psi} (1+q^{-1})^{-1} \cdot \prod_{\substack{q|dN_\psi \\ q\nmid(dN_\psi/N_\chi)}} (1-q^{-1})^{-1}.$$

We consider when we could have $q|dN_\psi$ with $q \nmid (dN_\psi/N_\chi)$. If q is split in K this happens iff exactly one of $\mathfrak{q}$ or $\overline{\mathfrak{q}}$ divides $\mathfrak{m}_\psi$ (since the local representation for g_ψ at q is the principal series representation associated to the local characters of ψ at $\mathfrak{q}$ and $\overline{\mathfrak{q}}$, which has conductor equal to the product $N_\mathfrak{q} N_{\overline{\mathfrak{q}}}$ of their conductors, but the central character has conductor $\leq \max\{N_\mathfrak{q}, N_{\overline{\mathfrak{q}}}\}$ with equality of the those conductors are different). If q is inert this never happens (since the conductor of the central character is at most the local conductor of ψ at $\mathfrak{q}$, but the conductor of g_ψ is the norm of that local conductor which is twice as large). If q is ramified (and odd by assumption), then if $q \nmid N_\psi$ we have $q \| dN_\psi$ and $q \| N_\chi$, but if $q|N_\psi$ then the power of q dividing dN_ψ is larger than the power that could possibly divide N_χ (at most the power dividing N_ψ).

Putting this together, we conclude that

$$\prod_{\substack{q|dN_\psi \\ q\nmid(dN_\psi/N_\chi)}} (1-q^{-1})^{-1} = \prod_{q|d, q\nmid N_\psi} (1-q^{-1})^{-1} \cdot \prod_{\substack{q|N_\psi \\ q=\mathfrak{q}\overline{\mathfrak{q}}\nmid\mathfrak{m}_\psi}} (1-q^{-1})^{-1},$$

and plugging this into our formulas gives the desired result. □

Finally, we use the result of Theorem 2.2.3 to conclude:

Corollary 2.4.5. *Let ψ be a Hecke character of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ (with the fundamental discriminant d odd) of weight $(m-1,0)$ and conductor $\mathfrak{m}_\psi$ with norm $N_\psi = N(\mathfrak{m}_\psi)$. Then we have*

$$\frac{\pi^2}{6}\frac{(4\pi)^m}{(m-1)!}\langle g_\psi, g_\psi\rangle = L(\psi^2(N\circ\overline{\chi}_\psi), m)L(\chi_K, 1) \cdot \prod_{q|N_\psi} L_q(\chi_K, 1)^{-1} \prod_{q|dN_\psi} (1+q^{-1})^{-1} \prod_{\substack{q|N_\psi \\ q=\mathfrak{q}\overline{\mathfrak{q}}\nmid\mathfrak{m}_\psi}} (1-q^{-1})^{-1}.$$

2.4.2 Ichino's formula in our case

We now apply Ichino's formula in the special case we want to consider, with one arbitrary modular form f and two CM forms g_φ, g_ψ coming from Hecke characters on the same imaginary quadratic field. In particular we make the following set of assumptions on these forms:

- f is a newform of some weight k, level N, and with trivial character.
- $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field of odd fundamental discriminant, and such that every prime dividing N splits in K. (In particular this means d is coprime to N).
- φ, ψ are Hecke characters of K of weights $(m-k-1,0)$ and $(m-1,0)$, respectively, for some integer $m > k$.

- The central characters χ_φ, χ_ψ (the finite-type parts of φ and ψ, restricted to $\mathbb{Z}$) are trivial. This forces the conductors of φ and ψ to be ideals generated by integers in $\mathbb{Z}$.

- The conductors of φ and ψ are coprime to N and d. Moreover, they are given by ℓ^{c_ℓ} and $c\ell^{c_\ell}$, respectively, and we have
  - c is coprime to Nd.
  - $\ell \nmid Ndc$ is a prime inert in K, and the local components of φ and ψ at ℓ are inverse to each other.

With these assumptions we will apply Ichino's formula for f, g_φ, and g_ψ. We see that the latter two modular forms have weights $k-m$ and m, respectively, and both have character χ_K, so the weight and character assumptions of Theorem 2.2.5 are satisfied. The levels of the forms g_φ and g_ψ are $\ell^{2c_\ell}d$ and $c^2\ell^{2c_\ell}d$, respectively, and we take the auxiliary integers to be $M_f = M_{g_\psi} = 1$ and $M_{g_\psi} = c^2N$. Ichino's formula then tells us

$$|\langle f(z)g_\varphi(c^2Nz), g_\psi(z)\rangle|^2 = \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2}2^{4m-4}(c^2N)^{m-k}} L(f\times g_\varphi \times \overline{g}_\psi, m-1) \prod I_q^{**}$$

where the product is over primes q dividing $dNc^2\ell^{2c_\ell}$. This gives us four cases of local integrals to deal with:

1. $q|d$: Here q is odd, $\pi_{f,q}$ is unramified, and the local representations $\pi_{\varphi,q}$ and $\pi_{\psi,q}$ are each principal series associated to a pair of an unramified character and a character of conductor q with quadratic ramification. By Proposition 2.3.16, we have $I_q^{**} = (1-q^{-2})^{-2}q^{-1}$.

2. $q|N$: In this case, $\pi_{f,q}$ is ramified (and we can't say much else about it since we aren't putting many assumptions on f) and the other two local representations

are unramified. Thus we're in the situation of Corollary 2.3.6, so $I_q^{**} = q^{-n_q}(1+q^{-1})^{-2}$ where q^{n_q} is the power of q dividing N.

3. $q|c$: In this case only $\pi_{\psi,q}$ is ramified (either a ramified principal series or supercuspidal) and the other two local representations are unramified. So again we're in the situation of Corollary 2.3.6 and $I_q^{**} = q^{-2n_q}(1+q^{-1})^{-2}$ where q^{n_q} is the power of q dividing c (and thus q^{2n_q} is the power of q dividing the conductor of $\pi_{\psi,q}$).

4. $q = \ell$: In this case $\pi_{f,q}$ is an unramified principal series, and $\pi_{\psi,q}$ and $\pi_{\psi,q}, \pi_{\varphi,q}$ are isomorphic supercuspidals of conductor ℓ^{2c_ℓ} which are invariant under twisting by the unramified quadratic character of $\mathbb{Q}_\ell^\times$. We can thus apply Proposition 2.3.15.

Thus we have:

$$\prod I_q^{**} = (dNc^2\ell^{2c_\ell})^{-1} \prod_{q|cNd\ell} (1+q^{-1})^{-2} \prod_{q|d} (1-q^{-1})^{-2} \cdot (\star)_\ell$$

for

$$(\star)_\ell = \left(\frac{(\alpha^{c_\ell+1} - \alpha^{-c_\ell-1}) - \ell^{-1}(\alpha^{c_\ell-1} - \alpha^{-c_\ell+1})}{\alpha - \alpha^{-1}} \right)^2$$

where α, α^{-1} are the roots of the Hecke polynomial for f at ℓ, i.e. satisfy $a_\ell(f) = (\alpha + \alpha^{-1})\ell^{(k-1)/2}$.

Chapter 3

An excursion through Hida theory

3.1 p-adic modular forms

In this chapter we describe the portions of Hida theory that we will need for our argument. In particular, we want to establish that if f is a fixed modular form and φ, ψ are Hecke characters varying suitably in families Φ, Ψ, we can construct a p-adic analytic function $\langle f \mathrm{g}_\Phi, \mathrm{g}_\Psi \rangle$ that explicitly interpolates the family of Petersson inner products $\langle f g_\varphi, g_\psi \rangle$. Then, in the next chapter we will use this together with our explicit version of Ichino's formula to give a formula explicitly relating $\langle f \mathrm{g}_\Phi, \mathrm{g}_\Psi \rangle$ to $\mathcal{L}_p^{BDP}(f, \Xi)$, where Ξ is a family of characters constructed from Φ and Ψ.

We will start in the first three sections of this chapter by recalling the foundations and fundamental results of the theory of p-adic and Λ-adic modular forms that we will need, in the generality and formalism we want to use. The remaining three sections we will describe the tools we need to construct $\langle f \mathrm{g}_\Phi, \mathrm{g}_\Psi \rangle$: we will recall Hida's construction of pairings that interpolate Petersson inner products in Section 3.4, we'll construct our families of CM forms explicitly in Section 3.5, and finally in Section 3.6 we'll show that an important computation that we need follows from the main conjecture of Iwasawa theory for Hecke characters on an imaginary quadratic field (proven by Rubin [Rub91]).

The theory of p-adic and Λ-adic modular forms was largely developed by Hida, and we follow his writings of the subject (especially [Hid88] and [Hid93], but also [Hid85], [Hid86a] and [Hid86b], as well as Wiles' paper [Wil88]). We approach the subject exclusively by treating these objects as formal q-expansions, and sweep the underlying theory of "geometric modular forms" under the rug. We will also always assume p is an odd prime for simplicity - large parts of the theory still work for $p = 2$ but statements of results become more complicated, largely because we repeatedly use the isomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \cong Z_{p-1} \times Z_{p^{r-1}}$ for odd primes (which is false for $p = 2$).

To start off, consider the spaces of classical modular forms $M_k(\Gamma, \chi) = M_k(\Gamma, \chi; \mathbb{C})$ or cusp forms $S_k(\Gamma, \chi) = S_k(\Gamma, \chi; \mathbb{C})$ for a weight k, a congruence subgroup Γ, and character χ (in the case of $\Gamma = \Gamma_0(N)$). For any subalgebra $A \subseteq \mathbb{C}$ we can define A-submodules $M_k(\Gamma, \chi; A)$ consisting of the forms with Fourier coefficients lying in A, and similarly $S_k(\Gamma, \chi; A)$ for cusp forms. One way to formalize this is to use the q-expansion principle to embed these spaces of modular forms into into an appropriate space $\mathbb{C}[\![q^{1/M}]\!]$, and then define $M_k(\Gamma, \chi; A)$ as $M_k(\Gamma, \chi; \mathbb{C}) \cap A[\![q^{1/M}]\!]$.

Once we've done this, the fundamental result on integrality of newform coefficients tells us that for any ring A containing the image of χ we have bases of M_k and S_k with coefficients in A. Therefore we can conclude

$$M_k(\Gamma, \chi; A) \otimes_A \mathbb{C} \cong M_k(\Gamma, \chi; \mathbb{C}) \qquad S_k(\Gamma, \chi; A) \otimes_A \mathbb{C} \cong S_k(\Gamma, \chi; \mathbb{C})$$

where these isomorphisms are induced by the obvious inclusion maps. Applying this, we can change our scalars to a p-adic field F, and end up with a space $M_k(\Gamma, \chi; F)$ that we can consider as having a common basis with $M_k(\Gamma, \chi; \mathbb{C})$.

Definition 3.1.1. Fix a weight k, a congruence subgroup Γ, and a character χ. If $F/\mathbb{Q}_p$ is a p-adic field containing the image of χ and $F_0 \subseteq F$ is a number field with

F as its completion (which also contains the image of χ), and we let $\mathcal{O}_F$ and $\mathcal{O}_{F_0}$ be the integer rings of F and F_0, respectively, then we can define

$$M_k(\Gamma, \chi; \mathcal{O}_F) = M_k(\Gamma, \chi; \mathcal{O}_{F_0}) \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F \qquad M_k(\Gamma, \chi; F) = M_k(\Gamma, \chi; F_0) \otimes_{F_0} F$$

One can then check that this space is independent of the choice of field F_0. Now, let $F_b[\![q^{1/M}]\!]$ be the subring of $F[\![q^{1/M}]\!]$ consisting of formal power series $\sum a_n q^{n/M}$ with $a_n \in F$, but such that $\{|a_n| : n \in \mathbb{Z}\}$ is a bounded set. The fact that each $M_k(\Gamma, \chi; \mathbb{C})$ has a basis of forms with algebraic integer Fourier coefficients implies that any element in $M_k(\Gamma, \chi; F)$ is a finite F-linear combination of such, so $M_k(\Gamma, \chi; F)$ in fact embeds into $F_b[\![q^{1/M}]\!]$. Then, on $F_b[\![q^{1/M}]\!]$ we can define a norm

$$\left| \sum_{n \in \mathbb{N}} a_n q^{m/N} \right|_p = \sup\{|a_n|_p : n \in \mathbb{Z}\};$$

this is a well-defined real number by our boundedness assumption, and it makes $F_b[\![q^{1/M}]\!]$ into a F-Banach space.

We will define the space of p-adic modular forms (over F) as a certain closed subspace of $F_b[\![q^{1/M}]\!]$, which will thus be a p-adic Banach space. Any individual space $M_k(\Gamma, \chi; F)$ is finite-dimensional and thus already closed; what we will do is take the closure of some sort of infinite-dimensional space of modular forms. There are two natural ways we can produce these: by varying the weight, or by varying the level. It turns out that both ways will generally produce the same closure! We start by varying the weight: we define

$$M(\Gamma, \chi; F) = M_{\leq\infty}(\Gamma, \chi; F) = \bigoplus_{j=0}^{\infty} M_j(\Gamma, \chi; F),$$

and similarly for cusp forms and/or $\mathcal{O}_F$-coefficients. By the q-expansion principle, these spaces still embed in $F[\![q^{1/M}]\!]$, and thus we can define:

Definition 3.1.2. Fix a congruence subgroup Γ, a character, and a p-adic field $F/\mathbb{Q}_p$ containing the values of χ. For $A = F, \mathcal{O}_F$ we define the spaces $\overline{M}(\Gamma, \chi; A)$ of *p-adic modular forms* and $\overline{S}(\Gamma, \chi; A)$ of *p-adic cusp forms* with coefficients in A as the closure of the space $M(\Gamma, \chi; A)$ or $S(\Gamma, \chi; A)$, respectively, in $F[\![q^{1/M}]\!]$ with the Banach space topology given above.

Equivalently, $\overline{M}(\Gamma, \chi; A)$ is the completion of $M(\Gamma, \chi; A)$ with respect to the norm induced from $F_b[\![q^{1/M}]\!]$, and for $A = F$ the spaces $\overline{M}(\Gamma, \chi; A)$ and $\overline{S}(\Gamma, \chi; A)$ are again F-Banach spaces. We are most interested in the case of $\Gamma = \Gamma_1(N)$, and we write $\overline{M}(N; A)$ to denote $\overline{M}(\Gamma_1(N); A)$ and similarly for $\overline{S}$. We will also occasionally need to work with the larger space $\overline{M}(\Gamma_1(N, M); A)$.

As mentioned, we could have also fixed the weight k and varied the level of our modular forms and taken a completion that way. However, by a theorem of Katz [Kat76] (using the theory of geometric modular forms), varying the p-part of the level turns out to be equivalent to varying the weight. In particular, we have:

Theorem 3.1.3. *As subspaces of $\mathcal{O}_F[\![q^{1/M}]\!]$, we have*

$$\overline{S}(\Gamma \cap \Gamma_1(p^r), \mathcal{O}_F) = \overline{S}(\Gamma, \mathcal{O}_F) \qquad \overline{M}(\Gamma \cap \Gamma_1(p^r), \mathcal{O}_F) = \overline{M}(\Gamma, \mathcal{O}_F)$$

for $\Gamma = \Gamma(N), \Gamma_1(N)$, or $\Gamma_1(N, M)$ with N, M prime to p.

Thus if we set

$$M_k(Np^\infty; A) = M_k(\Gamma_1(Np^\infty); A) = \varinjlim_r M_k(\Gamma_1(Np^r); A) = \bigcup_r M_k(\Gamma_1(Np^r); A),$$

the q-expansion principle gives us a natural embedding $M_k(Np^\infty; \mathcal{O}_F) \hookrightarrow \overline{M}(N; \mathcal{O}_F)$ for any N and k (and similarly for cusp forms).

3.1.1 Trace operators and the action of Λ

Now that we've defined spaces of p-adic modular forms, we can discuss various operators and functions on them. We will discuss Hecke operators for p-adic modular forms in the next section; in this one, we'll focus on two other classes of operators that are important.

Let $p \geq 3$ be a prime, and N be an integer coprime to p. To start off, we want to define an action of the group

$$\widehat{Z}_n = \varprojlim_r (\mathbb{Z}/Np^r\mathbb{Z})^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \cong (\mathbb{Z}/Np\mathbb{Z})^\times \times (1+p)^{\mathbb{Z}_p}$$

on $\overline{M}(N;F)$. We do this by first defining an action on $M_k(\Gamma_1(Np^r);F)$ for any k and any $r \geq 0$ by

$$\langle z \rangle f = z_p^k f|_k[\sigma_z] \qquad \sigma_z \in \mathrm{SL}_2(\mathbb{Z}), \sigma_z \equiv \begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix} \pmod{Np^r},$$

where $z \mapsto z_p$ under the projection $\widehat{Z}_n \to \mathbb{Z}_p^\times$; so this is a slightly modified version of the classical action of $(\mathbb{Z}/Np^r\mathbb{Z})^\times$ by diamond operators (hence the notation). We can then check that these actions are all compatible:

Proposition 3.1.4. *The action of $\widehat{Z}_n$ on the spaces $M_k(\Gamma_1(Np^r);F)$ are compatible in the sense that they all extend to a unique action on $\sum_{k,r} M_k(\Gamma_1(Np^r);F)$. Moreover, this extends to a continuous action of $\widehat{Z}_n$ on $\overline{M}(N;F)$. The subspaces $\overline{S}(N;F)$, $\overline{M}(N;\mathcal{O}_F)$, and $\overline{S}(N;\mathcal{O}_F)$ are invariant under this action.*

Since $\overline{M}(N;F)$ is a $\mathcal{O}_F$-module, this group action naturally gives it a $\mathcal{O}_F[\widehat{Z}_n]$-module structure, which in fact extends to a $\mathcal{O}_F[\![\widehat{Z}_n]\!]$-module structure. By just considering the direct factor $(1+p)^{\mathbb{Z}_p}$ of $\widehat{Z}_n$, we get that $\overline{M}(N;F)$ has a Λ-module structure for

$$\Lambda = \mathcal{O}_F[\![(1+p)^{\mathbb{Z}_p}]\!] \cong \mathcal{O}_F[\![\mathbb{Z}_p]\!] \cong \mathcal{O}_F[\![X]\!].$$

This module structure will be fundamental for the definition we will give of families of p-adic modular forms! We note that any integer d coprime to Np maps into Λ via the inclusion $(\mathbb{Z}/Np\mathbb{Z})^\times \hookrightarrow \widehat{Z}_N$ and then the projection $\widehat{Z}_N \twoheadrightarrow (1+p)^{\mathbb{Z}_p}$. We denote the image of such an operator as $\langle d\rangle_\Lambda$; note that this is *not* the same as $\langle d\rangle \in \widehat{Z}_N$. In fact, if $\widetilde{\omega} : \widehat{Z}_N \twoheadrightarrow (\mathbb{Z}/Np^r\mathbb{Z})^\times \hookrightarrow \mathcal{O}_F^\times$ is the natural character (serving the same purpose as the Teichmüller character for $N = 1$), we have $\langle d\rangle = \widetilde{\omega}(d)\langle d\rangle_\Lambda$.

One thing that the action of $\widehat{Z}_N$ does is lets us recover the character and weight of the modular form. In particular, if χ is a character of $(\mathbb{Z}/p^rN\mathbb{Z})^\times$ for some $r \geq 1$, then modular form $f \in M_k(Np^r, \chi)$ satisfies

$$\langle z\rangle f = z_p^k(\widetilde{\omega}^{-k}\chi)(z)f$$

for all $z \in \widehat{Z}_N$. Accordingly, we say that a p-adic modular form $f \in \overline{M}(N;F)$ has *weight k and character χ* if it satisfies this identify for all z. Clearly this is a necessary condition for the form to actually lie in $M_k(Np^r, \chi)$, but in general it is not sufficient - a p-adic modular form of weight k and character χ need not be classical of weight k and character χ. We also remark that we can make a number of variants on this definition; for instance we could more generally define the "weight" of a p-adic modular form by only looking at the action of a sufficiently deep subgroup of $(1+p)^{\mathbb{Z}_p}$, or we could define the "tame-at-p character" (independently of the weight) by looking at just the action on $(\mathbb{Z}/Np\mathbb{Z})^\times$.

In particular, if χ is a character of $(\mathbb{Z}/Np\mathbb{Z})^\times$ we can define the space $\overline{M}(N;\mathcal{O}_F)[\chi]$ as the subspace of $\overline{M}(N;\mathcal{O}_F)$ of forms that have tame-at-p character χ, i.e. such that $\langle z\rangle f = \chi(z)f$ for all $z \in (\mathbb{Z}/pN\mathbb{Z})^\times$. One reason for doing this is to define *trace operators*. If $M|N$ are two prime-to-p levels and χ is a character of $(\mathbb{Z}/Mp\mathbb{Z})^\times$, we know that classically we have a trace operator

$$\mathrm{tr} : M_k(Np^r, \chi) \to M_k(Mp^r, \chi)$$

given by $\mathrm{tr}(f) = \sum_\gamma \chi^{-1}(\gamma) f|[\gamma]$, where γ runs over a set of representatives of $\Gamma_0(Np^r)\backslash\Gamma_0(Mp^r)$. Since M, N are prime to p we can choose these representatives to satisfy $\gamma \equiv 1 \pmod{p}^r$; then the action of these γ's matches up with the action of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ discussed in Section 1.IV of [Hid88], which preserves the space $\overline{M}(N; \mathcal{O}_F)$. Thus our classical trace operators have a common extension $\mathrm{tr} : \overline{M}(N; \mathcal{O}_F) \to \overline{M}(M; \mathcal{O}_F)$. Similarly, we have a trace operator

$$\mathrm{tr} : \overline{M}(\Gamma_0(Np^r, M); \mathcal{O}_F) \to \overline{M}(Np^r; \mathcal{O}_F)$$

as discussed in Section 1.V of [Hid88] which is compatible with the trace operators on classical spaces.

3.1.2 Hecke operators

Finally, we define Hecke operators and their associated Hecke algebras for $\overline{M}(N; F)$ and its subspaces. The Hecke operators themselves can be defined from the ones we already have on the spaces of classical modular forms $M_k(\Gamma_1(Np); F)$, or equivalently by using the usual formula on q-expansions:

Proposition 3.1.5. *Fix an integer n. Then we can define a Hecke operator $T(n)$ on $\overline{M}(N; F)$ as the unique continuous extension of the usual Hecke operator $T(n)$ on the sum of all subspaces $M_k(\Gamma_1(Np^r), F)$ for $r > 0$. This can be equivalently described in terms of its Fourier coefficients by*

$$a(m, f|T(n)) = \sum_{d|(m,n),(d,Np)=1} d^{-1} a(mn/d^2, \langle d \rangle_\Lambda f).$$

where $f|d$ denotes the action of $d \in \mathbb{Z}_p^\times \subseteq \widehat{Z}_N$ discussed in the previous section. The subspaces $\overline{S}(N;F)$, $\overline{M}(N;\mathcal{O}_F)$, and $\overline{S}(N;\mathcal{O}_F)$ are invariant under this action.

We then define the Hecke algebra $\mathbb{T}(\overline{M}(N;F)) \subseteq \mathrm{End}_{F\text{-cont}}(\overline{M}(N;F))$ as the F-subalgebra generated by the Hecke operators. Evidently the restriction map from $\overline{M}(N;F)$ to any space $M_k(\Gamma_1(Np^r);F)$ induces a surjection of $\mathbb{T}(\overline{M}(N;F))$ onto $\mathbb{T}(M_k(\Gamma_1(Np^r);F))$ taking $T(n)$ to $T(n)$ for all n, and we can in fact check that $\mathbb{T}(\overline{M}(N;F))$ is the inverse limit of finite-dimensional algebras $\mathbb{T}(M_{\leq k}(\Gamma_1(Np^r);F))$ (varying k and/or r). The same statements hold for Hecke algebras of $\overline{S}(N;F)$, and also for replacing F with $\mathcal{O}_F$ in each case.

Next, we recall that for finite-dimensional spaces we have a perfect pairing

$$M_k(\Gamma_1(Np^r);\mathcal{O}_F) \times \mathbb{T}(M_k(\Gamma_1(Np^r),\mathcal{O}_F)) \to \mathcal{O}_F$$

given by $(f,t) \mapsto a(1,f|t)$. The same formula induces a perfect pairing $\overline{M}(N;\mathcal{O}_F) \times \mathbb{T}(\overline{M}(N;\mathcal{O}_F)) \to \mathcal{O}_F$, and thus we have isomorphisms

$$\mathbb{T}(\overline{M}(N;\mathcal{O}_F)) \cong \mathrm{Hom}_{\mathcal{O}_F}(\overline{M}(N;\mathcal{O}_F),\mathcal{O}_F),$$

$$\overline{M}(N;\mathcal{O}_F) \cong \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{T}(\overline{M}(N;\mathcal{O}_F)),\mathcal{O}_F);$$

and similarly for cusp forms; see Theorem 1.3 of [Hid88]. We will use this duality repeatedly in the next section.

Finally, we note that the automorphisms of $\overline{M}(N;\mathcal{O}_F)$ arising from the action of $\widehat{Z}_N$ defined in the previous section all lie in $\mathbb{T}(\overline{M}(N;\mathcal{O}_F))$; this can be deduced from checking that if $\ell \nmid Np$ is a prime then our formula for Hecke operators gives that the action of $\ell \in \widehat{Z}_N$ is given by the Hecke operator $\ell(T(\ell)^2 - T(\ell^2))$. Thus we have a natural map $\widehat{Z}_N \to \mathbb{T}(\overline{M}(N;\mathcal{O}_F))$, which extends to a homomorphism $\mathcal{O}_F[\![\widehat{Z}_N]\!] \to \mathbb{T}(\overline{M}(N;\mathcal{O}_F))$. In particular this makes $\mathbb{T}(\overline{M}(N;\mathcal{O}_F))$ into a Λ-algebra.

3.2 p-adic families of modular forms

Now that we've set up the basic theory of p-adic modular forms, in this section we develop the theory of p-adic families of them, following Hida. These families will be parametrized by the ring Λ we introduced in the previous section, or certain (finite flat) extensions of it. We start in the case of Λ itself.

3.2.1 Λ-adic modular forms

Recall that, given a finite extension $F/\mathbb{Q}_p$ we're working over, Λ was defined as $\mathcal{O}_F[\![\Gamma]\!]$ for $\Gamma = (1+p)^{\mathbb{Z}_p} \subseteq \mathbb{Z}_p^\times$ abstractly isomorphic to $\mathbb{Z}_p$. Moreover we know Λ is abstractly isomorphic to the formal power series ring $\mathcal{O}_F[\![X]\!]$; if we pick a topological generator γ of Γ (usually $\gamma = 1+p$) then the isomorphism is determined by $\gamma \leftrightarrow 1+X$.

Using the description of Λ as a power series ring, we know that the set of continuous $\mathcal{O}_F$-algebra homomorphisms $\mathrm{Hom}(\Lambda, \mathcal{O}_F)$ is in bijection with elements of the maximal ideal $\mathfrak{m}_F \subseteq \mathcal{O}_F$, by associating $x \in \mathfrak{m}_F$ to the homomorphism $\Lambda \to \mathcal{O}_F$ characterized by $X \mapsto x$. In the framework of rigid geometry, this means that Λ is the coordinate ring of the open unit disc, and its F-valued points are the homomorphisms $\Lambda \to \mathcal{O}_F$. Motivated by this, we think of of an element $f \in \Lambda$ as an analytic function on the open unit disc, which we can evaluate at a point $P \in \mathrm{Hom}(\Lambda, \mathcal{O}_F)$ by taking $P(f)$.

Given this setup, for an integer k we define a distinguished point $P_k \in \mathrm{Hom}(\Lambda, \mathcal{O}_F)$ by $P_k(X) = (1+p)^k - 1$, or equivalently $P_k(\gamma) = (1+p)^k$. Then, following the formalism of Wiles [Wil88], we define a Λ-adic modular form to be a formal q-expansion with coefficients in Λ, such that evaluating it at a point P_k gives a classical modular form of weight k.

Definition 3.2.1. A Λ-*adic modular form* of level N (for $p \nmid N$) and character χ (a Dirichlet character modulo Np) is a formal power series $\mathbb{f} = \sum A_n q^n \in \Lambda[\![q]\!]$ such that, for all but finitely many $k \geq 1$, the following is satisfied:

- The formal power series $\mathbb{f}_k = P_k(\mathbb{f}) = \sum P_k(A_n)q^n$ is in fact a classical modular form lying in the space $M_k(Np, \chi\omega^{-k}; \mathcal{O}_F)$.

If all but finitely many $\mathbb{f}_k$'s are actually cusp forms, we say $\mathbb{f}$ is a Λ-adic cusp form. We let $\mathbb{M}(N;\Lambda)$ denote the set of all Λ-adic modular forms of level N and character χ, and $\mathbb{S}(Np;\Lambda)$ the set of Λ-adic cusp forms; these are evidently sub-Λ-modules of $\Lambda[\![q]\!]$.

An alternative way to formalize this concept is through the idea of *measures*. This requires a bit of setup:

Definition 3.2.2. Let X be a (compact) topological space, and let $C(X;\mathcal{O}_F)$ be the compact p-adic Banach space of all continuous functions $X \to \mathcal{O}_F$ with the sup-norm. If M is a $\mathcal{O}_F$-Banach space, we define the space of *M-valued measures on X* as the space

$$\mathrm{Meas}(X;\mathcal{O}_F) = \mathrm{Hom}_{\mathcal{O}_F\text{-cont}}(C(X;\mathcal{O}_F), M).$$

This definition is by formal analogy with real-valued measure theory; a measure (in the classical sense) on a compact space is equivalently determined by the continuous $\mathbb{R}$-linear integration functional $C(X,\mathbb{R}) \to \mathbb{R}$. In the literature this analogy is sometimes emphasized by writing measures (in our sense) as $f \mapsto \int f d\mu$, but we'll just use $f \mapsto \mu(f)$ to denote the continuous homomorphism we're calling a "measure".

The reason that measures come up naturally in our context is that the ring $\Lambda = \mathcal{O}_F[\![\Gamma]\!]$ itself can be viewed as a space of them. We let $\log_\Gamma : (1+p)^{\mathbb{Z}_p} \to \mathbb{Z}_p$ be the isomorphism $(1+p)^x \mapsto x$; this is not equal to the usual p-adic logarithm but is a scalar multiple of it. Then, the following result is an easy consequence of Mahler's theorem (which says that all continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ can be written as a series $x \mapsto \sum a_k \binom{x}{k}$).

Lemma 3.2.3. *We have* $\mathrm{Meas}(\Gamma, \mathcal{O}_F) \cong \Lambda$*, via the map sending a power series* $A = \sum a_n X^n \in \mathcal{O}_F[\![X]\!] \cong \Lambda$ *to the measure* μ_A *that takes the function* $x \mapsto \log_\Gamma(x) \mapsto$

$\binom{\log_\Gamma(x)}{n}$ *to the value* a_n*. Under this isomorphism, the action of* $\gamma \in \Gamma$ *by multiplication on* Λ *corresponds to the action of* Γ *on* $\mathrm{Meas}(\Gamma, \mathcal{O}_F)$ *described by* $(\gamma \cdot \mu)(f) = \mu(x \mapsto f(\gamma x))$*.*

Proof. We prove the first part by showing the additive version, that $\mathcal{O}_F[\![X]\!] \cong \mathrm{Meas}(\mathbb{Z}_p, \mathcal{O}_F)$ via mapping $A = \sum a_n X^n$ to μ_A characterized by by $\mu_A(x \mapsto \binom{x}{n}) = a_n$. First of all such a μ_A is well-defined by Mahler's theorem; continuous functions $\mathbb{Z}_p \to \mathcal{O}_F$ can be uniquely written as $\sum b_n \binom{x}{n}$ with $b_n \in \mathcal{O}_f$ satisfying $|b_n|_p \to 0$, so we can define μ_A on the whole domain $C(\mathbb{Z}_p, \mathcal{O}_F)$ by

$$\mu_A\left(\sum b_n \binom{x}{n}\right) = \sum a_n b_n,$$

which converges because $|a_n| \leq 1$ and $|b_n| \to 0$. It's then straightforward to check linearity and continuity of μ_A, and then linearity and continuity of $A \mapsto \mu_A$. Injectivity follows because if $\mu_A = 0$ then necessarily $a_n = \mu_A(\binom{x}{n}) = 0$ and thus $A = 0$, and surjectivity because any μ is μ_A for $A = \sum \mu(\binom{x}{n})X^n$.

Translating back to the multiplicative situation gives $\Lambda \cong \mathrm{Meas}(\Gamma, \mathcal{O}_F)$ by the formula specified. Then, we note that topological generator $\gamma_p \in \Gamma$ corresponds to $(1+X) \in \mathcal{O}_F[\![X]\!]$ under our isomorphism, so the action of γ_p on Λ is given by taking $A = \sum a_n X^n$ to $(1+X)A = \sum (a_n + a_{n-1})X^n$. Then

$$\begin{aligned} \mu_{(1+X)A}\left(x \mapsto \binom{\log_\Gamma(x)}{n}\right) &= a_n + a_{n-1} \\ &= \mu_A\left(x \mapsto \binom{\log_\Gamma(x)}{n}\right) + \mu_A\left(x \mapsto \binom{\log_\Gamma(x)}{n-1}\right), \end{aligned}$$

which by basic identities involving binomial polynomials is

$$\mu_{(1+X)A}\left(x \mapsto \binom{\log_\Gamma(x)}{n}\right) = \mu_A\left(x \mapsto \binom{\log_\Gamma(x)+1}{n}\right) = \mu_A\left(x \mapsto \binom{\log_\Gamma(\gamma_p x)}{n}\right).$$

By continuity and linearity we get $\mu_{(1+X)A}(x \mapsto f(x))$ equals $\mu_A(x \mapsto f(\gamma_p x))$ for any x, and thus the action of Γ on Λ translates to the specified action of the topological generator γ_p and thus of all of Γ on $\mathrm{Meas}(\Gamma, \mathcal{O}_F)$. □

Using this isomorphism, we can then see that if A is an element of Λ, taking the specialization $P_m(A)$ is the same as evaluating the measure μ_A under the continuous function $\Gamma \to \mathcal{O}_F$ given by $x \mapsto x^m$. Furthermore, we can also check that the above isomorphism extends to an isomorphism

$$\mathrm{Meas}(\Gamma, \mathcal{O}_F[\![q]\!]) \cong \Lambda[\![q]\!],$$

again such that if $A \leftrightarrow \mu_A$ then the specialization $P_k(M) \in \mathcal{O}_F[\![q]\!]$ is equal to $\mu_A(x \mapsto x^m)$.

Thus, a Λ-adic modular form $\mathbb{f}$ (which is by definition an element of $\Lambda[\![q]\!]$) naturally corresponds to a $\Lambda[\![q]\!]$-valued measure $\mu_{\mathbb{f}}$ on Γ. Moreover, we know that the specializations $\mu_{\mathbb{f}}(x \mapsto x^k)$ actually lie in $\overline{M}(N; \mathcal{O}_F)$ for all $k \gg 0$. Using the following lemma, we can check that this means the whole domain maps into $\overline{M}(N; \mathcal{O}_F)$:

Lemma 3.2.4. *Let M be a Banach space over $\mathcal{O}_F$, and $M' \subseteq M$ a closed subspace that's saturated in the sense that if $m \in M$ satisfies $pm \in M'$, then we have $m \in M'$. If $\mu \in \mathrm{Meas}(\Gamma, M)$ is a measure such that for all $k \geq k_0$, we have $\mu(x \mapsto x^k) \in M'$, then in fact the image of μ lies in M' and thus $\mu \in \mathrm{Meas}(\Gamma, M')$.*

Proof. For convenience we shift from the multiplicative group Γ to the additive group $\mathbb{Z}_p$, and prove the equivalent statement for a measure $\mu \in \mathrm{Meas}(\mathbb{Z}_p, M)$ such that

$$\mu\big(x \mapsto (1+p)^{kx}\big) \in M'$$

for $k \geq k_0$. To get that $\mu(f) \in M'$ for all f, by Mahler's theorem it's sufficient to show that $\mu(\binom{x}{n}) \in M'$ for all n, since the binomial polynomials span a dense subspace and

μ is $\mathcal{O}_F$-linear. To show this, we will use the Mahler product decomposition

$$(1+p)^{kx} = \sum_{m=0}^{\infty} \binom{x}{m} ((1+p)^k - 1)^m.$$

We will also use the following claim about a congruence, and then proceed to show that $\mu(\binom{x}{n}) \in M'$ by induction on n.

Claim: For any N we have $(1+p)^{p^N} \equiv 1 \pmod{p^{N+1}}$ and $\not\equiv 1 \pmod{p^{N+2}}$. By the binomial expansion we have

$$(1+p)^{p^N} = \sum_{i=0}^{p^N} \binom{p^N}{i} p^i,$$

and by a theorem of Kummer it's known that the power of p dividing $\binom{p^N}{i}$ is $N - \mathrm{ord}_p(i)$. Since we certainly always have $\mathrm{ord}_p(i) \leq i-1$ for $i \geq 1$, we have that p^{N+1} divides $\binom{p^N}{i} p^i$ for such i, giving the congruence modulo p^{N+1}. On the other hand, the $i = 1$ term is exactly p^{N+1} and the $i \geq 2$ terms are divisible by p^{N+2} because $\mathrm{ord}_p(i) \leq i - 2$ for such i (since we're assuming $p \neq 2$), so we get that $(1+p)^{p^N} \equiv 1 + p^{N+1} \pmod{p^{N+2}}$.

Base case: $n = 0$. The function $\binom{x}{0}$ is just the constant function 1. By the Mahler product decomposition and the claim that we have $(1+p)^{xp^N} \equiv 1 \pmod{p^{N+1}}$ for any N; since $p^N \geq k_0$ for large enough N this lets is write 1 as a limit of functions that we know satisfy $\mu(f) \in M'$, as desired.

Inductive step: $n > 0$. To show that $\mu(\binom{x}{n}) \in M'$, it's sufficient to find a sequence of functions f_N with $\binom{x}{n} \equiv f_N(x) \pmod{p^N}$ (so $f_N \to \binom{x}{N}$ in $C(\Gamma, \mathcal{O}_F)$) and with $\mu(f_N) \in M'$ (so the limit $\mu(\binom{x}{n})$ also lies in M'). To construct our f_N for N such that $p^{N-1} \geq k_0$, we start with the Mahler product decomposition

$$(1+p)^{p^{N-1}x} = \sum_{m=0}^{\infty} \binom{x}{m} ((1+p)^{p^{N-1}} - 1)^m.$$

By assumption μ takes the function $(1+p)^{p^{N-1}x}$ into M', and by induction μ takes the functions $\binom{x}{m}$ into M' for $m < n$; rearranging we find we have

$$f = \binom{x}{n}((1+p)^{p^{N-1}} - 1)^n + \sum_{m=n+1}^{\infty} \binom{x}{m}((1+p)^{p^{N-1}} - 1)^m$$

for a function f satisfying $\mu(f) \in M'$. Now, by our claim above the term $(1+p)^{p^{N-1}} - 1$ is equal to p^N times a p-adic unit u. So we can rewrite this as

$$f = \binom{x}{n} u^n p^{Nn} + \sum_{m=n+1}^{\infty} \binom{x}{m} u^m p^{Nm}.$$

Dividing through we find that

$$u^{-1}\frac{f}{p^{Nn}} = \binom{x}{n} + \sum_{m=n+1}^{\infty} \binom{x}{m} u^{m-n} p^{N(m-n)}$$

is still a function $\mathbb{Z}_p \to \mathcal{O}_F$. Moreover, by our "saturated" hypothesis we conclude that μ still maps this into M'. So we can take $f_N = u^{-1}f/p^{Nn}$ and conclude that $\mu(f_N) \in M'$ and $f_N \equiv \binom{x}{n} \pmod{p^N}$ as desired. □

We can then apply this with $M = \mathcal{O}_F[\![q]\!]$ and $M' = \overline{M}(N; \mathcal{O}_F)$; we note that this space of p-adic modular forms is saturated because it's the intersection of a K-vector space $\overline{M}(N; K)$ with M. So if $\mathbb{f}$ is a Λ-adic form, we conclude that $\mu_{\mathbb{f}}$ is actually a $\overline{M}(N; \mathcal{O}_F)$-valued measure on Γ by the lemma. We can further analyze it by defining a $(\Lambda \times \Lambda)$-module structure on the module

$$\mathrm{Meas}(\Gamma, \overline{M}(N; \mathcal{O}_F)) \cong \mathrm{Hom}_{\mathcal{O}_F\text{-cont}}(C(\Gamma; \mathcal{O}_F), \overline{M}(N; \mathcal{O}_F))$$

induced by our Λ-actions on the spaces $C(\Gamma; \mathcal{O}_F)$ and $\overline{M}(N; \mathcal{O}_F)$; in particular for

$(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$ and a measure μ we define

$$((\gamma_1, \gamma_2) \cdot \mu)(x \mapsto f(x)) = \langle \gamma_2 \rangle \cdot \mu(x \mapsto f(\gamma_1 x)) \in \overline{M}(N; \mathcal{O}_F).$$

Then, if $\mu = \mu_{\mathbb{f}}$ for a Λ-adic modular form $\mathbb{f}$, we claim that the action of an element (γ, γ^{-1}) is trivial; to check this, note that evaluating $\mathbb{f}$ at $x \mapsto x^k$ gives us a classical modular form $\mathbb{f}_k$ for $k \gg 0$, and then evaluating $(\gamma, \gamma^{-1}) \cdot \mathbb{f}$ gives us

$$((\gamma, \gamma^{-1}) \cdot \mathbb{f})(x \mapsto x^k) = \langle \gamma^{-1} \rangle \mathbb{f}(x \mapsto \gamma^k x^k) = \gamma^k \langle \gamma^{-1} \rangle \mathbb{f}_k = \mathbb{f}_k$$

using linearity of both $\mathbb{f}$ and $\langle \gamma^{-1} \rangle$. So $(\gamma, \gamma^{-1}) \cdot \mathbb{f} = \mathbb{f}$ when evaluated at $x \mapsto x^k$ for $k \gg 0$, and because such functions span a dense subspace we can conclude $(\gamma, \gamma^{-1}) \cdot \mathbb{f} = \mathbb{f}$ as measures and thus Λ-adic modular forms. Since this is true for all $\gamma \in \Gamma$, we conclude that such an $\mathbb{f}$ is invariant under the antidiagonal copy of Λ in $\Lambda \times \Lambda$; we say it's "Λ-invariant" for short. Summing up:

Proposition 3.2.5. *If $\mathbb{f} \in \mathbb{M}(N, \chi)$ is a Λ-adic modular form, then the associated measure $\mu_{\mathbb{f}}$ is valued in $\overline{M}(N; \mathcal{O}_F)$ and is Λ-invariant. Thus we could equivalently define Λ-adic modular forms of this level and character as being Λ-invariant $\overline{M}(N; \mathcal{O}_F)$-valued measures μ such that the specializations $\mu(x \mapsto x^k)$ lie in $M_k(Np, \chi\omega^{-k}; \mathcal{O}_F)$ for all but finitely many $k \geq 2$.*

The space of Λ-invariant measures still naturally has an action of Λ (coming from the quotient of $\Lambda \times \Lambda$ by the antidiagonal Λ where the action is invariant); this is equivalently described by

$$(\gamma \cdot \mu)(x \mapsto f(x)) = \langle \gamma \rangle \mu(x \mapsto f(x)) = \mu(x \mapsto f(\gamma x)).$$

This resulting Λ-action on Λ-invariant measures corresponds to the natural Λ-action on Λ-adic modular forms coming from scalar multiplication.

The point of view of measures makes it clear that if $\mathbb{f}$ is a Λ-adic form, all of the specializations $\mathbb{f}_k = P_k(\mathbb{f}) = \mu_{\mathbb{f}}(x \mapsto x^k)$ satisfy the appropriate transformation property to be p-adic modular forms of weight k and character $\chi\omega^{-k}$. However they may not be classical forms!

Finally, we note that using measures makes it easy to define Hecke algebras for Λ-adic forms. In fact, the Hecke algebra $\mathbb{T}(\overline{M}(N; \mathcal{O}_F))$ naturally acts on $\mathbb{M}(N, \chi)$! We can define a pairing

$$\mathbb{T}(\overline{M}(N; \mathcal{O}_F)) \times \mathbb{M}(N, \chi) \to \Lambda$$

by mapping $(T, \mathbb{f})$ to $T\mathbb{f}$ where $\mu_{T\mathbb{f}}$ is determined in terms of $\mu_{\mathbb{f}}$ by $\mu_{T\mathbb{f}}(f) = T \cdot \mu_{\mathbb{f}}(f)$. This is evidently a Λ-invariant pairing so induces a map $\mathbb{T}(\overline{M}(N; \mathcal{O}_F)) \to \operatorname{End}_{\Lambda}(\mathbb{M}(N, \chi))$. We define the image of this map to be $\mathbb{T}(\mathbb{M}(N, \chi))$; it's generated by the operators $T(n)$ which we can check act on the q-expansions in $\Lambda[\![q]\!]$ by

$$a(m, \mathbb{f}|T(n)) = \sum_{d|(m,n),(d,Np)=1} \langle d \rangle_{\Lambda} d^{-1} a(mn/d^2, \mathbb{f}),$$

where now $\langle d \rangle_{\Lambda}$ is just treated as a scalar in Λ. In fact it's easy to see that the surjection $\mathbb{T}(\overline{M}(N; \mathcal{O}_F)) \twoheadrightarrow \mathbb{T}(\mathbb{M}(N, \chi))$ factors through to the Hecke algebra

$$\mathbb{T}(\overline{M}(N; \mathcal{O}_F)[\chi]) \cong \frac{\mathbb{T}(\overline{M}(N; \mathcal{O}_F))}{\big(\langle z \rangle - \chi(z) : (\mathbb{Z}/N\mathbb{Z})^\times\big)}.$$

for the submodule

$$\overline{M}(N; \mathcal{O}_F)[\chi] = \{f \in \overline{M}(N; \mathcal{O}_F) : \forall z \in (\mathbb{Z}/N\mathbb{Z})^\times, \langle z \rangle f = \chi(z) f\}.$$

However, we still won't necessarily get an equality here.

3.2.2 $\mathcal{I}$-adic modular forms

We now want to expand our discussion of Λ-adic forms in two ways: first, by allowing coefficients to lie in certain extensions $\mathcal{I} \supseteq \Lambda$ (which will ultimately be needed for what we want to construct), and secondly by dealing with specializations of forms in a more sophisticated way (which is helpful for making sense of the generalization from Λ to $\mathcal{I}$).

We start with the second topic. Recall that if we fix a topological generator γ of Γ, homomorphisms $\Lambda \to \mathcal{O}_F$ are in bijection with elements of $\mathfrak{m}_F$ by having an element $x \in \mathfrak{m}_F$ correspond to the unique homomorphism $\Lambda \to \mathcal{O}_F$ given by $\gamma \mapsto 1 + x$. We set up some notation:

Definition 3.2.6. We let $\mathcal{X}(\Lambda, \mathcal{O}_F)$ denote the set of all homomorphisms $P : \Lambda \to \mathcal{O}_F$, which is naturally in bijection with $\mathfrak{m}_F$ as above. Actually, any such point P can be thought of in three different ways that we'll pass between freely:

- As a $\mathcal{O}_F$-linear homomorphism $\Lambda \to \mathcal{O}_F$, characterized by $\gamma \mapsto x$
- As the kernel of such a homomorphism, which is a height-one prime ideal of Λ.
- As a generator of such a kernel, namely $\gamma - x$ (where x is the image of γ under the homomorphism).

With this set up, we can define a distinguished subset of $\mathcal{X}(\Lambda, \mathcal{O}_F)$.

Definition 3.2.7. We define $\mathcal{X}_{\text{alg}}(\Lambda, \mathcal{O}_F)$ as the subset of $\mathcal{X}(\Lambda, \mathcal{O}_F)$ consisting of all points $P_{k,\varepsilon} = \varepsilon(\gamma) \cdot (1+p)^k$ for $k \geq 0$ and $\varepsilon : \Gamma \to \mathcal{O}_F^\times$ a finite-order character. Given such a point $P = P_{k,\varepsilon}$, we write $k(P) = k$, $\varepsilon_P = \varepsilon$, and $r(P) = r$ where r is the conductor of ε (i.e. the kernel of ε is $\gamma^{p^{r-1}}\Gamma$).

Our definition of Λ-adic modular forms only mentioned the specializations at $P_k = P_{k,\varepsilon_0}$, for the trivial character ε_0. The Λ-invariance of the associated measure tells us that specializations at other points $P_{k,\varepsilon}$ would have the appropriate transformation property to be a p-adic modular form of the weight and character we'd expect, but it wouldn't let us conclude that these forms are classical. One could give a similar definition requiring classical behavior at some larger subset of the algebraic points; this more restrictive definition would give subspace of Λ-adic forms, and one can analyze its relation to the original space. However, this point is not particularly important for this thesis, so we will continue working with our original definition (only requiring classicality at all but finitely many of the points P_k).

Next, we consider enlarging our base ring Λ. One way is to expand from $\Lambda_F = \mathcal{O}_F[\![\Gamma]\!]$ to $\Lambda_L = \mathcal{O}_L[\![\Gamma]\!]$ for L/F a finite extension; this is not particularly interesting since everything we've done before works just as well over a different base field from F. More interesting is considering other sorts of extensions of Λ; the most general case one could reasonably work with would be to take finite flat extensions $\mathcal{I}/\Lambda$. We'll consider the following setup, as used by Hida in [Hid88].

Definition 3.2.8. For the following, we consider $\Lambda = \Lambda_F$ for a finite extension $F/\mathbb{Q}_p$, and let $\mathcal{F} = Q(\Lambda)$ be the quotient field. Then, we consider a finite extension $\mathcal{L}/\mathcal{F}$, and let $\mathcal{I}$ be the integral closure of Λ in $\mathcal{L}$. We further assume:

1. The relative algebraic closure of F in $\mathcal{L}$ is still F.

2. There are infinitely many points in the set $\mathcal{X}_{\text{alg}}(\mathcal{I}; \mathcal{O}_F)$ defined below.

Neither of the conditions are particularly serious, since they can always be made to hold by making F larger. (For the first property we just increase F to whatever its algebraic closure in $\mathcal{L}$ was; the second we'll discuss below). The set $\mathcal{X}_{\text{alg}}(\mathcal{I}; \mathcal{O}_F)$ mentioned is just defined as points of $\mathcal{I}$ "lying over" algebraic points of Λ:

Definition 3.2.9. We define $\mathcal{X}_{\text{alg}}(\mathcal{I}; \mathcal{O}_F)$ to be the set of points $P \in \mathcal{X}(\mathcal{I}; \mathcal{O}_F) = \text{Hom}(\mathcal{I}, \mathcal{O}_F)$ such that the restriction $P|_\Lambda$ lies in $\mathcal{X}(\Lambda, \mathcal{O}_F)$. We often abuse notation and let $P_{k,\varepsilon}$ denote any point of $\mathcal{X}(\mathcal{I}; \mathcal{O}_F)$ lying over the point $P_{k,\varepsilon}$ in $\mathcal{X}_{\text{alg}}(\Lambda, \mathcal{O}_F)$.

We note that there may not be a point of $\mathcal{X}(\mathcal{I};\mathcal{O}_F)$ lying over any given $P_{k,\varepsilon}$: $\Lambda \to \mathcal{O}_F$; however at worst there is always an extension $\mathcal{I} \to \mathcal{O}_M$ for some finite extension M/F of degree at most $[\mathcal{L}:\mathcal{F}]$. Since there's only finitely many extensions of bounded degree of a p-adic field, we can guarantee that $\mathcal{X}_{\text{alg}}(\mathcal{I};\mathcal{O}_F)$ contains a point lying over every $P_{k,\varepsilon}$ by replacing F by the composite of the (finitely many) fields M such that some $P \in \mathcal{X}_{\text{alg}}(\Lambda;\mathcal{O}_F)$ extends to $\mathcal{I} \twoheadrightarrow \mathcal{O}_M$. In particular this lets us guarantee that $\mathcal{X}_{\text{alg}}(\mathcal{I};\mathcal{O}_F)$ is infinite. We then make the following definition, entirely analogous to the one for Λ.

Definition 3.2.10. A *$\mathcal{I}$-adic modular form* of level N (for $p \nmid N$) and character χ (a Dirichlet character modulo Np) is a formal power series $\mathbb{f} = \sum A_n q^n \in \mathcal{I}[\![q]\!]$ such that, for all but finitely many $k \geq 1$, the following is satisfied:

- For any point $P_k \in \mathcal{X}_{\text{alg}}(\mathcal{I},\mathcal{O}_F)$ lying over $P_k \in \mathcal{X}_{\text{alg}}(\Lambda,\mathcal{O}_F)$, the formal power series $P_k(\mathbb{f}) = \sum P_k(A_n)q^n$ is in fact a classical modular form lying in the space $M_k(Np,\chi\omega^{-k};\mathcal{O}_F)$.

If all but finitely many $\mathbb{f}_k$'s are actually cusp forms, we say $\mathbb{f}$ is a $\mathcal{I}$-adic cusp form. We let $\mathbb{M}(N;\mathcal{I})$ denote the set of all $\mathcal{I}$-adic modular forms of level N and character χ, and $\mathbb{S}(N;\mathcal{I})$ the set of $\mathcal{I}$-adic cusp forms.

Again, we can define Hecke algebras for these spaces; the most convenient way for our purposes here is simply to define Hecke operators $T(n)$ on Fourier expansions in the same way we computed they worked for Λ; so given a $\mathcal{I}$-adic modular form $\mathbb{f}$, we define $\mathbb{f}|T(n)$ as the q-expansion

$$a(m,\mathbb{f}|T(n)) = \sum_{d|(m,n),(d,Np)=1} \langle d\rangle_\Lambda d^{-1} a(mn/d^2,\mathbb{f});$$

this $T(n)$ is compatible with classical Hecke operators under specialization at P_k by construction, and thus $\mathbb{f}|T(n)$ is indeed a $\mathcal{I}$-adic modular form again. We define the

Hecke algebras $\mathbb{T}(\mathbb{M}(N,\chi;\mathcal{I}))$ as the $\mathcal{I}$-subalgebra of $\mathrm{End}_{\mathcal{I}}(\mathbb{M}(N,\chi;\mathcal{I}))$ generated by the $T(n)$'s, and similarly for cusp forms.

We'd like to see that we have an action of $\mathbb{T}(\overline{M}(N;\mathcal{O}_F))$ that factors through the action of $\mathbb{T}(\mathbb{M}(N,\chi;\mathcal{I}))$, as we did in the case of $\mathcal{I}=\Lambda$. This seems harder to get in this generality since we don't have a direct description of $\mathcal{I}$-adic forms as measures valued in p-adic modular forms. One approach to obtaining this might be to studying the ordinary theory as described in the next section and prove the desired statement for the ordinary Hecke algebra, then see if that helps with the general case. Fortunately, in certain special cases this isn't necessary, and we can define an action of $\mathbb{T}(\overline{M}(N;\chi))$ more directly:

- If we know $\mathbb{M}(N,\chi;\mathcal{I})$ is actually equal to the naturally embedded submodule $\mathbb{M}(N,\chi)\otimes_{\Lambda}\mathcal{I}$, we can use the natural action of $\mathbb{T}(\overline{M}(N;\mathcal{O}_F))$ on this submodule.
- If we know $\mathcal{I}$ is itself isomorphic to a space of measures $\mathrm{Meas}(\Gamma',\mathcal{O}_F)$ for Γ' containing Γ as a finite-index subgroup, then we can realize $\mathcal{I}$-adic modular forms as $\overline{M}(N;\chi)$-valued measures and define an action of $\mathbb{T}(\overline{M}(N;\chi))$ that way.

For the cases we deal with in this thesis, we will always have $\mathcal{I}$ fall into the latter category and thus can define Hecke actions that way. Along the same lines, in these cases we can use our trace operators such as $\mathrm{tr}:\overline{M}(N;\mathcal{O}_F)[\chi]\to\overline{M}(M;\mathcal{O}_F)$ discussed in the previous section to obtain trace operators for $\mathcal{I}$-adic modular forms, by defining $\mathrm{tr}(\mathbb{f})$ by specifying its measure as $\mu_{\mathrm{tr}(\mathbb{f})}=\mathrm{tr}\circ\mu_{\mathbb{f}}$. So if $M|N$ are prime to p and χ is Dirichlet character for $(\mathbb{Z}/M\mathbb{Z})^\times$ we get a trace operator

$$\mathrm{tr}:\mathbb{M}(N,\chi;\mathcal{I})\to\mathbb{M}(M,\chi;\mathcal{I})$$

that's compatible with the classical ones on specializations, and along the same lines we get a trace operator

$$\mathrm{tr} : \mathbb{M}(\Gamma_0(N, M), \chi; \mathcal{I}) \to \mathbb{M}(N, \chi; \mathcal{I}).$$

3.3 The ordinary theory

Hida's theory of p-adic and Λ-adic modular forms largely focuses on *ordinary* forms. In this section we define what these forms are and recall some of Hida's results about them that we'll need to use. Throughout, we'll let $\mathcal{I}$ be an extension of Λ of the type defined in the previous section.

If f is a classical modular form which is an eigenform of the $T(p)$ operator with eigenvalue λ_p, we say f is *ordinary* (or *p-ordinary* to emphasize the prime) if λ_p is a p-adic unit. We work with this idea by using the *ordinary idempotent* operator.

Definition 3.3.1. If $\mathbb{T} = \mathbb{T}(M)$ is the Hecke algebra associated to a space of modular forms M over $\mathcal{O}_F$ or F (for F a p-adic field), we define its *ordinary idempotent* e as the unique idempotent $e \in \mathbb{T}$ such that $eT(p)$ is a unit in $e\mathbb{T}$ and $(1-e)T(p)$ is topologically nilpotent in $(1-e)T(p)$. We define the ordinary Hecke algebra $\mathbb{T}^{\mathrm{ord}}$ to be the direct factor $e\mathbb{T}$, and the ordinary subspace M^{ord} of M to be the image $e[M]$.

For classical spaces $M_k(N, \chi; \mathcal{O}_F)$, one can construct this e by noting that the Hecke algebra is finite-dimensional over F and thus decomposes as a finite product of local rings; thus e can just be the projection onto those local rings in which $T(p)$ acts as a unit. By taking inverse limits we can obtain an e for $\overline{M}(N; \mathcal{O}_F)$ and then we can further get one for $\mathbb{M}(N, \chi; \mathcal{I})$ by using the surjection of Hecke algebras we have. Alternatively, we can define $e = \lim_{n\to\infty} T(p)^{n!}$ (interpreted in an appropriate way in each of our contexts). However we define it, we note that the e's are compatible between inclusions of different spaces of p-adic modular forms, and commute with specializations of $\mathcal{I}$-adic forms (i.e. $P(e\mathbb{f}) = e \cdot P(\mathbb{f})$). Also, all of these statements are the same cusp forms in place of holomorphic modular forms.

The utility of working with ordinary forms is that there is a fundamental finiteness result due to Wiles [Wil88], that $\mathbb{M}^{\text{ord}}(N, \chi; \mathcal{I})$ and $\mathbb{S}^{\text{ord}}(N, \chi; \mathcal{I})$ are free of finite rank over $\mathcal{I}$. Building on this, we can get many nice statements about ordinary forms that fail in generality - for instance that when we specialize an ordinary $\mathcal{I}$-adic form at an algebraic point, we always get a classical form. Thus many of the subtleties of working with more general forms disappear, e.g. the difference between requiring classical specializations at all points of $\mathcal{X}_{\text{alg}}(\mathcal{I}; \mathcal{O}_F)$ or just the ones of the form P_k. Moreover, in some cases we can even work back to say new things about the general case! We don't concern ourselves too much with these results, since we won't use them directly; we just quote the following statement that we will need to apply.

Proposition 3.3.2. *The inclusion* $\mathbb{S}^{\text{ord}}(N, \chi; \Lambda) \otimes_\Lambda \mathcal{I} \hookrightarrow \mathbb{S}^{\text{ord}}(N, \chi; \mathcal{I})$ *is actually an equality. Thus we also get an isomorphism*

$$\mathbb{T}(\mathbb{S}^{\text{ord}}(N, \chi; \mathcal{I})) \cong \mathbb{T}(\mathbb{S}^{\text{ord}}(N, \chi; \Lambda)) \otimes_\Lambda \mathcal{I}.$$

For notational convenience we'll take $\mathbb{T}^{\text{ord}}(N, \chi; \mathcal{I})$ to denote $\mathbb{T}(\mathbb{S}^{\text{ord}}(N, \chi; \mathcal{I}))$, the Hecke algebra of the space of ordinary $\mathcal{I}$-adic forms; the above proposition combined with an earlier fact tells us that $\mathbb{T}(\overline{S}^{\text{ord}}(N; \mathcal{O}_F)) \otimes_\Lambda \mathcal{I}$ surjects onto $\mathbb{T}(\mathbb{S}^{\text{ord}}(N, \chi; \mathcal{I}))$. We will use this fact to translate results from [Hid88] (formulated in terms of homomorphisms out of $\mathbb{T}(\overline{S}^{\text{ord}}(N; \mathcal{O}_F)) \otimes_\Lambda \mathcal{I}$).

3.3.1 Ordinary $\mathcal{I}$-adic newforms

An important type of $\mathcal{I}$-adic modular form is one such that the specializations are classical newforms; we quote Hida's results on such forms from [Hid88] (translated into our setup). Naively, we might want to say $\mathbb{f}$ is a $\mathcal{I}$-adic newform if all of the specializations $P_{k,\varepsilon}(\mathbb{f})$ are actual newforms. However, this won't quite work; some of our specializations "should be" newforms of level N, but the $T(p)$ operator for such a prime-to-p level doesn't match up with the $T(p)$ operators we're using (which always are taken for levels $p^r N$ with $r \geq 1$). However, if we have a p-ordinary newform of level N, it turns out there's a canonical way to associate to it a form of level pN that's an eigenform for the Hecke algebra of that level.

Lemma 3.3.3. *Suppose $p \nmid N_0$ and $f \in S_k(N_0, \chi)$ is a p-ordinary newform. Then the polynomial*

$$x^2 - a_p(f)x + p^{k-1}\chi(p)$$

has roots α and β with $|\alpha|_p = 1$ and $|\beta|_p < 1$, respectively, and the space $U(f) = \mathbb{C}f(z) \oplus \mathbb{C}f(pz) \subseteq S_k(N_0 p, \chi)$ contains two forms

$$f^\sharp(z) = f(z) - \beta f(pz) \qquad\qquad f^\flat(z) = f(z) - \alpha f(pz)$$

which are eigenforms of $T(p) \in \mathbb{T}(S_k(N_0 p, \chi))$ with eigenvalues α and β, respectively.

In particular, the form $f^\sharp$ is p-ordinary; we call it the *p-stabilization* of the p-ordinary newform f.

Proof. By definition of f being an eigenform of the operator $T(p)$ of level N_0, we have

$$a_p f(z) = (T(p)f)(z) = p^{k-1}\chi(p)f(pz) + \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{z+i}{p}\right).$$

We want to solve for what constants γ the linear combination $g(z) = f(z) - \gamma f(pz)$ is an eigenform of the Hecke operator $T(p)$ of level N given by

$$(T(p)g)(z) = \frac{1}{p}\sum_{i=0}^{p-1} g\left(\frac{z+i}{p}\right).$$

Thus we want to solve for λ, γ such that

$$\lambda\big(f(z) - \alpha f(pz)\big) = \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{z+i}{p}\right) - \gamma\frac{1}{p}\sum_{i=0}^{p-1} f\left(p\frac{z+i}{p}\right).$$

Note that $f(p(z+b)/p) = f(z+i) = f(z)$, so the latter sum just reduces to $pf(z)$. Meanwhile, our original formula resulting from f being an eigenform tells us that the former sum is equal to $a_p f(z) - p^{k-1}\chi(p)f(pz)$. Thus we conclude $f(z) - \gamma f(pz)$ has eigenvalue λ iff we have the equality

$$\lambda f(z) - \lambda\gamma f(pz) = a_p f(z) - p^{k-1}\chi(p)f(pz) - \gamma f(z).$$

Since $f(z)$ and $f(pz)$ are linearly independent as functions $\mathbb{H} \to \mathbb{C}$, this holds iff λ, α satisfy the two equations

$$\lambda = a_p - \gamma \qquad\qquad \lambda\gamma = p^{k-1}\chi(p).$$

To solve this we multiply the former equation by α and use equality with the latter to conclude that we must have

$$a_p\gamma - \gamma^2 = \lambda\gamma = p^{k-1}\chi(p) \qquad \iff \qquad \gamma^2 - a_p\gamma + p^{k-1}\chi(p) = 0.$$

Since the latter is a quadratic equation in γ, it has two solutions, and we know one must not be a p-adic unit (since the product $p^{k-1}\chi(p)$ isn't) but the other one must be (since the sum a_p is); let β denote the non-unit root and α denote the unit root. Then we find that the forms $f^\sharp$ and $f^\flat$ we've defined are eigenforms with eigenvalues α and β, respectively, and these are the only possible ones (up to scalars) since we've found two distinct eigenvalues for a two-dimensional space, as desired. □

We can then quote Hida's characterization of $\mathcal{I}$-adic newforms, translated into our setup.

Theorem 3.3.4. *For an ordinary $\mathcal{I}$-adic eigenform $\mathbb{f} \in \mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I})$, the following are equivalent:*

- *$P_{k,\varepsilon}(\mathbb{f})$ is a newform of level $Np^{r(P)}$ for any element $P_{k,\varepsilon} \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_F)$ such that the p-part of $\varepsilon_P \chi \omega^{-k(P)}$ is nontrivial.*
- *$P_{k,\varepsilon}(\mathbb{f})$ is a newform of level $Np^{r(P)}$ for every element $P_{k,\varepsilon} \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I}; \mathcal{O}_F)$ such that the p-part of $\varepsilon_P \chi \omega^{-k(P)}$ is nontrivial.*

Moreover, if these conditions are satisfied and P is a point such that $\varepsilon_P \chi \omega^{-k(P)}$ is trivial (which forces $r(P) = 1$), then either $P(\mathbb{f})$ is actually a newform and $k(P) = 2$, or $P(\mathbb{f}) = f^\sharp$ is the p-stabilization of a p-ordinary newform f. Such a $\mathcal{I}$-adic newform induces a $\mathcal{I}$-algebra homomorphism

$$\lambda_{\mathbb{f}} : \mathbb{T}(\mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I})) \to \mathcal{I}$$

given by $T \mapsto a(1, f|T)$.

To obtain this from [Hid88] Theorem 4.1, we note that if we have a $\mathcal{I}$-adic eigenform then we can construct the map $\lambda_{\mathbb{f}}$ directly (the definition gives us something $\mathcal{I}$-linear for any $\mathcal{I}$-adic form $\mathbb{f}$, and the fact that it's an eigenform tells us it's an algebra homomorphism too). Precomposing with the surjection

$$\mathbb{T}(\overline{S}^{\mathrm{ord}}(N; \mathcal{O}_F)) \otimes_\Lambda \mathcal{I} \twoheadrightarrow \mathbb{T}(\mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I}))$$

gives us a $\mathcal{I}$-algebra homomorphism $\lambda : \mathbb{T}(\overline{S}^{\mathrm{ord}}(N; \mathcal{O}_F)) \otimes_\Lambda \mathcal{I} \to \mathcal{I}$ as in the statement of Hida's theorem, and the rest of the result follows.

3.3.2 Modules of congruence

Next, we introduce an important concept of the *module of congruence* of a newform (both classical and $\mathcal{I}$-adic), translating the results of Chapter 4 of [Hid88]. We start off with the classical case. Suppose that $f \in S_k^{\text{ord}}(Np^r, \chi; \mathcal{O}_F)$ is a p-stabilized newform (i.e. a newform of level divisible by p, or the p-stabilization of a newform of level not divisible by p). The classical theory of newforms lets us recover:

Proposition 3.3.5. *Given a p-stabilized newform $f \in S_k^{\text{ord}}(Np^r, \chi; \mathcal{O}_F)$ as above, let $\lambda_f : \mathbb{T}^{\text{ord}}(Np^r, \chi; F) \to F$ be the associated algebra homomorphism. Then we have a F-algebra decomposition*

$$\mathbb{T}_k^{\text{ord}}(Np^r, \chi; F) = \mathbb{T}(f)_F \oplus \mathbb{T}(f)_F^{\perp}$$

where $\mathbb{T}(f)_F^{\perp}$ is the kernel of λ_f, $\mathbb{T}(f)_F \cong F$, and the projection $\mathbb{T}^{\text{ord}}(Np^r, \chi; F) \to \mathbb{T}(f)$ corresponds to λ_f under this isomorphism.

The multiplicative identity in $\mathbb{T}(f)_F$ is an idempotent in $\mathbb{T}_k^{\text{ord}}(Np^r, \chi; F)$ that we denote 1_f. We also define:

Definition 3.3.6. Let f be a p-stabilized newform as above. Let $\mathbb{T}(f)_{\mathcal{O}}$ and $\mathbb{T}(\mathbb{f})_{\mathcal{O}}^{\perp}$ be the projections of $\mathbb{T}_k^{\text{ord}}(Np^r, \chi; \mathcal{O}_F)$ onto $\mathbb{T}(f)_F$ and $\mathbb{T}(f)_F^{\perp}$, respectively. Then define the *module of congruences* for f as the quotient $\mathcal{O}_F$-module

$$C(f) = \frac{\mathbb{T}(f)_{\mathcal{O}} \oplus \mathbb{T}(f)_{\mathcal{O}}^{\perp}}{\mathbb{T}_k^{\text{ord}}(Np^r, \chi; \mathcal{O}_F)}.$$

We can check that $C(f) \cong \mathcal{O}_F / H_f \mathcal{O}_F$ for some element $H_f \in \mathcal{O}_F$ (unique up to units), which we call the *congruence number* of f.

Next, we carry out the same process for $\mathcal{I}$-adic forms. Suppose $\mathbb{f}$ is an ordinary $\mathcal{I}$-adic newform of level N and character χ with $\lambda_{\mathbb{f}} : \mathbb{T}^{\text{ord}}(N, \chi; \mathcal{I}) \to \mathcal{I}$ the associated algebra homomorphism. Let $Q(\mathcal{I})$ be the quotient field of $\mathcal{I}$, and by abuse of notation also let $\lambda_{\mathbb{f}}$ denote the extended homomorphism $\mathbb{T}^{\mathrm{ord}}(N,\chi;\mathcal{I}) \otimes_{\mathcal{I}} Q(\mathcal{I}) \to Q(\mathcal{I})$. Hida proves a direct sum decomposition of this algebra (really of $\mathbb{T}(\mathbb{S}^{\mathrm{ord}}(N;\mathcal{O}_F)) \otimes_{\Lambda} Q(\mathcal{I})$, but it descends to the one we want):

Theorem 3.3.7. *Given a $\mathcal{I}$-adic newform $\mathbb{f}$ and the associated homomorphism $\lambda_{\mathbb{f}}$ as above, we have a $Q(\mathcal{I})$-algebra decomposition*

$$\mathbb{T}^{\mathrm{ord}}(N,\chi;\mathcal{I}) \otimes Q(\mathcal{I}) = \mathbb{T}(\mathbb{f})_Q \oplus \mathbb{T}(\mathbb{f})_Q^{\perp}$$

where $\mathbb{T}(\mathbb{f})_Q^{\perp}$ is the kernel of $\lambda_{\mathbb{f}}$, $\mathbb{T}(\mathbb{f})_Q \cong Q(\lambda)$, and the projection $\mathbb{T}^{\mathrm{ord}}(N,\chi;\mathcal{I}) \otimes Q(\mathcal{I}) \to \mathbb{T}(\mathbb{f})_Q$ corresponds to $\lambda_{\mathbb{f}}$ under this isomorphism.

As before, we let $1_{\mathbb{f}}$ denote the idempotent of $\mathbb{T}(\mathbb{f})_F$, and we also let $\mathbb{T}(\mathbb{f})_{\mathcal{I}}$ and $\mathbb{T}(\mathbb{f})_{\mathcal{I}}^{\perp}$ denote the images of $\mathbb{T}^{\mathrm{ord}}(N,\chi;\mathcal{I})$ under the projections to $\mathbb{T}(\mathbb{f})_Q$ and $\mathbb{T}(\mathbb{f})_Q^{\perp}$, respectively. Furthermore, we can check that these definitions process are compatible with specialization:

Proposition 3.3.8. *Suppose that $\mathbb{f}$ is a $\mathcal{I}$-adic newform as above, and that we fix a point $P \in \mathcal{X}(\mathcal{I};\mathcal{O}_F)_{\mathrm{alg}}$. Then the inclusion*

$$\mathbb{T}^{\mathrm{ord}}(N,\chi;\mathcal{I}) \hookrightarrow \mathbb{T}(\mathbb{f})_{\mathcal{I}} \oplus \mathbb{T}(\mathbb{f})_{\mathcal{I}}^{\perp}$$

induces an isomorphism when we localize at the prime ideal generated by P which, when we take a quotient by P, passes to the decomposition associated to $\mathbb{f}_P$ by Proposition 3.3.5. Thus $1_{\mathbb{f}}$ projects to $1_{\mathbb{f}_P}$ under the surjection from $\mathbb{T}^{\mathrm{ord}}(N,\chi;\mathcal{I})$ to the appropriate Hecke algebra for $\mathbb{f}_P$.

We can then define congruence modules for $\mathbb{f}$, and state Hida's theorem that they are compatible with the ones for the specializations $\mathbb{f}_P$. For technical reasons, we introduce the notation that

$$\widetilde{\mathbb{T}}(\mathbb{f})_{\mathcal{I}}^{\perp} = \bigcap_{\mathfrak{p}} (\mathbb{T}(\mathbb{f})_{\mathcal{I}}^{\perp})_{\mathfrak{p}}$$

where $\mathfrak{p}$ runs over all prime ideals of height 1 in $\mathcal{I}$, with this intersection taken inside of $\mathbb{T}(\mathbb{f})_Q^{\perp}$. Clearly $\mathbb{T}(\mathbb{f})_{\mathcal{I}}^{\perp} \subseteq \widetilde{\mathbb{T}}(\mathbb{f})_{\mathcal{I}}^{\perp}$.

Definition 3.3.9. Given a $\mathcal{I}$-adic newform $\mathbb{f}$ as above, define the *modules of congruences* for $\mathbb{f}$ as

$$\mathcal{C}_0(\mathbb{f};\mathcal{I}) = \frac{\mathbb{T}(\mathbb{f})_{\mathcal{I}} \oplus \mathbb{T}(\mathbb{f})_{\mathcal{I}}^{\perp}}{\mathbb{T}^{\mathrm{ord}}(N,\chi;\mathcal{I})} \qquad \mathcal{C}(\mathbb{f};\mathcal{I}) = \frac{\mathbb{T}(\mathbb{f})_{\mathcal{I}} \oplus \widetilde{\mathbb{T}}(\mathbb{f})_{\mathcal{I}}^{\perp}}{\mathbb{T}^{\mathrm{ord}}(N,\chi;\mathcal{I})}.$$

By the second isomorphism theorem, our modules of congruence (which are defined in terms of a Hecke algebra that's a quotient of the one Hida uses) are isomorphic to Hida's. Then, translating [Hid88] Theorem 4.6 into our setup gives:

Theorem 3.3.10. *Fix a $\mathcal{I}$-adic newform $\mathbb{f}$ as above, and let R be a local ring of $\mathbb{T}(\overline{S}^{\mathrm{ord}}(N;\mathcal{O}_F))$ through which $\lambda_{\mathbb{f}}$ factors. Suppose that R is Gorenstein, i.e. that $R \cong \mathrm{Hom}_{\mathcal{I}}(R,\mathcal{I})$ as an R-module. Then we have $\mathcal{C}_0(\mathbb{f};\mathcal{I}) = \mathcal{C}(\mathbb{f};\mathcal{I}) \cong \mathcal{I}/H_{\mathbb{f}}\mathcal{I}$ for a nonzero element $H_{\mathbb{f}} \in \mathcal{I}$, and for any $P \in \mathcal{X}_{alg}(\mathcal{I},\mathcal{O}_F)$ with $k(P) \geq 2$ we have a canonical isomorphism*

$$\mathcal{C}_0(\mathbb{f};\mathcal{I}) \otimes_{\mathcal{I}} (\mathcal{I}/P\mathcal{I}) \cong C(\mathbb{f}_P).$$

So, from now on, if we're given a $\mathcal{I}$-adic modular form $\mathbb{f}$ (for which the Gorenstein condition above is satisfied), we'll let $H_{\mathbb{f}}$ denote any element $\mathcal{I}$ such that $\mathcal{C}_0(\mathbb{f};\mathcal{I}) \cong \mathcal{I}/H_{\mathbb{f}}\mathcal{I}$ and call it a *congruence number* for $\mathbb{f}$. The theorem above says that for any P, the specialization $H_{\mathbb{f},P}$ we get by projecting $H_{\mathbb{f}}$ to $\mathcal{I}/P\mathcal{I}$ serves as a congruence number for $\mathbb{f}_P$, i.e. $H_{\mathbb{f},P} = H_{\mathbb{f}_P}$. However, there is some subtlety here: even though Hida's work gives us a way to realize $H_{\mathbb{f}_P}$ from a special value of an adjoint L-function for any given P, it's only determined up to a unit, and it's not immediately clear how to choose the units to fit the special values into a p-adic family. Instead, we just know that a family $H_{\mathbb{f}}$ exists and that $H_{\mathbb{f},P}$ is a p-adic unit times Hida's L-value formula. To be able to write $H_{\mathbb{f}}$ explicitly in terms of L-values amounts to showing we can construct a p-adic L-function interpolating the adjoint L-values in question.

In general this should be able to be recovered as a consequence of the modularity lifting apparatus developed by Wiles. In the special case we'll deal with (where $\mathbb{f}$ comes from a family of CM modular forms) we will show that the main conjecture of Iwasawa theory for imaginary quadratic fields (proven by Rubin) is enough to write $H_{\mathbb{f}}$ as an explicit p-adic L-function associated to a Hecke character.

3.4 Λ-adic Petersson inner products

Now that we've set up the basic framework of Λ-adic modular forms (and more generally $\mathcal{I}$-adic modular forms for certain extensions $\mathcal{I}/\Lambda$), in this section we'll recall results of Hida that allow us to define an element in $\mathcal{I}$ that interpolates the Petersson inner products of specializations of two $\mathcal{I}$-adic cusp forms. This will be our main tool from Hida theory that will let us take the identities we get from Ichino's formula and turn them into an identity of p-adic analytic functions.

Naively, one might want a pairing on the space of $\mathcal{I}$-adic cusp forms that interpolates the Petersson inner products, but this is much to hope for. Instead, if we're given an ordinary $\mathcal{I}$-adic newform $\mathbb{f}$ of level N, we can define a linear functional $\ell_{\mathbb{f}} : \mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I}) \to \mathcal{I}$ such that $\ell_{\mathbb{f}}(\mathbb{g})$ interpolates the Petersson inner products of $\mathbb{f}_P$ and $\mathbb{g}_P$. Hida constructs such forms in Section 7 of [Hid88], and we give a construction along the same lines.

Proposition 3.4.1. *Let $\mathbb{f} \in \mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I})$ be a $\mathcal{I}$-adic newform. Then we can define a linear functional $\ell_{\mathbb{f}} : \mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I}) \to \mathcal{I}$ by the formula*

$$\ell_{\mathbb{f}}(\mathbb{g}) = a(1, 1_{\mathbb{f}}\mathbb{g})H_{\mathbb{f}}$$

where $H_{\mathbb{f}}$ is the congruence number associated to $\mathbb{f}$ and $1_{\mathbb{f}}$ is the idempotent of the Hecke algebra associated to $\mathbb{f}$ (both as defined in the previous section).

Proof. The assignment $\mathrm{g} \mapsto a(1, 1_{\mathrm{f}}\mathrm{g})$ is evidently linear. The only issue is that by definition, 1_{f} is only an element of $\mathbb{T}^{\mathrm{ord}}(N, \chi; \mathcal{I}) \otimes Q(\mathcal{I})$, so we only get a linear functional $\mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I}) \to Q(\mathcal{I})$. However, by definition H_{f} is exactly what's needed to fix this; it annihilates the congruence module

$$\mathcal{C}_0(\mathrm{f}; \mathcal{I}) = \frac{\mathbb{T}(\mathrm{f})_{\mathcal{I}} \oplus \mathbb{T}(\mathrm{f})_{\mathcal{I}}^{\perp}}{\mathbb{T}^{\mathrm{ord}}(N, \chi; \mathcal{I})};$$

since 1_{f} lies in $\mathbb{T}(\mathrm{f})_{\mathcal{I}}$ by construction, $H_{\mathrm{f}} \cdot 1_{\mathrm{f}}$ is trivial in this quotient, i.e. $H_{\mathrm{f}} \cdot 1_{\mathrm{f}}$ lies in $\mathbb{T}^{\mathrm{ord}}(N, \chi; \mathcal{I})$. So $H_{\mathrm{f}} 1_{\mathrm{f}} \mathrm{g} \in \mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I})$ and so its first Fourier coefficient is actually in in $\mathcal{I}$. □

We remark that we can extend this to a linear functional $\mathbb{S}(N, \chi; \mathcal{I}) \to \mathcal{I}$ by precomposing it with the ordinary projector e; on this larger space it's thus given by

$$\ell_{\mathrm{f}}(\mathrm{g}) = a(1, e1_{\mathrm{f}}\mathrm{g}) H_{\mathrm{f}}.$$

For simplicity, we'll abuse notation and let $1_{\mathrm{f}} \in \mathbb{T}(N, \chi; \mathcal{I})$ denote $e1_{\mathrm{f}}$, since this just amounts to taking the image of $1_{\mathrm{f}} \in \mathbb{T}^{\mathrm{ord}}$ under the inclusion $\mathbb{T}^{\mathrm{ord}} \to \mathbb{T}$ coming from multiplication by e. If $P \in \mathcal{X}(\mathcal{I}, \mathcal{O}_F)_{\mathrm{alg}}$ then we know we have

$$\ell_{\mathrm{f}}(\mathrm{g})_P = a(1, 1_{\mathrm{f}_P} \mathrm{g}_P) H_{\mathrm{f},P}.$$

and the right-hand side is computed in the appropriate space of classical cusp forms (with F coefficients).

We also note that we can extend ℓ_{f} to be defined on even larger spaces of $\mathcal{I}$-adic forms by trace operators as discussed at the end of Section 3.2. In particular for any M coprime to p we have trace operators

$$\mathrm{tr} : \mathbb{S}^{\mathrm{ord}}(M \cdot N, \chi; \mathcal{I}) \to \mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I})$$

and

$$\mathrm{tr} : \mathbb{S}^{\mathrm{ord}}(\Gamma_0(N, M), \chi; \mathcal{I}) \to \mathbb{S}^{\mathrm{ord}}(N, \chi; \mathcal{I})$$

via summing over representatives of $\Gamma_0(MN)\backslash\Gamma_0(N)$ and $\Gamma_0(N, M)\backslash\Gamma_0(N)$, respectively. These specialize to the usual trace operators on spaces of classical modular forms, and in particular we note that if $\Gamma' \subseteq \Gamma$ are congruence subgroups and we consider $\mathrm{tr} : S_k(\Gamma', \chi) \to S_k(\Gamma, \chi)$, then for any $f \in S_k(\Gamma', \chi)$ and $g \in S_k(\Gamma, \chi)$ we have

$$\langle \mathrm{tr}(f), g\rangle = [\Gamma : \Gamma']\langle f, g\rangle$$

for our normalized Petersson inner products.

3.4.1 Computing $a(1, 1_f g)$

So, to write the values $\ell_{\mathbb{f}}(\mathrm{g})_P$ explicitly, we need to compute $a(1, 1_f g)H_f$ for the classical modular forms $f = \mathbb{f}_P$ and $g = \mathrm{g}_P$. We won't say anything about H_f for now, and instead focus the term $a(1, 1_f g)$. Since $\mathbb{f}$ is an ordinary $\mathcal{I}$-adic newform, f will always be either an ordinary newform or a p-stabilization of an ordinary newform; we'll need to deal with those two cases separately. The form g, meanwhile, can be any cusp form.

Our first observation is that the projection 1_f is characterized by the properties that $1_f f = f$, and $1_f h = 0$ if h is an eigenform of any Hecke operator t and has a different eigenvalue from f. The latter property follows from noting that if t is such an operator, and $th = \lambda' h$ with $\lambda_f(t) \neq \lambda'$, then we can compute

$$\lambda_f(t)1_f h = t \cdot (1_f h) = 1_f \cdot (th) = 1_f \cdot (\lambda' h) = \lambda' 1_f h$$

which forces $1_f h = 0$.

If we assume f actually has coefficients in a number field $F_0 \subseteq F$, and restrict to elements $g \in S_k(N,\chi;F_0)$ then we can compute $a(1,1_f g) \in F_0$ by computing this in $\mathbb{C}$ (where we can pass to an element $1_f \in \mathbb{T}_{\mathbb{C}}(S_k(N,\chi))$ characterized by the same property as before). Since $g \mapsto a(1,1_f g)$ is then a linear functional on $S_k(N,\chi)$, by duality for the Petersson inner product there exists a modular form $h \in S_k(N,\chi)$ such that $\langle g,h\rangle = a(1,1_f g)$ for all g. In fact, it's sufficient to find f' such that $g \mapsto \langle g,f'\rangle$ is a scalar multiple of $g \mapsto a(1,1_f g)$, and then comparing values at f gives us $a(1,1_f g) = \langle g,f'\rangle/\langle f,f'\rangle$. It will turn out that f' is either equal to or closely related to f (and its exact form will come out from our analysis). We start by checking:

Lemma 3.4.2. *Let $f \in S_k(N,\chi)$ be either a p-ordinary newform or the p-stabilization of an ordinary newform. Then the unique form $h \in S_k(N,\chi)$ such that $\langle g,h\rangle = a(1,1_f g)$ lies in the prime-to-N Hecke eigenspace $U(f) \subseteq S_k(N,\chi)$ of f.*

Proof. We know $S_k(N,\chi)$ decomposes as an orthogonal direct sum of prime-to-N Hecke eigenspaces, and in fact this is a decomposition of spaces $U(h')$ as h' ranges over all newforms of character χ and level dividing N. Moreover, we know 1_f annihilates these subspaces for every $h' \neq f$, and thus $a(1,1_f-)$ is trivial on all of them. The only way that $\langle -,h\rangle$ can be trivial on all of them is if h has no component in each of these spaces $U(h')$, i.e. $h \in U(f)$. □

We now split into our two cases.

Case 1: An actual newform. Suppose that $f \in S_k(N,\chi)$ is actually a newform of level N. Then the multiplicity one theorem tells us that $U(f)$ is one-dimensional, so h must be a scalar multiple of f itself. To find the right scalar we note that $a(1,1_f f) = a(1,f) = 1$ so we must have $\langle h,f\rangle = 1$; thus we get:

Corollary 3.4.3. *Let $f \in S_k(N,\chi)$ be a p-ordinary newform. Then for any $g \in S_k(N,\chi)$ we have*

$$a(1,1_f g) = \frac{\langle g,f\rangle}{\langle f,f\rangle}.$$

Case 2: The p-stabilization of a newform. This case is slightly more delicate. For convenience we change our notation a bit, letting f denote the original p-ordinary newform in $S_k(N_0, \chi)$ (for $p \nmid N_0$). If we take $N = N_0 p$, we know that $U(f) \subseteq S_k(N, \chi)$ is two-dimensional, spanned by $f(z)$ and $f(pz)$. Moreover, the operator $T(p)$ of level N has two eigenforms in this space with different eigenvalues; more specifically we take the notation that

$$x^2 - a_p(f)x + p^{k-1}\chi(p) = (x - \alpha_f)(x - \beta_f)$$

with $|\alpha_f|_p = 1$ and $|\beta_f|_p < 1$. By Lemma 3.3.3 the two eigenforms are

$$f^\sharp(z) = f(z) - \beta_f f(pz) \qquad\qquad f^\flat(z) = f(z) - \alpha_f f(z),$$

with eigenvalues α_f and β_f, respectively. Thus $f^\sharp$ spans the one-dimensional ordinary subspace of $U(f)$; and also $f^\sharp$ is what would actually arise as the specialization of a $\mathcal{I}$-adic newform.

So, we want to compute the linear functional $a(1, 1_{f^\sharp}-)$ on $S_k(N, \chi)$. We can pin this down exactly because we know it's zero on the orthogonal complement of the two-dimensional space $U(f)$, and we also know its behavior on U_f because $1_{f^\sharp} f^\sharp = f^\sharp$ (by definition) and $1_f f^\flat = 0$ (since $f^\flat$ has a $T(p)$-eigenvalue different from $f^\sharp$). So $a(1, 1_{f^\sharp}-)$ is (up to a scalar) equal to the form $\langle -, f' \rangle$ for some $f' \in U(f) = \mathbb{C}f(z) \oplus \mathbb{C}f(pz)$ that's orthogonal to $f^\flat$. Finding such a form is still a bit delicate, though; to do this we'll need to understand how to compute the Petersson inner products $\langle f(z), f(pz) \rangle$, $\langle f(pz), f(z) \rangle$, and $\langle f(pz), f(pz) \rangle$ in terms of $\langle f(z), f(z) \rangle = \langle f, f \rangle$. The starting point is computing the last product; this can be done in great generality by change-of-variables as follows.

Lemma 3.4.4. *Let f, g be two weight-k cusp forms (of any level). Then for any* *prime p and any integer i, we have*

$$\langle f(pz+i), g(pz+i)\rangle = p^{-k}\langle f(z), g(z)\rangle.$$

Proof. We can pick some common congruence subgroup $\Gamma(N)$ on which all of $f(z)$, $f(pz)$, $g(z)$, and $g(pz)$ are defined, with $p^2|N$ for simplicity. Then by definition we have

$$\langle f(pz+i), g(pz+i)\rangle = \frac{1}{[\overline{\Gamma}:\overline{\Gamma}(N)]}\int_{D(N)} f(pz+i)g(pz+i)\,\mathrm{Im}(z)^k d\mu(z)$$

$$= p^{-k}\frac{1}{[\overline{\Gamma}:\overline{\Gamma}(N)]}\int_{D(N)} f(pz+i)g(pz+i)\,\mathrm{Im}(pz+i)^k d\mu(z),$$

where $d\mu$ is the standard hyperbolic measure on $\mathbb{H}$ and $D(N)$ is a fundamental domain for $\Gamma(N)$. We can then use change-of-variables on the Möbius transformation $\sigma : z \mapsto pz+i$, under which $d\mu$ is invariant, to conclude

$$\langle f(pz+i), g(pz+i)\rangle = p^{-k}\frac{1}{[\overline{\Gamma}:\overline{\Gamma}(N)]}\int_{\sigma D(N)} f(z)g(z)\,\mathrm{Im}(z)^k d\mu(z).$$

Finally, we note that $\sigma D(N)$ is a fundamental domain for the group $\sigma\overline{\Gamma}(N)\sigma^{-1}$, which we can compute has the same index in $\overline{\Gamma}$ as $\overline{\Gamma}(N)$ does; thus the right hand side of our equation is $p^k\langle f(z), g(z)\rangle$ (computed for the congruence group $\sigma\overline{\Gamma}(N)\sigma^{-1}$). $\square$

Returning to our situation where f is an eigenform of level N_0 with $p \nmid N$, we can use the result above plus the Hecke relation for $T(p)f$ to work out what $\langle f(z), f(pz)\rangle$ and $\langle f(pz), f(z)\rangle$ are.

Lemma 3.4.5. *Let $f \in S_k(N_0, \chi)$ be a newform, and let $p \nmid N_0$ be a prime. Then we have*

$$\langle f(z), f(pz)\rangle = \frac{a_p}{p^{k-1}(p+1)}\langle f, f\rangle \qquad \langle f(pz), f(z)\rangle = \frac{\overline{\chi}(p)a_p}{p^{k-1}(p+1)}\langle f, f\rangle.$$

Proof. To compute these, we start by noting that f being an eigenform tells us

$$a_p f(z) = \chi(p)p^{k-1}f(pz) + \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{z+i}{p}\right).$$

Applying $\langle -, f(z)\rangle$ to both sides and using linearity in the first coordinate we get

$$a_p\langle f, f\rangle = \chi(p)p^{k-1}\langle f(pz), f(z)\rangle + \frac{1}{p}\sum_{i=0}^{p-1}\left\langle f\left(\frac{z+i}{p}\right), f(z)\right\rangle.$$

We then use the previous proposition, plus invariance of f under $z \mapsto z+1$, to tell us

$$\left\langle f\left(\frac{z+i}{p}\right), f(z)\right\rangle = p^k\left\langle f\left(\frac{(pz-i)+i}{p}\right), f(pz-i)\right\rangle = p^k\langle f(z), f(pz)\rangle,$$

and therefore reduce our equality above to

$$a_p\langle f, f\rangle = \chi(p)p^{k-1}\langle f(pz), f(z)\rangle + p^k\langle f(z), f(pz)\rangle.$$

An identical computation with $\langle f(z), -\rangle$ (using antilinearity in the second coordinate) tells us

$$\overline{a}_p\langle f, f\rangle = \overline{\chi}(p)p^{k-1}\langle f(z), f(pz)\rangle + p^k\langle f(pz), f(z)\rangle.$$

Also, we can apply the relation $\overline{a}_p = \overline{\chi}(p)a_p$, which follows from making the computation $T(p)^* = \chi(p)T(p)$ and then evaluating $\langle T(p)f, f\rangle = \langle f, \chi(p)T(p)f\rangle$.

Thus, we can solve for $\langle f(z), f(pz)\rangle$ and $\langle f(pz), f(z)\rangle$ from the two equations

$$\begin{aligned} a_p\langle f, f\rangle &= \chi(p)p^{k-1}\langle f(pz), f(z)\rangle + p^k\langle f(z), f(pz)\rangle \\ \overline{\chi}(p)a_p\langle f, f\rangle &= p^k\langle f(pz), f(z)\rangle + \overline{\chi}(p)p^{k-1}\langle f(z), f(pz)\rangle \end{aligned}$$

via linear algebra. For instance, multiplying the first equation through by $\overline{\chi}(p)p$ and subtracting the second from this gives

$$(\overline{\chi}(p)pa_p - \overline{\chi}(p)a_p)\langle f, f\rangle = (\overline{\chi}(p)p^{k+1} - \overline{\chi}(p)p^{k-1})\langle f(z), f(pz)\rangle,$$

which rearranges to

$$\langle f(z), f(pz)\rangle = \frac{\overline{\chi}(p)a_p(p-1)}{\overline{\chi}(p)p^{k-1}(p^2-1)}\langle f, f\rangle = \frac{a_p}{p^{k-1}(p+1)}\langle f, f\rangle.$$

A parallel calculation (or just taking the complex conjugate of this one) gives the formula for $\langle f(pz), f(z)\rangle$ as well. □

We can then complete our computation of $a(1, 1_f g)$ in the case when we're working with a p-stabilization of a newform.

Proposition 3.4.6. *Suppose $p \nmid N_0$ and $f \in S_k(N_0, \chi)$ is a p-ordinary newform, and let $\alpha = \alpha_f$ and $\beta = \beta_f$ be the roots of $x^2 - a_p(f)x + p^{k-1}\chi(p)$ with $|\alpha|_p = 1$ and $|\beta|_p < 1$, as before. Then the linear functional $a(1, 1_{f^\sharp}-)$ is given by*

$$a(1, 1_{f^\sharp} g) = \frac{\langle g, f^\natural\rangle}{\langle f^\sharp, f^\natural\rangle} \qquad\qquad f^\natural = f(z) - p\beta f(pz)$$

Proof. By the above, we know we can write $a(1, 1_{f^\sharp}-) = \langle -, h\rangle$ for a unique element $h \in \mathbb{C}f(z) \oplus \mathbb{C}f(pz)$, or $a(1, 1_{f^\sharp}-) = \langle -, h\rangle/\langle f^\sharp, h\rangle$ for any h in a unique one-dimensional subspace of this. Assuming the subspace isn't $\mathbb{C}f(pz)$ (which will be justified when we find a different subspace that works), we can take $h = f(z) - Cf(pz)$; since $\langle f^\sharp, h\rangle/\langle f^\sharp, h\rangle = 1 = a(1, 1_f-)$ trivially, we just need to choose C such that we have $\langle f^\flat, h\rangle = 0 = a(1, 1_{f^\sharp} f^\flat)$.

Thus, we want to solve the equation

$$\langle f^\flat, h\rangle = \langle f(z) - \alpha f(pz), f(z) - Cf(pz)\rangle = 0$$

for β. Expanding this by linearity we get

$$0 = \langle f(z), f(z)\rangle - \alpha\langle f(pz), f(z)\rangle - \overline{C}\langle f(z), f(pz)\rangle + \alpha\overline{C}\langle f(pz), f(pz)\rangle.$$

Substituting in the results of the previous two lemmas, we get

$$0 = \langle f, f\rangle - \alpha\frac{\overline{\chi}(p)a_p}{p^{k-1}(p+1)}\langle f, f\rangle - \overline{C}\frac{a_p}{p^{k-1}(p+1)}\langle f, f\rangle + \alpha\overline{C}p^{-k}\langle f, f\rangle.$$

Rearranging, this solves to

$$\overline{C} = \frac{1 - \alpha\frac{\overline{\chi}(p)a_p}{p^{k-1}(p+1)}}{\frac{a_p}{p^{k-1}(p+1)} - \alpha p^{-k}},$$

and it just remains to simplify this expression. We start by multiplying through by $p^k(p+1)$ to clear internal denominators:

$$\overline{C} = \frac{p^k(p+1) - \alpha\overline{\chi}(p)a_p p}{a_p p - \alpha(p+1)}.$$

Next, we substitute $a_p = \alpha + \beta$ and $\chi(p)p^{k-1} = \alpha\beta$ to get:

$$\overline{C} = \frac{p(p+1)\overline{\chi}(p)\alpha\beta - \alpha\overline{\chi}(p)(\alpha+\beta)p}{(\alpha+\beta)p - \alpha(p+1)}.$$

We then simply expand this out, noting that we can factor out $\overline{\chi}(p)\alpha p$ from the numerator:

$$\overline{C} = \frac{\overline{\chi}(p)\alpha p(p\beta + \beta - \beta - \alpha)}{p\beta + p\alpha - p\alpha - \alpha} = \overline{\chi}(p)\alpha p\frac{p\beta - \alpha}{p\beta - \alpha} = \overline{\chi}(p)\alpha p.$$

Finally, noting that by the Ramanujan conjecture for newforms we have $\overline{\chi}(p)\alpha = \overline{\beta}$, so we get that $C = p\beta$ is a solution to our equation, and thus $f^\natural = f(z) - p\beta f(pz)$ satisfies $\langle f^\flat, f^\natural\rangle = 0$. $\square$

3.4.2 Euler factors from p-stabilizations

From the previous section, we've seen that our element $\ell_{\mathbb{f}}(\mathbb{g}) \in \mathcal{I}$ interpolating Petersson inner products will have specializations of the form

$$\frac{\langle g, f^{\natural}\rangle}{\langle f^{\sharp}, f^{\natural}\rangle}$$

when we specialize at points P where $\mathbb{f}_P$ is the p-stabilization of a newform f. Similarly, g may also be p-stabilized. On the other hand, our automorphic formulas usually only involve the newforms themselves. So we'd like to relate the ratio above to one not involving p-stabilizations.

More specifically, to match the notation we'll use later on, we will change our notation to let $\mathbb{h}$ denote our $\mathcal{I}$-adic newform rather than $\mathbb{f}$, and we'll consider a specialization at a point P where we get a p-stabilization $h^{\sharp}$ of a p-ordinary newform h of prime-to-p level and weight m. We'll take the specialization of $\mathbb{g}$ at P to be of the form $fg^{\sharp}$ where f is a fixed newform of weight k (not assumed ordinary, and perhaps even having p dividing its level) and $g^{\sharp}$ is the p-stabilization of a p-ordinary newform g of prime-to-p level and weight $m-k$. We'll assume the usual relation on the central characters, that $\chi_f \chi_g = \chi_h$.

In fact, we'll need to consider a more general situation where we have newforms f_0, g_0, h_0 and f, g, h are translates of them of the form $f(z) = f_0(L_f z)$, $g(z) = g_0(L_g z)$, and $h(z) = h_0(L_h z)$ for L_f, L_g, L_h integers (dividing the LCM of the levels of f_0, g_0, h_0). In this case, the ratio we'll actually want to deal with is

$$\frac{\langle fg^{\sharp}, h^{\natural}\rangle}{\langle h_0^{\sharp}, h_0^{\natural}\rangle} = (*) \frac{\langle fg, h\rangle}{\langle h_0, h_0\rangle},$$

where we want to compute $(*)$ as an explicit constant. Carrying this out amounts to an exercise in manipulating Petersson inner products plus some linear algebra, along the lines of what we did to compute $h^\natural$ in the previous section. We will ultimately end up finding that $(*)$ can be viewed as a product of Euler factors at the prime p.

Since f_0, g_0, h_0 are newforms and thus eigenforms of the Hecke operators $T(p)$, we know they are invariant under $z \mapsto z+1$ and satisfy

$$(\alpha_f + \beta_f) f_0(z) = \alpha_f \beta_f f_0(pz) + \frac{1}{p} \sum_{i=0}^{p-1} f_0\left(\frac{z+i}{p}\right),$$

$$(\alpha_g + \beta_g) g_0(z) = \alpha_g \beta_g g_0(pz) + \frac{1}{p} \sum_{i=0}^{p-1} g_0\left(\frac{z+i}{p}\right),$$

$$(\alpha_h + \beta_h) h_0(z) = \alpha_h \beta_h h_0(pz) + \frac{1}{p} \sum_{i=0}^{p-1} h_0\left(\frac{z+i}{p}\right),$$

where α, β are the two roots of the appropriate Hecke polynomial. Since g_0 and h_0 are p-ordinary we can assume α_g, α_h are p-adic units and β_g, β_h are not. On the other hand, we don't assume anything about α_f, β_f; in fact one or both may even be zero depending on the level of f_0. Also, these Hecke equations will give us equations for $f(z)$, $g(z)$, and $h(z)$, but the form of the resulting equations depends on whether p divides the constants L_g and L_h.

Relating $\langle h_0^\sharp, h_0^\natural \rangle$ to $\langle h_0, h_0 \rangle$. We start by working with the denominator of the expression, since here the computation is an extension of what we've already done in the previous section. By definition, we have

$$\langle h_0^\sharp, h_0^\natural \rangle = \langle h_0(z) - \beta_h h_0(pz), h_0(z) - p\beta_h h_0(pz) \rangle$$

$$= \langle h_0(z), h_0(z) \rangle - \beta_h \langle h_0(pz), h_0(z) \rangle - p\overline{\beta}_h \langle h_0(z), h_0(pz) \rangle + p\beta_h \overline{\beta}_h \langle h_0(pz), h_0(pz) \rangle.$$

Letting $\langle h_0, h_0\rangle$ denote $\langle h_0(z), h_0(z)\rangle$, we can use the results of the previous section to express each of these Petersson inner products in terms of $\langle h_0, h_0\rangle$, getting

$$\langle h_0^\sharp, h_0^\natural\rangle = \langle h_0, h_0\rangle \left(1 - \frac{\overline{\chi}(p)\beta_h a_p(f)}{p^{k-1}(1+p)} - \frac{p\overline{\beta}_h a_p(f)}{p^{k-1}(1+p)} + \frac{p\beta_h\overline{\beta}_h}{p^k}\right).$$

Now, we can get rid of the complex conjugates via the Ramanujan conjecture: we know $\overline{\beta}_h = \overline{\chi}(p)\alpha_h$ and also $|\beta_h|^2 = p^{k-1}$ (so the rightmost term is identically 1). Thus the term in parentheses simplifies to

$$2 - \frac{\overline{\chi}(p)(\beta_h + p\alpha_h)a_p(f)}{p^{k-1}(1+p)}.$$

Using that $p^{k-1} = \overline{\chi}(p)\alpha_h\beta_h$, that $a_p(f) = \alpha_h + \beta_h$, and collecting everything with a common denominator gives

$$\frac{2(1+p)\alpha_h\beta_h - (\beta_h + p\alpha_h)(\alpha_h + \beta_h)}{\alpha_h\beta_h(1+p)} = \frac{\alpha_h\beta_h + p\alpha_h\beta_h - \beta_h^2 - p\alpha_h^2}{\alpha_h\beta_h(1+p)}$$

Cancelling and factoring gives

$$\frac{1 + p - \beta_h/\alpha_h - p\alpha_h/\beta_h}{1+p} = \frac{(1-\beta_h/\alpha_h)(1-p\alpha_h/\beta_h)}{(1+p)}.$$

So we conclude

$$\langle h_0^\sharp, h_0^\natural\rangle = \langle h_0, h_0\rangle \cdot \frac{(1-\beta_h/\alpha_h)(1-p\alpha_h/\beta_h)}{(1+p)}.$$

For our purposes later it's helpful to rearrange this as

Proposition 3.4.7. *With our setup as above, we have*

$$\langle h_0^\sharp, h_0^\natural\rangle = \langle h_0, h_0\rangle \cdot \frac{(-\alpha_h/\beta_h)(1-\beta_h/\alpha_h)(1-p^{-1}\beta_h/\alpha_h)}{(1+p^{-1})}.$$

Relating $\langle fg^\sharp, h^\natural\rangle$ to $\langle fg, h\rangle$. We now turn to the numerator. The idea is much the same as before, but the computation is a bit more involved since we have three different forms in our inner product rather than two equal ones. At this point we need to make a restriction that the level of f is coprime to p - a similar computation can be done if p does divide the level of f, but the details work out differently. In this case, L_f, L_g, L_h are all coprime to p; substituting in the appropriate multiple of z in our Hecke equations for f_0, g_0, h_0 give us

$$(\alpha_f+\beta_f)f(z) = \alpha_f\beta_f f(pz) + \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{z+i}{p}\right),$$

$$(\alpha_g+\beta_g)g(z) = \alpha_g\beta_g g(pz) + \frac{1}{p}\sum_{i=0}^{p-1} g\left(\frac{z+i}{p}\right),$$

$$(\alpha_h+\beta_h)h(z) = \alpha_h\beta_h h(pz) + \frac{1}{p}\sum_{i=0}^{p-1} h\left(\frac{z+i}{p}\right).$$

In particular, the sums have the same form because p is coprime to each L; so $L\cdot(z+i)/p = (Lz+Li)/p$ runs over the same values modulo $\mathbb{Z}$ as $(Lz+i)/p$ does for $0 \le i \le p-1$. Also, since p doesn't divide the level of f we know that α_f, β_f are both nonzero.

To start out with our computation of $\langle fg^\sharp, h^\natural\rangle$, note that by definition we have

$$\begin{aligned}\langle f(z)g^\sharp(z), h^\natural(z)\rangle &= \langle f(z)(g(z)-\beta_g g(pz)), h(z) - p\beta_h h(pz)\rangle\\ &= \langle f(z)g(z), h(z)\rangle - \beta_g\langle f(z)g(pz), h(z)\rangle\\ &\quad - p\overline{\beta}_h\langle f(z)g(z), h(pz)\rangle + p\beta_g\overline{\beta}_h\langle f(z)g(pz), h(pz)\rangle.\end{aligned}$$

Again, we need to express all of the inner products in the last expression in terms of $\langle fg, h\rangle = \langle f(z)g(z), h(z)\rangle$. To simplify notation, we let $\langle fg_p, h\rangle = \langle f(z)g(pz), h(z)\rangle$, $\langle fg_p, h_p\rangle = \langle f(z)g(pz), h(pz)\rangle$, and so on. The above expression contains $\langle fg_p, h\rangle$,

$\langle fg, h_p \rangle$, and $\langle fg_p, h_p \rangle$ (as well as $\langle fg, h \rangle$ itself); as in the computations earlier, we'll find linear equations relating these quantities to each other and ultimately letting us solve for each of them in terms of $\langle fg, h \rangle$.

We start out by using the Hecke eigenform equation for $f(z)$ plus first-coordinate linearity of the Petersson inner products to get

$$(\alpha_f + \beta_f)\langle f(z)g(z), h(z)\rangle = \alpha_f\beta_f\langle f(pz)g(z), h(z)\rangle + \frac{1}{p}\sum_{i=0}^{p-1}\left\langle f\left(\frac{z+i}{p}\right)g(z), h(z)\right\rangle.$$

The first inner product on the RHS is $\langle f_p g, h \rangle$. Moreover, for the ones in the sum, we can apply Lemma 3.4.4 and the fact our forms are translation-invariant by $\mathbb{Z}$ to conclude

$$\left\langle f\left(\frac{z+i}{p}\right)g(z), h(z)\right\rangle = p^m\langle f(z)g(pz-i), h(pz-i)\rangle = p^m\langle f(z)g(pz), h(pz)\rangle.$$

Summing up, we have the equation

$$(\alpha_f + \beta_f)\langle fg, h\rangle = \alpha_f\beta_f\langle f_p g, h\rangle + p^m\langle fg_p, h_p\rangle. \tag{F1}$$

To solve for $\langle f_p g, h \rangle$ and $\langle fg_p, h_p \rangle$ we need another equation relating them. It turns out we can get one from directly manipulating the Petersson inner product

$$\langle f_p g, h\rangle = \frac{1}{\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H})}\int_{\Gamma_0(N)\backslash\mathbb{H}} f(pz)g(z)\overline{h(z)}\,\mathrm{Im}(z)^k d\mu$$

itself, using the rules for modular transformations of our forms. In particular, fix some integer N divisible by the levels of f, g, and h, such that $p \nmid N$ (which we can do because our levels and our scalars L were all coprime to N). Then we can choose an integer b with $bN \equiv -1 \pmod{p}$, and then take integers a, d with $ad - bN = 1$

such that $p|a$. Doing this we get two matrices

$$\gamma_1 = \begin{bmatrix} a & b \\ N & d \end{bmatrix} \qquad \gamma_2 = \begin{bmatrix} a/p & b \\ N & dp \end{bmatrix}$$

that lie in $\Gamma_0(N)$. Since our forms are modular for this group we find

$$f(pz) = \overline{\chi}_f(dp)(cpz + dp)^{-k} f(\gamma_2 pz)$$

$$g(z) = \overline{\chi}_g(d)(cz + d)^{k-m} g(\gamma_1 z) \qquad h(z) = \overline{\chi}_h(d)(cz + d)^{-m} h(\gamma_1 z).$$

Since $\chi_f \chi_g = \chi_h$ by our assumptions on the characters of our forms, we find

$$f(pz)g(z)\overline{h(z)} = \overline{\chi}_f(p)p^{-k}|cz + d|^{-2m} f(\gamma_2 pz)g(\gamma_1 z)\overline{h(\gamma_1 z)}.$$

Furthermore, since we can easily check $\gamma_2 pz = 1/p \cdot \gamma_1 z$ and also we know $\text{Im}(z)/|cz + d|^2 = \text{Im}(\gamma_1 z)$, we get

$$\langle f_p g, h\rangle = \frac{\overline{\chi}_f(p)p^{-k}}{\text{vol}(\Gamma_0(N)\backslash\mathbb{H})} \int_{\Gamma_0(N)\backslash\mathbb{H}} f(1/p \cdot \gamma_1 z)g(\gamma_1 z)\overline{h(\gamma_1 z)}\,\text{Im}(\gamma_1 z)^k d\mu.$$

Since the measure is invariant under γ_1 this becomes

$$\langle f_p g, h\rangle = \overline{\chi}_f(p)p^{-k}\langle f(z/p)g(z), h(z)\rangle = \overline{\chi}_f(p)p^{m-k}\langle f g_p, h_p\rangle. \tag{F2}$$

Combining Equations (F1) and (F2) (and using that $\alpha_f\beta_f = \chi_p(f)p^{k-1}$) gives us

$$\langle f g_p, h_p\rangle = \frac{\alpha_f + \beta_f}{p^{m-1}(1 + p)}\langle f g, h\rangle.$$

Carrying out a similar Hecke-eigenvalue argument for g and h gives us two more equations

$$(\alpha_g + \beta_g)\langle fg, h\rangle = \alpha_g\beta_g\langle fg_p, h\rangle + p^m\langle f_p g, h_p\rangle, \tag{G1}$$

$$(\overline{\alpha}_h + \overline{\beta}_h)\langle fg, h\rangle = \overline{\alpha}_h\overline{\beta}_h\langle fg, h_p\rangle + p^m\langle f_p g_p, h\rangle. \tag{H1}$$

Also, carrying out similar Petersson inner product manipulations gives us another two equations

$$\langle fg_p, h\rangle = \overline{\chi}_g(p)p^{k-m}\langle f(z/p)g(z), h(z)\rangle = \overline{\chi}_g(p)p^k\langle f_p g, h_p\rangle. \tag{G2}$$

$$\langle fg, h_p\rangle = \chi_h(p)p^{-m}\langle f(z)g(z), h(z/p)\rangle = \chi_h(p)\langle f_p g, h_p\rangle. \tag{H2}$$

Combining Equations (G1) and (G2) gives

$$\langle fg_p, h\rangle = \overline{\chi}_g(p)\frac{\alpha_g + \beta_g}{p^{m-k-1}(1+p)}\langle fg, h\rangle,$$

and combining Equations (H1) and (H2) gives

$$\langle fg, h_p\rangle = \chi_h(p)\frac{\overline{\alpha}_h + \overline{\beta}_h}{p^{m-1}(1+p)}\langle fg, h\rangle.$$

We can now return to the original quantity we wanted to compute,

$$\langle fg^\sharp, h^\natural\rangle = \langle fg, h\rangle - \beta_g\langle fg_p, h\rangle - p\overline{\beta}_h\langle fg, h_p\rangle + p\beta_g\overline{\beta}_h\langle fg_p, h_p\rangle.$$

By substituting in our formulas from above, this equals

$$\langle fg, h\rangle\left(1 - \frac{\beta_g\overline{\chi}_g(p)(\alpha_g + \beta_g)}{p^{m-k-1}(1+p)} - \frac{p\overline{\beta}_h\chi_h(p)(\overline{\alpha}_h + \overline{\beta}_h)}{p^{m-1}(1+p)} + \frac{p\beta_g\overline{\beta}_h(\alpha_f + \beta_f)}{p^{m-1}(1+p)}\right).$$

To begin simplifying the expression in parentheses, we use that $\chi_g(p)p^{m-k-1} = \alpha_g\beta_g$ and $\chi_h(p)p^{m-1} = \alpha_h\beta_h$ and also (from the Ramanujan conjecture) that $\beta_h = \chi_h(p)\overline{\alpha}_h$

and $\alpha_h = \chi_h(p)\overline{\beta}_h$; this gives us

$$\left(1 - \frac{\beta_g(\alpha_g + \beta_g)}{\alpha_g\beta_g(1+p)} - \frac{p\alpha_h(\alpha_h + \beta_h)}{\alpha_h\beta_h(1+p)} + \frac{p\beta_g\alpha_h(\alpha_f + \beta_f)}{\alpha_h\beta_h(1+p)}\right).$$

Simplifying and factoring out $p/(1+p) = 1/(1+p^{-1})$ we get that this is equal to

$$\frac{1}{1+p^{-1}}\left((1+p^{-1}) - p^{-1}\left(1 + \frac{\beta_g}{\alpha_g}\right) - \left(\frac{\alpha_h}{\beta_h} + 1\right) + \frac{\beta_g(\alpha_f + \beta_f)}{\beta_h}\right).$$

Collecting terms we get

$$\frac{1}{1+p^{-1}}\left(-\frac{\alpha_h}{\beta_h} - p^{-1}\frac{\beta_g}{\alpha_g}\frac{\beta_g\alpha_f}{\beta_h} + \frac{\beta_g\beta_f}{\beta_h}\right).$$

We now make one more substitution to get rid of the p^{-1}: we have

$$p^{-1} = \frac{p^{m-2}}{p^{m-1}} = \frac{\chi_f(p)p^{k-1}\chi_g(p)p^{m-k-1}}{\chi_h(p)p^{m-1}} = \frac{\alpha_f\beta_f\alpha_g\beta_g}{\alpha_h\beta_h}.$$

Substituting this back in and factoring out $(-\alpha_h/\beta_h)$ we get our expression is equal to

$$\frac{(-\alpha_h/\beta_h)}{1+p^{-1}}\left(1 + \frac{\alpha_f\beta_f\beta_g^2}{\alpha_h^2} - \frac{\beta_g\alpha_f}{\alpha_h} + \frac{\beta_g\beta_f}{\alpha_h}\right).$$

We can then factor this remaining term, letting us conclude:

Proposition 3.4.8. *Suppose we're in the above setup (in particular with the level of f coprime to p). Then we have*

$$\langle fg^\sharp, h^\natural\rangle = \langle fg, h\rangle\frac{(-\alpha_h/\beta_h)}{1+p^{-1}}\left(1 - \frac{\beta_g\alpha_f}{\alpha_h}\right)\left(1 - \frac{\beta_g\beta_f}{\alpha_h}\right).$$

3.5 Families of Hecke characters and CM forms

In this section we discuss how to construct $\mathcal{I}$-adic families of Hecke characters we'll be concerned with, as well as the associated $\mathcal{I}$-adic CM forms. Here is where we may be forced to actually use an extension $\mathcal{I}$ rather than Λ itself, due to the p-part of the class group of our imaginary quadratic field K.

3.5.1 Families of Hecke characters

What we mean by a $\mathcal{I}$-adic family of Hecke characters is a continuous homomorphism $\Psi : \mathbb{I}_K/K^\times \to \mathcal{I}^\times$; given this, if we take a point $P \in \mathcal{X}(\mathcal{I};\mathcal{O}_F)$ and view it as a homomorphism $\mathcal{I} \to \mathcal{O}_F$, the composition $P \circ \Psi$ (the "specialization at P") becomes a p-adic Hecke character $\mathbb{I}_K/K^\times \to \mathcal{O}_F^\times \subseteq \overline{\mathbb{Q}}_p^\times$. In particular, we'll want Ψ to be such that if we specialize at $P_{k,\varepsilon} \in \mathcal{X}(\mathcal{I};\mathcal{O}_F)$ we end up with an algebraic Hecke character with an infinity type determined by k (say, $(k,0)$ or $(k,-k)$).

To define such a family, we want to work from a $\mathbb{Z}_p$-extension of fields K_∞/K. In particular we want an extension unramified outside of p, so we can start by looking at the maximal such extension $K^{(p)}/K$. By class field theory, we can establish that we have an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (\mathcal{O}_{\mathfrak{p}}^\times \times \mathcal{O}_{\bar{\mathfrak{p}}}^\times)/\mathcal{O}_K^\times & \longrightarrow & \mathbb{I}_K/C^{(p)} & \longrightarrow & \mathrm{Cl}_K & \longrightarrow & 1 \\ & & \updownarrow{\scriptstyle \mathrm{Art}_K} & & \updownarrow{\scriptstyle \mathrm{Art}_K} & & \updownarrow{\scriptstyle \mathrm{Art}_K} & & \\ 1 & \longrightarrow & \mathrm{Gal}(K^{(p)}/H) & \longrightarrow & \mathrm{Gal}(K^{(p)}/K) & \longrightarrow & \mathrm{Gal}(H/K) & \longrightarrow & 1 \end{array},$$

where H is the Hilbert class field of K. Since $\mathrm{Gal}(K^{(p)}/H)$ is isomorphic to

$$\frac{\mathcal{O}_{\mathfrak{p}}^\times \times \mathcal{O}_{\bar{\mathfrak{p}}}^\times}{\mathcal{O}_K^\times} \cong \frac{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times}{\mathcal{O}_K^\times} \cong \frac{(\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times}{\delta[\mathcal{O}_K^\times]} \times (1+p)^{\mathbb{Z}_p} \times (1+p)^{\mathbb{Z}_p},$$

we can take a field H_∞/H such that $\mathrm{Gal}(H_\infty/H)$ corresponds to the quotient leaving the factor of $(1+p)^{\mathbb{Z}_p}$ from inside of $\mathcal{O}_{\mathfrak{p}}^\times$. Then H_∞/H is a $\mathbb{Z}_p$-extension that's unramified outside of $\mathfrak{p}$, and we have

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (1+p)^{\mathbb{Z}_p} & \longrightarrow & \mathbb{I}_K/C_{H_\infty} & \longrightarrow & \mathrm{Cl}_K & \longrightarrow & 1 \\ & & \updownarrow{\scriptstyle \mathrm{Art}_K} & & \updownarrow{\scriptstyle \mathrm{Art}_K} & & \updownarrow{\scriptstyle \mathrm{Art}_K} & & \\ 1 & \longrightarrow & \mathrm{Gal}(H_\infty/H) & \longrightarrow & \mathrm{Gal}(H_\infty/K) & \longrightarrow & \mathrm{Gal}(H/K) & \longrightarrow & 1 \end{array}.$$

Thus $\mathrm{Gal}(H_\infty/K)$ is an abelian group that's an extension of a finite abelian group by a copy of $\mathbb{Z}_p$. This means that the extension itself decomposes as a direct product of a copy of $\mathbb{Z}_p$ (containing, but not necessarily equal to, the $\mathbb{Z}_p$ we're extending by) with a finite abelian group that's a direct factor of $\mathrm{Gal}(H/K)$. If we let H_0/K be the subfield fixed by the finite abelian direct factor, and K_∞ the subfield of H_∞ fixed by that factor, we get $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$ and a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (1+p)^{\mathbb{Z}_p} & \longrightarrow & \mathbb{I}_K/C_{K_\infty} & \longrightarrow & (\mathrm{Cl}_K)_0 & \longrightarrow & 1 \\ & & \updownarrow{\scriptstyle \mathrm{Art}_K} & & \updownarrow{\scriptstyle \mathrm{Art}_K} & & \updownarrow{\scriptstyle \mathrm{Art}_K} & & \\ 1 & \longrightarrow & \mathrm{Gal}(K_\infty/H) & \longrightarrow & \mathrm{Gal}(K_\infty/K) & \longrightarrow & \mathrm{Gal}(H_0/K) & \longrightarrow & 1 \end{array}.$$

where $(\mathrm{Cl}_K)_0$ is the quotient of Cl_K corresponding to H_0, of order p^a. Here, both rows are abstractly isomorphic to extensions of the form $0 \to \mathbb{Z}_p \to \mathbb{Z}_p \to \mathbb{Z}/p^e\mathbb{Z} \to 0$.

Given this, we can abstractly extend the inclusion $(1+p)^{\mathbb{Z}_p} \hookrightarrow \overline{\mathbb{Q}}_p^\times$ to a character $\alpha : \mathbb{I}_K/C_{K_\infty} \to \overline{\mathbb{Q}}_p^\times$, i.e. a Hecke character. By restricting back to the behavior on $\mathcal{O}_{\mathfrak{p}}^\times \times \mathcal{O}_{\overline{\mathfrak{p}}}^\times$ in $\mathbb{I}_K$, we find that by construction α is algebraic with infinity-type $(1,0)$, conductor $\mathfrak{p}$, and finite-type $\omega_{\mathfrak{p}}^{-1}$ (where $\omega_{\mathfrak{p}}$ is the Teichmüller character on $(\mathcal{O}/\mathfrak{p})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times$). We also define $\beta = \alpha(\alpha^c)^{-1}$, where α^c is α precomposed with the action of complex conjugation on $\mathbb{I}_K$; thus β is algebraic with infinity-type $(1,-1)$, conductor $\mathfrak{p}\overline{\mathfrak{p}}$, and finite-type $\omega_{\mathfrak{p}}^{-1}\omega_{\overline{\mathfrak{p}}}$.

We remark that we quite specified α uniquely (unless the group $(\mathrm{Cl}_K)_0$ is trivial and we haven't actually extended our original map) - there are actually p^c different ways to abstractly extend the homomorphism $(1+p)^{\mathbb{Z}_p} \to$ to $\mathbb{I}/C_{K_\infty} \to \overline{\mathbb{Q}}_p^\times$, which differ by the p^c characters of $\mathbb{I}/C_{K_\infty}$ that are trivial on $(1+p)^{\mathbb{Z}_p}$. There's no canonical way to choose a "best" one, and fortunately it won't matter which one we take.

Now, $\mathbb{I}_K/C_{K_\infty} \cong \mathbb{Z}_p$ and it contains the original $(1+p)^{\mathbb{Z}_p} \cong \mathbb{Z}_p$ with index p^c; restricted to this subgroup α is continuous and injective. Thus α itself is continuous (since it's continuous on an open subgroup) and injective (because the kernel needs to intersect $(1+p)^{\mathbb{Z}_p}$ trivially, and no nontrivial subgroup does that). Thus we can conclude it maps isomorphically onto a subgroup $(1+\pi)^{\mathbb{Z}_p} \subseteq \overline{\mathbb{Q}}_p^\times$ for some element π such that $(1+\pi)^{p^e} = (1+p)$. By picking a finite extension $F/\mathbb{Q}_p$ containing this element π, we can view α and β as characters $\mathbb{I}_K/K^\times \to \mathcal{O}_F^\times$.

Now, working over our finite extension $F/\mathbb{Q}_p$ (which we may enlarge as needed later on), recall that $\Lambda = \Lambda_F$ is defined as $\mathcal{O}[\![\Gamma]\!]$ where Γ is the group $(1+p)^{\mathbb{Z}_p}$ (with distinguished topological generator $\gamma_p = 1+p$). We then define $\mathcal{I} = \mathcal{O}[\![\Gamma']\!]$ where Γ' is the group $(1+\pi)^{\mathbb{Z}_p}$ (with distinguished topological generator $\gamma_\pi = 1+\pi$) that contains Γ as a subgroup of index p^a; thus $\mathcal{I} \supseteq \Lambda$ is a finite (as a module) Λ-algebra. It's straightforward to check that $\mathcal{I}$ is an extension as considered in Definition 3.2.8; we can view it as a power series ring $\mathcal{O}_F[\![Y]\!]$ with Λ embedded inside as $\mathcal{O}_F[\![(1+Y)^{p^c}-1]\!]$, and then check that $\mathcal{I}$ is indeed the integral closure of Λ inside the Laurent series ring $\mathcal{O}_F(\!(Y)\!)$, and F is the relative algebraic closure of $\mathbb{Q}_p$ in it. To see that $\mathcal{X}(\mathcal{I}, \mathcal{O}_F)$ has infinitely many elements, we can in fact define a canonical extension of each point $P_k \in \mathcal{X}(\mathcal{I}, \mathcal{O}_F)$ by mapping γ_π to $(1+\pi)^k$.

Then, we can define a $\mathcal{I}$-adic family of Hecke characters $\mathcal{A} : \mathbb{I}_K/K^\times \to \mathcal{I}^\times$ as the composition of $\alpha : \mathbb{I}_K/K^\times \to \Gamma'$ with the tautological embedding $\Gamma' \hookrightarrow \mathcal{O}_F[\![\Gamma']\!]$. Continuity is immediate, as both α and the tautological embedding are continuous by construction. Moreover, we have the specialization property

$$P_m \circ \mathcal{A} = \alpha^m.$$

This is again essentially immediate from the definition; P_m composed with the tautological embedding gives the map $\Gamma' \to \Gamma'$ defined by $x \mapsto x^m$, so $P_m \circ \mathcal{A}$ is thus $y \mapsto \alpha(y)^m$. Similarly, we define $\mathcal{B} = \mathcal{A}(\mathcal{A}^c)^{-1}$ which has specializations $P_m \circ \mathcal{B} = \beta^m$.

In addition to working with the canonical extension of P_m to $\mathcal{X}(\mathcal{I}, \mathcal{O}_F)$ defined above, for the purposes of the theory of $\mathcal{I}$-adic modular forms we'll have to work with *all* extensions of $P_m : \Lambda \to \mathcal{O}_F$ to $P'_m : \mathcal{I} \to \mathcal{O}_F$. It's easy to see that an arbitrary such extension is of the form $\gamma_\pi \mapsto \zeta(1+\pi)^m$ for ζ a p^c-th root of unity. Then if we let $\varepsilon'_\zeta : \Gamma' \to \overline{\mathbb{Q}}_p^\times$ be the character taking $\varepsilon'_\zeta(\gamma_\pi) = \zeta$ we get that $P'_m \circ \mathcal{A} = (\varepsilon'_\zeta \circ \alpha)\alpha^m$. Note that the characters ε'_ζ are exactly the extensions of the trivial character on Γ to Γ', and that the compositions $(\varepsilon'_\zeta \circ \alpha)$ is an unramified finite-order Hecke character (because α takes the embedded copy of $\widehat{\mathcal{O}}_K \subseteq \mathbb{I}_K$ to Γ, which is killed by ε'_ζ).

The families of Hecke characters we'll ultimately want to use will then be translates of $\mathcal{A}$ or $\mathcal{B}$ by a fixed Hecke character. For instance, if we fix a Hecke character $\xi_0 \in \Sigma_{cc}(\mathfrak{N}, c)$ of infinity-type $(a-1, -a+k+1)$, we can define the family Ξ mentioned in Theorem 1.3.3 as $\Xi = \xi_0\beta^{-a}\mathcal{B}$; then Ξ is indeed a family such that $\xi_m = P_m \circ \Xi$ has infinity-type $(m-1, -m+k+1)$ for any m, and moreover satisfies $\xi_a = \xi_0$ and that that $\xi_m \in \Sigma_{cc}(\mathfrak{N}, c)$ for $m \equiv a \pmod{p-1}$ (since then ξ_m is given by ξ_a times a power of the unramified character β^{p-1}). For the purposes of the next section, on the other hand, we'll be interested in fixing a finite-order character ψ and defining families along the lines of $\Psi = \psi\alpha^{-1}\mathcal{A}$ with specializations $P_m \circ \Psi = \psi\alpha^{m-1}$ of infinity-type $(m-1, 0)$.

3.5.2 Families of CM forms

Now that we've shown how to define families of Hecke characters for our imaginary quadratic field K, we want to build from them an associated family of CM forms. Recall that a Hecke character of infinity-type $(m-1,0)$ leads to a holomorphic newform of weight m. So if we want to build a $\mathcal{I}$-adic CM form g_Ψ such that $P_m \circ \mathrm{g}_\Psi$ is a weight-m CM form, the natural choice is to take Ψ to be of the form suggested at the end of the previous section (with $P_m \circ \Psi$ of weight $(m-1,0)$) and try to construct g_Ψ so that $P_m(\mathrm{g}_\Psi) = g_{P_m \circ \Psi}$.

We need to be a little careful, of course. So suppose we start with an ideal $\mathfrak{m}$ prime to p, and a finite-order character ψ of conductor either $\mathfrak{m}$ or $\mathfrak{m}\mathfrak{p}$. In particular this forces the $\mathfrak{p}$-part of the finite-type of ψ to be of the form $\omega_{\mathfrak{p}}^{a-1}$ for some residue class a modulo $p-1$. Then we define $\Psi = \psi\alpha^{-1}\mathcal{A}$, so $\psi_m = P_m \circ \Psi$ is given by $\psi\alpha^{m-1}$, and find that ψ_m has conductor $\mathfrak{m}$ when $m \equiv a \pmod{p-1}$, and conductor $\mathfrak{m}\mathfrak{p}$ otherwise.

Next we need write down what the CM newform associated to the Hecke character ψ_m is. There's a slight wrinkle here: we've constructed ψ_m as a p-adic Hecke character, while the formula in Section 2.4 for defining classical CM forms is done in terms of complex Hecke characters. So we need to transfer the algebraic p-adic character $\psi_{m,\mathbb{Q}_p} : \mathbb{I}_K/K^\times \to \overline{\mathbb{Q}}_p^\times$ to an algebraic complex character $\psi_{m,\mathbb{C}} : \mathbb{I}_I/K^\times \to \mathbb{C}^\times$. We recall from Section 1.2 that if α is an idele, these characters are related by

$$\psi_m(\alpha) = \psi_{m,\mathbb{Q}_p}(\alpha) = (\iota_p \circ \iota_\infty^{-1})\big(\psi_{m,\mathbb{C}}(\alpha)\alpha_\infty^a \overline{\alpha}_\infty^b\big)\alpha_{\mathfrak{p}}^{-a}\alpha_{\overline{\mathfrak{p}}}^{-b}.$$

In particular, we're interested in treating $\psi_{m,\mathbb{C}}$ as a classical Hecke character and evaluating it ideals $\mathfrak{a}$. This amounts to evaluating it at an adele $\alpha = (\alpha_v)$ where $\alpha_v = \pi_v^{\mathrm{ord}_v(\mathfrak{a})}$ for finite places and $\alpha_v = 1$ for infinite places. Thus we find (suppressing the embeddings):

$$\psi_m(\alpha) = \psi_{m,\mathbb{C}}(\alpha)(\pi_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})})^{-(m-1)}.$$

If we abuse notation and just write $\mathfrak{a}$ for this adele (even though it isn't canonically associated to α because it involved a choice of uniformizers), and choose $\pi_{\mathfrak{p}} = p$, we get

$$\psi_m(\mathfrak{a}) = \psi_{m,\mathbb{Q}_p}(\mathfrak{a})p^{(m-1)\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})}.$$

Then, the CM newform associated to $\psi_{m,\mathbb{C}}$ is given by

$$g_{\psi_m} = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m})=1} \psi_{m,\mathbb{C}}(\mathfrak{a})q^{N(\mathfrak{a})} = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m})=1} \psi_m(\mathfrak{a})p^{(m-1)\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})}q^{N(\mathfrak{a})} \in S_m(dN(\mathfrak{m}), \chi_K\chi_{\psi_m})$$

in the case when $m \equiv a \pmod{p-1}$ and

$$g_{\psi_m} = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \psi_{m,\mathbb{C}}(\mathfrak{a})q^{N(\mathfrak{a})} = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \psi_m(\mathfrak{a})p^{(m-1)\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})}q^{N(\mathfrak{a})} \in S_m(dN(\mathfrak{m})p, \chi_K\chi_{\psi_m})$$

when $m \not\equiv a \pmod{p-1}$. We note that in both cases the form is p-ordinary, as the coefficient of q^p is $\psi_m(\overline{\mathfrak{p}})$ (a unit) or $\psi_m(\overline{\mathfrak{p}}) + \psi_m(\mathfrak{p})p^{m-1} \equiv \psi_m(\overline{\mathfrak{p}})$ in the two cases.

At this point, we can see that $g_{\psi,m}$ is written as $\sum P_m(\Psi(\mathfrak{a}))p^{(n-1)\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})}q^{N(\mathfrak{a})}$, so we'd like to simply define $\mathrm{g}_\Psi \in \mathcal{I}[\![q]\!]$ by the formula $\sum \Psi(\mathfrak{a})p^{(n-1)\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})}q^{N(\mathfrak{a})}$ so that when we specialize at P_m we get $g_{\psi,m}$. The only concern is what index set of $\mathfrak{a}$'s we want to sum over - but the characterization of $\mathcal{I}$-adic newforms in Theorem 3.3.4 makes this clear: we want to go with the sum from the case $m \not\equiv a \pmod{p-1}$, where we already have a power of p in the level. Thus we define

$$\mathrm{g}_\Psi = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \Psi(\mathfrak{a})p^{(m-1)\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})}q^{N(\mathfrak{a})} \in \mathcal{I}[\![q]\!].$$

We see that for $m \not\equiv a \pmod{p-1}$ we have that $P_m(\mathrm{g}_\Psi) = g_{\psi_m}$ is a newform in $S_m(dN(\mathfrak{m})p, \chi_K\chi_\psi\omega^m)$, so g_Ψ has the right behavior to be a $\mathcal{I}$-adic newform in

$\mathbb{S}(dN(\mathfrak{m}), \chi_K\chi_\psi\omega^{-a}; \mathcal{I})$. The only thing that remains to check is what $P_m(\mathrm{g}_\Psi)$ is for $m \equiv a \pmod{p-1}$. Fortunately, here it's what we'd expect too, the p-stabilization $g^\sharp_{\psi_m}$. To see this we note that the roots of the Hecke polynomial at p for g_{ψ_m} are $\psi_m(\mathfrak{p})p^{m-1}$ and $\psi_m(\overline{\mathfrak{p}})$, respectively, and the latter one is clearly a unit while the former isn't. So we have $g^\sharp_{\psi_m} = g_{\psi,m}(z) - \psi_m(\mathfrak{p})p^{m-1}g_{\psi,m}(pz)$. But the latter term is just

$$\psi_m(\mathfrak{p})p^{m-1} \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m})=1} \psi_m(\mathfrak{a})p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})}q^{N(\mathfrak{a})p} = \sum_{\mathfrak{a}\mathfrak{p}:(\mathfrak{a},\mathfrak{m})=1} \psi_m(\mathfrak{a}\mathfrak{p})p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a}\mathfrak{p})}q^{N(\mathfrak{a}\mathfrak{p})},$$

so subtracting it off from the sum over $\mathfrak{a}$ coprime to $\mathfrak{m}$ that defines $g_{\psi_m}(z)$ removes the terms corresponding to ideals divisible by $\mathfrak{p}$, and we conclude

$$g^\sharp_{\psi_m} = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \psi_m(\mathfrak{a})p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})}q^{N(\mathfrak{a})} = P_m(\mathrm{g}_\Psi).$$

So we get that $P_m(\mathrm{g}_\Psi)$ is of the form we want for our canonical extensions $P_m : \mathcal{I} \to \mathcal{O}_F$. Since the definition of a $\mathcal{I}$-adic modular form we're using requires we study *all* extensions, we need to consider $P'_m(\mathrm{g}_\Psi)$ for the arbitrary extension P'_m taking γ_π to $\zeta(1+\pi)^m$ as discussed in the previous section. But we know $P'_m \circ \Phi$ will just give us ψ_m times an unramified Hecke character ε', so $P_m(\mathrm{g}_\Psi)$ is either $g_{\varepsilon'\psi_m}$ or $g^\sharp_{\varepsilon'\psi_m}$, depending on the residue class of m. Either way it is a p-adic modular form of the appropriate weight and level, so g_Ψ is indeed a $\mathcal{I}$-adic modular form. Summing up:

Proposition 3.5.1. *Let $\mathfrak{m}$ be an ideal coprime to p, and let ψ be a finite-order Hecke character of conductor $\mathfrak{m}$ or $\mathfrak{m}\mathfrak{p}$ such that the $\mathfrak{p}$-part of the finite-type of ψ is $\omega_\mathfrak{p}^{a-1}$. Then we can construct a family of Hecke characters $\Psi : \mathbb{I}_K/K^\times \to \mathcal{I}$ such that $P_m \circ \Psi = \psi\alpha^{m-1}$, and an associated $\mathcal{I}$-adic CM newform*

$$\mathrm{g}_\Psi = \sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \Psi(\mathfrak{a})p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})}q^{N(\mathfrak{a})} \in \mathbb{S}^{\mathrm{ord}}(dN(\mathfrak{m}), \chi_K\chi_\psi; \mathcal{I})$$

such that $P_m(\mathrm{g}_\Psi) = g_{\psi_m}$ *for* $m \not\equiv a \pmod{p-1}$ *and* $P_m(\mathrm{g}_\Psi) = g^\sharp_{\psi_m}$ *for* $m \equiv a$ $\pmod{p-1}$.

We will also need a variant of this construction that will give us $\mathcal{I}$-adic forms interpolating the product of a fixed modular form f with a varying CM form. So suppose we have $f = \sum a_n q^n \in M_k(N, \chi_f)$, and similarly to before we fix $\mathfrak{m}$ prime to p along with a finite-order Hecke character φ of conductor $\mathfrak{m}$ or $\mathfrak{m}\mathfrak{p}$ such that the $\mathfrak{p}$-part of the finite-type of ψ is $\omega_\mathfrak{p}^b$. Then we can define a family $\Phi = \varphi\alpha^{-k-1}\mathcal{A}$ such that $P_m \circ \Phi = \varphi\alpha^{m-k-1}$ (which we denote φ_{m-k}) has its corresponding CM form $g_{P_m\circ\Phi}$ of weight $m-k$, so $f \cdot g_{P_m\circ\Phi}$ has the appropriate weight m. Then, along the same lines as before we can define a family

$$f\mathrm{g}_\Phi = \left(\sum_{n=1}^{\infty} a_n q^n\right)\left(\sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \Phi(\mathfrak{a}) p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})} q^{N(\mathfrak{a})}\right) \in \mathbb{S}(dN\cdot N(\mathfrak{m}), \chi_f\chi_K\chi_\varphi; \mathcal{I})$$

such that the specializations are $P_m(f\mathrm{g}_\Phi) = f\cdot g_{\varphi_{m-k}}$ for $m \not\equiv b \pmod{p-1}$ and $P_m(f\mathrm{g}_\Phi) = f\cdot g^\sharp_{\varphi_{m-k}}$ for $m \equiv b \pmod{p-1}$. (Here we take the p-stabilization of $g_{\varphi_{m-k}}$ since by construction it's a p-ordinary CM form, but leave f alone since we're not assuming anything about it).

In fact we'll need a slight modification of this construction - rather than interpolating the product $f(z)g_{\varphi_{m-k}}(z)$ itself, we'll want to interpolate translates such as like $f(M_1 z/L)g_{\varphi_{m-k}}(M_2 z/L)$ for M_1, M_2, L positive integers. Of course, changing from z to Mz/L just changes the q-expansion by replacing q with $q^{Mz/L}$, so it's easy to modify our series defining $f\mathrm{g}_\Phi$ to account for this. So, we get:

Proposition 3.5.2. *Let* $f = \sum a_n q^n \in M_k(N, \chi_f)$ *be a modular form,* $\mathfrak{m}$ *be an ideal coprime to* p*, and let* ψ *be a finite-order Hecke character of conductor* $\mathfrak{m}$ *or* $\mathfrak{m}\mathfrak{p}$ *such that the* $\mathfrak{p}$*-part of the finite-type of* ψ *is* $\omega_\mathfrak{p}^{a-1}$*. Also fix positive integers* M_1, M_2, L*. Then we can construct a family of Hecke characters* $\Phi : \mathbb{I}_K/K^\times \to \mathcal{I}$ *such that*

$P_m \circ \Phi = \psi\alpha^{m-k-1}$, *and an associated* $\mathcal{I}$*-adic modular form*

$$f\mathrm{g}_\Phi^* = \left(\sum_{n=1}^{\infty} a_n q^{M_1 n/L}\right)\left(\sum_{\mathfrak{a}:(\mathfrak{a},\mathfrak{m}\mathfrak{p})=1} \Phi(\mathfrak{a}) p^{(m-1)\mathrm{ord}_\mathfrak{p}(\mathfrak{a})} q^{M_2 N(\mathfrak{a})/L}\right)$$

such that $P_m(\mathrm{g}_\Psi) = f(M_1 z/L) g_{\varphi_m}(M_2 z/L)$ *for* $m \not\equiv a \pmod{p-1}$ *and* $P_m(\mathrm{g}_\Psi) = f(M_1 z/L) g^\sharp_{\varphi_{m-k}}(M_2 z/L)$ *for* $m \equiv a \pmod{p-1}$.

In this generality we can just say that $f\mathrm{g}_\Phi^*$ lies in the space

$$\mathbb{S}(\Gamma_0(dNM_1M_2N(\mathfrak{m}), L), \chi_f\chi_K\chi_\varphi),$$

though in the cases we'll actually deal with we'll be able to have much tighter control over the level.

3.6 The module of congruences for a CM family

Finally, the last bit of Λ-adic theory we'll need to use is an explicit computation of the congruence number H_g in the case $\mathrm{g} = \mathrm{g}_\Psi$ is a $\mathcal{I}$-adic CM newform as constructed in the previous section. As we've mentioned, this computation is essentially a consequence of the main conjecture of Iwasawa theory for imaginary quadratic fields, which was proven by Rubin [Rub91]. The result is that H_g is essentially a p-adic L-function associated to an anticyclotomic family of Hecke characters closely related to Ψ; the connection between the main conjecture and the computation of H_g is studied by Hida and Tilouine in [HT93] and [HT94].

The p-adic L-function associated to Hecke characters of an imaginary quadratic field (or more generally of a CM field). These L-functions were first constructed by Katz [Kat78] and thus are referred to as *Katz p-adic L-functions*. Katz's original paper covers the case of p-power conductors; Hida and Tilouine [HT93] work out the construction in full generality (see also Hida's exposition in Chapter 9 of [Hid13]).

We'll give the necessary setup to state the special case of this construction that we'll actually use.

The main thing we need to do is to define the appropriate periods for algebraicity results. We actually define a pair $(\Omega_\infty, \Omega_p)$ of a complex and a p-adic period, following Katz and Hida-Tilouine. The idea is to fix an auxiliary prime-to-p conductor c, and take A_0 to be the elliptic curve defined by $A_0(\mathbb{C}) = \mathbb{C}/\mathcal{O}_c$, which we can show is defined over a number field. We then pick a nowhere-vanishing differential ω defined over this number field. Then, working over $\mathbb{C}$ we have a standard differential ω_∞ coming from the standard coordinate on $\mathbb{C}$ passed to $\mathbb{C}/\mathcal{O}_c$. Working over $\mathcal{O}_{\mathbb{C}_p}$ we can get an isomorphism between $\widehat{\mathbb{G}}_m$ and the formal completion $\widehat{\mathcal{A}}_0$ of a good integral model $\mathcal{A}_0$ of A_0, well-defined up to p-adic units; from this we get a differential ω_p induced by the standard period dt/t on $\widehat{\mathbb{G}}_m$. We can then define our periods:

Definition 3.6.1. Given a fixed differential ω defined over $\mathcal{O}_c$ as above, we define two periods $\Omega_\infty \in \mathbb{C}$ and $\Omega_p \in \mathbb{C}_p$ by

$$\omega = \Omega_\infty \omega_\infty \qquad\qquad \omega = \Omega_p \omega_p.$$

The pair $(\Omega_\infty, \Omega_p)$ is evidently well-defined up to algebraic scalars. We usually take the convention that we choose our scalar so that Ω_p is a p-adic unit; thus $(\Omega_\infty, \Omega_p)$ is well-defined up to p-adic units of $\overline{\mathbb{Q}}$ (via the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ that we've fixed since the beginning). Also, we note that while we started with a conductor c (and defined our elliptic curve A_0 as the one having CM by the order $\mathcal{O}_c$ of $\mathcal{O}_K$), the dependence on c is entirely illusory: since there's a prime-to-p isogeny between the resulting curves for any two such choices of c, we can choose the same pair of periods $(\Omega_\infty, \Omega_p)$ to arise from both cases.

With our periods constructed, we can state the interpolation property that characterizes the Katz p-adic L-function, and quote the existence result of this L-function.

In full generality, the Katz L-function can be defined to interpolate special values of Hecke L-functions where we can vary the character over all things in a certain range of conductors; we will be interested in a specialization that gives us something defined on a domain similar to Λ, which interpolates the special values in a particular anticyclotomic family of characters. In particular we'll consider a family $\Phi : \mathbb{I}_K \to \mathcal{I}$ constructed as in the last section to satisfy $P_m \circ \Phi = \varphi_m = \varphi\beta^{m-1}$, where β is the anticyclotomic character of weight $(1, -1)$ and finite-type $\omega_{\mathfrak{p}}^{-1}\omega_{\overline{\mathfrak{p}}}$, and φ is a fixed finite-order Hecke character. More specifically, we'll ask that φ has prime-to-p conductor $\mathfrak{c}$, and that the p-part of its finite-type is $\omega_{\mathfrak{p}}^{a}\omega_{\overline{\mathfrak{p}}}^{-a}$ for a residue class a mod $p-1$. Thus φ_m will be unramified at p (and have conductor exactly $\mathfrak{c}$) for $m \equiv a \pmod{p-1}$.

So, the interpolation property we want is at points P_m for $m \equiv a \pmod{p-1}$. There is a slight additional wrinkle: even though our family Φ of Hecke characters are defined over $\mathcal{I}$, the p-adic L-function will need to be defined over a larger ring; to deal with the periods we need to extend scalars to $\mathcal{O}_F^{\mathrm{ur}}$, the integer ring of the maximal unramified extension F^{ur}/F. So we define $\mathcal{I}^{\mathrm{ur}} = \mathcal{I} \otimes_{\mathcal{O}_F} \mathcal{O}_F^{\mathrm{ur}} = \mathcal{O}_F^{\mathrm{ur}}[\![\Gamma']\!]$. On this larger ring, our distinguished specializations $P_m : \mathcal{I} \to \mathcal{O}_F$ have a unique $\mathcal{O}_F^{\mathrm{ur}}$-linear extension to $\mathcal{I}^{\mathrm{ur}} \to \mathcal{O}_F^{\mathrm{ur}}$ that we also denote P_m, characterized in the usual way (by mapping the distinguished topological generator γ_π to $(1+\pi)^m$). Then:

Theorem 3.6.2. *Suppose Φ is an anticyclotomic family as described above, so that for $m \equiv a \pmod{p-1}$ we have that φ_m has infinity-type $(m-1, -m+1)$ and finite-type φ_{fin} of conductor $\mathfrak{c}$ coprime to p, for the character φ we started with. Then, for $m \geq 3$ satisfying $m \equiv a \pmod{p-1}$ we can define $L_{\mathrm{alg}}^{Katz}(\varphi_m, 1) \in \overline{\mathbb{Q}}$ and $L_p^{Katz}(\varphi_m, 1) \in \mathbb{C}_p$ satisfying*

$$\frac{L_p^{Katz}(\varphi_m, 1)}{\Omega_p^{2m-2}(1-\varphi_m(\overline{\mathfrak{p}})p^{-1})(1-\varphi_m(\mathfrak{p})^{-1})} = L_{\mathrm{alg}}^{Katz}(\varphi_m, 1) = \frac{(m-1)!\pi^{m-2}}{2^{m-2}\sqrt{d}^{m-2}} \frac{L(\varphi_m, 1)}{\Omega_\infty^{2m-2}},$$

and there exists an element $\mathcal{L}_p^{Katz}(\Phi) \in \mathcal{I}^{ur}$ *such that specializing at* P_m *for such* m *gives* $L_p^{Katz}(\varphi_m, 1)$.

Next, we connect this back to the problem of finding the congruence number associated to a $\mathcal{I}$-adic family g_Ψ where Ψ is a family with $P_m \circ \Psi = \psi_m = \psi\alpha^{m-1}$ of infinity-type $(m-1, 0)$ (where here α has infinity-type $(1, 0)$ and finite-type $\omega_{\mathfrak{p}}^{-1}$). As we've said, the Iwasawa main conjecture for the imaginary quadratic field K is enough to compute this; the details of how the main conjecture allows us to compute H_{g_Ψ} are worked out one way in [HT93] and [HT94], and can also be recovered from the Galois cohomology computations underlying Wiles' modularity lifting machinery. The computation tells us that H_{g_Ψ} is essentially the Katz L-function associated to the family $\Phi = \Psi(\Psi^c)^{-1}$ (where c denotes precomposition with complex conjugation). Since $\alpha(\alpha^c)^{-1} = \beta$, this Φ is the same as the Φ discussed above associated to $\varphi = \psi(\psi^c)^{-1}$.

Theorem 3.6.3. *Suppose that our family* Ψ *is such that the CM forms* g_{ψ_m} *have residually irreducible* p*-adic Galois representations. Then the element* $H_{\mathrm{g}_\Psi} \in \mathcal{I}$*, such that the congruence module of* g_Ψ *is isomorphic to* $\mathcal{I}/H_{\mathrm{g}_\Psi}\mathcal{I}$*, is equal to*

$$h_K \left(\prod_{q | N(\mathfrak{c})} L_q(\psi, 1)^{-1} L_q(1, \chi_K)^{-1} \right) \mathcal{L}_p^{Katz}(\Psi(\Psi^c)^{-1}) \in \mathcal{I}^{ur}$$

up to a unit of $\mathcal{I}^{ur}$*. Here* h_K *is the class number of* K *and* ψ *has conductor* $\mathfrak{c}$.

Then, combining this with the construction of Section 3.4 gives us:

Theorem 3.6.4. *Suppose that we have a family* $\Psi : \mathbb{I}_K \to \mathcal{I}$ *as above such that the forms* g_{ψ_m} *have residually irreducible Galois representation; let* g_Ψ *be the associated CM newform in* $\mathbb{S}^{\mathrm{ord}}(dN(\mathfrak{m}), \chi_K\chi_\psi; \mathcal{I})$*, and let* $\mathbb{f} \in \mathbb{S}(dN(\mathfrak{m}), \chi_K\chi_\psi; \mathcal{I})$ *be any* $\mathcal{I}$*-adic* *form. Then there exists an element* $\langle \mathbb{f}, \mathrm{g}_\Psi \rangle \in \mathcal{I}^{\mathrm{ur}}$ *characterized by*

$$P_m(\langle \mathbb{f}, \mathrm{g}_\Psi \rangle) = a(1, 1_{\mathrm{g}_\Psi} \mathbb{f}) \cdot \mathcal{L}_p^{Katz}(\Psi(\Psi^c)^{-1}) \cdot h_K \left(\prod_{q | N(\mathfrak{c})} L_q(\psi, 1)^{-1} L_q(1, \chi_K)^{-1} \right).$$

We remark that the hypothesis about residually irreducible Galois representations is automatically satisfied if Ψ is ramified at any inert prime, since then even the induced local representation at that prime is residually irreducible. Also, along this chain of reasoning we implicitly use the fact that the local ring of the Hecke algebra through which $\lambda_{\mathrm{g}_\Psi}$ factors is Gorenstein (as needed in Theorem 3.3.10 to conclude the existence of H_{g_Ψ} in the first place); this Gorenstein condition follows from modularity lifting work, e.g. from [Wil95], Corollary 2 to Theorem 2.1.

Chapter 4

The main calculation

4.1 Setup

In this chapter we proceed with the main calculation of the thesis. Our setup is that we start with an odd prime p, a classical newform f with trivial central character, an imaginary quadratic field K, and an appropriate anticyclotomic p-adic family Ξ of Hecke characters on K such that the BDP p-adic L-function $\mathcal{L}_p^{BDP}(f, \Xi)$ of Theorem 1.3.3 is defined. We want to prove an identity relating this to pairings of Hida families $\langle f \mathrm{g}_\Phi, \mathrm{g}_\Psi \rangle$ (for two families of characters Φ and Ψ we'll construct from Ξ). The precise assumptions on our setup will be stated in Section 4.1.3, after we've gone over an outline of what we want to do and which hypotheses naturally arise up in our approach.

First of all, by construction we know $\mathcal{L}_p^{BDP}(f, \Xi)$ lies in $\Lambda^{\mathrm{ur}} = \mathcal{O}_F^{\mathrm{ur}}[\![\Gamma]\!] \cong \mathcal{O}_F^{\mathrm{ur}}[\![X]\!]$ for $\Gamma = (1+p)^{\mathbb{Z}_p}$. Likewise $\langle f \mathrm{g}_\Phi, \mathrm{g}_\Psi \rangle$ lies in $\mathcal{I}^{\mathrm{ur}} = \mathcal{O}_F[\![\Gamma']\!]$ for $\Gamma' = (1+\pi)^{\mathbb{Z}_p}$ containing Γ with p-power index. To prove an equality involving these elements in $\mathcal{I}^{\mathrm{ur}}$, it's sufficient to prove equality for infinitely many specializations by the following lemma. (This can be recovered as a consequence of the Weierstrass Preparation Theorem because $\mathcal{I}^{\mathrm{ur}}$ is abstractly isomorphic to $\mathcal{O}_F^{\mathrm{ur}}[\![X]\!]$; alternatively it can be proven by commutative algebra using that $\mathcal{I}^{\mathrm{ur}}$ is Noetherian of dimension 2 and the points P correspond to prime ideals of height 1).

Lemma 4.1.1. *If $A, B \in \mathcal{I}^{\mathrm{ur}}$ are two elements such that $P(A) = P(B)$ for infinitely many points $P \in \mathcal{X}(\mathcal{I}^{\mathrm{ur}}; \mathcal{O}_F^{\mathrm{ur}})$, then $A = B$.*

Thus, to prove the equality we want it's sufficient to prove that it's true under an infinite family of specializations $P_m : \mathcal{I}^{\mathrm{ur}} \to \mathcal{O}_F^{\mathrm{ur}}$. By how we've constructed the elements we're studying, the specialization $P_m(\langle f \mathrm{g}_\Phi, \mathrm{g}_\Psi \rangle)$ is a quantity related to a Petersson inner product $\langle f g_{\varphi,m-k}, g_{\psi,m} \rangle$, while for certain m the specialization $P_m(\mathcal{L}_p^{BDP}(f, \Xi))$ is the BDP L-value $L_p^{BDP}(f, \xi_m^{-1})$, which comes from the Rankin L-value $L(0, f, \xi_m^{-1})$. So, our strategy will be to use an identity relating this Petersson inner product to this Rankin L-value to establish an identity of specializations $P_m(\langle f \mathrm{g}_\Phi, \mathrm{g}_\Psi \rangle)$ and $P_m(\mathcal{L}_p^{BDP}(f, \Xi))$ for infinitely many m (in fact, for all m in a particular arithmetic progression).

The identity we'll use, of course, will be Ichino's formula. Our classical version of this (Theorem 2.2.5) tells us that if f, g, h are newforms of weights k, $m-k$, and m, respectively, and their characters satisfy $\chi_f \chi_g = \chi_h$ then we have an identity of the form

$$|\langle f^\dagger g^\dagger, h^\dagger \rangle|^2 = C \cdot L(m-1, f \times g \times \overline{h})$$

for some explicit constant C (where $f^\dagger(z) = f(M_f z)$, $g^\dagger(z) = g(M_g z)$, and $h^\dagger(z) = h(M_h z)$ for certain integers M_f, M_g, M_h). In our situation we want to take f to be some fixed newform of weight k with trivial central character, while $g = g_{\varphi_{m-k}}$ and $h = g_{\psi_m}$ will be the CM forms associated to Hecke characters φ_{m-k} and ψ_m of an imaginary quadratic field K of weights $(m-k-1, 0)$ and $(m-1, 0)$, respectively.

Our more precise version of Ichino's formula (from Section 2.4.2) tells us

$$\begin{aligned}|\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle|^2 &= C_m \cdot L(m-1, f \times g_{\varphi,m-k} \times g_{\psi,m}) \\ &= C_m \cdot L(0, f, \varphi_{m-k}\psi_m^{-1}) \cdot L(0, f, \varphi_{m-k}^{-1}\psi_m^{-1}N^{m-k-1}, \chi_{\varphi_{m-k}}).\end{aligned}$$

Our families of Hecke characters will be set up so that $\varphi_{m-k}\psi_m^{-1}$ is actually a constant (independent of m), so the first L-factor is constant in the family of formulas we get as m varies. The constants C_m will turn out to be well-behaved as well, so our formula really will be able to be manipulated to become an identity of specializations of Λ.

4.1.1 Calibrating the families of characters

We still need to be a bit more careful about what families of characters Φ and Ψ we'll consider, and for what values of m we want to study the formula we have. First of all, we recall that our families of characters Φ and Ψ are specified by fixing finite-order characters φ and ψ and then setting

$$\varphi_{m-k} = \varphi\alpha^{m-k-1} \qquad \psi_m = \psi\alpha^{m-1}$$

where α is a Hecke character of weight $(1,0)$, conductor $\mathfrak{p}$, and finite part $\omega_{\mathfrak{p}}^{-1}$ (where $\omega_{\mathfrak{p}}$ is the Teichmüller character for the $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$). The $\mathbb{Z}$-parts of the finite types of these characters are then given by

$$\chi_{\varphi_{m-k}} = \chi_{\varphi}\omega^{-(m-k-1)} \qquad \chi_{\psi_m} = \chi_{\psi}\omega^{-(m-1)}.$$

Then, when $m \geq k+1$ we know that the associated CM forms lie in the spaces

$$g_{\varphi,m-k} \in S_{m-k}(dM_{\varphi}p, \chi_{\varphi}\omega^{-(m-k-1)}\chi_K) \qquad g_{\psi,m} \in S_m(dM_{\psi}p, \chi_{\psi}\omega^{-(m-1)}\chi_K)$$

where M_φ and M_ψ are the norms of the conductors of φ and ψ. (The p in the levels accounts for the fact that $\varphi_{m-k} = \varphi\alpha^{m-k-1}$ could be ramified at $\mathfrak{p}$ even if φ is not, and similarly for ψ_m). Ichino's formula will only apply if we have $\chi_f \chi_{g_{\varphi,m-k}} = \chi_{g_\psi,m}$, i.e.

$$\chi_f \cdot \chi_\varphi \omega^{-(m-k-1)} \chi_K = \chi_\psi \omega^{-(m-1)} \chi_K,$$

or equivalently $\chi_\varphi \omega^k = \chi_\psi$, using the assumption that χ_f is trivial. In this case, Ichino's formula plus our formal factorization of the L-function tells us

$$|\langle f g^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle|^2 = C_m \cdot L(0, f, \varphi\psi^{-1}\alpha^{-k}) \\ \cdot L(0, f, \varphi^{-1}\psi^{-1}\alpha^{-(2m-k-2)} N^{m-k-1}, \chi_\varphi \omega^{-(m-k-1)}).$$

In the study of the BDP L-function we don't want to have a Dirichlet character twist, so we'd like to choose φ and ψ so that $\chi_\varphi \omega^{-(m-k-1)}$ is the trivial character. Of course we can't guarantee this for all m; the best we can do is make it trivial on some arithmetic progression of weights $m \equiv a \pmod{p-1}$. So, if we fix some such a representing a residue class (taking $0 \leq a < p-1$ for convenience), we require $\chi_\varphi \omega^{-(m-k-1)} = \chi_\varphi \omega^{-(a-k-1)} = 1$ for $m \equiv a \pmod{p-1}$. This fully determines what χ_φ has to be, and thus fully determines χ_ψ by the equation from before.

So, we conclude that if our finite-order Hecke characters φ, ψ satisfy

$$\chi_\varphi = \omega^{a-k-1} \qquad \chi_\psi = \omega^{a-1}$$

then for $m > k$ satisfying $m \equiv a \pmod{p-1}$ Ichino's formula gives us an identity

$$|\langle f g^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle|^2 = C_m \cdot L(0, f, \varphi\psi^{-1}\alpha^{-k}) \cdot L(0, f, \varphi^{-1}\psi^{-1}\alpha^{-(2m-k-2)} N^{m-k-1}).$$

For convenience of notation we define

$$\eta = \varphi^{-1}\psi\alpha^k \qquad\qquad \xi_m = \varphi\psi\alpha^{2m-k-2}N^{-(m-k-1)},$$

and then our formula becomes

$$|\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle|^2 = C_m \cdot L(0,f,\eta^{-1}) \cdot L(0,f,\xi_m^{-1}).$$

We can see that η has weight $(k,0)$ and ξ_m has weight $(m-1,-m+k+1)$, and thus lie in the correct range of weights for $L(0,f,\eta^{-1})$ and $L(0,f,\xi_m^{-1})$ to be the L-values interpolated by the BDP L-function. We remark that while the formula (and the interpolation formulas for $\mathcal{L}^{BDP}$ and $\langle f\mathrm{g}_\Phi, \mathrm{g}_\Phi\rangle$) only hold if $m \geq k+1$, the character ξ_m is defined for any $m \equiv a \pmod{p-1}$, and ultimately we'll interested in specializing our p-adic L-function outside of the range of interpolation.

4.1.2 Specifying the characters more precisely

We now go into a bit more depth about exactly what φ and ψ will be. Ultimately, we want the products η and ξ_m to fit into the sets of characters $\Sigma^{(2)}_{cc}(\mathfrak{N},c)$ that [BDP13] study (with all of the ξ_m's having the same auxiliary constant c, but η perhaps having a different one).

So, suppose we fix an integer $\underline{a} \equiv a \pmod{p-1}$, a constant c (assumed to be coprime to p, d, and N), and a character $\xi_{\underline{a}} \in \Sigma_{cc}(\mathfrak{N},c)$ of weight $(\underline{a}-1, -\underline{a}+k+1)$. We want to define ψ and φ so that $\xi_{\underline{a}} = \varphi\psi\alpha^{2\underline{a}-k-2}N^{-(\underline{a}-k-1)}$ actually equals the character $\xi_{\underline{a}} \in \Sigma_{cc}(\mathfrak{N},c)$ that we've fixed, and also so that $\eta = \varphi^{-1}\psi\alpha^k$ lies in some $\Sigma^{(2)}_{cc}(\mathfrak{N},c')$ (we note that all of the characters ξ_m for $m \equiv a \pmod{p-1}$ can be written as $\xi_{\underline{a}}$ times an unramified anticyclotomic character so automatically will lie in $\Sigma_{cc}(\mathfrak{N},c)$). We remark that we'll be especially interested in taking $0 \leq \underline{a} \leq k$ so that $m = \underline{a}$ falls outside of the range of interpolation and thus the value $L_p^{BDP}(f,\xi_{\underline{a}}^{-1}) = P_{\underline{a}}(\mathcal{L}^{BDP}(\Xi))$

will have an interpretation as a p-adic regulator.

To construct φ and ψ satisfying the conditions we want (all of those stated, plus some more that will be useful for technical reasons later) we will first define characters ψ_{fin} and φ_{fin} of quotients $(\mathcal{O}_K/\mathfrak{m})^\times$ (for moduli $\mathfrak{m}$ to be determined precisely as we go along) and then show that we can define actual Hecke characters φ and ψ with these desired finite parts. We can always enlarge the modulus $\mathfrak{m}$ to some ideal $M\mathcal{O}_K$ for $M \in \mathbb{Z}$, and then the Chinese remainder theorem any character of $(\mathcal{O}_K/M\mathcal{O}_K)^\times$ splits up as a product of characters of $(\mathcal{O}_K/q^{e_q}\mathcal{O}_K)^\times$ for the finitely many primes $q|M$. So, we will build up ψ_{fin} and φ_{fin} by specifying how we want to define $\psi_{\text{fin},q}$ and $\varphi_{\text{fin},q}$ on $(\mathcal{O}_K/q^{e_q}\mathcal{O}_K)^\times$ for various primes q.

Case 1: $q = p$. Given how we want to define η and ξ_m in terms of φ and ψ, we will have

$$\eta_{\text{fin},p} = \varphi_{\text{fin},p}^{-1}\psi_{\text{fin},p}\omega_{\mathfrak{p}}^{-k} \qquad\qquad \xi_{m,\text{fin},p} = \varphi_{\text{fin},p}\psi_{\text{fin},p}\omega_{\mathfrak{p}}^{-(2m-k-2)}.$$

Since we want η and ξ_m to lie in the domain of the BDP L-function, they should have conductor coprime to p and thus $\eta_{\text{fin},p}$ and $\xi_{m,\text{fin},p}$ should be trivial! So we need to specify $\varphi_{\text{fin},p}$ and $\psi_{\text{fin},p}$ to make this so. We can simply take $\varphi_{\text{fin},p}$ and $\psi_{\text{fin},p}$ to be unramified at $\overline{\mathfrak{p}}$, but we need to include some powers of the character $\omega_{\mathfrak{p}}$ to compensate for those in the equations above. In particular we want to set $\varphi_{\text{fin},\mathfrak{p}} = \omega_{\mathfrak{p}}^A$ and $\psi_{\text{fin},\mathfrak{p}} = \omega_{\mathfrak{p}}^B$ for some exponents A, B such that

$$\eta_{\text{fin},\mathfrak{p}} = \varphi_{\mathfrak{p}}^{-1}\psi_{\mathfrak{p}}(\omega_{\mathfrak{p}})^{-k} = \omega_{\mathfrak{p}}^{-A+B-k} = 0$$

and

$$\chi_{m,\text{fin},\mathfrak{p}} = \varphi_{\mathfrak{p}}\psi_{\mathfrak{p}}(\omega_{\mathfrak{p}})^{-(2m-k-2)} = \omega_{\mathfrak{p}}^{A+B-2m+k+2} = 0$$

when $m \equiv a \pmod{p-1}$. Solving these equations we find we can take $A = a-k-1$ and $B = a-1$. We note that upon restricting to $\mathbb{Z}$ we will indeed get ω^{a-k-1} for

φ and ω^{a-1} for ψ, which are what we wanted the p-parts of the associated Dirichlet characters to be!

Case 2: $q|N$. For η and ξ_m to fit into the domain of the BDP L-function we need $\eta_{\mathrm{fin},q} = \varphi_{\mathrm{fin},q}^{-1}\psi_{\mathrm{fin},q}$ and $\xi_{m,\mathrm{fin},q} = \varphi_{\mathrm{fin},q}\psi_{\mathrm{fin},q}$ to both be trivial (a condition that's forced on us by definition of them being in $\Sigma_{cc}(\mathfrak{N}, c)$ with f having trivial character). So we simply take both $\psi_{\mathrm{fin},q}$ and $\varphi_{\mathrm{fin},q}$ to be trivial.

Case 3: $q|c$. Again, here we can make a simple choice, taking $\psi_{\mathrm{fin},q} = \xi_{\underline{a},\mathrm{fin},q}$ and $\varphi_{\mathrm{fin},q}$ to be trivial.

Case 4: An auxiliary prime ℓ. So far, we've specified ψ_{fin} and φ_{fin} to have conductors dividing pcN such that the characters η and ξ_m we end up with from them will lie in the domain of the BDP L-function. We could stop there, but again for technical reasons it will be important to allow ourselves to change things at one more prime. So we pick an auxiliary prime ℓ that is coprime to $2pdcN$ and that is inert in K. We'll then allow ourselves to take an arbitrary anticyclotomic character ν (that we'll pick later) of some ℓ-power conductor ℓ^{c_ℓ}, i.e. a primitive character of $(\mathcal{O}_K/\ell^{c_\ell}\mathcal{O}_K)^\times$ that's trivial on $(\mathbb{Z}/\ell^{c_\ell}\mathbb{Z})^\times$. We then take $\psi_{\mathrm{fin},\ell} = \nu$ and $\varphi_{\mathrm{fin},\ell} = \nu^{-1}$, so $\xi_{m,\mathrm{fin},\ell}$ is trivial but $\eta_{\mathrm{fin},\ell} = \nu^2$.

The final Hecke character So, combining the above four cases, we've specified characters ψ_{fin} and φ_{fin} of conductors dividing $\mathfrak{p}Nc\ell$. Since most of the things we specified were set up to be trivial on the embedded copy of $\mathbb{Z}$ in $\mathcal{O}_K$, we find that the restrictions of ψ_{fin} and φ_{fin} to $\mathbb{Z}$ are ω^{a-1} and ω^{a-k-1}, respectively.

Moreover, the basic theory of classical Hecke characters tells us that a character of $(\mathcal{O}_K/\mathfrak{m})^\times$ is realized as the finite part of a character of trivial weights iff it's trivial on the units of $\mathcal{O}_K$. Since we're working with an imaginary quadratic field and we're avoiding the exceptional cases of $K = \mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$, $\mathcal{O}_K^\times$ is just $\{\pm 1\}$ so we conclude that ψ_{fin} and φ_{fin} can be extended to Hecke characters iff they are trivial on -1. But we know $\psi_{\text{fin}}(-1) = \omega^{a-1}(-1) = (-1)^{a-1}$ and $\varphi_{\text{fin}}(-1) = \omega^{a-k-1}(-1) = (-1)^{a-k-1}$, and these are equal to 1 iff $a - 1 \equiv 0 \pmod 2$.

So, given this assumption on the parity of a, we know that we can find finite-order Hecke characters ψ and φ with finite parts given by the characters ψ_{fin} and φ_{fin} that we've specified, and in fact we're free to choose them so that $\xi_{\underline{a}} = \varphi\psi\alpha^{2\underline{a}-k-2}N^{-(\underline{a}-k-1)}$ equals $\underline{\xi}_{\underline{a}}$.

4.1.3 Summary of the setup

We summarize what our setup is so far, and add in the precise hypotheses we'll work under. Our inputs will be the following data:

- An odd prime p.
- A newform f of weight k, level N, and trivial character (note this forces k to be even). We assume that our prime p is coprime to the level N.
- An imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ with $d \equiv 3 \pmod 4$ and $d \geq 7$, such that all primes dividing pN split in K. Given this we take $\mathfrak{N}$ to be a cyclic ideal of norm N in K.
- A residue class modulo $p-1$, represented by $a \in \{0, \ldots, p-1\}$, such that $a-1$ is even.
- A weight $\underline{a} \equiv a \pmod{p-1}$ and a character $\underline{\xi}_{\underline{a}} \in \Sigma_{cc}(\mathfrak{N}, c)$ of weight $(a-1, -a+k+1)$ for some integer c coprime to $2pdN$.
- An auxiliary prime ℓ not dividing $2pcdN$ that is inert in K, and an auxiliary character ν of $(\mathcal{O}_K/\ell^{c_\ell}\mathcal{O}_K)^\times$ of conductor ℓ^{c_ℓ} and which is trivial on $(\mathbb{Z}/\ell^{c_\ell}\mathbb{Z})^\times$.

From this setup, we showed we could define finite-order Hecke characters φ and ψ of K such that if we defined $\eta = \varphi^{-1}\psi\alpha^k$ and $\xi_m = \varphi\psi\alpha^{2m-k-2}N^{-(m-k-1)}$ for $m \equiv a \pmod{p-1}$ then we had:

- The conductor of φ is either ℓ^{c_ℓ} or $\mathfrak{p}\ell^{c_\ell}$.
- The conductor of ψ is either $c\ell^{c_\ell}$ or $\mathfrak{p}c\ell^{c_\ell}$.
- Each ξ_m lies in $\Sigma_{cc}(\mathfrak{N}, c)$, and $\xi_{\underline{a}}$ is exactly equal to the $\underline{\xi_{\underline{a}}}$ we specified.
- The character η lies in $\Sigma_{cc}^{(2)}(\mathfrak{N}, c\ell^{c_\ell})$.
- The ℓ-component of the finite parts of ψ, φ, and η are ν, ν^{-1}, and ν^2, respectively.

We then let Φ, Ψ be the families with specializations $P_m \circ \Phi = \varphi_m = \varphi\alpha^{m-1}$ and $P_m \circ \Psi = \psi_m = \psi\alpha^{m-1}$ that we've constructed in Section 3.5.1. We let $g_{\psi,m} = g_{\psi_m}$ and $g_{\varphi,m-k} = g_{\varphi_{m-k}}$ be the associated CM forms of weights m and $m-k$, respectively.

4.2 The equation from Ichino's formula

We now recall the results of Chapter 2, on our explicit form of Ichino's formula. In particular, taking the results stated in Section 3.4 we find (for $g^\dagger_{\varphi,m-k}(z) = g_{\varphi,m-k}(Nc^2z)$) we have

$$|\langle fg^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle|^2 = C_m^{(1)} L(0, f, \eta^{-1}) L(0, f, \xi_m^{-1})$$

for

$$C_m^{(1)} = \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2}2^{4m-4}(c^2N)^{m-k}dNc^2\ell^{2c_\ell}} \cdot \prod_{q|cNd\ell}(1+q^{-1})^{-2}\prod_{q|d}(1-q^{-1})^{-2} \cdot (\star)_\ell$$

and

$$(\star)_\ell = \left(\frac{(\alpha^{c_\ell+1} - \alpha^{-c_\ell-1}) - \ell^{-1}(\alpha^{c_\ell-1} - \alpha^{-c_\ell+1})}{\alpha - \alpha^{-1}}\right)^2.$$

We start our manipulation of this formula by getting rid of the absolute value on the left hand side. What we need to do is take the complex conjugate

$$\overline{\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle} = \frac{1}{V}\int \overline{f(z)}\overline{g^{\dagger}_{\varphi,m-k}(Nc^2 z)} g_{\psi,m}(z)\,\mathrm{Im}(z)^m d\mu$$

(where, as usual, the integral is over a fundamental domain of an appropriate congruence subgroup and is normalized by the volume V of the domain, and is with respect to the invariant metric $d\mu$). We make this computation in more generality:

Proposition 4.2.1. *Let f, g, h be three newforms of weights $k, m-k, m$ and levels N_f, N_g, N_h, respectively. Let $N_{fgh} = \mathrm{lcm}(N_f, N_g, N_h)$, let M_f, M_g, M_h and L_f, L_g, L_h be positive integers such that $M_f N_f L_h = M_g N_g L_g = M_h N_h L_h = N_{fgh}$. Then we have*

$$\overline{\langle f(M_f z) g(M_g z), h(M_h z)\rangle} = -w_f w_g \overline{w}_h \frac{L_f^{k/2} L_g^{(m-k)/2} L_h^{m/2}}{M_f^{k/2} M_g^{(m-k)/2} M_h^{m/2}} \langle f(L_f z) g(L_g z), h(L_h z)\rangle$$

where w_f, w_g, w_h are the Atkin-Lehner eigenvalues for each newform.

Proof. Recall that if $f = \sum a_n q^n$ we let $f_\rho = \sum \overline{a}_n q^n$ be the modular form with the conjugate q-expansion. It's then easy to check that $\overline{f(z)} = f_\rho(-\overline{z})$ and likewise for the other forms, so we can rewrite the integral defining this Petersson inner product as

$$\frac{1}{V}\int f_\rho(-M_f \overline{z}) g_\rho(-M_g \overline{z}) \overline{h_\rho(-M_h \overline{z})}\,\mathrm{Im}(-\overline{z})^m d\mu.$$

Changing variables $z \leftrightarrow -\overline{z}$ gets that this is equal to

$$-\frac{1}{V}\int f_\rho(M_f z) g_\rho(M_g z) \overline{h_\rho(M_h z)}\,\mathrm{Im}(z)^m d\mu.$$

Now, we further know that f_ρ is the newform associated to the Atkin-Lehner involution W_{N_f} applied to f; letting w_f denote the eigenvalue for W_{N_f} (following the convention in [BDP13]) we have $f_\rho(z) = w_f W_{N_f}(f)$, and similarly for the other forms.

So our integral is

$$-\frac{w_f w_g \overline{w}_h}{V}\int W_{N_f}(f)(M_f z)W_{N_g}(g)(M_g z)\overline{W_{N_h}(h)(M_h z)}\,\mathrm{Im}(z)^m d\mu.$$

Now, W_{N_f} comes from applying the Möbius transformation $z \mapsto -1/N_f z$ and thus by definition

$$W_{N_f}(f)(z) = N_f^{k/2}(N_f z)^{-k} f(-1/N_f z) = N_f^{-k/2} z^{-k} f(-1/N_f z),$$

and similarly for g and h. Applying this we get our integral above is equal to

$$\frac{-w_f w_g \overline{w}_h}{N_f^{k/2} M_f^k N_g^{(m-k)/2} M_g^{m-k} N_h^{m/2} M_h^m} \cdot \frac{1}{V}\int f\left(\frac{-1}{N_f M_f z}\right) g\left(\frac{-1}{N_g M_g z}\right)\overline{h\left(\frac{-1}{N_h M_h z}\right)}\frac{\mathrm{Im}(z)^m}{|z|^{2m}}d\mu.$$

Using invariance under the transformation $z \mapsto -1/z$ we get that this integral is equal to

$$\int f\left(\frac{z}{N_f M_f}\right) g\left(\frac{z}{N_g M_g}\right)\overline{h\left(\frac{z}{N_h M_h}\right)}\,\mathrm{Im}(z)^m d\mu,$$

and thus our original expression $\overline{\langle f(M_f z)g(M_g z), h(M_h z)\rangle}$ equals

$$\frac{-w_f w_g \overline{w}_h}{N_f^{k/2} M_f^k N_g^{(m-k)/2} M_g^{m-k} N_h^{m/2} M_h^m}\left\langle f\left(\frac{z}{N_f M_f}\right) g\left(\frac{z}{N_g M_g}\right), h\left(\frac{z}{N_h M_h}\right)\right\rangle.$$

Finally, scaling the inner product by N_{fgh} gives us

$$\frac{-w_f w_g \overline{w}_h N_{fgh}^m}{N_f^{k/2} M_f^k N_g^{(m-k)/2} M_g^{m-k} N_h^{m/2} M_h^m}\langle f(L_f z)g(L_g z), h(L_h z)\rangle,$$

which simplifies to the expression in the statement. □

In our case, the levels are $N_f = N$, $N_g = d\ell^{2c_\ell}$, and $N_h = dc^2\ell^{2c_\ell}$, so $N_{fgh} =$

$Ndc^2\ell^{2c_\ell}$. Also, $M_f = M_h = 1$ while $M_g = Nc^2$, so we get $L_f = dc^2\ell^{2c_\ell}$, $L_g = 1$, and $L_h = N$, so we conclude that

$$\overline{\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle}$$

equals

$$-w_f w_\varphi \overline{w}_\psi \frac{d^{k/2}c^k\ell^{kc_\ell}\cdot 1\cdot N^{m/2}}{1\cdot N^{(m-k)/2}c^{m-k}\cdot 1}\langle f(dc^2\ell^{2c_\ell}z)g_{\varphi,m-k}(z), g_{\psi,m}(Nz)\rangle.$$

Writing $f^{\ddagger}(z) = f(dc^2\ell^{2c_\ell}z)$ and $g^{\ddagger}_{\psi,m}(z) = g_{\psi,m}(Nz)$ we can express this as

$$-w_f w_\varphi \overline{w}_\psi \frac{d^{k/2}\ell^{kc_\ell}N^{k/2}}{c^{m-2k}}\langle f^{\ddagger}g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle.$$

So, we conclude that we have

$$\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle\langle f^{\ddagger}g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle = C^{(2)}_m L(0,f,\eta^{-1})L(0,f,\xi_m^{-1})$$

for

$$C^{(2)}_m = \frac{-1}{w_f w_\varphi \overline{w}_\psi}\frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2}2^{4m-4}N^{m+1-k/2}d^{k/2+1}c^{m+2}\ell^{(k+2)c_\ell}} \cdot \prod_{q|cNd\ell}(1+q^{-1})^{-2}\prod_{q|d}(1-q^{-1})^{-2}\cdot(\star)_\ell.$$

4.3 Defining the $\mathcal{I}$-adic families

Now that we have an equation of the form

$$\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle\langle f^{\ddagger}g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle = C^{(2)}_m L(0,f,\eta^{-1})L(0,f,\xi_m^{-1}),$$

we go about picking our $\mathcal{I}$-adic families that give us elements of $\mathcal{I}$ interpolating the Petersson inner products on the left-hand side.

We've shown how to construct the $\mathcal{I}$-adic forms we'll need in Section 3.5. First of all, in Proposition 3.5.1 we showed how to construct a $\mathcal{I}$-adic newform g_Ψ that corresponds to the family Ψ we have, which has specializations $P_m(\mathrm{g}_\Psi)$ given by $g^\sharp_{\psi,m}$ if $m \equiv a \pmod{p-1}$ and $g_{\psi,m}$ otherwise. This is an ordinary $\mathcal{I}$-adic newform of level $dc^2\ell^{2c_\ell}$ and character ω^a, and thus it induces a linear functional

$$\ell_{\mathrm{g}_\Psi} : \mathbb{S}(dc^2\ell^{2c_\ell}, \omega^a; \mathcal{I}) \to \mathcal{I}$$

interpolating Petersson inner products as in Section 3.4. Moreover, for the case of a $\mathcal{I}$-adic CM newform we showed in Theorem 3.6.4 that we could get a more explicit interpolation (pinning down the congruence number exactly as a Katz p-adic L-function) at the expense of expanding from $\mathcal{I}$ to $\mathcal{I}^{\mathrm{ur}}$.

For the input vectors, we'll use the construction of Proposition 3.5.2. In the first case take $f\mathrm{g}^\dagger_\Phi$ to be the $\mathcal{I}$-adic family with specializations

$$P_m(f\mathrm{g}^\dagger_\Phi) = f(z)g^\sharp_{\varphi.m-k}(c^2Nz)$$

for $m \equiv a$. We can see that this is a form in $\mathbb{S}(Ndc^2\ell^{2c_\ell}, \omega^a; \mathcal{I})$ and thus we can take its trace down to our earlier space; accordingly, we define

$$\langle f\mathrm{g}^\dagger_\Phi, \mathrm{g}_\Psi\rangle = \langle \mathrm{tr}(f\mathrm{g}^\dagger_\Phi), \mathrm{g}_\Psi\rangle \in \mathcal{I}^{\mathrm{ur}}$$

as in Theorem 3.6.4. Similarly, for the second case we define $f\mathrm{g}^\ddagger_\Psi$ as the $\mathcal{I}$-adic family with specializations

$$P_m(f\mathrm{g}^\ddagger_\Phi) = f\left(\frac{dc^2\ell^{2c_\ell}z}{N}\right) g^\sharp_{\varphi.m-k}\left(\frac{z}{N}\right).$$

We can check this is a form in $\mathbb{S}(\Gamma_0(dc^2\ell^{2c_\ell}, N), \omega^a; \mathcal{I})$, and again we can trace it down to the level of g_Ψ and define

$$\langle f\mathrm{g}_\Phi^\ddagger, \mathrm{g}_\Psi\rangle = \langle \mathrm{tr}(f\mathrm{g}_\Phi^\ddagger), \mathrm{g}_\Psi\rangle \in \mathcal{I}^{\mathrm{ur}}.$$

Now, we consider the specializations of these elements of $\mathcal{I}^{\mathrm{ur}}$ at points P_m for $m \equiv a \pmod{p-1}$. By definition we have that $P_m(\langle f\mathrm{g}_\Phi^\dagger, \mathrm{g}_\Psi\rangle)$ is equal to the quantity

$$\langle fg_{\varphi,m-k}^\dagger, g_{\psi,m}\rangle_p = \frac{\langle \mathrm{tr}^\dagger(fg_{\varphi,m-k}^\sharp), g_{\psi,m}^\sharp\rangle}{\langle g_{\psi,m}^\sharp, g_{\psi,m}^\natural\rangle} \cdot L_p^{Katz}(\psi_m(\psi_m^c)^{-1}, 1) \cdot h_K \cdot \prod_{q|c\ell} L_q(1, \chi_K)^{-1}.$$

Similarly, $P_m(\langle f\mathrm{g}_\Phi^\ddagger, \mathrm{g}_\Psi\rangle)$ is equal to

$$\langle fg_{\varphi,m-k}^\ddagger, g_{\psi,m}\rangle_p = \frac{\langle \mathrm{tr}^\ddagger(fg_{\varphi,m-k}^\sharp), g_{\psi,m}^\sharp\rangle}{\langle g_{\psi,m}^\sharp, g_{\psi,m}^\natural\rangle} \cdot L_p^{Katz}(\psi_m(\psi_m^c)^{-1}, 1) \cdot h_K \cdot \prod_{q|c\ell} L_q(1, \chi_K)^{-1},$$

Before proceeding further, we want to analyze these traces. For $\langle fg_{\varphi,m-k}^\dagger, g_{\psi,m}\rangle_p$, we're tracing down from $\Gamma_0(Ndc^2\ell^{2c_\ell}p)$ to $\Gamma_0(dc^2\ell^{2c_\ell}p)$; computing the index we conclude

$$\langle \mathrm{tr}^\ddagger(fg_{\varphi,m-k}^\sharp), g_{\psi,m}^\sharp\rangle = \langle fg_{\varphi,m-k}^{\dagger\sharp}, g_{\psi,m}^\sharp\rangle N \prod_{q|N}(1+q^{-1}).$$

In $\langle fg_{\varphi,m-k}^\ddagger, g_{\psi,m}\rangle_p$, we're going from $\Gamma_0(dc^2\ell^{2c_\ell}p, N)$ to $\Gamma_0(dc^2\ell^{2c_\ell}p)$, and here we get

$$\langle \mathrm{tr}^\ddagger(fg_{\varphi,m-k}^\sharp), g_{\psi,m}^\sharp\rangle = \langle f^\ddagger(z/N) g_{\varphi,m-k}^\sharp(z/N), g_{\psi,m}^{\ddagger\sharp}(z/N)\rangle N \prod_{q|N}(1+q^{-1}).$$

Rescaling the inner product we get that this is equal to

$$\langle f^\ddagger g_{\varphi,m-k}^\sharp, g_{\psi,m}^{\ddagger\sharp}\rangle N^{m+1} \prod_{q|N}(1+q^{-1}).$$

Putting this together, we find that $\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_p$ is equal to

$$\frac{\langle fg^{\dagger\sharp}_{\varphi,m-k}, g^{\sharp}_{\psi,m}\rangle}{\langle g^{\sharp}_{\psi,m}, g^{\natural}_{\psi,m}\rangle} \cdot L_p^{Katz}(\psi_m(\psi_m^c)^{-1}, 1) \cdot h_K N \cdot \prod_{q|c\ell} L_q(1,\chi_K)^{-1} \cdot \prod_{q|N}(1+q^{-1})$$

and that $\langle fg^{\ddagger}_{\varphi,m-k}, g_{\psi,m}\rangle_p$ is equal to

$$\frac{\langle f^{\ddagger} g^{\sharp}_{\varphi,m-k}, g^{\ddagger\sharp}_{\psi,m}\rangle}{\langle g^{\sharp}_{\psi,m}, g^{\natural}_{\psi,m}\rangle} \cdot L_p^{Katz}(\psi_m(\psi_m^c)^{-1}, 1) \cdot h_K N^{m+1} \cdot \prod_{q|c\ell} L_q(1,\chi_K)^{-1} \cdot \prod_{q|N}(1+q^{-1}),$$

Finally, we define the associated algebraic parts, defining $\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_{\mathrm{alg}}$ as

$$\frac{\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle}{\langle g_{\psi,m}, g_{\psi,m}\rangle} \cdot L_{\mathrm{alg}}^{Katz}(\psi_m(\psi_m^c)^{-1}, 1) \cdot h_K N \cdot \prod_{q|c\ell} L_q(1,\chi_K)^{-1} \cdot \prod_{q|N}(1+q^{-1})$$

and $\langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle_{\mathrm{alg}}$ as

$$\frac{\langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle}{\langle g_{\psi,m}, g_{\psi,m}\rangle} \cdot L_{\mathrm{alg}}^{Katz}(\psi_m(\psi_m^c)^{-1}, 1) \cdot h_K N^{m+1} \cdot \prod_{q|c\ell} L_q(1,\chi_K)^{-1} \cdot \prod_{q|N}(1+q^{-1}).$$

These are algebraic numbers that are related to $\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_p$ and $\langle fg^{\ddagger}_{\varphi,m-k}, g_{\psi,m}\rangle_p$, respectively, by the periods and Euler factors in the definition of the Katz p-adic L-function, plus some Euler factors coming from changing our ratio of Petersson inner products.

At this point we're ready to carry out our strategy. We've started with a formula of the form

$$\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle\langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle = C_m \cdot L(0,f,\eta^{-1}) \cdot L(0,f,\xi_m^{-1}).$$

By multiplying by periods and other factors, we want to rearrange this to be

$$\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_{\mathrm{alg}}\langle f^{\ddagger}g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle_{\mathrm{alg}} = C'_m \cdot L^*_{\mathrm{alg}}(0,f,\eta^{-1}) \cdot L^{BDP}_{\mathrm{alg}}(0,f,\xi_m^{-1})$$

as an equality in $\overline{\mathbb{Q}} \subseteq \mathbb{C}$. We can then re-embed this as an equality in $\overline{\mathbb{Q}} \subseteq \mathbb{C}_p$ and further rearrange it to

$$\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_p\langle f^{\ddagger}g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle_p = C''_m \cdot L^*_p(f,\eta^{-1}) \cdot L^{BDP}_p(f,\xi_m^{-1}),$$

Since $L^*_p(0,f,\eta^{-1})$ independent of m and C''_m will be something we can describe explicitly as a function of m (and see is p-adic analytic), we'll be able to use this family of equalities to conclude an equality

$$\langle f\mathrm{g}^{\dagger}_{\Phi}, \mathrm{g}_{\Psi}\rangle\langle f\mathrm{g}^{\ddagger}_{\Phi}, \mathrm{g}_{\Psi}\rangle = \mathcal{C}'' \cdot L^*_p(f,\eta^{-1}) \cdot \mathcal{L}^{BDP}_p(f,\Xi)$$

in $\mathcal{I}$, which is what we want.

4.4 The equation for algebraic values

Now that we have our statement of Ichino's formula as an equation of transcendental numbers in $\mathbb{C}$, we want to manipulate it to get ourselves an equation of algebraic numbers in $\overline{\mathbb{Q}}$. This amounts to dividing both sides by Ω_∞^{4m-4} and pushing around the factors of π and the algebraic numbers we have to turn the factors $L(0,f,\eta^{-1})L(0,f,\xi_m^{-1})$ on the RHS to $L^{BDP}_{\mathrm{alg}}(0,f,\eta^{-1})L^{BDP}_{\mathrm{alg}}(0,f,\xi_m^{-1})$ (as defined in Theorem 1.3.1), and the factor $\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle\langle f^{\ddagger}g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle$ on the LHS the LHS to $\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_{\mathrm{alg}}\langle f^{\ddagger}g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle_{\mathrm{alg}}$ (as defined towards the end of Section 4.3). The bulk of this computation is taking the definition of $\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_{\mathrm{alg}}$ and rewriting it in terms of the algebraic number $\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle/\Omega_\infty^{2m-2}$, and likewise for

$\langle f^\ddagger g_{\varphi,m-k}, g^\ddagger_{\psi,m}\rangle_{\text{alg}}$.

4.4.1 Simplifying $\langle f g^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle_{\text{alg}}$ and $\langle f^\ddagger g_{\varphi,m-k}, g^\ddagger_{\psi,m}\rangle_{\text{alg}}$

.

Recall that by definition, $\langle f g^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle_{\text{alg}}$ is

$$\frac{\langle f g^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle}{\langle g_{\psi,m}, g_{\psi,m}\rangle} \cdot L^{Katz}_{\text{alg}}(\psi_m(\psi^c_m)^{-1}, 1) \cdot h_K N \cdot \prod_{q|c\ell} L_q(1, \chi_K)^{-1} \cdot \prod_{q|N}(1+q^{-1}).$$

Also by definition (from Theorem 3.6.2) we have

$$L^{Katz}_{\text{alg}}(\psi_m(\psi^c_m)^{-1}, 1) = \frac{(m-1)!\pi^{m-2}}{2^{m-2}\sqrt{d}^{m-2}} \frac{L(\psi_m(\psi^c_m)^{-1}, 1)}{\Omega^{2m-2}_\infty}.$$

The way we'll simplify this is by using Corollary 2.4.5 to see that the $\langle g_{\psi,m}, g_{\psi,m}\rangle$ in the denominator and the $L(\psi_m(\psi^c_m)^{-1}, 1)$ in the numerator will cancel out (up to simpler factors), leaving us with what we want: $\langle f g_{\varphi,m-k}, g_{\psi,m}\rangle$ in the numerator, Ω^{2m-2}_∞ in the denominator, and a leftover factor involving π and algebraic numbers. In particular, rearranging the corollary in question tells us

$$\frac{\pi^2}{6} \frac{(4\pi)^m}{(m-1)!} \prod_{q|dc\ell}(1+q^{-1}) = \frac{L(\psi^2_m, m)L(\chi_K, 1)}{\langle g_{\psi,m}, g_{\psi,m}\rangle} \prod_{q|c\ell} L_q(\chi_K, 1)^{-1}.$$

So we want to rearrange our expression for $\langle f g_{\varphi,m-k}, g_{\psi,m}\rangle_{\text{alg}}$ to get the RHS of this equation to appear in it. We already have the Petersson inner product in the denominator and the product of local factors $L_q(\chi_K, 1)^{-1}$. By the analytic class number formula, the value $L(1, \chi_K)$ is essentially accounted for by the class number h_K we have; precisely, we have $h_K = L(1, \chi_K) \cdot \sqrt{d}/\pi$. Finally, for the Hecke L-value, we note that we in fact have

$$L(\psi^2_m, m) = L(\psi_m(\psi^c_m)^{-1} N^{m-1}, m) = L(\psi_m(\psi^c_m)^{-1}, 1),$$

with the second equality following from how the norm shifts L-values, and the first because ψ_m^2 and $\psi_m(\psi_m^c)^{-1}N^{m-1}$ are in fact equal. This equality of characters amounts to the fact $\psi_m\psi_m^c = N^{m-1}$, which is true because we've constructed ψ_m to be of weight $(m-1, 0)$ and have $\psi_{m,\text{fin}}$ trivial on the embedded copy of $\mathbb{Z}$ - so we view it as a classical Hecke character and evaluate it on any ideal $\mathfrak{a}$, we know

$$(\psi_m\psi_m^c) = \psi_m(\mathfrak{a})\psi_m(\overline{\mathfrak{a}}) = \psi_m(N(\mathfrak{a})) = \psi_{m,\text{fin}}(N(\mathfrak{a}))N(\mathfrak{a})^{m-1} = N(\mathfrak{a})^{m-1}.$$

So, we can rearrange our formula for $\langle fg^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle_{\text{alg}}$ to get

$$\frac{\langle fg^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle}{\Omega_\infty^{2m-2}} \frac{(m-1)!\pi^{m-3}N}{2^{m-2}\sqrt{d}^{m-3}} \cdot \frac{L(\psi_m(\psi_m^c)^{-1}, 1)L(1, \chi_K)}{\langle g_{\psi,m}, g_{\psi,m}\rangle} \prod_{q|c\ell} L_q(1, \chi_K)^{-1} \prod_{q|N}(1+q^{-1}),$$

which we then conclude is equal to

$$\frac{\langle fg^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle}{\Omega_\infty^{2m-2}} \frac{(m-1)!\pi^{m-3}N}{2^{m-2}\sqrt{d}^{m-3}} \cdot \frac{\pi^2}{6} \frac{(4\pi)^m}{(m-1)!} \prod_{q|Ndc\ell} (1+q^{-1}).$$

Collecting terms we conclude

$$\langle fg^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle_{\text{alg}} = \frac{\langle fg^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle}{\Omega_\infty^{2m-2}} \cdot \frac{2^{m+1}\pi^{2m-1}N}{3\sqrt{d}^{m-3}} \prod_{q|Ndc\ell} (1+q^{-1}).$$

We can make an identical computation to also conclude

$$\langle f^\ddagger g_{\varphi,m-k}, g^\ddagger_{\psi,m}\rangle_{\text{alg}} = \frac{\langle f^\ddagger g_{\varphi,m-k}, g^\ddagger_{\psi,m}\rangle}{\Omega_\infty^{2m-2}} \cdot \frac{2^{m+1}\pi^{2m-1}N^{m+1}}{3\sqrt{d}^{m-3}} \prod_{q|Ndc\ell} (1+q^{-1}).$$

4.4.2 The resulting equation

We now return to our equation from Ichino's formula. Dividing both sides by Ω_∞^{4m-4} we get

$$\frac{\langle f g^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle}{\Omega_\infty^{2m-2}} \frac{\langle f^\ddagger g_{\varphi,m-k}, g^\ddagger_{\psi,m}\rangle}{\Omega_\infty^{2m-2}} = C_m^{(2)} \frac{L(0,f,\eta^{-1})}{\Omega_\infty^{2k}} \frac{L(0,f,\xi_m^{-1})}{\Omega_\infty^{4m-2k-4}}$$

where $C_m^{(2)}$ is the same constant from earlier:

$$C_m^{(2)} = \frac{-1}{w_f w_\varphi \overline{w}_\psi} \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2}2^{4m-4}N^{m+1-k/2}d^{k/2+1}c^{m+2}\ell^{(k+2)c_\ell}} \cdot \prod_{q|cNd\ell}(1+q^{-1})^{-2}\prod_{q|d}(1-q^{-1})^{-2}\cdot(\star)_\ell.$$

To get the LHS to be $\langle f g^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle_{\text{alg}}\langle f^\ddagger g_{\varphi,m-k}, g^\ddagger_{\psi,m}\rangle_{\text{alg}}$ we need to multiply by

$$\frac{2^{2m+2}\pi^{4m-2}N^{m+2}}{3^2\sqrt{d}^{2m-6}}\prod_{q|Ndc\ell}(1+q^{-1})^2.$$

Also, on the RHS we want to assemble the constants

$$C(f,\eta,c\ell^{c_\ell}) = (k-1)!\left(\frac{2\pi}{c\ell^{c_\ell}\sqrt{d}}\right)^{k-1}\left(\frac{\ell+1}{\ell-1}\right),$$

$$C(f,\xi_m,c) = (m-k-1)!(m-2)!\left(\frac{2\pi}{c\sqrt{d}}\right)^{2m-k-3}$$

used in the definition of $L^*_{\text{alg}}(0,f,\eta^{-1})$ and $L^{BDP}_{\text{alg}}(0,f,\xi_m^{-1})$. Rearranging, we get:

$$\langle f g^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle_{\text{alg}}\langle f^\ddagger g_{\varphi,m-k}, g^\ddagger_{\psi,m}\rangle_{\text{alg}} = C_m^{(3)}\cdot\frac{C(f,\eta,c\ell^{c_\ell})L(0,f,\eta^{-1})}{\Omega_\infty^{2k}}\frac{C(f,\xi_m,c)L(0,f,\xi_m^{-1})}{\Omega_\infty^{4m-2k-4}}$$

for

$$C_m^{(3)} = \frac{-1}{w_f w_\varphi \overline{w}_\psi}\frac{c^{m-6}}{2^{4m-2}}\frac{N^{k/2+1}}{\sqrt{d}^k\ell^{3c_\ell}}\prod_{q|d}(1-q^{-1})^{-2}\cdot(\star\star)_\ell,$$

where we absorb the new Euler factors at ℓ into $(\star\star)_\ell = (\star)_\ell(\ell-1)/(\ell+1)$.

Finally, we need to finish manipulating the factors on the right into algebraic parts of L-values. Since η is not part of a p-adic family, we will just use the quantity $C(f,\eta,c\ell^{c_\ell})L(0,f,\eta^{-1})/\Omega_\infty^{2k}$ we already have, which we've previously said is denoted by $L^*_{\text{alg}}(0,f,\eta^{-1})$. On the other hand, ξ_m will be part of a family so we do want to get $L^{BDP}_{\text{alg}}(0,f,\xi_m^{-1})$, which requires us to add in a factor of

$$w(f,\xi_m) = w_f \frac{(-N)^{m-k/2-1}}{b_N^{2m-k-2}} \psi_m(\mathfrak{b})\varphi_{m-k}(\mathfrak{b})$$

where $\mathfrak{b}$ is an ideal coprime to Nc and b_N is an element such that $\mathfrak{b}\mathfrak{N} = (b_N)$. We can manipulate this further to see that in our case it's constant with respect to m, and in fact equal to $i^{k-1}w_f N^{k/2}/\underline{\xi}_{\underline{a}}(\mathfrak{N})$. Similarly we can check that $w_\varphi \overline{w}_\psi$ (originally the product of root numbers of $g_{\varphi,m-k}$ and $g_{\psi,n}$, though we suppressed the m from the notation) is a product of Gauss sums that's also independent of m:

$$w_\varphi \overline{w}_\psi = \frac{1}{c\ell^{2c_\ell}} G(\nu^{-1})\overline{G(\nu\underline{\xi}_{\underline{a},\text{fin}})}$$

where

$$G(\nu^{-1}) = \left(\sum_{a\in\mathcal{O}_K/\ell^{c_\ell}\mathcal{O}_K} \nu^{-1}(a)\exp(2\pi i\,\text{tr}(a/\sqrt{d}\ell^{c_\ell}))\right),$$

$$G(\nu\underline{\xi}_{\underline{a}}) = \left(\sum_{b\in\mathcal{O}_K/c\ell^{c_\ell}\mathcal{O}_K} \nu^{-1}(a)\exp(2\pi i\,\text{tr}(b/\sqrt{d}c\ell^{c_\ell}))\right).$$

Ultimately this means we can write

$$\langle fg^\dagger_{\varphi,m-k}, g_{\psi,m}\rangle_{\text{alg}} \langle f^\ddagger g_{\varphi,m-k}, g^\ddagger_{\psi,m}\rangle_{\text{alg}} = C_m^{(4)} \cdot L^*_{\text{alg}}(0,f,\eta^{-1}) L^{BDP}_{\text{alg}}(0,f,\xi_m^{-1})$$

where

$$C_m^{(4)} = \zeta_0 \frac{c^{m-6}}{2^{4m-2}} \frac{N^{k/2+1}}{\sqrt{d}^k \ell^{3c_\ell}} \prod_{q|d} (1-q^{-1})^{-2} \cdot (\star\star)_\ell \qquad \zeta_0 = \frac{i^{k+1} \cdot (N^{k/2}/\underline{\xi}_{\underline{a}}(\mathfrak{N}))}{(G(\nu^{-1})/\ell^{c_\ell}) \cdot (\overline{G(\nu \underline{\xi}_{\underline{a},\mathrm{fin}})}/c\ell^{c_\ell})}.$$

4.5 The equation for p-adic values

The next step is to get from our equation to algebraic values to one for p-adic values. We can start by multiplying by Ω_p^{4m-4} to get the appropriate p-adic periods:

$$\frac{\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_{\mathrm{alg}}}{\Omega_p^{-(2m-2)}} \frac{\langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle_{\mathrm{alg}}}{\Omega_p^{-(2m-2)}} = C_m^{(4)} \frac{L^*_{\mathrm{alg}}(0,f,\eta^{-1})}{\Omega_p^{-2k}} \frac{L^{BDP}_{\mathrm{alg}}(0,f,\xi_m^{-1})}{\Omega_p^{-(4m-2k-4)}}.$$

Then, we just need to deal with the Euler factors. First of all, $L^*_{\mathrm{alg}}(0,f,\eta^{-1})\Omega_p^{2k}$ is already $L_p^*(f,\eta^{-1})$ by definition, so we don't need to do anything with that. To get from $L^{BDP}_{\mathrm{alg}}(0,f,\xi_m^{-1})$ to $L_p^{BDP}(f,\xi_m^{-1})$, the Euler factor we need to multiply is

$$(1-\xi_m^{-1}(\overline{\mathfrak{p}})a_p + \xi_m^{-2}(\overline{\mathfrak{p}})\chi_f(p)p^{k-1})^2 = (1-\xi_m^{-1}(\overline{\mathfrak{p}})\alpha_f)^2(1-\xi_m^{-1}(\overline{\mathfrak{p}})\beta_f)^2.$$

Now, by definition we have

$$\xi_m(\overline{\mathfrak{p}}) = \psi_m(\overline{\mathfrak{p}})\varphi_{m-k}(\overline{\mathfrak{p}})N(\overline{\mathfrak{p}})^{-(m-k-1)} = \psi_m(\overline{\mathfrak{p}})\varphi_{m-k}(\overline{\mathfrak{p}})p^{-(m-k-1)};$$

using that φ_{m-k} has infinity-type $(m-k-1,0)$ and trivial central character we can rewrite this as $\xi_m(\overline{\mathfrak{p}}) = \psi_m(\overline{\mathfrak{p}})/\varphi_{m-k}(\mathfrak{p})$ and thus by definition we have

$$L_p^{BDP}(f,\xi_m^{-1}) = \left(1 - \frac{\varphi_{m-k}(\mathfrak{p})}{\psi_m(\overline{\mathfrak{p}})}\alpha_f\right)^2 \left(1 - \frac{\varphi_{m-k}(\mathfrak{p})}{\psi_m(\overline{\mathfrak{p}})}\beta_f\right)^2 \frac{L^{BDP}_{\mathrm{alg}}(0,f,\xi_m^{-1})}{\Omega_p^{-(4m-2k-4)}}.$$

The next step is to compare $\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_{\mathrm{alg}}$ to $\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_p$ and likewise $\langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle_{\mathrm{alg}}$ to $\langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle_p$. From the definitions in Section 4.3, we see that the difference is that e.g. $\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_{\mathrm{alg}}$ has a factor of

$$\frac{\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle}{\langle g_{\psi,m}, g_{\psi,m}\rangle} \cdot L^{Katz}_{\mathrm{alg}}(\psi_m(\psi^c_m)^{-1}, 1)$$

while $\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_p$ has in its place a factor of

$$\frac{\langle fg^{\dagger\sharp}_{\varphi,m-k}, g^{\sharp}_{\psi,m}\rangle}{\langle g^{\sharp}_{\psi,m}, g^{\natural}_{\psi,m}\rangle} \cdot L^{Katz}_{p}(\psi_m(\psi^c_m)^{-1}, 1).$$

Now, this Katz p-adic L-value is defined by

$$L^{Katz}_{\mathrm{alg}}(\psi_m(\psi^c_m)^{-1}, 1)\Omega_p^{2m-2}\left(1-\frac{\psi_m(\overline{\mathfrak{p}})}{\psi_m(\mathfrak{p})}p^{-1}\right)\left(1-\frac{\psi_m(\overline{\mathfrak{p}})}{\psi_m(\mathfrak{p})}\right).$$

On the other hand, from Proposition 3.4.7 we know

$$\langle g^{\sharp}_{\psi,m}, g^{\natural}_{\psi,m}\rangle = \langle g_{\psi,m}, g_{\psi,m}\rangle \cdot \frac{(-\alpha/\beta)(1-\beta/\alpha)(1-p^{-1}\beta/\alpha)}{(1+p^{-1})},$$

where α and β are the roots of the Hecke polynomial for $g_{\psi,m}$ with α a p-adic unit and β not. Since the two roots are $\psi_m(\mathfrak{p})$ and $\psi_m(\overline{\mathfrak{p}})$, we conclude $\alpha = \psi_m(\overline{\mathfrak{p}})$ and $\beta = \psi_m(\mathfrak{p})$ since ψ_m has infinity-type $(m-1, 0)$ and thus we get powers of p from $\mathfrak{p}$ but not $\overline{\mathfrak{p}}$. So, this means we have:

$$\langle g^{\sharp}_{\psi,m}, g^{\natural}_{\psi,m}\rangle = \langle g_{\psi,m}, g_{\psi,m}\rangle \frac{-\psi_m(\overline{\mathfrak{p}})/\psi_m(\mathfrak{p})}{(1+p^{-1})}\left(1-\frac{\psi_m(\overline{\mathfrak{p}})}{\psi_m(\mathfrak{p})}p^{-1}\right)\left(1-\frac{\psi_m(\overline{\mathfrak{p}})}{\psi_m(\mathfrak{p})}\right).$$

The latter two factors are exactly the Euler factors from the Katz p-adic L-function;

so we conclude

$$\frac{\langle f g^{\dagger\sharp}_{\varphi,m-k}, g^{\sharp}_{\psi,m}\rangle}{\langle g^{\sharp}_{\psi,m}, g^{\natural}_{\psi,m}\rangle} \cdot L_p^{Katz}(\psi_m(\psi_m^c)^{-1},1)$$
$$= \frac{\langle f g^{\dagger\sharp}_{\varphi,m-k}, g^{\sharp}_{\psi,m}\rangle}{\langle g_{\psi,m}, g_{\psi,m}\rangle} \cdot \frac{L_{\mathrm{alg}}^{Katz}(\psi_m(\psi_m^c)^{-1},1)}{\Omega_p^{-(2m-2)}} \cdot \frac{(1+p^{-1})}{(-\psi_m(\overline{\mathfrak{p}})/\psi_m(\mathfrak{p}))}.$$

Finally we need to compare $\langle f g^{\dagger\sharp}_{\varphi,m-k}, g^{\sharp}_{\psi,m}\rangle$ to $\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle$. This is done by Proposition 3.4.8, where we get

$$\langle f g^{\dagger\sharp}_{\varphi,m-k}, g^{\sharp}_{\psi,m}\rangle = \frac{\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle(1+p^{-1})}{(-\psi_m(\overline{\mathfrak{p}})/\psi_m(\mathfrak{p}))}\left(1-\frac{\varphi_{m-k}(\mathfrak{p})\alpha_f}{\psi_m(\overline{\mathfrak{p}})}\right)\left(1-\frac{\varphi_{m-k}(\mathfrak{p})\beta_f}{\psi_m(\overline{\mathfrak{p}})}\right).$$

Substituting this back in we find

$$\frac{\langle f g^{\dagger\sharp}_{\varphi,m-k}, g^{\sharp}_{\psi,m}\rangle}{\langle g^{\sharp}_{\psi,m}, g^{\natural}_{\psi,m}\rangle} \cdot L_p^{Katz}(\psi_m(\psi_m^c)^{-1},1)$$
$$= \frac{\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle}{\langle g_{\psi,m}, g_{\psi,m}\rangle} \cdot \frac{L_{\mathrm{alg}}^{Katz}(\psi_m(\psi_m^c)^{-1},1)}{\Omega_p^{-(2m-2)}}\left(1-\frac{\varphi_{m-k}(\mathfrak{p})\alpha_f}{\psi_m(\overline{\mathfrak{p}})}\right)\left(1-\frac{\varphi_{m-k}(\mathfrak{p})\beta_f}{\psi_m(\overline{\mathfrak{p}})}\right)$$

and thus conclude

$$\left\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\right\rangle_p = \frac{\left\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\right\rangle_{\mathrm{alg}}}{\Omega_p^{-(2m-2)}}\left(1-\frac{\varphi_{m-k}(\mathfrak{p})\alpha_f}{\psi_m(\overline{\mathfrak{p}})}\right)\left(1-\frac{\varphi_{m-k}(\mathfrak{p})\beta_f}{\psi_m(\overline{\mathfrak{p}})}\right).$$

An identical argument gives

$$\left\langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\right\rangle_p = \frac{\left\langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\right\rangle_{\mathrm{alg}}}{\Omega_p^{-(2m-2)}}\left(1-\frac{\varphi_{m-k}(\mathfrak{p})\alpha_f}{\psi_m(\overline{\mathfrak{p}})}\right)\left(1-\frac{\varphi_{m-k}(\mathfrak{p})\beta_f}{\psi_m(\overline{\mathfrak{p}})}\right).$$

Multiplying these two together leaves us with the same Euler factors as in the BDP L-function, so we conclude

$$\left\langle f g^{\dagger}_{\varphi,m-k}, g_{\psi,m}\right\rangle_p \left\langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\right\rangle_p = C_m^{(4)} \cdot L_p^*(f,\eta^{-1}) L_p^{BDP}(f,\xi_m^{-1}).$$

4.6 The final result

So, we've proven

$$\langle fg^{\dagger}_{\varphi,m-k}, g_{\psi,m}\rangle_p \langle f^{\ddagger} g_{\varphi,m-k}, g^{\ddagger}_{\psi,m}\rangle_p = C^{(4)}_m \cdot L^*_p(f,\eta^{-1}) L^{BDP}_p(f,\xi_m^{-1}),$$

where $C^{(4)}_m$ is an algebraic number with its only dependence on m being that it has a factor of $c^{m-6}/2^{4m-2}$. Since $p \nmid 2c$ this can be realized as a specialization of an element of Λ on our residue class of m's. Thus we conclude:

Theorem 4.6.1. *Let $f, K, a, \xi_{\underline{a}}, \varphi, \psi$ be as defined in Section 4.1.3. Then we have an equality in $\mathcal{I}^{ur}[1/p]$*

$$\langle f\mathrm{g}^{\dagger}_{\Phi}, \mathrm{g}_{\Psi}\rangle \langle f\mathrm{g}^{\ddagger}_{\Phi}, \mathrm{g}_{\Psi}\rangle = \mathcal{C} \cdot L^*_p(f,\eta^{-1}) \cdot \mathcal{L}^{BDP}_p(f,\Xi)$$

such that $\mathcal{C} \in \mathcal{I}[1/p]$ is specified by

$$P_m(\mathcal{C}) = \left(\frac{c}{2^4}\right)^m \cdot \prod_{q|d}(1-q^{-1})^{-2} \cdot \frac{4\zeta_0 N^{k/2+1}(\star\star)_\ell}{c^6\sqrt{d}^k \ell^{3c_\ell}},$$

for $m \equiv a \pmod{p-1}$, where

$$\zeta_0 = \frac{i^{k+1} \cdot (N^{k/2}/\xi_{\underline{a}}(\mathfrak{N}))}{(G(\nu^{-1})/\ell^{c_\ell}) \cdot (\overline{G(\nu\xi_{\underline{a},\mathrm{fin}})}/c\ell^{c_\ell})}$$

is a fixed root of unity and

$$(\star\star)_\ell = \left(\frac{\ell-1}{\ell+1}\right)\left(\frac{(\alpha^{c_\ell+1}-\alpha^{-c_\ell-1}) - \ell^{-1}(\alpha^{c_\ell-1}-\alpha^{-c_\ell+1})}{\alpha-\alpha^{-1}}\right)^2$$

is an Euler factor at our auxiliary prime ℓ.

As a consequence of this, we obtain a result about congruences between p-adic L-functions. Suppose $f, f' \in S_k(N)$ are newforms of the same level and character, such that $f \equiv f' \pmod{p^n}$ (in the sense that the Fourier coefficients are all congruent modulo that power, after being embedded in $\overline{\mathbb{Q}}_p$). It's then immediate that $f\mathrm{g}_\Phi^\dagger \equiv f'\mathrm{g}_\Phi^\dagger \pmod{p^n}$ (since all of the coefficients involve the same $\mathcal{I}$-linear combinations of those of f and f') and then that $\langle f\mathrm{g}_\Phi^\dagger, \mathrm{g}_\Psi\rangle \equiv \langle f'\mathrm{g}_\Phi^\dagger, \mathrm{g}_\Psi\rangle$ (because this comes from applying a $\mathcal{I}$-integral Hecke operator and taking the first Fourier coefficient, then multiplying by the same H_{g_Ψ} in both cases).

So, if $f \equiv f' \pmod{p^n}$ we can conclude that our equations of the above theorem for f, f' are congruent modulo p^n. Applying this to the right-hand sides and cancelling term independent of f in our coefficient $\mathcal{C}$, we conclude

$$(\star\star)_{\ell,f} L_p^*(f, \eta^{-1}) \cdot \mathcal{L}_p^{BDP}(f, \Xi) \equiv (\star\star)_{\ell,f'} L_p^*(f', \eta^{-1}) \cdot \mathcal{L}_p^{BDP}(f', \Xi)$$

modulo p^n. (We note that everything we're cancelling is a p-adic unit, except maybe $\prod_{q|d}(1-q^{-1})^{-2}$, but this could only contain negative powers of p and thus dividing by it would only *increase* the congruence.) Finally, we note $(\star\star)_{\ell,f} \equiv (\star\star)_{\ell,f'}$ because the only dependence on f is in the root α of the Hecke polynomial, which is congruent modulo p^n. Assuming we chose one (and hence both) of these L-factor to be units mod p, we can cancel them and conclude:

Corollary 4.6.2. *Let $f, f' \in S_k(N)$ be newforms such that $f \equiv f' \pmod{p^n}$. Letting η and Ξ be as above, with the prime ℓ chosen so that $(\star\star)_\ell$ is a unit, we have*

$$L_p^*(f, \eta^{-1}) \cdot \mathcal{L}_p^{BDP}(f, \Xi) \equiv L_p^*(f', \eta^{-1}) \cdot \mathcal{L}_p^{BDP}(f', \Xi) \pmod{p^n}.$$

Finally, we remark that we expect it is not so difficult to prove $L_p^*(f, \eta^{-1}) \equiv L_p^*(f', \eta^{-1})$ directly, using Rankin-Selberg unfolding; by using this argument and some nonvanishing results due to Hsieh [Hsi14] we expect to be able to obtain that if $f \equiv f' \pmod{p^n}$ then $\mathcal{L}_p^{BDP}(f, \Xi) \equiv \mathcal{L}_p^{BDP}(f', \Xi) \pmod{p^n}$.

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