Certainty Equivalent and No Arbitrage: a Reconciliation via Duality Theory

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Abstract

We describe a general principle for the valuation problem in incomplete financial markets that reconciles the "utility" and "martingale" approaches.

We provide a general criterion, for selecting one equivalent martingale measure, that requires minimizing an appropriate functional which depends on investors utility and we give sufficient conditions for the existence of the martingale measure that minimizes this functional. We then show that most existing financial criteria for pricing in incomplete markets are particular cases of our approach.

Keywords: *Certainty Equivalent, No Arbitrage, Incomplete Markets, Asset Pricing, Duality Theory, Minimal Martingale Measures, Relative Entropy.*

1 Introduction

One of the most important results in Mathematical Finance is the application of one simple but very powerful principle: the absence of arbitrage opportunities. Let $(\Omega, F, \mathcal{F}, P)$ be a complete filtered probability space and $\mathcal{M}_e$

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(resp. $\mathcal{M}$) be the set of equivalent (resp. $P-$absolutely continuous) martingale measures for the stochastic processes representing discounted prices of the available financial securities.

The No Arbitrage Principle (in its various forms, see for example [3], [6], [12], [14]) is equivalent to the existence of a $Q \in \mathcal{M}_e$ and prescribes that a pricing functional has to be represented by an equivalent martingale measure. In complete markets, where $\mathcal{M}_e$ is a singleton, this is sufficient to provide a satisfactory preference free answer for the valuation problem.

However, in incomplete markets when no perfect hedging of claims is possible the set $\mathcal{M}_e$ is not any more a singleton, as it was in complete markets, and we are left for any given non replicable claim only with an interval of possible prices, all compatible with the no arbitrage condition, so that no more "pricing by arbitrage" is possible.

So the problem is to determine a criterion for selecting one martingale measure in $\mathcal{M}_e$. In the mathematical finance literature one commonly introduces a sort of "distance" $d(Q, P)$ between probabilities and then selects the measure $Q^* \in \mathcal{M}_e$, if it exists, that minimizes this distance:

$$d(Q^*, P) = \min_{Q \in \mathcal{M}_e} d(Q, P). \qquad (\bullet)$$

This approach is taken for example in Föllmer and Schweizer [8], Schweizer [19], Delbaen and Schachermayer [7], Frittelli [10].

Since in incomplete markets there are claims that carry an intrinsic unhedgeble component of risk, it seems natural to take into consideration the preference structure of investors and so to reintroduce a description of the agent via his utility function. This was done in different ways by Karatzas et all [16] and Davis [4].

Note that, different from existing literature, we will not focuse our attention on trading policies (replicating or super-replicating trading strategies). We are closer to an "actuarial" approach than a hedger one. For our purposes, all relevant information about the structure of the securities market is already contained in the set $\mathcal{M}$.

Our approach is as follows: suppose $Q \in \mathcal{M}$ is a given price functional compatible with the no arbitrage condition, let

$$\varphi = \frac{dQ}{dP}$$

and let u be a utility function (see Sec. 2 for details and assumptions), then we define

$$V(x,\varphi) = \left\{ \sup_{w \in L^\infty} E[u(w)] \quad \text{subject to: } E[\varphi w] \leq x \right\}.$$

$V(x,\varphi)$ represents the maximum expected utility that an agent can reach in this market with an initial endowment x when the prices of the claims are given by Q.

We select the measure $Q^* \in \mathcal{M}$ that solves the following problem:

(A) *minimize among all martingale measures the functional $V(x,\varphi)$:*

$$V(x,\varphi^*) = \min_{Q \in \mathcal{M}} V(x,\varphi).$$

We interpret Q^* as the "projection" of P on the set $\mathcal{M}$ where the "distance" depends on u and is given by $V(x,\varphi)$.

The economic intuition beyond our approach is that if the market is not too far from equilibrium then the maximum expected utility attainable with a fixed initial amount should not be too large.

A further *economic justification of our approach, based on the notion of certainty equivalent,* will be given in Sec.4.

In general the minimizing measure Q^* is absolutely continuous with respect to P. However, in Sec.4 *we provide conditions on the utility function that guarantee that Q^* is equivalent to P.*

Our first result *is to explicitly compute in a simple form $V(x,\varphi)$, applying Fenchel Duality Theorem and Legendre transform* (see Th.2 and the examples in Sec.5).

We then show that *most existing financial criteria* in the form ($\bullet$) (as for examples the Optimal Variance measure [7] and the Minimal Entropy measure [10]) for pricing in incomplete markets *are special cases of our approach* for suitably selected utility functions. All these particular cases and others are discussed in Sec.5.

In particular when u is the exponential utility function of the form $u(x) = -e^{-x}$ our choice of the pricing measure is the minimal entropy martingale measure (for a detailed study of this measure see Frittelli [9], [10], [11], and

Miyahara [18]). Our theorem also provides a direct economic interpretation of the relative entropy $H(Q,P)$:

$$H(Q,P) = \sup_{E_Q[w]=0} \left\{ u^{-1} E_P\left[u(w)\right] \right\}$$

is the maximum certainty equivalent of all contingent claims having zero prices under Q.

Let us now pose a different problem which is related with the one studied by Davis [4]. Consider an agent with utility u and initial endowment x. Under the only information of no arbitrage, the appropriate budget constraint of the agent is that all prices compatible with the no arbitrage principle should be less than its initial endowment. His problem is then to:

(B) *maximize the expected utility in the class of claims having no arbitrage price less than or equal to x:*

$$\sup_{w \in L^\infty} E[u(w)] \quad \text{subject to: } \max_{Q \in \mathcal{M}} E[\varphi w] = x$$

We then show that *problem* **(B)** *and problem* **(A)** *are in fact equivalent*; indeed, under suitable hypotheses, we prove by convex duality argument similar to the one applied before that (Th.3):

$$\begin{aligned} V(x) &\triangleq \left\{ \sup_{w \in L^\infty} E[u(w)] \text{ subject to: } \max_{Q \in \mathcal{M}} E[\varphi w] = x \right\} \\ &= \min_{Q \in \mathcal{M}} V(x,\varphi) = V(x,\varphi^*). \end{aligned}$$

The minimal "distance" $V(x,\varphi)$ between the measure P and the measure Q ($\varphi = \frac{dQ}{dP}$) is the maximum expected utility attainable in "the worst case".

Note that in particular cases we prove (see Th.3) the *existence of the measure $\varphi^* \in \mathcal{M}$ minimizing $V(x,\varphi)$*.

The paper is organized as follows.

In Sec. 2 we explain our notations, state the main assumptions on the set of (martingale) measures M, illustrate our definition of utility functions, and prove preliminary results from duality theory.

In Sec. 3 we state and prove our main theorems. The financial interpretation of these results is provided in Sec. 4. In Sec. 5 we discuss some examples of the application of our results and show that most financial criteria are particular cases of our approach.

2 Notations and Preliminaries

Throughout the paper $(\Omega, \mathcal{F}, P)$ is a given probability space. We denote with $L^1_+ = \{f \in L^1(\Omega, \mathcal{F}, P) : f \geq 0 \ P-a.s.\}$ the positive cone in $L^1 = L^1(\Omega, \mathcal{F}, P)$, with $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ the space of essentially bounded random variables on $(\Omega, \mathcal{F}, P)$ and with $L^\infty(\mathcal{D})$ its convex subset of random variables having values in an open interval $\mathcal{D} \subseteq \mathbb{R}$. Let $E[\cdot]$, with no subscript, denote the expected value with respect to the probability P. Let V be a vector space and (V, V^*) be a dual system where V^* is a subspace of the algebraic dual of V. Let f (resp. g) be a convex (resp. concave) functional defined on V. With f^* (resp. g^*) we denote the convex (resp. concave) conjugate functional of f defined by:

$$\begin{aligned} f^* &: V^* \to \mathbb{R}, \ f^*(z) = \sup_{w \in L^\infty} \{E[zw] - f(w)\}, \\ g^* &: V^* \to \mathbb{R}, \ g^*(z) = \inf_{w \in L^\infty} \{E[zw] - g(w)\}. \end{aligned}$$

We denote the indicator function of a set $S \subseteq V$ with

$$\delta(w|S) = \begin{cases} 0 & w \in S \\ +\infty & w \notin S \end{cases},$$

the convex conjugate of the indicator function $\delta(\cdot|S)$ with $\delta^*_S : V^* \to \mathbb{R}$, the negative polar S^- of a cone $S \subseteq V$ with

$$S^- \triangleq \{s^* \in V^* : < s, s^* > \leq 0 \quad \forall s \in S\},$$

the negative bipolar S^{--} of S with

$$S^{--} \triangleq \left\{s \in V : < s, s^* > \leq 0 \quad \forall s^* \in S^-\right\}.$$

Since S^- is a cone then the convex conjugate $\delta^*_{S^-}$ coincides with the indicator function $\delta(\mu|S^{--})$ of the bipolar S^{--}.

A topology υ on V is *compatible* with the dual system (V, V^*) if V^* coincides with $(V, \upsilon)^*$. The weak topology $\sigma(V, V^*)$ (resp. the Mackey topology $\tau(V, V^*)$) on V is the weakest (resp. the strongest) topology compatible with the dual system (V, V^*).

The closure of a convex set $S \subset V$ is the same in any topology compatible with the dual system (V, V^*).

If $S \subseteq V$ is a convex cone closed in any topology compatible with the duality (V, V^*) then the bipolar V^{--} coincides with V (Grothendieck Th.5, Sec. 9, Ch.2).

The norm dual space of $L^\infty(\Omega, \mathcal{F}, P)$ is $ba = ba(\Omega, \mathcal{F}, P)$, the set of bounded finite additive set functions on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P. We will mainly be interested in the $\sigma(ba, L^\infty)$ and the $\tau(L^1, L^\infty)$ topologies.

In all this section with $M \subseteq L^1_+$ we denote a given fixed convex cone $(0 \in M)$ closed in the L^1-norm topology.

In the following theorems we will make use either one of the following two

Assumptions:

1) $M \subseteq L^1_+ \subseteq ba$ is closed in the $\sigma(ba, L^\infty)$ topology of ba.

2) $U : L^\infty(\mathcal{D}) \to \mathbb{R} \cup \{-\infty\}$ is a concave functional continuous, at some point $w_0 \in L^\infty(\mathcal{D})$, in the Mackey topology $\tau(L^\infty, L^1)$ on L^∞.

Remark 1 *If the state space Ω is finite, M is finite dimensional and Assumption 1) is satisfied. Note that in general the image of L^1 through the natural embedding $\varkappa : L^1 \to ba$ is a dense subspace of ba in the $\sigma(ba, L^\infty)$ topology. This suggests that Assumption 1) is quite strong. However, the Vitali-Hahn-Saks theorem guarantees that M is already $\sigma(ba, L^\infty)-$sequential closed. Hence, any assumption that guarantees that the $\sigma(ba, L^\infty)$ closure of M coincides with the $\sigma(ba, L^\infty)$ sequential closure of M is sufficient for assumption 1) to hold.*

The proof of the following Lemma is based on the previous remarks on the bipolar cone.

Lemma 1 *a) Under assumption **1)** we have: $\delta^*_{M^-}(\mu) = \delta(\mu|M)$, $\mu \in ba$.*

*b) Under assumption **2)** we have: $\delta^*_{M^-}(z) = \delta(z|M)$, $z \in L^1$ and the epigraph $[U, L^\infty]$ has non empty interior in the product topology of $L^\infty \times \mathbb{R}$ where L^∞ is endowed with the $\tau(L^\infty, L^1)$ topology.*

Theorem 1 *Let $U : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R} \cup \{-\infty\}$ be a concave functional and assume that $\sup_{w \in M^-} U(x + w) < +\infty$. Then, under either assumption 1) or 2), for any $x \in \mathcal{D}$*

$$\sup_{w \in M^-} U(x + w) = \min_{z \in M \cap A_x} -U^*(z) + xE[z],$$

the minimum is attained and $A_x = \{z \in L^1 : \sup_{w \in L^\infty} \{U(x + w) - E[zw]\} < +\infty\}$.

Proof. The proof is just an application of Fenchel Duality Theorem (see Luenberger [17], Th. 1, Ch. 7.12) which states that if $f : C \to \mathbb{R}$ is convex, $g : D \to \mathbb{R}$ is concave, $C = D = L^\infty$, either the epigraph $[f, C]$ or $[g, D]$ has non empty interior in the product topology of $L^\infty \times \mathbb{R}$ then

$$\sup_{w \in C \cap D} g(w) - f(w) = \min_{z \in C^* \cap D^*} f^*(z) - g^*(z)$$

where $C^* \triangleq \{\mu \in ba : f^*(\mu) < +\infty\}$, $D^* \triangleq \{\mu \in ba : g^*(\mu) > -\infty\}$.

Take $f(w) = \delta(w|M^-)$, $g(w) = U(x + w)$. Then we get $g^*(z) = U^*(z) - xE[z]$.

If assumption 1) holds we work with the dual pairing (ba, L^∞).

Then $[f, C]$ has non empty interior in the product topology of $L^\infty \times \mathbb{R}$ since the random variable $w = -1$ $(P - a.s.)$ is an interior point of M^- in the norm topology of L^∞. From Lemma 1 a) we know that $f^*(\mu) = \delta(\mu|M)$ and so:

$$C^* = \{\mu \in ba : \delta(\mu|M) < +\infty\} = M$$

$$C^* \cap D^* = \left\{ z \in L^1 : \sup_{w \in L^\infty} \{U(x + w) - E[zw]\} < +\infty \right\} \cap M.$$

which is our thesis.

If assumption 2) holds we work with the dual pairing (L^1, L^∞). Consider the Mackey topology $\tau(L^\infty, L^1)$ on L^∞. Then by Lemma 1 b) we know that $[g, D]$ has non empty interior in the product topology of $L^\infty \times \mathbb{R}$ and that $f^*(z) = \delta(z|M)$, $z \in L^1$ and we get the thesis. ■

2.1 Utility functions

We now state three assumptions on the utility functions that we consider satisfied throughout the paper.

Let $\mathcal{D} = (\alpha, \beta)$, $-\infty \leq \alpha < \beta \leq +\infty$, be an open interval of $\mathbb{R}$.
1) $u : \mathcal{D} \to \mathbb{R}$ is an increasing, concave, real differentiable function. We extend u to $\mathbb{R}$ by setting

$$u(x) = -\infty \text{ for } x < \alpha,\ x > \beta,\ u(\alpha) = \lim_{x \to \alpha^+} u(x),\ u(\beta) = \lim_{x \to \beta^-} u(x);$$

with this extension $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave upper semicontinuous.
2) $u' : \mathcal{D} \to (0, +\infty)$ defines a bijection between $\mathcal{D}$ and $(0, +\infty)$.
3) $I : (0, +\infty) \to \mathcal{D}$ defined by $I(y) = (u')^{-1}(y)$ is differentiable.

We have:

$$I(0^+) = \beta,\ I(+\infty) = \alpha,$$

and we convene that $yI(y) = 0$ for $y = 0$.

Remark 2 *Examples of utility functions that satisfy these assumptions are:*

a) the exponential utility $u(x) = -e^{-x}$, $\mathcal{D} = \mathbb{R}$

b) the log utility $u(x) = \ln x$, $\mathcal{D} = (0, +\infty)$

c) the HARA class of utilities: $u(x) = \frac{p}{1-p}|x|^p$, $\mathcal{D} = (0, +\infty)$ *if* $-\infty < p < 1$, $p \neq 0$, $\mathcal{D} = (-\infty, 0)$ *if* $p > 1$.

The Fenchel-Legendre transform u^* of u is given by:

$$u^*(y) = \inf_{x \in \mathcal{D}} \{xy - u(x)\} = -u(I(y)) + yI(y) \text{ for } y \in (0, +\infty),$$

since for each $y \in (0, +\infty)$ the minimum is always attained by $x = I(y)$, and $u^*(0) = -\sup u(x) = -\lim_{x \to \beta} u(x) = -u(\beta)$.

Define

$$U : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R} \text{ by } U(w) = E[u(w)].$$

Lemma 2 *Let* $z \in L^1$, $z \geq 0$ $P - a.s.$ *then:*

$$U^*(z) = E[u^*(z)] = E[-u(I(z)) + zI(z)].$$

Proof. Let $z \in L^1$, $z \geq 0$ $P - a.s.$ and note that $\forall w \in L^\infty$

$$w(\omega)z(\omega) - u(w(\omega)) \geq \inf_{x \in \mathbb{R}} \{E[xz(\omega) - u(x)]\} \triangleq u^*(z(\omega)),\ P - a.s..$$

Therefore, $\forall w \in L^\infty$, $E[w(\omega)z(\omega) - u(w(\omega))] \geq E[u^*(z(\omega))]$ and so:

$$U^*(z) \triangleq \inf_{w \in L^\infty} \{E[wz - u(w)]\} \geq E[u^*(z(\omega))].$$

Define $B_n = \{x \in \mathbb{R} : |x| \leq n\}$ $\forall n \geq 1$ and

$$u_n^*(y) = \inf_{x \in B_n} \{xy - u(x)\}, \ y \in [0, +\infty).$$

Then $u_n^*(z(\omega)) \downarrow u^*(z(\omega))$ $P-a.s.$ In order to apply Fatou lemma to the sequence $u_n^*(z)$ we need to show that there exists an integer n_0 and an integrable function F such that $u_{n_0}^*(z) \leq F$, $P-a.s.$. Define $g : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by $g(\omega, x) = xz(\omega) - u(x)$. Then $g(\omega, \cdot) : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is $P-a.s.$ lower semi continuous and, from the measurable selection theorem, $\forall n \geq 1$ there exists a measurable mapping $\bar{x}_n : \Omega \to B_n$ such that

$$g(\omega, \bar{x}_n(\omega)) = \min_{x \in B_n} g(\omega, x),$$

which means:

$$\bar{x}_n(\omega)z(\omega) - u(\bar{x}_n(\omega)) = u_n^*(z(\omega)).$$

Our assumptions on u guarantee that there exists a $\hat{x} \in L^\infty$ such that $u(\hat{x}) \in L^1$ (just take $\hat{x} = c$ with $\alpha < c < \beta$). Therefore for any $n \geq \|\hat{x}\|_\infty$ we have $u_n^*(z(\omega)) = \inf_{x \in B_n} \{xz(\omega) - u(x)\} \leq \{\hat{x}(\omega)z(\omega) - u(\hat{x}(\omega))\} \in L^1$.

Since $\bar{x}_n \in L^\infty$, $\forall n \geq 1$, we have:

$$\begin{aligned} U^*(z) &\triangleq \inf_{w \in L^\infty} \{E[wz - u(w)]\} \leq \lim_{n \to +\infty} E[\bar{x}_n z - u(\bar{x}_n)] \\ &= \lim_{n \to +\infty} E[u_n^*(z(\omega))] \leq E[u^*(z(\omega))] \text{ by Fatou. } \blacksquare \end{aligned}$$

3 Duality results

Fix now a probability measure Q absolutely continuous with respect to P. Set $\varphi = \frac{dQ}{dP}$, $\varphi \in L^1$, $\varphi \geq 0$ $P-a.s.$, define $\mathcal{M} = \{\varphi\}$ and let M be the positive cone generated by $\mathcal{M}$ and note that M is closed in the weak topology $\sigma(ba, L^\infty)$. Note that for each given $x \in \mathcal{D}$, if $\varphi I(y\varphi) \in L^1$ for all $y \in (0, +\infty)$, the equation $E[\varphi I(y\varphi)] = x$ has a unique solution $y \in (0, +\infty)$.

Definition 1 *Let*

$$\mathcal{Y}_\varphi : \mathcal{D} \to (0, +\infty) \text{ be such that: } \quad E[\varphi I(\mathcal{Y}_\varphi(x)\varphi)] = x \ \forall x \in \mathcal{D}.$$

For each $x \in \mathcal{D}$ set:

$$V(x, \varphi) = \left\{ \sup_{w \in L^\infty(\mathcal{D})} E[u(w)] \text{ subject to: } E[\varphi w] \leq x \right\}.$$

Theorem 2 *For any $x \in \mathcal{D}$, if $V(x,\varphi) < +\infty$ then*

$$V(x,\varphi) = \min_{y\in(0,+\infty)} \left\{E[u(I(y\varphi)) - y\varphi I(y\varphi)] + xy\right\}.$$

If also $zI(z) \in L^1\ \forall z \in M$, then

$$V(x,\varphi) = E[u(I(\mathcal{Y}_\varphi(x)\varphi))].$$

Proof. The convex cone M is weakly closed and so it satisfies the assumptions in Th. 1. Let $\mathcal{D}_x = (\alpha - x, \beta - x)$ and set:

$$S(x) = \{w \in L^\infty(\mathcal{D}) :\ E[\varphi w] \le x\}$$

$$T(x) = \{y \in L^\infty : y = x + u,\ u \in L^\infty(\mathcal{D}_x) :\ E[zu] \le 0\ \forall z \in M\}.$$

It is easy to verify that $S(x) = T(x)$ and therefore:

$$\begin{aligned}
V(x,\varphi) &\triangleq \left\{\sup_{w\in L^\infty(\mathcal{D})} E[u(w)] \text{ subject to : } E[\varphi w] \le x\right\} && (1)\\
&= \sup_{w\in S(x)} E[u(w)] = \sup_{y\in T(x)} E[u(y)] = \sup_{u\in M^- \cap L^\infty(\mathcal{D}_x)} E[u(x+u)] && (2)\\
&= \sup_{u\in M^-} E[u(x+u)] = \min_{z\in M\cap A_x} -U^*(z) + xE[z] && (3)\\
&= \min_{z\in\mathcal{A}} E[u(I(z)) - zI(z)] + xE[z] && (4)
\end{aligned}$$

where $\mathcal{A} = \{z \in M : E[u(I(z)) - zI(z)] < +\infty\}$. Equality 2-3 holds since $u(x) = -\infty$ for $x < \alpha$, $x > \beta$; since $V(x,\varphi) < +\infty$, theorem 1 justifies the equality in line 3; lemma 2 justifies equality 3-4. Since $z = 0$ can't be the minimum and each $z \neq 0$ is equal to $y\varphi$ for $y \in (0,+\infty)$ we get:

$$\begin{aligned}
V(x,\varphi) &= \min_{y\in(0,+\infty)} E[u(I(y\varphi)) - y\varphi I(y\varphi)] + xy && (5)\\
&= E[u(I(\mathcal{Y}_\varphi(x)\varphi))], && (6)
\end{aligned}$$

where the last equality is true since the minimum in eq. 5 is achieved by $y = \mathcal{Y}_\varphi(x)$. Indeed from the first order condition we get the equation

$$E[\varphi I(y\varphi)] = x$$

that has the unique, non zero, solution $y = \mathcal{Y}_\varphi(x)$ for each $x \in \mathcal{D}$. ■

We now consider an arbitrary convex cone M in L^1_+.

Definition 2 *We denote with $\mathcal{M}$ the intersection of M with the unit ball of L^1:*

$$\mathcal{M} = M \cap \left\{\varphi \in L^1 : E[\varphi] = 1\right\}.$$

For $x \in \mathcal{D}$ set:

$$V(x) = \left\{ \sup_{w \in L^\infty(\mathcal{D})} E[u(w)] \text{ subject to: } \max_{\varphi \in \mathcal{M}} E[\varphi w] = x \right\},$$

$$\mathcal{M}^* = \{\varphi \in \mathcal{M} : V(x, \varphi) < +\infty\}.$$

Theorem 3 *Let $M \subseteq L^1_+$ be a convex cone ($0 \in M$) closed in the norm topology of L^1. Let $x \in \mathcal{D}$ and assume that $\mathcal{M}^* \neq \emptyset$. Then, under either assumption 1) or 2):*

$$\begin{aligned} V(x) &\triangleq \left\{ \sup_{w \in L^\infty(\mathcal{D})} E[u(w)] \text{ subject to: } \max_{\varphi \in \mathcal{M}} E[\varphi w] = x \right\} \\ &= \min_{\varphi \in \mathcal{M}^*} \left\{ \sup_{w \in L^\infty(\mathcal{D})} E[u(w)] \text{ subject to: } E[\varphi w] \leq x \right\} \\ &= \min_{\varphi \in \mathcal{M}^*} V(x, \varphi), \end{aligned}$$

and the minimum is attained.

Proof. Let $\mathcal{D}_x = (\alpha - x, \beta - x)$ and set:

$$S(x) = \{w \in L^\infty(\mathcal{D}) : \ E[\varphi w] \leq x \ \forall \varphi \in \mathcal{M}\}$$

$$T(x) = \{y \in L^\infty : y = x + u, \ u \in L^\infty(\mathcal{D}_x), \ E[zu] \leq 0 \ \forall z \in M\}.$$

It is easy to verify that $S(x) = T(x)$ and therefore:

$$\begin{aligned} V(x) &= \left\{ \sup_{w \in L^\infty(\mathcal{D})} E[u(w)] \text{ subject to} : \max_{\varphi \in \mathcal{M}} E[\varphi w] = x \right\} && (7) \\ &= \sup_{w \in S(x)} E[u(w)] = \sup_{y \in T(x)} E[u(y)] = \sup_{u \in M^- \cap L^\infty(\mathcal{D}_x)} E[u(x+u)] && (8) \\ &= \sup_{u \in M^-} E[u(x+u)] = \min_{z \in M \cap A_x} -U^*(z) + xE[z] && (9) \\ &= \min_{z \in \mathcal{A}} E[u(I(z)) - zI(z)] + xE[z] && (10) \\ &= \min_{y\varphi \in \mathcal{A}: \varphi \in \mathcal{M}, \ y > 0} E[u(I(y\varphi)) - y\varphi I(y\varphi)] + xE[y\varphi] && (11) \end{aligned}$$

where $\mathcal{A} = \{z \in M : E[u(I(z)) - zI(z)] < +\infty\}$. The equality 8-9 holds since $u(x) = -\infty$ for $x < \alpha$, $x > \beta$; since $\mathcal{M}^* \neq \emptyset$ $\exists \psi \in \mathcal{M} : V(x) \leq V(x, \psi) < +\infty$, so theorem 1 justifies the equality in line 9; lemma 2 justifies the equality 9-10; equality 10-11 holds since M is a cone and $z = 0$ can't be the minimum.

Note that if (y^*, φ^*) realizes the minimum in eq. 11 then

$$V(x) = \min_{y \in (0,+\infty)} E[u(I(y\varphi^*)) - y\varphi^* I(y\varphi^*)] + xy = V(x, \varphi^*)$$

so that certainly $\varphi^* \in \mathcal{M}^*$. Then (from eq. 7-11)

$$\begin{aligned} V(x) &= \min_{\substack{y\varphi \in \mathcal{A}: \\ \varphi \in \mathcal{M},\ y \in (0,+\infty)}} E[u(I(y\varphi)) - y\varphi I(y\varphi)] + xy && (12) \\ &= \min_{\varphi \in \mathcal{M}^*} \left\{ \min_{y \in (0,+\infty)} E[u(I(y\varphi)) - y\varphi I(y\varphi)] + yx \right\} && (13) \\ &= \min_{\varphi \in \mathcal{M}^*} V(x, \varphi) && (14) \\ &= \min_{\varphi \in \mathcal{M}^*} \left\{ \sup_{w \in L^\infty(\mathcal{D})} E[u(w)] \text{ subject to: } E[\varphi w] \leq x \right\} && (15) \end{aligned}$$

where equality 13-14 comes from Th. 2. ■

4 Financial Interpretation

Let $\mathcal{T} \subseteq [0, +\infty)$ represent the set of trading dates and let χ be a family of adapted stochastic processes on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathcal{T}}, P)$ that represent deflated price processes of the securities available for trading in the market.

In this section we denote with $\mathcal{M}$ (resp. $\mathcal{M}^{loc}$) the set of martingale (resp. local martingale) probability measures for χ absolutely continuous with respect to P. It is convenient to identify the probability measures $Q \in \mathcal{M}$ with their densities $\varphi = \frac{dQ}{dP} \in L^1$.

If χ is a family of bounded (resp. locally bounded) stochastic processes, the fundamental theorem of asset pricing guarantees that the set $\mathcal{M}$ (resp. $\mathcal{M}^{loc}$) is not empty, provided that the market admits no free lunch (see [6] and [12] for details).

Moreover, one may easily check the following:

Lemma 3 *If χ is a family of bounded (resp. locally bounded) processes then $\mathcal{M}$ (resp. $\mathcal{M}^{loc}$) is closed in L^1.*

So if Assumption 2) is satisfied we may directly apply Theorems 3 and 2 to the convex cone M generated by $\mathcal{M}$ (or $\mathcal{M}^{loc}$). Otherwise, we need to check whether Assumption 1) is satisfied for M.

4.1 Certainty equivalent

The economic interpretation of our approach is enlighted by one of the oldest criteria for the valuation of financial claims (or lotteries): the well known notion of *certainty equivalent* originally formulated by Daniel Bernoulli [2] and deeply studied also by De Finetti [5]. The assumed monotonicity of the utility function allows us to express the certainty equivalent as the transformed expected value of the claim w under the reference measure P: $c(w) = u^{-1}(E_P[u(w)])$.

However, consider the certainty equivalent itself as a pricing rule is problematic since the certainty equivalent is obviously not a *linear* functional on the space of contingent claims so that arbitrage opportunities may again arise.

Shortly it will be clear that our approach presented in the previous sections is equivalent to *approximate the certainty equivalent under the restriction of the no arbitrage principle.*

For a given pricing measure $Q \in \mathcal{M}_e$ and a given utility function u consider the functional defined on a suitable domain of L^∞ given by

$$c(w) - E_Q[w] = u^{-1}E_P[u(w)] - E_Q[w].$$

In this definition we are considering and comparing two components: the artificial world of martingale pricing generated by the no arbitrage principle and the world of (subjective) probability assignments and preference relations. In order to approximate the certainty equivalent we would like to reduce this difference as much as it is possible under the constraint that we want to end up with a linear pricing functional compatible with the no arbitrage condition.

Definition 3

$$\delta(x,\varphi) = \sup_{w \in L^\infty(\mathcal{D}): E[\varphi w] = x} \left\{ u^{-1}E[u(w)] - E[\varphi w] \right\}.$$

$\delta(x,\varphi)$ is the difference between the certainty equivalent and the martingale price in the *worst* possible case.

The simple relation between $\delta(x,\varphi)$ and $V(x,\varphi)$ is given in the following:

Proposition 1 *If $\mathcal{M}^* \neq \emptyset$, $x \in \mathcal{D}$, we have:*

a) $\varphi^* \in \arg\min V(x,\varphi) \Leftrightarrow \varphi^* \in \arg\min \delta(x,\varphi)$,

b) $\delta(x,\varphi) = u^{-1}(V(x,\varphi)) - x$ *and, if* $\varphi I(y\varphi) \in L^1 \ \forall y \in (0,+\infty)$,

$$\delta(x,\varphi) = u^{-1}\left(E\left[u\left(I\left(\mathcal{Y}_\varphi(x)\varphi\right)\right)\right]\right) - x,$$

c) when the minimum exists we have: $\min_{\varphi\in\mathcal{M}} \delta(x,\varphi) = u^{-1}(V(x)) - x$.

Proof. The proof follows from Th.3 and Th.2 and from the monotonicity of the utility function which guarantees that:

$$\delta(x,\varphi) = \sup_{w\in L^\infty(\mathcal{D}):E[\varphi w]\leq x} \left\{u^{-1}(E[u(w)]) - E[\varphi w]\right\}. \blacksquare$$

For our purposes, we may work indifferently with $\delta(x,\varphi)$ or $V(x,\varphi)$ and so in the following examples we will consider the "distance" $\delta(x,\varphi)$. This has the advantage of providing simple formulas, since in $\delta(x,\varphi)$ we are comparing quantities expressed in the same "units".

4.1.1 The $\delta(x,\varphi)$ "distance"

In the following remarks we state some elementary properties of $\delta(x,\varphi)$ as a function of φ.

Remark 3 *Let $a,b,c,k \in \mathbb{R}$, $a > 0, k \neq 0$, set $u_{a,b,c,k}(x) = ku(ax+b)+c$. Then $\delta_{a,b,c,k}(x,\varphi) = \frac{1}{a}\delta(ax+b,\varphi)$ where the supremum are taken in the proper domain. Note that if u is increasing and concave then also $u_{a,b,c,k}$ is increasing and concave whenever $a > 0$, $k > 0$. This remark allows us to suppress the dependence of the utility function from the irrelevant parameters and to modify the domain of u.*

Proof. It follows directly from the computation:

$$\begin{aligned}\delta_{a,b,c,k}(x,\varphi) &= \sup_{E_Q[w]=x} \frac{1}{a}\left\{u^{-1}(E_P[u(aw+b)]) - E_Q[aw+b]\right\} \\ &= \frac{1}{a}\sup_{E_Q[y]=ax+b} \delta(y,\varphi) = \frac{1}{a}\delta(ax+b,\varphi)\end{aligned}$$

where the first supremum is taken in $L^\infty(\Omega, \mathcal{F}, P; \mathcal{D})$, the second in $L^\infty(\Omega, \mathcal{F}, P; \mathcal{D}')$, with $\mathcal{D}' = (a\alpha + b, a\beta + b)$, $\mathcal{D} = (\alpha, \beta)$, and the mapping $w \to y = aw + b$ is a bijection between $L^\infty(\Omega, \mathcal{F}, P; \mathcal{D})$ and $L^\infty(\Omega, \mathcal{F}, P; \mathcal{D}')$. ■

In the following three remarks we state some conditions on u and on φ under which $\delta(x, \varphi)$ is finite or it is equal to $+\infty$.

Remark 4 *Let $\mathcal{D}$ be bounded from below and $x \in \mathcal{D}$. Then for all $\varphi = \frac{dQ}{dP} \geq \varepsilon > 0$ we have $\delta(x, \varphi) < +\infty$.*

Proof. Without loss of generality let $u : (0, +\infty) \to \mathbb{R}$. We have

$$E_P[\varphi w] \geq \varepsilon E_P[w] \ \forall w \in L^\infty(\Omega, \mathcal{F}, P; (0, +\infty))$$

and so, by Jensen inequality,

$$E_P[u(w)] \leq u(E_P[w]) \leq u(\frac{1}{\epsilon} E_P[\varphi w]) = u(\frac{1}{\epsilon} E_Q[w]).$$

Therefore: $u^{-1}\left(E_P\left[u(w)\right]\right) - E_Q\left[w\right] \leq \left(\frac{1}{\varepsilon} - 1\right) E_Q[w]$ and $\delta(x, \varphi) \leq \left(\frac{1}{\varepsilon} - 1\right) x$. ■

Remark 5 *The previous remark applies to probabilities Q equivalent to P. We now show that if the utility u is unbounded from above and if $Q << P$ but Q is not equivalent to P then $\delta(x, \varphi) = +\infty$ for every $x \in \mathcal{D}$.*

Proof. Let $u : (\alpha, +\infty) \to (a, +\infty)$, $a, \alpha \geq -\infty$, be a utility such that $u(x) \to +\infty$ for $x \to +\infty$ and let $A \in \mathcal{F}$ such that $Q(A) = 0$ and $P(A) > 0$. Take $w_n = x$ on A^c and $w_n = n$ on A. Then $E_Q[w_n] = x \ \forall n$, $E_P[u(w_n)] = u(x)P(A^c) + u(n)P(A) \to +\infty$ so that also $c(w_n) - E_Q\left[w_n\right] \to +\infty$. ■

Remark 5 implies that if u is unbounded from above and there exists $Q^* \in \mathcal{M}$ minimizing $\delta(x, \varphi)$, then Q^* is equivalent to P.

Remark 6 *If $u : (0, +\infty) \to (-\infty, 0)$ is bounded from above then $\delta(x, \varphi) < +\infty$ ($Q << P$) for every $x > 0$.*

Proof. By contradiction, suppose that $\sup c(z) = +\infty$. Then there exists a sequence $z_n \geq 0$ $P-a.s.$ such that $E_Q[z_n] = x$ and $c(z_n) \to +\infty$. Hence $E_P[u(z_n)] \to 0^-$ and, eventually passing to subsequences, $P(\omega : u(z_n) \to 0^-) = 1 = P(\omega : z_n \to +\infty) = Q(\omega : z_n \to +\infty)$, since $Q << P$. By Fatou, $+\infty = E_Q[\lim z_n] \leq \liminf E_Q[z_n]$ which is a contradiction. ■

Two cases arise frequently: when $c(w)$ is additive with respect to constants and when $c(w)$ is positively homogeneous. If u is exponential we are in the first case, while for HARA and *log* utilities we are in the second (for an explicit characterization of the associative mean having these properties - "medie associative traslative, medie associative omogenee" - we refer to De Finetti [5]).

Remark 7 *a) If the certainty equivalent c is positively homogeneous (i.e. $c(xw) = xc(w)$, $x \in \mathbb{R}$, $x>0$) then*

$$\delta(x, \varphi) = x\delta(1, \varphi).$$

b) If the certainty equivalent c is additive respect to constants (i.e. $c(w+k) = c(w) + k$, $k \in \mathbb{R}$) then

$$\delta(x_1, \varphi) = \delta(x_2, \varphi) \ \forall x_1, x_2 \in \mathcal{D}.$$

Therefore, it is possible in both cases to choose the measure Q^* minimizing $\delta(x, \varphi)$ independently from x.

Proof.

$$\begin{aligned}
\delta(1, \varphi) &= \sup_{w \in L^\infty(\mathcal{D}): E[\varphi w]=1} \left\{u^{-1}E\left[u(w)\right] - E\left[\varphi w\right]\right\} \\
&= \sup_{w \in L^\infty(\mathcal{D}): E[\varphi x w]=x} \frac{1}{x}\left\{u^{-1}E\left[u(xw)\right] - E\left[\varphi x w\right]\right\} \\
&= \frac{1}{x}\sup_{z \in L^\infty(\mathcal{D}): E[\varphi z]=x} \left\{u^{-1}E\left[u(z)\right] - E\left[\varphi z\right]\right\} = \frac{1}{x}\delta(x, \varphi).
\end{aligned}$$

$$\begin{aligned}
\delta(x_1, \varphi) &= \sup_{E_Q[w]=x_1} \{c(w) - E_Q[w]\} \\
&= \sup_{E_Q[w+x_2-x_1]=x_2} \{c(w + x_2 - x_1) - E_Q[w + x_2 - x_1]\} \\
&= \delta(x_2, \varphi) \ \forall x_1, x_2 \in \mathcal{D}. \ ■
\end{aligned}$$

In general $\delta(x, \varphi)$ is not a metric. However, note the following:

Remark 8 *1)* $\delta(x,\varphi) \geq 0 \ \forall Q << P \ \forall x \in \mathcal{D}$.

2) If $Q = P$, *then* $\delta(x,\varphi) = 0$.

3) If $\delta(x,\varphi) = 0 \ \forall x : 0 < x < x_0$, *then* $Q = P$.

Therefore, if c *is positively homogeneous or additive we have, for* $x \neq 0$, $\delta(x,\varphi) = 0 \Leftrightarrow Q = P$

Proof.

1) Let $w = x \in L^{\infty}(\mathcal{D})$, then $\delta(x,\varphi) \geq u^{-1}E_P\left[u(x)\right] - E_Q\left[x\right] = x - x = 0$.

2) If $Q = P$ Jensen inequality assures $E_P\left[u(w)\right] \leq u\left(E_P[w]\right)$ and hence $\delta(x,\varphi) = 0$ for every x.

3) Without loss of generality (see remark 3) we assume that $0 \in \mathcal{D}$, $u(0) = 0$, and $0 < u'(0) < +\infty$. If $\delta(x,\varphi) = 0$ for each $0 < x < x_0$ then $E_P\left[u(w)\right] \leq u\left(E_Q[w]\right) \ \forall w \in \mathcal{D} : 0 \leq E_Q[w] \leq x_0$. For every $A \in \mathcal{F}$, let $w_A = \varepsilon 1_A$. If $\varepsilon < x_0$, we get $P(A)u(\varepsilon) \leq u(\varepsilon Q(A))$. By passing to the limit as $\varepsilon \to 0$ we get $P(A) \leq Q(A)$. Since A is arbitrary, it follows $P = Q$. ■

5 Examples

In the following examples, thanks to remark 3, we will suppress the dependence of the utility functions from the irrelevant parameters and set $\varphi = \frac{dQ}{dP}$. We will compute $\delta(x,\varphi)$ with the formula given in Proposition 1 b):

$$\delta(x,\varphi) = u^{-1}\left(E\left[u\left(I\left(\mathcal{Y}_{\varphi}(x)\varphi\right)\right)\right]\right) - x. \tag{16}$$

5.1 Relative Entropy

5.1.1 Exponential utility

For exponential utilities, as already stated, the certainty equivalent c is additive respect to constants and so $\delta(x,\varphi) = \delta(0,\varphi) \ \forall x \in \mathbb{R}$.

Proposition 2 *Let* $u(x) = -e^{-x}$, $x \in (-\infty, +\infty)$. *Then*

$$\delta(x,\varphi) = \delta(0,\varphi) = H(Q,P) \quad \forall x \in \mathbb{R},$$

where $H(Q,P)$ *is the relative entropy defined by*

$$H(Q,P) = \begin{cases} E_P[\frac{dQ}{dP}\ln\frac{dQ}{dP}] & \text{if } Q << P \\ +\infty & \text{otherwise} \end{cases}. \tag{17}$$

Proof. We provide a direct proof of this proposition based on an alternative characterization of the relative entropy. We have:

$$c(w) - E_Q[w] = -\ln E_P[\exp(-w)] - E_Q[w].$$

Since $L^\infty(\Omega, \mathcal{F}, P)$ is a linear space,

$$\begin{aligned} \sup_{w \in L^\infty} c(w) - E_Q[w] &= \sup_{w \in L^\infty} c(-w) - E_Q[-w] \\ &= \sup_{w \in L^\infty} \{-\ln E_P[\exp(w)] + E_Q[w]\} = H(Q,P) \end{aligned} \quad (18)$$

where the last equality follows from the functional form of the relative entropy (see for example [15] Theorem 1.4.4).
From eq. 18 we know that $H(Q,P) = \sup_{w \in L^\infty} \{c(w) - E_Q[w]\} \geq \delta(0,\varphi)$. Let $w_n \in L^\infty : \{c(w_n) - E_Q[w_n]\} \to H(Q,P)$, and let $z_n = w_n - E_Q[w_n]$; then $E_Q[z_n] = 0$ and also $\{c(z_n) - E_Q[z_n]\} = \{c(w_n) - E_Q[w_n]\} \to H(Q,P)$ hence $\delta(0,\varphi) \geq H(Q,P)$. ■

This means that the entropy

$$H(Q,P) = \sup_{E_Q[w]=0} \left\{u^{-1} E_P\left[u(w)\right]\right\}$$

is the maximum certainty equivalent of all contingent claims having zero prices under Q.

Remark 9 *If we apply formula 16 to the utility function $u(x) = -e^{-x}$, $x \in \mathbb{R}$ we get the same conclusion: $\delta(x,\varphi) = u^{-1}\left(E\left[u\left(I\left(\mathcal{Y}_\varphi(x)\varphi\right)\right)\right]\right) - x = H(Q,P)$.*

In the case of exponential utility a sufficient condition for the existence of Q^*, weaker than assumption 1), is that $\mathcal{M}$ is closed in variation. We refer to the paper by Frittelli [10] for a detailed discussion on the existence and uniqueness and characterization of the minimal entropy martingale measure Q^*.

5.1.2 Log utility

Let $u(x) = \ln x$, $x \in (0, +\infty)$. Then

$$\delta(x,\varphi) = x\left\{e^{-E[\ln \varphi]} - 1\right\} \geq 0.$$

So minimizing δ is equivalent to maximizing $E[\ln \frac{dQ}{dP}]$ which is equivalent to minimizing the relative entropy $H(P,Q) = E[\ln \frac{dP}{dQ}] = -E[\ln \frac{dQ}{dP}]$ for $Q \sim P$.

5.2 HARA class of Utility functions

Consider the HARA class of Utility functions:

$$u(x) = \frac{p}{1-p}|x|^p, \text{ where } \begin{cases} \mathcal{D} = (0, +\infty) & \text{if } -\infty < p < 1, p \neq 0 \\ \mathcal{D} = (-\infty, 0) & \text{if } p > 1 \end{cases}.$$

Then, letting $q \in \mathbb{R} : 1/p + 1/q = 1$,

$$\delta(x, \varphi) = x \left\{ \left(E\left[\varphi^q\right] \right)^{-\frac{1}{q}} - 1 \right\} \geq 0, \ x \in \mathcal{D}.$$

So, for $p > 0$, $p \neq 1$, minimizing δ is equivalent to minimizing $E\left[\varphi^q\right]$. For $p < 0$, minimizing δ is equivalent to maximizing $E\left[\varphi^q\right]$.

5.2.1 Kakutani-Hellinger Integral

1. For $p < 0$ we maximize $E\left[\varphi^q\right]$. Note that in this case $0 < q = \frac{p}{p-1} < 1$ and $E\left[\varphi^q\right] = H(q, Q, P)$ is the Hellinger integral of order q.

2. For $p = -1$ we get $u(x) = -\frac{1}{2}\frac{1}{x}$, $x \in (0, +\infty)$,

$$\delta(x, \varphi) = x \left[\frac{1}{\left(E\left[\sqrt{\varphi}\right] \right)^2} - 1 \right] \geq 0,$$

$q = \frac{1}{2}$ and $H(\frac{1}{2}, Q, P) = E\left[\sqrt{\varphi}\right]$. In this case minimizing δ is equivalent to maximizing the Hellinger integral of order $\frac{1}{2}$ i.e. minimizing the Kakutani-Hellinger distance $\rho(Q, P)$ given by

$$\rho^2(Q, P) = 1 - H(\frac{1}{2}, Q, P).$$

5.2.2 Variance

1. For $p = 2$ we get $u(x) = -2x^2$, $x \in (-\infty, 0)$, and

$$\delta(x, \varphi) = x \left[\frac{1}{\sqrt{E[\varphi^2]}} - 1 \right] \geq 0.$$

Therefore minimizing δ is equivalent to minimizing $E[\varphi^2]$ (since $x < 0$), or the variance of $(\frac{dQ}{dP})$.

Henceforth, our minimizing measure Q^* coincides with the variance optimal measure Q^{var} (see [19] and [7] for a detailed study of this issue).

2. For $p = \frac{2}{3}$ we get $u(x) = \sqrt[3]{x^2}$, $x \in (0, +\infty)$, and

$$\delta(x, \varphi) = x \left[\sqrt{E\left[\frac{1}{\varphi^2}\right]} - 1 \right] \geq 0.$$

Therefore minimizing δ is equivalent to minimizing $E[(\frac{dP}{dQ})^2]$, or the variance of $(\frac{dP}{dQ})$ for $Q \sim P$.

5.2.3 Square root utility

For $p = \frac{1}{2}$ we get $u(x) = \sqrt{x}$, $x \in (0, +\infty)$, and:

$$\delta(x, \varphi) = x \left[E\left[\frac{1}{\varphi}\right] - 1 \right] \geq 0.$$

Therefore, minimizing δ is equivalent to minimizing $E[\frac{dP}{dQ}]$ i.e. the expectation of $(\frac{dP}{dQ})$ for $Q \sim P$.

References

[1] Bewley, T.F. (1972): "Existence of equilibria in Economies with infinitely Many Commodities", *J.E.T.*, 4, 514-540.

[2] Bernoulli, D. (1738): "Specimen Theoriae Novae de Mensura Sortis", *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 5, 175-192.

[3] Dalang, R.C., A. Morton, and W. Willinger (1990): "Equivalent martingale measures and no arbitrage in stochastic securities market models," *Stoc. & Stoc. Rep.*, 29, 185-201.

[4] Davis, M.H.A. (1994): "A general option pricing formula," Preprint, Imperial College, London.

[5] De Finetti, B. (1931): "Sul concetto di media", *Giornale degli Economisti*, 2, 369-396, 1931.

[6] Delbaen, F., and W. Schachermayer (1994): "A general version of the fundamental theorem of asset pricing," *Mathematische. Ann.*, 300, 463-520.

[7] Delbaen, F., and W. Schachermayer (1996): "The variance optimal martingale measure for continuous processes", *Bernoulli*, 9, 81-105.

[8] Föllmer, H., and M. Schweizer (1991): "Hedging of Contingent Claims under Incomplete Information," in M.H.A. Davis and R.J. Elliott (eds), *Applied Stochastic Analysis*, Stochastics Monographs, Vol. 5, Gordon and Breach, London/New York.

[9] Frittelli, M. (1995): "Minimal Entropy Criterion for Pricing in One Period Incomplete Markets," Working Paper, Dip. Met. Quant., Univ. of Brescia, Italy.

[10] Frittelli, M. (1996): "The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets" , Working Paper #20, Dip. Matematica, Univ. Milano (to appear in Math. Finance).

[11] Frittelli, M., and P. Lakner (1996): "Counterexamples for the existence of the minimal entropy martingale measure" Working Paper, Dip. Met. Quant., Univ. of Brescia, Italy.

[12] Frittelli, M., and P. Lakner (1994): "Almost sure characterization of martingales," *Stoc. & Stoc. Rep.*, 49, 181-190.

[13] Grothendieck, A. (1973): "Topological vector spaces", Gordon and Breach Sc. Publ.

[14] Harrison, J.M., and S.R. Pliska (1981): "Martingales and stochastic integrals in the theory of continuous trading," *Stoc. Proc. Appl.*, 11, 215-260.

[15] Ihara, S "Information Theory", World Scientific, 1993.

[16] Karatzas I., Shreve S., Lehoczky J., Xu G., (1991):"Martingale and duality methods for utility maximization in an incomplete market", *SIAM J. Control and Opti*m., 29, 702-730.

[17] Luenberger, D.G. (1969): "Optimization by vector space methods", Wiley Sons Inc.

[18] Miyahara, Y. (1996): "Canonical martingale measures of incomplete assets markets", Working Paper, Nagoya City Univ.

[19] Schweizer, M. (1995): "Variance-Optimal Hedging in Discrete Time", Mathematics of Operation Research 20, 1-32.