Putting Order in Risk Measures

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Abstract

This paper introduces a set of axioms that define convex risk measures. Duality theory provides the representation theorem for these measures and the link with pricing rules.

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1 Introduction

In finance, we are often exposed to risk in capital, whether as investors, traders or corporations. It seems therefore useful to *quantify* the riskiness of our position and hence to decide if it is *acceptable* or not. For this reason, several classes of risk measures were proposed in literature.

In a *static* setting, Value at Risk (see, for instance, Duffie and Pan [10]) and coherent risk measures (see Artzner et al. [2], [3] and Delbaen [8]) are often considered, while *dynamic* measures were proposed, among others, by Cvitanic & Karatzas [6] and Wang [20].

In this paper, we will focus on static measures of risk and we will analyze the basic properties of them. For simplicity, we will consider market models without interest rates; it is immediate, however, to extend all definitions and results to the "real" case, by appropriately discounting.

One possible "tool" to measure risk, very popular in practice, is the Value at Risk. If we fix in advance a future date T, Value at Risk at the probability level α (denoted by VaR_α) is the opposite of the $\alpha-$quantile of the final net worth in T. In other words, if VaR_α is negative, then it represents the maximal amount of money we could loose with a given probability α.

There are several criticisms to VaR in literature (see, for example, Artzner, et al. [2], [3] and the recent paper by Frey and McNeil [12]). First of all, we

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notice that VaR is a model dependent measure of risk because, by definition, it depends on the initial reference probability.

Moreover, as we will see later, it seems natural to look for a measure of risk which is "sensitive" to diversification, i.e. a measure which decreases when we diversify. Unfortunately, in general VaR fails to satisfy this property and, even for sums of independent risky positions, its behavior is not as we would expect.

The above criticism led Artzner et al. [2], [3] to introduce an axiomatic definition of coherent measures of risk in a finite probability space, later extended in general spaces by Delbaen [8]. Furthermore, they were able to show that these measures admit a particular representation.

In this paper we prove that an analogous representation still holds if we considerably weaken the axioms of coherence. We then characterize those "representable" risk measures.

Let X be an ordered locally convex topological vector space. The space X is interpreted as the "habitat" of the financial positions whose riskiness has to be quantified. We assume that X is endowed with a topology τ for which X and its topological dual space X', formed with all continuous linear functional on X, form a dual system. To be more concrete (but most of the results hold in an ordered locally convex T.V.S.) in this paper we'll assume that $X = L^p(\Omega, \mathcal{F}, P)$, $1 \leq p \leq +\infty$, and $X' \subseteq L^1(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space. For those interested mainly in finite-dimensional aspects, the only example that needs to be kept in mind is: $X = X' = \mathbb{R}^n$, where $|\Omega| = n$ and $\mathbb{R}^n$ is endowed with the euclidean norm. Other examples are: $X = L^p(\Omega, \mathcal{F}, P)$ and $X' = L^q(\Omega, \mathcal{F}, P)$ where $p \in (1, +\infty)$, p and q are conjugate, and τ is the norm topology in $L^p(\Omega, \mathcal{F}, P)$; or $X = L^\infty(\Omega, \mathcal{F}, P)$ and $X' = L^1(\Omega, \mathcal{F}, P)$ and $\tau = \sigma(L^\infty, L^1)$. We denote with $\mathbf{1}$ the random variable $P - a.s.$ equal to 1, with " $\leq$ " the natural preorder on the vector space X given by inequalities that hold $P - a.s.$ The set X'_+ of positive continuous linear functional on X is given by

$$X'_+ = \{x' \in X' \mid x'(x) \leq 0 \ \forall x \in X : x \leq 0\},$$

and

$$\mathcal{Z} = \left\{x' \in X'_+ : x'(\mathbf{1}) = 1\right\}$$

is the set of probability densities in X'. The Radon Nikodym theorem allows us to identify probability densities $x' \in \mathcal{Z}$ with the associated probability measures P' by setting $\frac{dP'}{dP} = x'$. Therefore we'll use indifferently

$$x'(x) = E_P[x'x] = E_{P'}[x], \ \text{if } x' \in \mathcal{Z}.$$

A *risk measure* is a functional $\rho : X \to \mathbb{R}$ satisfying certain axioms that we are going to introduce.

For the financial interpretation we recall that if $\rho(x)$ is negative, then the position x is *acceptable* and $\rho(x)$ represents the maximal amount which the investor can withdraw without changing the acceptability of x. On the other hand, if $\rho(x)$ is positive, then x is *unacceptable* and $\rho(x)$ represents the minimal extra cash which the investor has to add to the initial position x to make it acceptable.

For the formulation of the axioms it is more opportune to work with $\pi(x) \triangleq \rho(-x)$ than with $\rho(x)$. The reason of this different "choice" of notation is that with $\rho(-x)$ there will be the "right" sign in the monotonicity and translability axioms.

Throughout the paper we'll often require that the following rather weak, even though basic, assumption holds true.

Assumption (A)
The functional $\pi : X \to \mathbb{R}$ is finite valued, convex and lower semi-continuous, i.e.:
$\pi(\alpha x + (1-\alpha)y) \leq \alpha\pi(x) + (1-\alpha)\pi(y), \forall x, y \in X, \forall \alpha \in [0,1]$, *(convexity);*
the set $\{x \in X : \pi(x) \leq \alpha\}$ is closed in X for all $\alpha \in \mathbb{R}$ (lower semi-continuity).

Remark 1 *If $|\Omega| = n$, $X = X' = \mathbb{R}^n$, then the assumption that π is convex on X already implies the continuity of π. Henceforth, the assumption of lower semi-continuity is superfluous if $X = \mathbb{R}^n$ and it is relevant only for infinite dimensional spaces.*

We present some plausible explanations for Assumption (A).

First note that the formulations of convexity and lower semi-continuity are the same for $\pi(x)$ or $\rho(x) = \pi(-x)$, i.e.: π is convex and lower semi-continuous iff ρ is convex and lower semi-continuous.

As we will see in Remark 8 below, under the assumption $\pi(\mathbf{0}) = 0$, convexity of π implies that:

$(*)$ $\pi(\alpha x) \leq \alpha\pi(x)$, $\forall \alpha \in [0,1]$, $\forall x \in X$;

$(**)$ $\pi(\alpha x) \geq \alpha\pi(x)$, $\forall \alpha \geq 1$, $\forall x \in X$.

Both inequalities can be interpreted via liquidity arguments. The latter seems reasonable since, when α becomes large, the whole position (αx) is less liquid than α singular positions x. When α is small, the opposite inequality must hold for specular reasons.

Furthermore, we notice that *convexity encourages diversification of risk* with proportions of positions x and y. From the "liquidity condition" (*) and from the consideration that positions αx and $(1-\alpha)y$ jointly taken can "offset each other", it seems natural to assume that their joint risk measure is less than or equal to the weighted sum of the positions x and y taken singularly.

The lower semi-continuity property guarantees that the limit position of a sequence (or net) of acceptable positions remains acceptable.

We present a list of (not independent) axioms for the functional π. We'll discuss them in relation to their financial interpretation and to the representation property.

Axioms

(b) positivity: $x \leq 0 \Rightarrow \pi(x) \leq \pi(\mathbf{0})$, $\forall x \in X$;

(b$'$) monotonicity: $x \leq y \Rightarrow \pi(x) \leq \pi(y)$, $\forall x, y \in X$;

(c) $\pi(\mathbf{0}) = 0$;

(d) constancy: $\pi(\boldsymbol{\alpha}) = \alpha$ $\forall \alpha \in \mathbb{R}$;

(e) sublinearity:

$\pi(ax) = a\pi(x)$, $\forall a \geq 0$, $\forall x \in X$ (positive homogeneity);

$\pi(x+y) \leq \pi(x) + \pi(y), \forall x, y \in X$ (subadditivity);

(f) normality: i) $\pi(\mathbf{1}) = 1$;

ii) $\pi(-\mathbf{1}) = -1$;

(g) translability: $\pi(x+a) = \pi(x) + a$, $\forall a \in \mathbb{R}$, $\forall x \in X$.

Notice that under the assumption **(A),** the axioms (b) and (b$'$) are equivalent (see Remark 8). Suppose that $\pi : X \to \mathbb{R}$ satisfies $\pi(\mathbf{0}) = 0$. Then it is an easy exercise (see Remark 9) to show that which ever two axioms, among convexity subadditivity positive homogeneity, hold true then the other one also holds true. We selected the convexity axiom as the basic and most relevant one. Besides its financial interpretation suggested above, convexity alone (with lower semi-continuity) is sufficient to guarantee the representation property that we are looking for (see Theorem 5). However, if we add either the subadditivity or the positive homogeneity axiom to convexity we automatically end up into the class of sublinear risk measures, that was considered by Artzner et al. [2] for the definition of coherent risk measures.

Definition 2 *(see Artzner et al. [2], [3] and Delbaen [8]) A functional $\rho : X \to \mathbb{R}$ is a* coherent risk measure *if $\pi(x) \triangleq \rho(-x)$ satisfies axioms (b), (e) and (g).*

The motivations for the axioms defining coherent risk measures are the following.

(e) Sublinearity. Subadditivity has an easy interpretation. Let us suppose that we own two positions which jointly have a positive measure of risk. Hence, we have to add extra cash to obtain a "neutral" position. If the subadditivity did not hold, then, in order to deposit less extra cash, it would be sufficient for us to separate in two accounts our positions. Roughly speaking, it seems reasonable to have a *discount* when we "buy" several positions.

We notice that subadditivity implies that $\rho(nx) \leq n\rho(x)$ for every $n \in \mathbb{N}$ and $x \in X$. The opposite inequality is imposed by the positive homogeneity axiom. However, this last axiom may not be necessary: indeed only properties (*) and (**) are supported by liquidity arguments and follow directly from convexity.

(b), (b$'$) Monotonicity. As it seems obvious to expect, the monotonicity of ρ implies that if two final net worth are such that $x \leq y$, then their risk measures have to satisfy $\rho(x) \geq \rho(y)$ (note that the opposite inequality holds, because of the financial interpretation of the risk measure and, consequently, because of the setting $\rho(x) = \pi(-x)$).

(g) Translability. We notice that translability implies

$$\rho(x + \rho(x)) = \pi(-x - \pi(-x)) = \pi(-x) - \pi(-x) = 0.$$

That is, when we add $\rho(x)$ to the initial position x, we obtain a "neutral" position. We'll see later that the sublinearity implies that (d), (f) and (g) are equivalent. Of course, the financial meaning of axioms (d) and (f) is self evident.

Delbaen [8] proved the following representation theorem of coherent risk measures.

Theorem 3 ***[Delbaen; Theorem 2.3; [8]]***
Let $X = L^{\infty}(\Omega, \mathcal{F}, P)$ and let $X' = ba(\Omega, \mathcal{F}, P)$ be the space of all bounded, finitely additive and $P-$ absolutely continuous set functions defined on $(\Omega, \mathcal{F})$. If $\rho : X \to \mathbb{R}$ is a coherent risk measure then there exists a convex $\sigma(X', X)$-closed set $\mathcal{P} \subseteq \mathcal{Z}$ such that

$$\rho(x) = \pi(-x) = \sup_{\mu \in \mathcal{P}} E_{\mu}[-x], \quad \forall x \in L^{\infty}. \tag{1}$$

We notice that the previous theorem provides the characterization of coherent risk measures in terms of finitely additive functionals. By adding a continuity property (called the Fatou property) or equivalently the hypothesis that the set of acceptable positions $\{x \in L^{\infty} : \rho(x) \leq 0\}$ is $\sigma\left(L^{\infty}, L^{1}\right)$ $-$closed, one gets the same representation (1), where the finitely additive measures in *ba* are replaced by the countably additive measures, i.e.: by probability measures, and hence $\mathcal{P} \subseteq \mathcal{Z} \subseteq L^{1}$ (see Theorem 3.2, Delbaen [8]).

The financial importance of the representation above is that *any* coherent risk measure $\rho(x)$ can be obtained as the supremum of the expected loss $E_{\mu}[-x]$ over a set $\mathcal{P}$ of "generalized scenarios". Such measures generalize, for example, the Standard Portfolio Analysis of Risk method, where the maximum is calculated over 14 standard scenarios and 2 "extreme" scenarios.

In this paper we show that the functional form of the representation above depends *only* on the *sublinearity* of the functionals, and not on the other axioms, required for the definition of a coherent risk measure, see also Frittelli [14] for further details. *Moreover, we'll show a representation theorem for the larger class of convex risk measures*, which is basically the well known 1-1 correspondence between convex closed functions and their conjugate functions. The notion of convex risk measure was introduced by D. Heath [15] for measures of risk defined on a finite set Ω.

Definition 4 *Let $\pi : X \to \mathbb{R}$. If there exist a convex function $F : X' \to \mathbb{R} \cup \{+\infty\}$ and a non empty convex set $\mathcal{P} \subseteq X'$ such that*

$$\pi(x) = \sup_{x' \in \mathcal{P}} \{x'(x) - F(x')\} < +\infty \ \forall x \in X, \tag{I}$$

then we say that π is representable *or that $\rho(x) = \pi(-x)$ is a* convex risk measure*.*

We'll prove (see Corollary 7) the following

Theorem 5 *1) A functional $\pi : X \to \mathbb{R}$ is representable if and only if it is convex and lower semi-continuous.*

2) A functional $\pi : X \to \mathbb{R}$ is representable with $F = 0$ on $\mathcal{P}$ if and only if it is sublinear and lower semi-continuous.

For the representation of a convex risk measure, from (I) we get

$$\rho(x) = \pi(-x) = \sup_{x' \in \mathcal{P}} \{x'(-x) - F(x')\} < +\infty, \; x \in X. \tag{2}$$

If we interpret F as a correction term and $x'(-x)$ as the "expected loss", we see that ρ is the maximum "corrected expected loss" over a set of generalized scenarios, where the corrections depend on scenarios. Of course, the interpretation of $x'(-x)$ as "expected loss" holds only if $x' \in \mathcal{P} \subseteq \mathcal{Z}$, since in this case x' can be identified with $\frac{dP'}{dP}$, $x'(-x) = E_{P'}[-x]$ and

$$\rho(x) = \pi(-x) = \sup_{P' \in \mathcal{P} \subseteq \mathcal{Z}} \{E_{P'}[-x] - F(P')\}. \tag{3}$$

We wish to stress that one can decide to impose or not other axioms in addition to convexity and lower-semi-continuity, according to what one is interested in. The assumption of these further axioms only modifies the functional F and the set $\mathcal{P}$ over which the supremum is taken.

In particular (see Corollary 7 and Remarks 8 and 10 for the precise statements):

- For convex lower semi-continuous functionals:
  - in the representation (I) we have:
    - positivity $\longleftrightarrow \mathcal{P} \subseteq X'_+$;
    - $\pi(\mathbf{0}) = 0 \longleftrightarrow \inf_{x' \in \mathcal{P}} F(x') = 0$;
    - constancy $\longleftrightarrow \mathcal{P} \subseteq \{x' \in X' : x'(\mathbf{1}) = 1\}$ and $\inf_{x' \in \mathcal{P}} F(x') = 0$;
    - positivity and constancy $\longleftrightarrow \mathcal{P} \subseteq \mathcal{Z}$ and $\inf_{x' \in \mathcal{P}} F(x') = 0$;
  - if $\pi(\mathbf{0}) = 0$, then the translability and the constancy axioms are equivalent.
- For sublinear lower semi-continuous functionals:
  - in the representation (I) we have $F = 0$ on $\mathcal{P}$ and:
    - positivity $\longleftrightarrow \mathcal{P} \subseteq X'_+$;
    - positivity and translability $\longleftrightarrow \mathcal{P} \subseteq \mathcal{Z}$, (this is exactly the case of coherent risk measures; see eq. (3), where $F = 0$ on $\mathcal{P}$);
  - the translability, the constancy and the normality axioms are all equivalent.

The functional representation in equation (3) suggests an interesting link between non-linear pricing functionals in incomplete markets and risk measures. From this comparison we will also show two examples: the first is a class of convex risk measures that are not sublinear and the second consists of a class of sublinear risk measures that are not coherent.

It is well known that in incomplete market models it is convenient to embed the contingent claim pricing theory in a utility maximization framework. We'll briefly mention the notion of the dynamic certainty equivalent, introduced by Frittelli in [13], which defines, for each claim $x \in X = L^{\infty}$, a price compatible with the no arbitrage principle and with the preferences of the investor. For unexplained notations and concepts one may consult [13]. We will consider a stochastic incomplete model of a security market with no arbitrage opportunities. Henceforth, the set $\mathcal{M} \subseteq \mathcal{Z}$ of martingale probability measures absolutely continuous with respect to P is not empty and has an infinite number of elements (as usual we identify $x' \in \mathcal{Z}$ with the probability P' such that $x' = \frac{dP'}{dP}$).

For each, possibly non attainable, claim $x \in X$ there is an interval of prices that are compatible with the absence of arbitrage. The maximum price $\widehat{x} \in \mathbb{R}$ in this interval (which also corresponds to the super replication price of x), is given by

$$\widehat{x} = \sup\left\{E_{P'}[x] \mid P' \in \mathcal{M}\right\}.$$

An agent may buy claims and/or invest in the security market with the objective of maximizing the expected utility from terminal wealth. If $x_0 \in \mathbb{R}$ is the initial endowment, the agent can buy any claim $z \in X$ having maximum price $\widehat{z} \in \mathbb{R}$ less than or equal to x_0. If u is a concave increasing utility function, then the maximum attainable utility from $x_0 \in \mathbb{R}$ is given by

$$U_0(x_0) \triangleq \sup\left\{E_P[u(z)] \mid z \in X,\ E_{P'}[z] \leq x_0\ \forall P' \in \mathcal{M}\right\}.$$

Analogously, we may define the function $U : X \to \mathbb{R} \cup \{+\infty\}$

$$U(x) \triangleq \sup\left\{E_P[u(z)] \mid z \in X,\ E_{P'}[z] \leq E_{P'}[x]\ \forall P' \in \mathcal{M}\right\},$$

which represents the maximum attainable utility from the claim $x \in X$. Then *the dynamic certainty equivalent* $\mathbb{E}(x)$ *of* $x \in X$ *is defined as the real number* x_0 *satisfying the equation:*

$$U_0(x_0) = U(x).$$

Many results, regarding the existence, the uniqueness, the interpretation and the characterization of $\mathbb{E}(x)$ can be found in [13]. Here we only mention two cases which will provide the examples mentioned above.

First, consider the exponential utility function $u(y) = -e^{-y}$. Then the dynamic certainty equivalent $\mathbb{E}(x)$ can be computed by the formula:

$$\mathbb{E}(x) = \inf_{P' \in \mathcal{M}} \left\{E_{P'}[x] + H(P', P)\right\} - \inf_{P' \in \mathcal{M}} H(P', P),\ x \in X, \tag{4}$$

where H is the *Relative Entropy* defined by:

$$H(P', P) \triangleq \ E_P\left[\frac{dP'}{dP} \ln\left(\frac{dP'}{dP}\right)\right], \quad \text{for } P' \ll P\ .$$

$H(P', P)$ is, more or less, the conjugate U^* of the function U (for details and precise formulation see [13]).

Defining $\rho(x) \triangleq -\mathbb{E}(x)$ we deduce from (4):

$$\rho(x) = \sup_{P' \in \mathcal{M}} \{E_{P'}[-x] - H(P', P)\} + \inf_{P' \in \mathcal{M}} H(P', P) = \sup_{P' \in \mathcal{M}} \{E_{P'}[-x] - F(P')\}, \tag{5}$$

where $F : L^1 \to \mathbb{R} \cup \{+\infty\}$, given by

$$F(x') = E_P[x' \ln(x')] - \inf_{x' \in \mathcal{M}} E_P[x' \ln(x')], \; x' = \frac{dP'}{dP}, \tag{6}$$

is convex and satisfies $\inf_{x' \in \mathcal{M}} F(x') = 0$. Letting $\mathcal{P} \triangleq \mathcal{M} \subseteq \mathcal{Z}$ and comparing (3) and (5) we see that $\pi(x) \triangleq \rho(-x)$ is a representable (convex and not in general sublinear) functional which satisfies the positivity and constancy axioms ($\mathcal{P} \subseteq \mathcal{Z}$ and $\inf_{x' \in \mathcal{P}} F(x') = 0$).

Furthermore, from equations (6) and (5) we see that the functional F is, except for a constant, equal to $H(P', P)$, the conjugate of U.

This suggests that in the representation (3) of a given convex risk measure with $\mathcal{P} \subseteq \mathcal{Z}$, the correction term F could be determined by the preferences of the investor, while the set $\mathcal{P}$ of possible scenarios could be exogenously determined, for example by some regulatory institution or by the market itself, as in the case of the set of martingale measures.

It was also shown, in Bellini and Frittelli [4] and in Delbaen et al. [9], that the problem of maximizing the expected exponential utility, from the terminal wealth and a certain given claim $x \in X$, can be expressed, via duality, as the maximization over the set of martingale measures of the expected value of the claim x, minus an entropic penalty term. This is also the interpretation of the last term in equation (5), except for the algebraic sign of the claim x, that depends on the interpretation of x as a loss or as a profit.

The other case of interest is when the utility function u belongs to the power HARA class or is the logarithmic utility. In these cases, it was shown in [13] that the dynamic certainty equivalent (defined only for $x \in X_+$) is given by

$$\mathbb{E}(x) = \inf_{P' \in \mathcal{M}} \{G_u(P') \cdot E_{P'}[x]\}, \tag{7}$$

where $G_u : \mathcal{M} \to \mathbb{R}$ satisfies

$$\inf_{P' \in \mathcal{M}} G_u(P') = 1 \tag{8}$$

and it depends on the utility u through the conjugate of the function U.

Let $\mathcal{P} \triangleq \left\{x' \in X'_+ : x' = \frac{dP'}{dP} G_u(P'), \text{ for } P' \in \mathcal{M}\right\}$ and define, for $x \in X$, $\rho(x) \triangleq -\mathbb{E}(x)$. Then we have from (7):

$$\rho(x) = \sup_{x' \in \mathcal{P}} \{x'(-x)\}. \tag{9}$$

Comparing (9) and (2) with $F = 0$ on $\mathcal{P}$, we see that $\pi(x) \triangleq \rho(-x)$ is a representable sublinear functional which satisfies the positivity axiom ($\mathcal{P} \subseteq X'_+$), but not the translability axiom g), or equivalently the normality axiom f). Indeed, from (7) and (8) we see that $\pi(x) = -\mathbb{E}(-x)$ only satisfies the normality axiom (f) ii): $\pi(-\mathbf{1}) = -1$, but not also (f) i): $\pi(\mathbf{1}) = 1$. Therefore, $\rho(x) = \pi(-x) = -\mathbb{E}(x)$ is not a coherent risk measure.

From the definition of $\mathbb{E}(x)$ it is not possible to guarantee that $\sup_{P' \in \mathcal{M}} G_u(P') = 1$. This latter condition, equivalent to $\pi(\mathbf{1}) = 1$, together with (8) would imply $G_u(P') \equiv 1$ and henceforth the coherence of ρ. The factor G_u can be seen as a dilation or a squeeze of a coherent risk measure. The same remarks, made for the exponential utility, in relation to the interpretation of F, hold for the factor G_u, since also G_u is determined by the preferences of the investors.

Another example of a convex risk measure, can be found in the recent work of Carr et al. [5], where the notion of an acceptable position is based on a finite number of inequalities determined by a finite number of probability measures and constants, called "floors". From their definition of acceptability it can be deduced that a position x is acceptable if

$$\sup_{i=1,\ldots,m} \{E_{P_i}[-x] + f_i\} \leq 0, \qquad (\circ)$$

where the P_i could be either valuation measures (used for pricing) or stress ones, and the floors f_i are real numbers such that $f_i = 0$, if P_i is a valuation measure, and $f_i \leq 0$, if P_i is a stress one.

We have already seen that if $\pi(\mathbf{0}) = 0$ then we have $\inf_{x' \in \mathcal{P}} F(x') = 0$, in the representation (I) of π. Therefore, by setting $F(P_i) \triangleq -f_i \geq 0$, we notice that ($\circ$) is a particular case of (3) and that the floors in ($\circ$) have the same role as $-F$ in (3).

Finally we recall that there are notable links between coherent risk measures, convex games, distorted probabilities, Choquet integral representation and insurance prices; the interested reader can find the definitions and results about convex games in Delbaen [7], [8] and Schmeidler [18]; about Choquet integration and insurance prices in Wang et al. [19] and in Artzner [1]; about risk measures and insurance premium principles in Landsman & Sherris [16].

Delbaen [8] showed that, if $v : \mathcal{F} \to \mathbb{R}_+$ is a convex game such that $v(\Omega) = 1$ and if $\mathcal{C}(v) \subseteq ba(\Omega, \mathcal{F}, P)$ stands for the core of v, then

$$\rho(x) = \sup_{\mu \in \mathcal{C}(v)} E_\mu[-x] \qquad (10)$$

is a coherent risk measure. Viceversa, such a measure ρ comes from a convex game if and only if it satisfies a comonotonicity axiom.

Furthermore, from this representation, it follows that coherent risk measures are strongly linked with the insurance prices proposed by Wang et al. [19]. It could be interesting to weaken some of the axioms, without loosing the representation property, as we have done for coherent risk measures.

After having submitted the paper, we become aware of the preprint by Follmer and Schied [11] that also discusses the representation property of convex measures of risk and provides many relevant results, some which are quite similar to ours.

2 Representation of Convex Risk Measures

The representation of the functional satisfying assumption (A) is an immediate consequence of duality theory. Recall that the conjugate f^* and the biconjugate f^{**} of a convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ are given by:

$$\begin{aligned} f^* &: X' \to \mathbb{R} \cup \{+\infty\}, \; f^*(x') \triangleq \sup_{x \in X} \{x'(x) - f(x)\}, \\ f^{**} &: X \to \mathbb{R} \cup \{+\infty\}, \; f^{**}(x) \triangleq \sup_{x' \in X'} \{x'(x) - f^*(x')\}. \end{aligned}$$

The following Theorem is the well known 1-1 correspondence between closed convex functions π on X and closed convex function π^* on X'. The proof can be found, for example, in Rockafellar [17] Th.5.

Theorem 6 *If π satisfies assumption (A) then*

$$\pi = \pi^{**}, \qquad \textit{i.e.: } \pi(x) = \sup_{x' \in X'} \{x'(x) - \pi^*(x')\}. \tag{11}$$

Corollary 7 *.*
i) π satisfies (A) iff π is representable.
ii) π satisfies (A) and (b) iff π is representable and $\mathcal{P} \subseteq X'_+$.
iii) π satisfies (A) and (c) iff π is representable and $\inf_{x' \in \mathcal{P}} F(x') = 0$.
iv) π satisfies (A) and (d) iff π is representable, $\mathcal{P} \subseteq \{x' \in X' : x'(\mathbf{1}) = 1\}$ and $\inf_{x' \in \mathcal{P}} F(x') = 0$.
v) π satisfies (A), (b), (d) iff π is representable, $\mathcal{P} \subseteq \mathcal{Z}$ and $\inf_{x' \in \mathcal{P}} F(x') = 0$.
vi) if in assumption (A) the convexity axiom is replaced by the stronger axiom (e) (sublinearity), then in any of the above items i) ii) iii) iv) or v) the representation of π given in (I) holds with $F(x') = 0 \; \forall x' \in \mathcal{P}$.

Proof. As a preliminary observation note that if π is representable then it is finite valued and convex on X and, since $\mathcal{P} \subseteq X'$, lower semi-continuous; so assumption (A) holds true.

i) Theorem 6 implies that

$$-\infty < \pi(x) = \sup_{x' \in X'} \{x'(x) - \pi^*(x')\} = \sup_{x' \in X' : \pi^*(x') < +\infty} \{x'(x) - \pi^*(x')\}, \tag{12}$$

and (I) holds with $F = \pi^*$ and $\mathcal{P} = \{x' \in X' : \pi^*(x') < +\infty\} \neq \emptyset$.

ii) Let π satisfies (A). From $\pi(x) \leq \pi(\mathbf{0}) \; \forall x \in X : x \leq 0$, and from (12) we deduce:

$$x'(x) \leq \pi^*(x') + \pi(\mathbf{0}) \quad \forall x \in X : x \leq 0, \; \forall x' \in X' : \pi^*(x') < +\infty,$$

or, since $\pi^*(x') + \pi(\mathbf{0})$ is finite and independent from x,

$$x'(x) \leq 0 \quad \forall x \in X : x \leq 0,\ \forall x' \in X' : \pi^*(x') < +\infty,$$

and therefore $x' \in X' : \pi^*(x') < +\infty$ implies $x' \in X'_+$. Then condition (I) holds with $F = \pi^*$ and $\mathcal{P} = \left\{x' \in X'_+ : \pi^*(x') < +\infty\right\}$. Viceversa, let $x \in X : x \leq 0$. Then for all $x' \in \mathcal{P} \subseteq X'_+$ we have $x'(x) \leq 0$ and therefore $\pi(x) \triangleq \sup_{x' \in \mathcal{P}} \{x'(x) - F(x')\} \leq \sup_{x' \in \mathcal{P}} \{-F(x')\} \triangleq \pi(\mathbf{0})$.

iii) From (12) and from $\pi(\mathbf{0}) = 0$ we deduce

$$\pi(\mathbf{0}) = \sup\{-\pi^*(x') \mid x' \in X' : \pi^*(x') < +\infty\} = 0.$$

The thesis follows with $F = \pi^*$ and $\mathcal{P} = \{x' \in X' : \pi^*(x') < +\infty\}$.

iv) From (12) and from $\pi(\boldsymbol{\alpha}) = \alpha$, $\forall \alpha \in \mathbb{R}$, we deduce

$$\pi(\boldsymbol{\alpha}) = \sup_{x' \in X' : \pi^*(x') < +\infty} \{\alpha x'(\mathbf{1}) - \pi^*(x')\} = \alpha,\ \forall \alpha \in \mathbb{R}.$$

Therefore

$$\alpha[x'(\mathbf{1}) - 1] \leq \pi^*(x')\ \forall x' \in X' : \pi^*(x') < +\infty,\ \forall \alpha \in \mathbb{R}.$$

Since $\mid \alpha \mid$ may become arbitrarily large, $x' \in X' : \pi^*(x') < +\infty$ implies $x'(\mathbf{1}) = 1$. Condition (I) holds with $F = \pi^*$ and $\mathcal{P} = \{x' \in X' : x'(\mathbf{1}) = 1 \text{ and } \pi^*(x') < +\infty\}$. Since $\pi(\mathbf{0}) = 0$, from iii) we know that $\inf_{x' \in \mathcal{P}} F(x') = 0$. Viceversa, if $\mathcal{P} \subseteq \{x' \in X' : x'(\mathbf{1}) = 1\}$ and $\inf_{x' \in \mathcal{P}} F(x') = 0$ then: $\pi(\boldsymbol{\alpha}) \triangleq \sup_{x' \in \mathcal{P}} \{x'(\boldsymbol{\alpha}) - F(x')\} = \alpha - \inf_{x' \in \mathcal{P}} F(x') = \alpha$.

v) follows as in (ii) and (iv).

vi) The sublinearity of π implies the convexity of π and hence the representation in (12) holds. The sublinearity also implies $\pi(\mathbf{0}) = 0$, and therefore $\pi^*(x') \triangleq \sup_{x \in X} \{x'(x) - \pi(x)\} \geq 0$ for all $x' \in X'$. If $x' \in X'$ and $\bar{x} \in X$ satisfy $[x'(\bar{x}) - \pi(\bar{x})] > 0$ then

$$\pi^*(x') \triangleq \sup_{x \in X} \{x'(x) - \pi(x)\} \geq \sup_{\lambda > 0} \{x'(\lambda\bar{x}) - \pi(\lambda\bar{x})\} = \sup_{\lambda > 0} \{\lambda[x'(\bar{x}) - \pi(\bar{x})]\} = +\infty.$$

Therefore:

$$\pi^*(x') < +\infty \Rightarrow x'(x) - \pi(x) \leq 0\ \forall x \in X \Rightarrow \pi^*(x') \leq 0.$$

and so $F = \pi^* = 0$ on $\mathcal{P} \triangleq \{x' \in X' : \pi^*(x') < +\infty\}$. Viceversa, suppose that π is representable with $F = 0$ and $\mathcal{P} \subseteq X'$, i.e.:

$$\pi(x) = \sup_{x' \in \mathcal{P}} \{x'(x)\} < +\infty\ \forall x \in X.$$

Then π is lower semi-continuous and satisfies (e) (and consequently (A)). If $\mathcal{P} \subseteq X'_+$, then π satisfies also (b). If $\mathcal{P} \subseteq \{x' \in X' : x'(\mathbf{1}) = 1\}$ then π satisfies also (d). If $\mathcal{P} \subseteq \mathcal{Z}$, then π satisfies (b) and (d). ■

Let us recall some simple consequences of the assumption of convexity or sublinearity. We'll assume that $\pi(\mathbf{0}) = 0$, which is quite natural from the financial point of view.

Remark 8 *Assume that $\pi : X \to \mathbb{R}$ satisfies $\pi(\mathbf{0}) = 0$.*

If π is convex then:

h) $\pi(\alpha x) \leq \alpha \pi(x)$, $\forall \alpha \in [0,1]$, $\forall x \in X$;

l) $\pi(\alpha x) \geq \alpha \pi(x)$, $\forall \alpha \in (-\infty, 0] \cup [1 + \infty)$, $\forall x \in X$;

m) $\pi(x - y) \geq -\pi(-x) - \pi(y)$, $\forall x, y \in X$.

If π is convex and positive then:

n) $x \geq 0 \Rightarrow \pi(x) \geq 0$, $\forall x \in X$.

If π is convex and lower semi-continuous then axioms (b) (positivity) and (b′) (monotonicity) are equivalent.

If π is convex and lower semi-continuous then axioms (d) (constancy) and (g) (translability) are equivalent.

Proof. h) For any $\alpha \in [0,1]$ and $x \in X$, we have

$$\pi(\alpha x) = \pi(\alpha x + (1 - \alpha)\mathbf{0}) \leq \alpha \pi(x) + (1 - \alpha)\pi(\mathbf{0}) = \alpha \pi(x).$$

l) For any $x \in X$ we have:

$$0 = \pi(\mathbf{0}) = \pi\left(\frac{1}{2}x + \frac{1}{2}(-x)\right) \leq \frac{1}{2}\pi(x) + \frac{1}{2}\pi(-x),$$

and therefore

$$\pi(x) \geq -\pi(-x). \tag{13}$$

From h) we get, for any $\alpha > 1$,

$$\pi(x) = \pi\left(\frac{1}{\alpha}\alpha x\right) \leq \frac{1}{\alpha}\pi(\alpha x).$$

For any $\alpha \in [-1, 0)$ we get

$$\pi(\alpha x) \geq -\pi(-\alpha x) \geq \alpha \pi(x),$$

where the first inequality comes from (13) and the second one from h), since $(-\alpha) \in (0, 1]$.
If $\alpha < -1$, then $\frac{1}{\alpha} \in (-1, 0)$ and from the previous case we get

$$\pi(x) = \pi\left(\frac{1}{\alpha}\alpha x\right) \geq \frac{1}{\alpha}\pi(\alpha x).$$

m) For any $x, y \in X$, we have

$$\pi\left(-\frac{1}{2}y\right) = \pi\left(-\frac{1}{2}x + \frac{1}{2}(x - y)\right) \leq \frac{1}{2}\left[\pi(-x) + \pi(x - y)\right],$$

$$\pi(x - y) \geq 2\pi\left(-\frac{1}{2}y\right) - \pi(-x) \geq -\pi(y) - \pi(-x),$$

where the last inequality is due to property l).

n) If $x \geq 0$ then $\pi(-x) \leq 0$ and, from (13), $-\pi(x) \leq \pi(-x) \leq 0$.

Obviously (b′) implies (b). To show the converse, from $\sup_{x'\in\mathcal{P}} A(x') - \sup_{x'\in\mathcal{P}} B(x') \leq \sup_{x'\in\mathcal{P}} \{A(x') - B(x')\}$ and the representation given in (I), we get

$$\begin{aligned}\pi(x) - \pi(y) &= \sup_{x'\in\mathcal{P}} \{x'(x) - F(x')\} - \sup_{x'\in\mathcal{P}} \{x'(y) - F(x')\} \\ &\leq \sup_{x'\in\mathcal{P}} \{x'(x-y)\} \leq 0,\end{aligned}$$

since $\mathcal{P} \subseteq X'_+$ (from Corollary 7 ii)) and $x - y \leq 0$.

To prove the last sentence, suppose (d) is satisfied. From (I) and Corollary 7 iv), it follows that, for any $x \in X$ and $\gamma \in \mathbb{R}$,

$$\begin{aligned}\pi(x+\boldsymbol{\gamma}) &= \sup_{x'\in\mathcal{P}\subseteq\{x'\in X':x'(\mathbf{1})=1\}} \{x'(x+\boldsymbol{\gamma}) - F(x')\} \\ &= \sup_{x'\in\mathcal{P}\subseteq\{x'\in X':x'(\mathbf{1})=1\}} \{x'(x) + \boldsymbol{\gamma} - F(x')\} = \pi(x) + \gamma.\end{aligned}$$

Viceversa, if (g) is satisfied, then (d) follows immediately by taking $x = 0$. ■

Remark 9 *If $\pi : X \to \mathbb{R}$ satisfies $\pi(\mathbf{0}) = 0$, which ever two axioms, among convexity (CO) subadditivity (SA) positive homogeneity (PH), hold true, then the other one holds true as well. Note that PH implies $\pi(\mathbf{0}) = 0$, but SA and CO don't. For example, if $\pi : L^\infty \to \mathbb{R}$ is defined by $\pi(x) = E_P[x] + 1$ then $\pi(0) = 1$, SA and CO are satisfied but not PH.*

Proof. SB and PH implies CO: $\pi(\alpha x + (1-\alpha)y) \leq \pi(\alpha x) + \pi((1-\alpha)y) = \alpha\pi(x) + (1-\alpha)\pi(y)$.

PH and CO implies SA: $\frac{1}{2}\pi(x+y) = \pi\left(\frac{1}{2}x + \frac{1}{2}y\right) \leq \frac{1}{2}\pi(x) + \frac{1}{2}\pi(y)$.

SA and CO implies PH: Let $\alpha \geq 0$ and $\alpha = [\alpha] + mant(\alpha)$, where $[\alpha] \in \mathbb{N}$ and $mant(\alpha) \in [0,1)$. From SA, by induction, we have:

$$\pi(nx) \leq n\pi(x), \ \forall n \in \mathbb{N}. \tag{14}$$

Therefore:

$$\begin{aligned}\pi(\alpha x) &= \pi([\alpha]x + mant(\alpha)x) \leq \pi([\alpha]x) + \pi(mant(\alpha)x) \\ &\leq [\alpha]\pi(x) + mant(\alpha)\pi(x) = \alpha\pi(x), \ \forall \alpha \geq 0,\end{aligned}$$

where the last inequality follows from (14) and h). From l) we deduce:

$$\pi(\alpha x) = \alpha\pi(x), \ \forall \alpha \geq 1.$$

Hence, for $\alpha \in (0,1)$,

$$\pi(x) = \pi(\frac{1}{\alpha}\alpha x) = \frac{1}{\alpha}\pi(\alpha x).$$

■

Remark 10 *Assume that $\pi : X \to \mathbb{R}$ is sublinear. Then π is convex, $\pi(\mathbf{0}) = 0$ and:*

p) $\pi(ax) \geq a\pi(x)$, $\forall a \in \mathbb{R}$, $\forall x \in X$;
q) $\pi(x-y) \geq \pi(x) - \pi(y)$, $\forall x, y \in X$;
r) $\pi(x) - \pi(-y) \leq \pi(x+y) \leq \pi(x) + \pi(y)$, $\forall x, y \in X$.

Axioms (d) (constancy), (f) (normality) and (g) (translability) are equivalent. Axioms (b) (positivity) and (b′) (monotonicity) are equivalent.

Proof. Property p) follows from l) and positive homogeneity. The subadditivity of π implies q), since

$$\pi(x) = \pi(x - y + y) \leq \pi(x - y) + \pi(y).$$

r) follows from the subadditivity of π and from q), replacing y with $-y$. The conditions $\pi(\mathbf{1}) = 1$ and $\pi(-\mathbf{1}) = -1$ are equivalent to $\pi(\boldsymbol{\alpha}) = \alpha$, for all $\alpha \in \mathbb{R}$, and so (f)$\Leftrightarrow$(d). If π satisfies (f) then (g) follows from condition r), taking $y = a$. Viceversa, if (g) is satisfied then $\pi(\mathbf{1}) = 1$ and $\pi(-\mathbf{1}) = -1$ follow by taking in (g) $x = \pm\mathbf{1}$ and $a = \mp 1$.

(b′) follows from (b) and q). ■

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