Conditionally Evenly Convex Sets and Evenly Quasi-Convex Maps

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Abstract

Evenly convex sets in a topological vector space are defined as the intersection of a family of open half spaces. We introduce a generalization of this concept in the conditional framework and provide a generalized version of the bipolar theorem. This notion is then applied to obtain the dual representation of conditionally evenly quasi-convex maps.

1 Introduction

A subset $C$ of a topological vector space is evenly convex if it is the intersection of a family of open half spaces, or equivalently, if every $x \notin C$ can be separated from $C$ by a continuous linear functional. Obviously an evenly convex set is necessarily convex. This idea was firstly introduced by Fenchel [Fe52] aimed to determine the largest family of convex sets $C$ for which the polarity $C = C^{00}$ holds true. It is well known that in the framework of incomplete financial markets the Bipolar Theorem is a key ingredient when we represent the super replication price of a contingent claim in terms of the class of martingale measures. Recently evenly convex sets and in particular evenly quasi-concave real valued functions have been considered by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio in the context of Decision Theory [CV09] and Risk Measures [CV10]. Evenly quasiconcavity is the weakest notion that enables, in the static setting, a complete quasi-concave duality, which is a key structural property regarding the dual representation of the behavioral preferences and Risk Measures. Similarly Drapeau and Kupper [DK10] obtained a complete static quasi-convex duality under slightly different conditions of the risk preferences structure that is strictly related to the notion of evenly convexity.

In a conditional framework, as for example when $\mathcal{F}$ is a sigma algebra containing the sigma algebra $\mathcal{G}$ and we deal with $\mathcal{G}$-conditional expectation, $\mathcal{G}$-conditional sublinear expectation, $\mathcal{G}$-conditional risk measure, the analysis of the duality theory is more delicate. We may consider conditional maps $\rho : E \to L^p(\Omega, \mathcal{F}, \mathbb{P})$ defined either on vector spaces (i.e. $E = L^p(\Omega, \mathcal{F}, \mathbb{P})$) or on $L^0$-modules (i.e. $E = L^0_\mathcal{G}(\mathcal{F}) := \{yx \mid y \in L^0(\Omega, \mathcal{G}, \mathbb{P}) \text{ and } x \in L^p(\Omega, \mathcal{F}, \mathbb{P}) \}$).

As described in details by Filipovic, Kupper and Vogelpoth [FKV09], [FKV10] and by Guo [Gu10] the $L^0$-modules approach (see also Section 3 for more details) is a very powerful tool for the analysis of conditional maps and their dual representation.

In this paper we show that in order to achieve a conditional version of the representation of evenly quasi-convex maps a good notion of evenly convexity is crucial. We introduce the concept of a conditionally evenly convex set, which is tailor made for the conditional setting, in a framework that exceeds the module setting alone, so that will be applicable in many different context.

In Section 2 we provide the characterization of evenly convexity (Theorem 11 and Proposition 33) and state the conditional version of the Bipolar Theorem (Theorem 12). Under additional topological assumptions, we show that conditionally convex sets that are closed or open are conditionally evenly convex (see Section 4, Proposition 23). As a consequence, the conditional evenly

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quasiconvexity of a function, i.e. the property that the conditional lower level sets are evenly convex, is a weaker assumption than quasiconvexity and lower (or upper) semicontinuity.

In Section 3 we apply the notion of conditionally evenly convex set to the the dual representation of evenly quasiconvex maps, i.e. conditional maps \( \rho : E \to L^0(\Omega, \mathcal{F}, \mathbb{P}) \) with the property that the conditional lower level sets are evenly convex. We prove in Theorem 17 that an evenly quasiconvex regular map \( \pi : E \to L^0(\mathcal{G}) \) can be represented as

\[
\pi(X) = \sup_{\mu \in \mathcal{L}(E, L^0(\mathcal{G}))} \mathcal{R}(\mu(X), \mu),
\]

where

\[
\mathcal{R}(Y, \mu) := \inf_{\xi \in \mathcal{E}} \{ \pi(\xi) \mid \mu(\xi) \geq Y \}, \quad Y \in L^0(\mathcal{G}),
\]

\( E \) is a topological \( L^0 \)-module and \( \mathcal{L}(E, L^0(\mathcal{G})) \) is the module of continuous \( L^0 \)-linear functionals over \( E \).

The proof of this result is based on a version of the hyperplane separation theorem and not on some approximation or scalarization arguments, as it happened in the vector space setting (see [FM11]). By carefully analyzing the proof one may appreciate many similarities with the original demonstration in the static setting by Penot and Volle [PV90]. One key difference with [PV90], in addition to the conditional setting, is the continuity assumption needed to obtain the representation (1). We work, as in [CV09], with evenly quasiconvex functions, an assumption weaker than quasiconvexity and lower (or upper) semicontinuity. As explained in [FM11] the representation of the type (1) is a cornerstone in order to reach a robust representation of Quasi-convex Risk Measures or Acceptability Indexes.

2 On Conditionally Evenly Convex sets

The probability space \((\Omega, \mathcal{G}, \mathbb{P})\) is fixed throughout this paper. Whenever we will discuss conditional properties we will always make reference, even without explicitly mentioning it in the notations - to conditioning with respect to the sigma algebra \( \mathcal{G} \).

We denote with \( L^0 := L^0(\Omega, \mathcal{G}, \mathbb{P}) \) the space of \( \mathcal{G} \)-measurable random variables that are \( \mathbb{P} \) a.s. finite, whereas by \( L^0 \) the space of extended random variables which may take values in \( \mathbb{R} \cup \{\infty\} \). We remind that all equalities/inequalities among random variables are meant to hold \( \mathbb{P} \)-a.s. As the expected value \( E_\mathbb{P}[\cdot] \) is mostly computed w.r.t. the reference probability \( \mathbb{P} \), we will often omit \( \mathbb{P} \) in the notation. For any \( A \in \mathcal{G} \) the element \( 1_A \in L^0 \) is the random variable a.s. equal to 1 on \( A \) and 0 elsewhere. In general since \((\Omega, \mathcal{G}, \mathbb{P})\) are fixed we will always omit \( \mathbb{P} \). We define \( L^0_+ = \{ X \in L^0 \mid X \geq 0 \} \) and \( L^0_{0+} = \{ X \in L^0 \mid X > 0 \} \).

The essential \((\mathbb{P} \text{ almost surely})\) supremum ess \( \sup_\lambda (X_\lambda) \) of an arbitrary family of random variables \( X_\lambda \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \) will be simply denoted by \( \sup_\lambda (X_\lambda) \), and similarly for the essential infimum (see [FS04] Section A.5 for reference).

Definition 1 (Dual pair)

A dual pair \((E, E', \langle \cdot, \cdot \rangle)\) consists of:

1. \((E, +)\) (resp. \((E', +)\)) is any structure such that the formal sum \( x1_A + y1_{AC} \) belongs to \( E \) (resp. \( x'1_A + y'1_{AC} \in E' \)) for any \( x, y \in E \) (resp. \( x', y' \in E' \)) and \( A \in \mathcal{G} \) with \( \mathbb{P}(A) > 0 \) and there exists an null element \( 0 \in E \) (resp. 0 \in \( E' \)) such that \( x + 0 = x \) for all \( x \in E \) (resp. \( x' + 0 = x' \) for all \( x' \in E' \)).

2. A map \( \langle \cdot, \cdot \rangle : E \times E' \to L^0 \) such that

\[
\langle x1_A + y1_{AC}, x' \rangle = \langle x, x' \rangle 1_A + \langle y, x' \rangle 1_{AC}
\]

\[
\langle x, x'1_A + y'1_{AC} \rangle = \langle x, x' \rangle 1_A + \langle x, y' \rangle 1_{AC}
\]

\( \langle 0, x' \rangle = 0 \) and \( \langle x, 0 \rangle = 0 \)
for every $A \in \mathcal{G}$, $\mathbb{P}(A) > 0$ and $x, y \in E$, $x', y' \in E'$.

Clearly in many applications $E$ will be a class of random variables (as vector lattices, or $L^0$-modules as in the Examples 3 and 25) and $E'$ is a selection of conditional maps, for example conditional expectations, sublinear conditional expectations, conditional risk measures.

We recall from [FKV09] an important type of concatenation:

**Definition 2 (Countable Concatenation Hull).**

(CSet) A subset $C \subset E$ has the countable concatenation property if for every countable partition $\{A_n\}_n \subseteq \mathcal{G}$ and for every countable collection of elements $\{x_n\}_n \subset C$ we have $\sum_n 1_{A_n} x_n \in C$.

Given $C \subset E$, we denote by $C^{cc}$ the countable concatenation hull of $C$, namely the smallest set $C^{cc} \supseteq C$ which satisfies (CSet):

$$C^{cc} = \left\{ \sum_n 1_{A_n} x_n \mid x_n \in C, \{A_n\}_n \subseteq \mathcal{G} \text{ is a partition of } \Omega \right\}.$$

These definitions can be plainly adapted to subsets of $E'$.

The action of an element $\xi' = \sum_m 1_{B_m} x'_m$ in $(E')^{cc}$ over $\xi = \sum_n 1_{A_n} x_n \in E^{cc}$ is defined as

$$\langle \xi, \xi' \rangle = \left\langle \sum_n 1_{A_n} x_n, \sum_m 1_{B_m} x'_m \right\rangle = \sum_n \sum_m (x_n, x'_m) 1_{A_n \cap B_m},$$

and does not depend on the representation of $\xi' \in (E')^{cc}$ and $\xi \in C^{cc}$.

**Example 3** Let $\mathcal{F}$ be a sigma algebra containing $\mathcal{G}$. Consider the vector space $E := L^p(\mathcal{F}) := L^p(\Omega, \mathcal{F}, \mathbb{P})$, for $p \geq 1$. If we compute the countable concatenation hull of $L^p(\mathcal{F})$ we obtain exactly the $L^0$-module

$$L^0_{\mathcal{G}}(\mathcal{F}) := \{yx \mid y \in L^0(\mathcal{G}) \text{ and } x \in L^p(\mathcal{F})\}$$

as introduced in [FKV09] and [FKV10] (see Example 25 for more details).

Similarly, the class of conditional expectations $\mathcal{E} = \{E[\cdot | Z] | Z \in L^q(\Omega, \mathcal{F}, \mathbb{P})\}$ and $\frac{1}{p} + \frac{1}{q} = 1$ can be identified with the space $L^q(\mathcal{F})$. Hence the countable concatenation hull $\mathcal{E}^{cc}$ will be exactly $L^0_{\mathcal{G}}(\mathcal{F})$, the dual $L^0$-module of $L^0_{\mathcal{G}}(\mathcal{F})$.

If $E$ (or $E'$) does not fulfill (CSet) we can always embed the theory in its concatenation hull and henceforth we make the following:

**Assumption:** In the sequel of this paper we always suppose that both $E$ and $E'$ satisfies (CSet).

We recall that a subset $C$ of a locally convex topological vector space $V$ is evenly convex if it is the intersection of a family of open half spaces, or equivalently, if every $x \notin C$ can be separated from $C$ by a continuous real valued linear functional. As the intersection of an empty family of half spaces is the entire space $V$, the whole space $V$ itself is evenly convex.

However, in order to introduce the concept of conditional evenly convex set (with respect to $\mathcal{G}$) we need to take care of the fact that the set $C$ may present some components which degenerate to the entire $E$. Basically it might occur that for some $A \in \mathcal{G}$

$$C 1_A = E 1_A,$$

i.e., for each $x \in E$ there exists $\xi \in C$ such that $\xi 1_A = x 1_A$. In this case there are no chances of finding an $x \in E$ satisfying $1_A C \cap 1_A \{x\} = \emptyset$ and consequently no conditional separation may occur. It is clear that the evenly convexity property of a set $C$ is meaningful only on the set where $C$ does not coincide with the entire $E$. Thus we need to determine the maximal $\mathcal{G}$-measurable set on which $C$ reduces to $E$. To this end, we set the following notation that will be employed many times.
Notation 4 Fix a set $C \subseteq E$. As the class $\mathcal{A}(C) := \{ A \in \mathcal{G} \mid C 1_A = E 1_A \}$ is closed with respect to countable union, we denote with $A_C$ the $\mathcal{G}$-measurable maximal element of the class $\mathcal{A}(C)$ and with $D_C$ the (P-a.s. unique) complement of $A_C$ (see also the Remark 30). Hence $C 1_{A_C} = E 1_{A_C}$.

We now give the formal definition of conditionally evenly convex set in terms of intersections of hyperplanes in the same spirit of [Fe52].

**Definition 5** A set $C \subseteq E$ is conditionally evenly convex if there exist $\mathcal{L} \subseteq E'$ (in general non-unique and empty if $C = E$) such that

$$C = \bigcap_{x' \in \mathcal{L}} \{ x \in E \mid \langle x, x' \rangle < Y_{x'} \text{ on } D_C \} \text{ for some } Y_{x'} \in L^0. \quad (3)$$

**Remark 6** Notice that for any arbitrary $D \in \mathcal{G}$, $\mathcal{L} \subseteq E'$ the set

$$C = \bigcap_{x' \in \mathcal{L}} \{ x \in E \mid \langle x, x' \rangle < Y_{x'} \text{ on } D \} \text{ for some } Y_{x'} \in L^0$$

is evenly convex, even though in general $D_C \subseteq D$.

**Remark 7** We observe that since $E$ satisfies (CSet) then automatically any conditionally evenly convex set satisfies (CSet). As a consequence there might exist a set $C$ which fails to be conditionally evenly convex, since does not satisfy (CSet), but $C^{cc}$ is conditionally evenly convex. Consider for instance $E = L^1_0(F), E' = L^\infty_0(F)$, endowed with the pairing $\langle x, x' \rangle = E[x x'|\mathcal{G}]$. Fix $x' \in L^\infty_0(F), Y \in L^0(\mathcal{G})$ and the set

$$C = \{ x \in L^1_0(F) \mid E[x x'|\mathcal{G}] < Y \}.$$

Clearly $C$ is not conditionally evenly convex since $C \subsetneq C^{cc}$; on the other hand

$$C^{cc} = \{ x \in L^1_0(F) \mid E[x x'|\mathcal{G}] < Y \}$$

which is by definition evenly convex.

**Remark 8** Recall that a set $C \subseteq E$ is $L^0$-convex if $\Lambda x + (1 - \Lambda)y \in C$ for any $x, y \in C$ and $\Lambda \in L^0$ with $0 \leq \Lambda \leq 1$.

Suppose that all the elements $x' \in E'$ satisfy:

$$\langle \Lambda x + (1 - \Lambda)y, x' \rangle \leq \Lambda \langle x, x' \rangle + (1 - \Lambda) \langle y, x' \rangle, \text{ for all } x, y \in E, \Lambda \in L^0: 0 \leq \Lambda \leq 1.$$

If $E$ is $L^0-$convex then every conditionally evenly convex set is also $L^0-$convex.

In order to separate one point $x \in E$ from a set $C \subseteq E$ in a conditional way we need the following definition:

**Definition 9** For $x \in E$ and a subset $C$ of $E$, we say that $x$ is outside $C$ if $1_A \{ x \} \cap 1_A C = \emptyset$ for every $A \in \mathcal{G}$ with $A \subseteq D_C$ and $\mathbb{P}(A) > 0$.

This is of course a much stronger requirement than $x \notin C$.

**Definition 10** For $C \subseteq E$ we define the polar and bipolar sets as follows

$$C^o := \{ x' \in E' \mid \langle x, x' \rangle < 1 \text{ on } D_C \text{ for all } x \in C \},$$

$$C^{cc} := \{ x \in E \mid \langle x, x' \rangle < 1 \text{ on } D_C \text{ for all } x' \in C^o \} = \bigcap_{x' \in C^o} \{ x \in E \mid \langle x, x' \rangle < 1 \text{ on } D_C \}.$$
We now state the main results of this note about the characterization of evenly convex sets and the Bipolar Theorem. Their proofs are postponed to the Section 4.

**Theorem 11** Let \((E, E', \langle \cdot, \cdot \rangle)\) be a dual pairing introduced in Definition 1 and let \(C \subseteq E\). The following statements are equivalent:

1. \(C\) is conditionally evenly convex.
2. \(C\) satisfies (CSet) and for every \(x\) outside \(C\) there exists \(x' \in E'\) such that 
   \[ \langle \xi, x' \rangle < \langle x, x' \rangle \text{ on } D_C, \forall \xi \in C. \]

**Theorem 12 (Bipolar Theorem)** Let \((E, E', \langle \cdot, \cdot \rangle)\) be a dual pairing introduced in Definition 1 and assume in addition that the pairing \(\langle \cdot, \cdot \rangle\) is \(L^0\)-linear in the first component i.e.

\[ \langle \alpha x + \beta y, x' \rangle = \alpha \langle x, x' \rangle + \beta \langle x, x' \rangle \]

for every \(x' \in E', x, y \in E, \alpha, \beta \in L^0\). For any \(C \subseteq E\) such that \(0 \in C\) we have:

1. \(C^\circ = \{ x' \in E' \mid \langle x, x' \rangle < 1 \text{ on } D_C \text{ for all } x \in C \}\)
2. The bipolar \(C^{\circ\circ}\) is a conditionally evenly convex set containing \(C\).
3. The set \(C\) is conditionally evenly convex if and only if \(C = C^{\circ\circ}\).

Suppose that the set \(C \subseteq E\) is a \(L^0\)-cone, i.e. \(\alpha x \in C\) for every \(x \in C\) and \(\alpha \in L^0_{++}\). In this case, it is immediate to verify that the polar and bipolar can be rewritten as:

\[ C^\circ = \{ x' \in E' \mid \langle x, x' \rangle \leq 0 \text{ on } D_C \text{ for all } x \in C \} \]
\[ C^{\circ\circ} = \{ x \in E \mid \langle x, x' \rangle \leq 0 \text{ on } D_C \text{ for all } x' \in C^\circ \}. \] (4)

### 3 On Conditionally Evenly Quasi-Convex maps

Here we state the dual representation of conditional evenly quasiconvex maps of the Penot-Volle type which extends the results obtained in [FM11] for topological vector spaces. We work in the general setting outlined in Section 2. The additional basic property that is needed is regularity.

**Definition 13** A map \(\pi : E \to \bar{L}^0\) is (REG) regular if for every \(x_1, x_2 \in E\) and \(A \in \mathcal{G}\),

\[ \pi(x_1 1_A + x_2 1_{A^c}) = \pi(x_1) 1_A + \pi(x_2) 1_{A^c}. \]

**Remark 14** (On REG) It is well known that (REG) is equivalent to:

\[ \pi(x 1_A) 1_A = \pi(x) 1_A, \forall A \in \mathcal{G}, \forall x \in E. \]

Under the countable concatenation property it is even true that (REG) is equivalent to countably regularity, i.e.

\[ \pi \left( \sum_{i=1}^{\infty} x_i 1_{A_i} \right) = \sum_{i=1}^{\infty} \pi(x_i) 1_{A_i} \text{ on } \bigcup_{i=1}^{\infty} A_i \]

if \(x_i \in E\) and \(\{A_i\}\) is a sequence of disjoint \(\mathcal{G}\) measurable sets. Indeed \(x := \sum_{i=1}^{\infty} x_i 1_{A_i} \in E\) and \(\sum_{i=1}^{\infty} \pi(x_i) 1_{A_i} \in \bar{L}^0\). (REG) then implies \(\pi(x) 1_{A_i} = \pi(x 1_{A_i}) 1_{A_i} = \pi(x 1_{A_i}) 1_{A_i} = \pi(x) 1_{A_i}. \)
Let $\pi : E \to \bar{L}^0$ be (REG). There might exist a set $A \in \mathcal{G}$ on which the map $\pi$ is infinite, in the sense that $\pi(\xi)1_A = +\infty 1_A$ for every $\xi \in E$. For this reason we introduce
\[ \mathcal{M} := \{ A \in \mathcal{G} \mid \pi(\xi)1_A = +\infty 1_A \forall \xi \in E \}. \]
Applying Lemma 36 in Appendix with $F := \{ \pi(\xi) \mid \xi \in E \}$ and $Y_0 = +\infty$ we can deduce the existence of two maximal sets $T_\pi \in \mathcal{G}$ and $\Upsilon_\pi \in \mathcal{G}$ for which $P(T_\pi \cap \Upsilon_\pi) = 0$, $P(T_\pi \cup \Upsilon_\pi) = 1$ and
\begin{align}
\pi(\xi) &= +\infty \quad \text{on } \Upsilon_\pi \quad \text{for every } \xi \in E, \\
\pi(\zeta) &< +\infty \quad \text{on } T_\pi \quad \text{for some } \zeta \in E. \quad (5)
\end{align}

**Definition 15** A map $\pi : E \to \bar{L}^0(\mathcal{G})$ is

(QCO) **conditionally quasiconvex** if $U_Y = \{ \xi \in E \mid \pi(\xi)1_{T_\pi} \leq Y \}$ are $L^0$-convex (according to Remark 8) for every $Y \in \bar{L}^0(\mathcal{G})$.

(EQC) **conditionally evenly quasiconvex** if $U_Y = \{ \xi \in E \mid \pi(\xi)1_{T_\pi} \leq Y \}$ are conditionally evenly convex for every $Y \in \bar{L}^0(\mathcal{G})$.

**Remark 16** For $\pi : E \to \bar{L}^0(\mathcal{G})$ the quasiconvexity of $\pi$ is equivalent to the condition
\[ \pi(\Lambda x_1 + (1 - \Lambda)x_2) \leq \pi(x_1) \vee \pi(x_2), \quad (6) \]
for every $x_1, x_2 \in E$, $\Lambda \in L^0(\mathcal{G})$ and $0 \leq \Lambda \leq 1$. In this case the sets $\{ \xi \in E \mid \pi(\xi)1_D < Y \}$ are $L^0(\mathcal{G})$-convex for every $Y \in \bar{L}^0(\mathcal{G})$ and $D \in \mathcal{G}$ (This follows immediately from (6)).
Moreover under the further structural property of Remark 8 we have that (EQC) implies (QCO).
We will see in the $L^0$-modules framework that if the map $\pi$ is either lower semicontinuous or upper semicontinuous then the reverse implication holds true (see Proposition 23, Corollary 26 and Proposition 27).

We now state the main result of this Section.

**Theorem 17** Let $(E, E', (\cdot, \cdot))$ be a dual pairing introduced in Definition 1. If $\pi : E \to \bar{L}^0(\mathcal{G})$ is (REG) and (EQC) then
\[ \pi(x) = \sup_{x' \in E'} \mathcal{R}((x, x'), x'), \quad (7) \]
where for $Y \in \bar{L}^0(\mathcal{G})$ and $x'$,
\[ \mathcal{R}(Y, x') := \inf_{\xi \in E} \{ \pi(\xi) \mid (\xi, x') \geq Y \}. \quad (8) \]

4 **Conditional Evenly convexity in $L^0$-modules**

This section is inspired by the contribution given to the theory of $L^0$-modules by Filipovic et al. [FKV09] on one hand and on the other to the extended research provided by Guo from 1992 until today (see the references in [Gu10]).

The following Proposition 23 shows that the definition of a conditionally evenly convex set is the appropriate generalization, in the context of topological $L^0$ module, of the notion of an evenly convex subset of a topological vector space, as in both setting convex (resp. $L^0$-convex) sets that are either closed or open are evenly (resp. conditionally evenly) convex. This is a key result that allows to show that the assumption (EQC) is the weakest that allows to reach a dual representation of the map $\pi$.

We will consider $L^0$, with the usual operations among random variables, as a partially ordered ring and we will always assume in the sequel that $\tau_0$ is a topology on $L^0$ such that $(L^0, \tau_0)$ is a topological ring. We do not require that $\tau_0$ is a linear topology on $L^0$ (so that $(L^0, \tau_0)$ may not be a topological vector space) nor that $\tau_0$ is locally convex.
Definition 18 (Topological $L^0$-module) We say that $(E, \tau)$ is a topological $L^0$-module if $E$ is a $L^0$-module and $\tau$ is a topology on $E$ such that the module operation
\[(i) \quad (E, \tau) \times (E, \tau) \to (E, \tau), \quad (x_1, x_2) \mapsto x_1 + x_2, \]
\[(ii) \quad (L^0, \tau_0) \times (E, \tau) \to (E, \tau), \quad (\gamma, x_2) \mapsto \gamma x_2 \]
are continuous w.r.t. the corresponding product topology.

Definition 19 (Duality for $L^0$-modules) For a topological $L^0$-module $(E, \tau)$, we denote
\[E^* := \{x^*: (E, \tau) \to (L^0, \tau_0) \mid x^* \text{ is a continuous module homomorphism}\}. \tag{9} \]
It is easy to check that $(E, E^*, \langle \cdot, \cdot \rangle)$ is a dual pair, where the pairing is given by $\langle x, x^* \rangle = x^*(x)$. Every $x^* \in E^*$ is $L^0$-linear in the following sense: for all $\alpha, \beta \in L^0$ and $x_1, x_2 \in E$
\[x^*(\alpha x_1 + \beta x_2) = \alpha x^*(x_1) + \beta x^*(x_2). \]
In particular, $x^*(x_11_A + x_21_{A^c}) = x^*(x_1)1_A + x^*(x_2)1_{A^c}$.

Definition 20 A map $\| \cdot \|: E \to L^0_+$ is a $L^0$-seminorm on $E$ if
\[(i) \quad \|\gamma x\| = |\gamma|\|x\| \quad \text{for all } \gamma \in L^0 \quad \text{and } x \in E, \]
\[(ii) \quad \|x_1 + x_2\| \leq \|x_1\| + \|x_2\| \quad \text{for all } x_1, x_2 \in E. \]

The $L^0$-seminorm $\| \cdot \|$ becomes a $L^0$-norm if in addition
\[(iii) \quad \|x\| = 0 \implies x = 0. \]
We will consider families of $L^0$-seminorms $\mathcal{Z}$ satisfying in addition the property:
\[\sup \{\|x\| \mid \|x\| \in \mathcal{Z}\} = 0 \text{ iff } x = 0, \tag{10} \]

As clearly pointed out in [Gu10], one family $\mathcal{Z}$ of $L^0$-seminorms on $E$ may induce on $E$ more than one topology $\tau$ such that $\{x_\alpha\}$ converges to $x$ in $(E, \tau)$ iff $\|x_\alpha - x\|$ converges to 0 in $(L^0, \tau_0)$ for each $\| \cdot \| \in \mathcal{Z}$. Indeed, also the topology $\tau_0$ on $L^0$ play a role in the convergence.

Definition 21 ($L^0$-module associated to $\mathcal{Z}$) We say that $(E, \mathcal{Z}, \tau)$ is a $L^0$-module associated to $\mathcal{Z}$ if:
1. $\mathcal{Z}$ is a family of $L^0$-seminorms satisfying $(10)$,
2. $(E, \tau)$ is a topological $L^0$-module,
3. A net $\{x_\alpha\}$ converges to $x$ in $(E, \tau)$ iff $\|x_\alpha - x\|$ converges to 0 in $(L^0, \tau_0)$ for each $\| \cdot \| \in \mathcal{Z}$.

Remark 2.2 in [Gu10] shows that any random locally convex module over $\mathbb{R}$ with base $(\Omega, \mathcal{G}, \mathbb{P})$, according to Definition 2.1 [Gu10], is a $L^0$-module $(E, \mathcal{Z}, \tau)$ associated to a family $\mathcal{Z}$ of $L^0$-seminorms, according to the previous definition.

Proposition 23 holds if the topological structure of $(E, \mathcal{Z}, \tau)$ allows for appropriate separation theorems. We now introduce two assumptions that are tailor made for the statements in Proposition 23, but in the following subsection we provide interesting and general examples of $L^0$-module associated to $\mathcal{Z}$ that fulfill these assumptions.

Separation Assumptions Let $E$ be a topological $L^0$-module, let $E^*$ be defined in $(9)$ and let $C_0 \subseteq E$ be nonempty, $L^0$-convex and satisfy (CSet).

S-Open If $C_0$ is also open and $\{x\}1_A \cap C_01_A = \emptyset$ for every $A \in \mathcal{G}$ s.t. $P(A) > 0$, then there exists $x^* \in E^*$ s.t. $x^*(x) > x^*(\xi) \quad \forall \xi \in C_0$.

S-Closed If $C_0$ is also closed and $\{x\}1_A \cap C_01_A = \emptyset$ for every $A \in \mathcal{G}$ s.t. $P(A) > 0$, then there exists $x^* \in E^*$ s.t. $x^*(x) > x^*(\xi) \quad \forall \xi \in C_0$. 

7
Lemma 22.

1. Let $E$ be a topological $L^0$-module. If $C, E_i \subseteq E$, $i = 1, 2$, are open and non-empty and $A \in G$, then the set $C_1A + C_2A^\circ$ is open.

2. Let $(E, Z, \tau)$ be $L^0$-module associated to $Z$. Then for any net $\{x_\alpha\} \subseteq E$, $x \in E$, $\eta \in E$ and $A \in G$

$$\xi_\alpha \xrightarrow{\tau} \xi \implies (\xi_\alpha 1_A + \eta 1_{A^\circ}) \xrightarrow{\tau} (\xi 1_A + \eta 1_{A^\circ}).$$

Proof. 1. To show this claim let $x := x_1 1_A + x_2 1_{A^\circ}$ with $x_i \in C_i$ and let $U_0$ be a neighborhood of 0 satisfying $x_1 + U_0 \subseteq C_i$. Then the set $U := (x_1 + U_0) 1_A + (x_2 + U_0) 1_{A^\circ} = x + U_0 1_A + U_0 1_{A^\circ}$ is contained in $C_1A + C_2A^\circ$ and it is a neighborhood of $x$, since $U_0 1_A + U_0 1_{A^\circ}$ contains $U_0$ and is therefore a neighborhood of 0.

2. Observe that a seminorm satisfies $\|1_A (\xi_\alpha - \xi)\| = 1_A \|\xi_\alpha - \xi\| \leq \|\xi_\alpha - \xi\|$ and therefore, by condition 3. in Definition 21 the claim follows. In particular, $\xi_\alpha \xrightarrow{\tau} \xi \implies (\xi_\alpha 1_A + \eta 1_{A^\circ}) \xrightarrow{\tau} (\xi 1_A + \eta 1_{A^\circ})$.

Proposition 23 Let $(E, Z, \tau)$ be $L^0$-module associated to $Z$ and suppose that $C \subseteq E$ satisfies (CSet).

1. Suppose that the strictly positive cone $L^0_+$ is $\tau_0$-open and that there exist $x'_0 \in E^*$ and $x_0 \in E$ such that $x'_0(x_0) > 0$. Under Assumption S-Open, if $C$ is open and $L^0_+$-convex then $C$ is conditionally evenly convex.

2. Under Assumption S-Closed, if $C$ is closed and $L^0_+$-convex then it is conditionally evenly convex.

Proof. 1. Let $C \subseteq E$ be open, $L^0_+$-convex, $C \neq \emptyset$ and let $A_C \in G$ be the maximal set given in the Notation 4, being $D_C$ its complement. Suppose that $x$ is outside $C$, i.e. $x \in E$ satisfies $\{x\} 1_A \cap C 1_A = \emptyset$ for every $A \in G$, $A \subseteq D_C$, $P(A) > 0$. Define the $L^0_+$-convex set

$$E := \{x \in E | x'_0(x) > x'_0(x_0)\} = (x'_0)^{-1}(x'_0(x) + L^0_+)$$

and notice that $\{x\} 1_A \cap E 1_A = \emptyset$ for every $A \in G$. As $L^0_{++}$ is $\tau_0$-open, $E$ is open in $E$. As $x'_0(x_0) > 0$, then $(x + x_0) \in E$ and $E$ is non-empty.

Then the set $C_0 = C 1_{D_C} + E 1_{A_C}$ is $L^0_+$-convex, open (by Lemma 22) and satisfies $\{x\} 1_A \cap C_0 1_A = \emptyset$ for every $A \in G$ s.t. $P(A) > 0$. Assumption S-Open guarantees the existence of $x^* \in E^*$ s.t. $x^*(x) > x^*(x_0) \forall \xi \in C_0$, which implies $x^*(x) > x^*(x_0)$ on $D_C$, $\forall \xi \in C$. Hence, by Theorem 11, $C$ is conditionally evenly convex.

2. Let $C \subseteq E$ be closed, $L^0_+$-convex, $C \neq \emptyset$ and suppose that $x \in E$ satisfies $\{x\} 1_A \cap C 1_A = \emptyset$ for every $A \in G$, $A \subseteq D_C$, $P(A) > 0$. Let $C_0 = C 1_{D_C} + \{x + \varepsilon\} 1_{A_C}$ where $\varepsilon \in L^0_{++}$. Clearly $C_0$ is $L^0_+$-convex. In order to prove that $C_0$ is closed consider any net $\xi_\alpha \xrightarrow{\tau} \xi$, $\{\xi_\alpha\} \subseteq C_0$. Then $\xi_\alpha = Z_\alpha 1_{D_C} + (x_\alpha + \varepsilon) 1_{A_C}$, with $Z_\alpha \in C$, and $(x_\alpha + \varepsilon) 1_{A_C} = \xi_1 1_{A_C}$. Take any $\eta \in C$. As $C$ is $L^0_+$-convex, $\xi_\alpha 1_{D_C} + \eta 1_{A_C} = Z_\alpha 1_{D_C} + \eta 1_{A_C} \in C$ and, by Lemma 22, $\xi_\alpha 1_{D_C} + \eta 1_{A_C} \xrightarrow{\tau} \xi 1_{D_C} + \eta 1_{A_C} = Z 1_{D_C} + \eta 1_{A_C} = Z \in C$, as $C$ is closed. Therefore, $\xi = Z 1_{D_C} + (x + \varepsilon) 1_{A_C} \in C_0$. Since $C_0$ is closed, $L^0_+$-convex and $\{x\} 1_A \cap C_0 1_A = \emptyset$ for every $A \in G$, assumption S-Closed guarantees the existence of $x^* \in E^*$ s.t. $x^*(x) > x^*(x_0) \forall \xi \in C_0$, which implies $x^*(x) > x^*(x_0)$ on $D_C$, $\forall \xi \in C$. Hence, by Theorem 11, $C$ is conditionally evenly convex.

Proposition 24 Let $(E, Z, \tau)$ and $E^*$ be respectively as in definitions 19 and 21, and let $\tau_0$ be a topology on $L^0_+$ such that the positive cone $L^0_+$ is closed. Then any conditionally evenly convex $L^0_+$-cone containing the origin is closed.

Proof. From (20) and the bipolar Theorem 12 we know that

$$C = C^{++} = \bigcap_{x \in C^*} \{x \in E | \langle x, x' \rangle \leq 0 \text{ on } D_C\}.$$
We only need to prove that $S_{Y^*} = \{ x \in E \mid \langle x, x' \rangle \leq 0 \text{ on } D_C \}$ is closed for any $x' \in C^*$. Let $x_\alpha \in S_{Y^*}$ be a net such that $x_\alpha \to x$. Since $x' \in E^*$ is continuous we have $Y_\alpha \coloneqq \langle x_\alpha, x' \rangle \to Y \coloneqq \langle x, x' \rangle$, with $Y_\alpha \leq 0$ on $D_C$. We surely have that $x_\alpha 1_{D_C} \to x 1_{D_C}$ which implies that $Y_\alpha 1_{D_C} \to Y 1_{D_C}$. Since $-Y_\alpha 1_{D_C} \in L^0_+$ for every $\alpha$ and $L^0_+$ is closed we conclude that $Y = \langle x, x' \rangle \leq 0$ on $D_C$. \hfill \blacksquare

4.1 On $L^0$-module associated to $Z$ satisfying S-Open and S-Closed

Based on the results of Guo [Gu10] and Filipovic et al. [FKV09], we show that a family of seminorms on $E$ may induce more than one topology on the $L^0$-module $E$ and that these topologies satisfy the assumptions S-Open and S-Closed.

These examples are quite general and therefore supports the claim made in the previous section about the relevance of conditional evenly convex sets. A concrete and significant example, already introduced in Section 2, is provided next. To help the reader in finding further details we use the same notations and definitions given in [FKV09] and [Gu10].

**Example 25** ([FKV10]) Let $\mathcal{F}$ be a sigma algebra containing in $\mathcal{G}$ and consider the generalized conditional expectation of $\mathcal{F}$-measurable non negative random variables: $E[\cdot | \mathcal{G}] : L^0_+ (\Omega, \mathcal{F}, \mathbb{P}) \to L^0_+ (\Omega, \mathcal{G}, \mathbb{P})$

$$E[x|\mathcal{G}] := \lim_{n \to +\infty} E[x \wedge n|\mathcal{G}] .$$

Let $p \in [1, \infty]$ and consider the $L^0$-module defined as

$$L^p_0(\mathcal{G}) (\mathcal{F}) := \{ x \in L^0 (\Omega, \mathcal{F}, \mathbb{P}) \mid \| x \|_p \in L^0 (\Omega, \mathcal{G}, \mathbb{P}) \}$$

where $\| \cdot \|_p$ is the $L^0$-norm assigned by

$$\| x \|_p := \left\{ \begin{array}{ll} \inf \{ y \in L^0 (\mathcal{G}) \mid y \geq |x| \} & \text{if } p < +\infty \\ |x|^{\frac{1}{p}} & \text{if } p = +\infty \end{array} \right. \quad (11)$$

Then $L^p_0 (\mathcal{G}) (\mathcal{F})$ becomes a $L^0$-normed module associated to the norm $\| \cdot \|_p$ having the product structure:

$$L^p_0 (\mathcal{G}) (\mathcal{F}) = L^0 (\mathcal{G}) L^p (\mathcal{F}) = \{ xy \mid y \in L^0 (\mathcal{G}), \ x \in L^p (\mathcal{F}) \} .$$

For $p < \infty$, any $L^0$-linear continuous functional $\mu : L^p_0 (\mathcal{G}) (\mathcal{F}) \to L^0$ can be identified with a random variable $z \in L^0_0 (\mathcal{G}) (\mathcal{F})$ as $\mu (\cdot) = E[z \cdot | \mathcal{G}]$ where $\frac{1}{p} + \frac{1}{q} = 1$. So we can identify $E^*$ with $L^q_0 (\mathcal{G}) (\mathcal{F})$.

The two different topologies on $E$ depend on which topology is selected on $L^0$: either the uniform topology or the topology of convergence in probability. The two topologies on $E$ will collapse to the same one whenever $\mathcal{G} = \sigma (\emptyset)$ is the trivial sigma algebra, but in general present different structural properties.

We set:

$$\| x \|_S := \sup \{ \| x \| \mid \| x \| \in S \}$$

for any finite subfamily $S \subset Z$ of $L^0_0$-seminorms. Recall from the assumption given in equation (10) that $\| x \|_S = 0$ if and only if $x = 0$.

**The uniform topology $\tau_\circ$** [FKV09]. In this case, $L^0$ is equipped with the following uniform topology. For every $\varepsilon \in L^0_+$, the ball $B_\varepsilon := \{ Y \in L^0 \mid | Y | \leq \varepsilon \}$ centered in 0 in $L^0$ gives the neighborhood basis of 0. A set $V \subset L^0$ is a neighborhood of $Y \in L^0$ if there exists $\varepsilon \in L^0_+$ such that $Y + B_\varepsilon \subset V$. A set $V$ is open if it is a neighborhood of all $Y \in V$. A net converges in this topology, namely $Y_N \to Y$ if for every $\varepsilon \in L^0_+$ there exists $N$ such that $|Y_N - Y| < \varepsilon$ for every $N > N$. In this case the space $(L^0, \| \cdot \|)$ loses the property of being a topological vector space. In this topology the positive cone $L^0_+$ is closed and the strictly positive cone $L^0_+ (\cdot)$ is open.

Under the assumptions that there exists an $x \in E$ such that $x 1_A \neq 0$ for every $A \in \mathcal{G}$ and that the topology $\tau$ on $E$ is Hausdorff, Theorem 2.8 in [FKV09] guarantees the existence of $x_0 \in E$.
A family \( Z \) of \( L^0 \)-seminorms on \( E \) induces a topology on \( E \) in the following way. For any finite \( S \subset Z \) and \( \varepsilon \in L^0_+ \) define
\[
    U_{S,\varepsilon} := \{ x \in E \mid \|x\|_S \leq \varepsilon \}
\]
\[
    U := \{ U_{S,\varepsilon} \mid S \subset Z \text{ finite and } \varepsilon \in L^0_+ \}.
\]
\( U \) gives a convex neighborhood base of 0 and it induces a topology on \( E \) denoted by \( \tau_c \). We have the following properties:

1. \((E, Z, \tau_c)\) is a \((L^0, |\cdot|)\)-module associated to \( Z \), which is also a locally convex topological \( L^0 \)-module (see Proposition 2.7 [Gu10]),
2. \((E, Z, \tau_c)\) satisfies S-Open and S-Closed (see Theorems 2.6 and 2.8 [FKV09]),
3. Any topological \((L^0, |\cdot|)\) module \((E, \tau)\) is locally convex if and only if \( \tau \) is induced by a family of \( L^0 \)-seminorms, i.e. \( \tau \equiv \tau_c \), (see Theorem 2.4 [FKV09]).

A probabilistic topology \( \tau_{e,\lambda} \) [Gu10] The second topology on the \( L^0 \)-module \( E \) is a topology of a more probabilistic nature and originated in the theory of probabilistic metric spaces (see [SS83]).

Here \( L^0 \) is endowed with the topology \( \tau_{e,\lambda} \) of convergence in probability and so the positive cone \( L^0_+ \) is \( \tau_{e,\lambda} \)-closed. According to [Gu10], for every \( e, \lambda \in \mathbb{R} \) and a finite subfamily \( S \subset Z \) of \( L^0 \)-seminorms we let
\[
    V_{S, e, \lambda} := \{ x \in E \mid P(\|x\|_S < e) > 1 - \lambda \}
\]
\[
    V := \{ U_{S, e, \lambda} \mid S \subset Z \text{ finite, } e > 0, 0 < \lambda < 1 \}.
\]
\( V \) gives a neighborhood base of 0 and it induces a linear topology on \( E \), also denoted by \( \tau_{e,\lambda} \) (indeed if \( E = L^0 \) then this is exactly the topology of convergence in probability). This topology may not be locally convex, but has the following properties:

1. \((E, Z, \tau_{e,\lambda})\) becomes a \((L^0, \tau_{e,\lambda})\)-module associated to \( Z \) (see Proposition 2.6 [Gu10]),
2. \((E, Z, \tau_{e,\lambda})\) satisfies S-Closed (see Theorems 3.6 and 3.9 [Gu10]).

Therefore Proposition 23 can be applied.

5 On Conditionally Evenly Quasi-Convex maps on \( L^0 \)-module

As an immediate consequence of Proposition 23 we have that lower (resp. upper) semicontinuity and quasiconvexity imply evenly quasiconvexity of \( \rho \). From Theorem 17 we then deduce the representation for lower (resp. upper) semicontinuous quasiconvex maps.

(LSC) A map \( \pi : E \rightarrow \bar{L}^0(G) \) is lower semicontinuous if for every \( Y \in L^0 \) the lower level sets
\[
    U_Y = \{ x \in E \mid \pi(x) 1_{T_x} \leq Y \}
\]
are \( \tau \)-closed.

Corollary 26 Let \((E, Z, \tau)\) and \( E' = E^* \) be respectively as in definitions 19 and 21, satisfying S-Closed.
If \( \pi : E \rightarrow \bar{L}^0(G) \) is (REG), (QCO) and (LSC) then (7) holds true.

In the upper semicontinuous case we can say more (the proof is postponed to Section 6).

(USC) A map \( \pi : E \rightarrow \bar{L}^0(G) \) is upper semicontinuous if for every \( Y \in L^0 \) the lower level sets
\[
    U_Y = \{ x \in E \mid \pi(x) 1_{T_x} < Y \}
\]
are \( \tau \)-open.
Proposition 27 Let \((E, \mathcal{Z}, \pi)\) and \(E' = E^*\) be respectively as in Proposition 23 statement 1, satisfying S-Open.
If \(\pi : E \to L^0(\mathcal{G})\) is (REG), (QCO) and (USC) then
\[
\pi(x) = \max_{x^* \in E^*} \mathcal{R}(x, x^*), x^*).
\]

In Theorem 17, \(\pi\) can be represented as a supremum but not as a maximum. The following corollary shows that nevertheless we can find a \(\mathcal{R}(x, x^*), x^*\) arbitrary close to \(\pi(x)\).

Corollary 28 Under the same assumption of Theorem 17 or Corollary 26, for every \(\varepsilon \in L_{+}^0\) there exists \(x_\varepsilon^* \in E^*\) such that
\[
\pi(x) - \mathcal{R}(x, x_\varepsilon^*), x_\varepsilon^*) < \varepsilon \text{ on the set } \{\pi(x) < +\infty\}.
\]

Proof. The statement is a direct consequence of the inequalities (28) through (29) of Step 3 in the proof of Theorem 17. \(\blacksquare\)

6 Proofs

Notation 29 The condition 1 \(\{\eta\} \cap 1 \mathcal{A} \mathcal{C} \neq \emptyset\) is equivalent to: \(\exists \xi \in \mathcal{C} \text{ s.t. } 1 \mathcal{A} \eta = 1 \mathcal{A} \xi\).

For \(\eta \in E, B \in \mathcal{G}\) and \(C \subseteq E\) we say that
\[
\eta \text{ is outside } |B| C \text{ if } \forall A \subseteq B, A \in \mathcal{G}, \mathbb{P}(A) > 0, 1 \mathcal{A} \{\eta\} \cap 1 \mathcal{A} C = \emptyset.
\]
If \(\mathbb{P}(B) = 0\) then \(\eta\) is outside|\(B| C\) is equivalent to \(\eta \in C\). Recall that \(\mathcal{A} \mathcal{C}\) is the maximal set of \(\mathcal{A} \mathcal{C} = \{B \in \mathcal{G} \mid 1 \mathcal{A} E = 1 \mathcal{A} C\}\), \(D \mathcal{C}\) is the complement of \(\mathcal{A} \mathcal{C}\) and that \(\eta\) is outside \(C\) if \(\eta\) is outside|\(D \mathcal{C} C\).

Remark 30 By Lemma 2.9 in [FKV09], we know that any non-empty class \(\mathcal{A}\) of subsets of a sigma algebra \(\mathcal{G}\) has a supremum \(\text{ess} \sup \{\mathcal{A}\} \in \mathcal{G}\) and that if \(\mathcal{A}\) is closed with respect to finite union (i.e. \(A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A}\)) then there is a sequence \(A_n \in \mathcal{A}\) such that \(\text{ess} \sup \{\mathcal{A}\} = \bigcup_{n \in \mathbb{N}} A_n\). Obviously, if \(\mathcal{A}\) is closed with respect to countable union then \(\text{ess} \sup \{\mathcal{A}\} = \bigcup_{n \in \mathbb{N}} A_n := A_M \in \mathcal{A}\) is the maximal element in \(\mathcal{A}\).

For our proofs we need a simplified version of a result proved by Guo (Theorem 3.13, [Gu10]) concerning hereditarily disjoint stratification of two subsets. We reformulate his result in the following

Lemma 31 Suppose that \(C \subseteq E\) satisfies \(1 \mathcal{A} C + 1 \mathcal{A} \mathcal{C} \subseteq C\), for every \(A \in \mathcal{G}\). If there exists \(x \in E\) with \(x \notin \mathcal{C}\) then there exists a set \(H := H_{\mathcal{C}, x} \in \mathcal{G}\) such that \(\mathbb{P}(H) > 0\) and
\[
1_{\Omega \setminus H} \{x\} \cap 1_{\Omega \setminus H} C \neq \emptyset \quad (14)
\]
\[x \text{ is outside } |H| C \quad (15)
\]
The two above conditions guarantee that \(H_{\mathcal{C}, x}\) is the largest set \(D \in \mathcal{G}\) such that \(x\) is outside|\(D| C\).

Lemma 32 Suppose that \(\mathcal{C}\) satisfies (CSet).
1. If \(x \notin \mathcal{C}\) then the set \(H_{\mathcal{C}, x}\) defined in Lemma 31 satisfies \(H_{\mathcal{C}, x} \subseteq D\mathcal{C}\) and so \(\mathbb{P}(D\mathcal{C}) \geq \mathbb{P}(H_{\mathcal{C}, x}) > 0\).
2. If \(x\) is outside \(\mathcal{C}\) then \(\mathbb{P}(H_{\mathcal{C}, x}) > 0\) and \(H_{\mathcal{C}, x} = D\mathcal{C}\).
3. If \(x \notin \mathcal{C}\) then
\[
\mathcal{C}' \setminus \mathcal{C} := \{y \in E \mid y \text{ is outside } \mathcal{C}\} \neq \emptyset.
\]
Proof. 1. Lemma 31 shows that $\mathbb{P}(H_{C,x}) > 0$. Since $1_{A_e}E = 1_{A_e}C$, if $x \notin C$ we necessarily have: $\mathbb{P}(H_{C,x} \cap A_e) = 0$ and therefore $H_{C,x} \subseteq D_C$.

2. If $x$ is outside $C$, then $x$ is outside $D_C$ and $x \notin C$. The thesis follows from $H_{C,x} \subseteq D_C$ and the fact that $H_{C,x}$ is the largest set $D \in \mathcal{G}$ for which $x$ is outside $D_C$.

3. is a consequence of Lemma 35 (see Appendix) item 1. ■

Proof of Theorem 11. (1)$\Rightarrow$(2). Let $\mathcal{L} \subseteq E', Y_x \in L^0$ and let

$$C := \bigcap_{x' \in \mathcal{L}} \{ \xi \in E \mid \langle \xi, x' \rangle < Y_{x'} \text{ on } D_C \},$$

which clearly satisfies $C^{cc} = C$. By definition, if there exists $x \in E$ s.t. $x$ is outside $C$ then $1_{A_e} \{ x \} \cap 1_{A_e}C = \emptyset \forall A \subseteq D_C$, $A \in \mathcal{G}$, $\mathbb{P}(A) > 0$, and therefore by the definition of $C$ there exists $x' \in \mathcal{L}$ s.t. $\langle x, x' \rangle \geq Y_{x'}$ on $D_C$. Hence: $\langle x, x' \rangle > \langle \xi, x' \rangle$ on $D_C$ for all $\xi \in C$.

(2)$\Rightarrow$(1) We are assuming that $C$ is $(\text{CSet})$, and there exists $x \in E$ s.t. $x \notin C$ (otherwise $C = E$). From (28) we know that $\chi = \{ y \in E \mid y$ is outside $C \}$ is nonempty. By assumption, for all $y \in \chi$ there exists $\xi_y \in E'$ such that $\langle \xi, \xi_y \rangle < \langle y, \xi_y \rangle$ on $D_C$, $\forall \xi \in C$. Define

$$B_y := \{ \xi \in E \mid \langle \xi, \xi_y \rangle < \langle y, \xi_y \rangle \text{ on } D_C \}.$$

$B_y$ clearly depends also on the selection of the $\xi'_y \in E'$ associated to $y$ and on $C$, but this notation will not cause any ambiguity. We have: $C \subseteq B_y$ for all $y \in \chi$, and $C \subseteq \bigcap_{y \in \chi} B_y$. We now claim that $x \notin C$ implies $x \notin \bigcap_{y \in \chi} B_y$, thus showing

$$C = \bigcap_{y \in \chi} B_y = \bigcap_{\xi_y \in \mathcal{L}} \{ \xi \in E \mid \langle \xi, \xi_y \rangle < Y_{\xi_y} \text{ on } D_C \},$$

where $\mathcal{L} := \{ \xi'_y \in E' \mid y \in \chi \}$, $Y_{\xi_y} := \langle y, \xi'_y \rangle \in L^0$, and the thesis is proved.

Suppose that $x \notin C$, then, by Lemma 31, $x$ is outside $H \ C$, where we set for simplicity $H = H_{C,x}$. Take any $y \in \chi \neq \emptyset$ and define $y_0 := x1_H + y1_{\Omega\setminus H} \in \chi$. Take $B_{y_0} = \{ \xi \in E \mid \langle \xi, \xi_{y_0} \rangle < \langle y_0, \xi_{y_0} \rangle \text{ on } D_C \}$ where $\xi_{y_0} \in E'$ is the element associated to $y_0$. If $x \in B_{y_0}$ then we would have: $\langle x, \xi_{y_0} \rangle < \langle y_0, \xi_{y_0} \rangle = \langle x, \xi_{y_0} \rangle$ on $H \subseteq D_C$, by Lemma 32 item 1, which is a contradiction, since $\mathbb{P}(H) > 0$. Hence $x \notin B_{y_0} \supseteq \bigcap_{y \in \chi} B_y$. ■

Proposition 33 Under the same assumptions of Theorem 11, the following are equivalent:

1. $C$ is conditionally evenly convex
2. for every $x \in E$, $x \notin C$, there exists $x' \in E'$ such that

$$\langle \xi, x' \rangle < \langle x, x' \rangle \text{ on } H_{C,x} \forall \xi \in C,$$

where $H_{C,x}$ is defined in Lemma 31.

Proof. (1)$\Rightarrow$(2): We know that $C$ satisfies $(\text{CSet})$. As $x \notin C$, from (28) and Lemma 31 we know that there exists $y \in E$ s.t. $y$ is outside $C$ and that $H = H_{C,x}$ satisfies $\mathbb{P}(H) > 0$. Define $\bar{x} = x1_H + y1_{\Omega\setminus H}$. Then $\bar{x}$ is outside $C$ and by Theorem 11 item 2 there exists $x' \in E'$

$$\langle \xi, x' \rangle < \langle \bar{x}, x' \rangle \text{ on } D_C, \forall \xi \in C.$$

This implies the thesis since $\langle x, x' \rangle = \langle x, x' \rangle 1_H + \langle y, x' \rangle 1_{\Omega\setminus H}$ and $H \subseteq D_C$.

(2)$\Rightarrow$(1): We show that item 2 of Theorem 11 holds true. This is trivial since if $x$ is outside $C$ then $x \notin C$ and $H_{C,x} = D_C$. ■

Proof of Theorem 12. Item (1) is straightforward; the fact that $C^\infty$ is conditionally evenly convex follows from the definition; the proof of $C \subseteq C^\infty$ is also obvious. We now suppose that $C$ is
conditionally even convex and show the reverse inequality $C^\infty \subseteq C$. By contradiction let $x \in C^\infty$ and $x \notin C$. As $C$ is conditionally even convex we apply Proposition 33 and find $x' \in E^*$ such that
\[ \langle \xi, x' \rangle < \langle x, x' \rangle \] on $H_{C,x}$ for all $\xi \in C$.
Since $0 \in C$, $0 = \langle 0, x' \rangle < \langle x, x' \rangle$ on $H := H_{C,x}$. Take any $x_1' \in C^\circ$ (which is clearly not empty) and set $y' := \frac{x_1'}{\langle x_1', x_1 \rangle} 1_H + x_1' 1_{H^\perp}$. Then $y' \in E^*$ and $\langle \xi, y' \rangle < 1$ on $D_C$ for all $\xi \in C$. This implies $y' \in C^\circ$. In addition, $(x, y') = 1$ on $H \subseteq D_C$ which is in contradiction with $x \in C^\infty$. 

**General properties of $\mathcal{R}(Y, \mu)$**

Following the path traced in [FM11], we adopt the module framework the proofs of the foremost properties holding for the function $\mathcal{R} : L^0(\mathcal{G}) \times E^* \rightarrow L^0(\mathcal{G})$ defined in (8). Let the effective domain of the function $\mathcal{R}$ be:
\[
\Sigma_\mathcal{R} := \{(Y, \mu) \in L^0(\mathcal{G}) \times E^* \mid \exists \xi \in E \text{ s.t. } \mu(\xi) \geq Y\}. \tag{18}
\]

**Lemma 34** Let $\mu \in E^*$, $X \in E$ and $\pi : E \rightarrow L^0(\mathcal{G})$ satisfy (REG).

i) $\mathcal{R}(\cdot, \mu)$ is monotone non decreasing.

ii) $\mathcal{R}(\Lambda \mu(X), \Lambda \mu) = \mathcal{R}(\mu(X), \mu)$ for every $\Lambda \in L^0(\mathcal{G})$.

iii) For every $Y \in L^0(\mathcal{G})$ and $\mu \in E^*$, the set 
\[ A_\mu(Y) := \{ \pi(\xi) \mid \xi \in E, \mu(\xi) \geq Y \} \]
is downward directed in the sense that for every $\pi(\xi_1), \pi(\xi_2) \in A_\mu(Y)$ there exists $\pi(\xi^*) \in A_\mu(Y)$ such that $\pi(\xi^*) \leq \min\{\pi(\xi_1), \pi(\xi_2)\}$.

iv) For every $A \in \mathcal{G}$, $(Y, \mu) \in \Sigma_\mathcal{R}$ 
\[ \mathcal{R}(Y, \mu) 1_A = \inf_{\xi \in E} \{ \pi(\xi) 1_A \mid Y \geq \mu(X) \} \tag{19} \]
\[ = \inf_{\xi \in E} \{ \pi(\xi) 1_A \mid Y 1_A \geq \mu(X 1_A) \} = \mathcal{R}(Y 1_A, \mu) 1_A \tag{20} \]

v) For every $X_1, X_2 \in E$
(a) $\mathcal{R}(\mu(X_1), \mu) \wedge \mathcal{R}(\mu(X_2), \mu) = \mathcal{R}(\mu(X_1) \wedge \mu(X_2), \mu)$
(b) $\mathcal{R}(\mu(X_1), \mu) \vee \mathcal{R}(\mu(X_2), \mu) = \mathcal{R}(\mu(X_1) \vee \mu(X_2), \mu)$

vi) The map $\mathcal{R}(\mu(X), \mu)$ is quasi-affine with respect to $X$ in the sense that for every $X_1, X_2 \in E$, $\Lambda \in L^0(\mathcal{G})$ and $0 \leq \Lambda \leq 1$, we have 
\[ \mathcal{R}(\mu(\Lambda X_1 + (1-\Lambda)X_2), \mu) \geq \mathcal{R}(\mu(X_1), \mu) \wedge \mathcal{R}(\mu(X_2), \mu) \] (quasiconvexity).
\[ \mathcal{R}(\mu(\Lambda X_1 + (1-\Lambda)X_2), \mu) \leq \mathcal{R}(\mu(X_1), \mu) \vee \mathcal{R}(\mu(X_2), \mu) \] (quasiconvexity).

vii) $\inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu_1) = \inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu_2)$ for every $\mu_1, \mu_2 \in E^*$.

**Proof.**

i) and ii) follow trivially from the definition.

iii) The set $\{ \pi(\xi) \mid \xi \in E, \mu(\xi) \geq Y \}$ is clearly downward directed. Thus there exists a sequence $\{\xi_m\}_{m=1}^\infty \in E$ such that 
\[ \mu(\xi_m) \geq Y \quad \forall m \geq 1, \quad \pi(\xi_m) \downarrow \mathcal{R}(Y, \mu) \quad \text{as } m \uparrow \infty. \]

Now let $\mathcal{R}(Y, \mu) < \alpha$: consider the sets $F_m = \{ \pi(\xi_m) < \alpha \}$ and the partition of $\Omega$ given by $G_1 = F_1$ and $G_m = F_m \setminus G_{m-1}$. Since we assume that $E$ satisfies (CSet) and from the property (REG) we get:
\[ \xi = \sum_{m=1}^\infty \xi_m 1_{G_m} \in E, \quad \mu(\xi) \geq Y \quad \text{and} \quad \pi(\xi) < \alpha. \]

iv), v) and vi) follow as in [FM11].

(vii) Notice that $\mathcal{R}(Y, \mu) \geq \inf_{\xi \in E} \pi(\xi)$, $\forall Y \in L^0(\mathcal{G})$, implies: $\inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu) \geq \inf_{\xi \in E} \pi(\xi)$. On the other hand, $\pi(\xi) \geq \mathcal{R}(\mu(\xi), \mu) \geq \inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu)$, $\forall \xi \in E$, implies: $\inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu) \leq \inf_{\xi \in E} \pi(\xi)$.
Proof of Theorem 17. Let \( \pi : E \to \tilde{L}^0(\mathcal{G}) \). There might exist a set \( A \subseteq \mathcal{G} \) on which the map \( \pi \) is constant, in the sense that \( \pi(\xi)1_A = \pi(\eta)1_A \) for every \( \xi, \eta \in E \). For this reason we introduce
\[
A := \{ B \subseteq \mathcal{G} \mid \pi(\xi)1_B = \pi(\eta)1_B \ \forall \xi, \eta \in E \}.
\]
Applying Lemma 36 in Appendix with \( F := \{ (\pi(\xi) - \pi(\eta)) | \xi, \eta \in E \} \) (we consider the convention \( +\infty - \infty = 0 \) and \( Y_0 = 0 \)) we can deduce the existence of two maximal sets \( A \subseteq \mathcal{G} \) and \( A' \subseteq \mathcal{G} \) for which \( P(A \cap A^c) = 0, P(A \cup A^c) = 1 \) and
\[
\begin{align*}
\pi(\xi) &= \pi(\eta) \quad \text{on } A, \quad \forall \xi, \eta \in E, \\
\pi(\zeta_1) &< \pi(\zeta_2) \quad \text{on } A^c, \quad \forall \zeta_1, \zeta_2 \in E.
\end{align*}
\]
Recall that \( \Upsilon_\pi \subseteq \mathcal{G} \) is the maximal set on which \( \pi(\xi)1_{\Upsilon_\pi} = +\infty1_{\Upsilon_\pi} \) for every \( \xi \in E \) and \( T_\pi \) its complement. Notice that \( \Upsilon_\pi \subseteq A \).

Fix \( x \in E \) and \( G = \{ \pi(x) < +\infty \} \). For every \( \varepsilon \in L^0_{+}(\mathcal{G}) \) we set
\[
Y_\varepsilon := 01_{\Upsilon_\pi} + (\pi(x)1_{A \setminus \Upsilon_\pi} + (\pi(x) - \varepsilon)1_{G \cap A'} + \varepsilon1_{G \cap A^c})
\]
and for every \( \varepsilon \in L^0(\mathcal{G}^+) \) we set
\[
C_\varepsilon = \{ \xi \in E | \pi(\xi)1_{T_\pi} \leq Y_\varepsilon \}.
\]

Step 1: on the set \( A, \pi(x) = R(\langle x, x' \rangle, x') \) for any \( x' \in E' \) and the representation
\[
\pi(x)1_A = \max_{x' \in E'} R(\langle x, x' \rangle, x')1_A
\]
trivially holds true on \( A \).

Step 2: by the definition of \( Y_\varepsilon \) we deduce that if \( C_\varepsilon = \emptyset \) for every \( \varepsilon \in L^0_{+} \) then \( \pi(x) \leq \pi(\xi) \) on the set \( A' \) for every \( \xi \in E \) and \( \pi(x)1_{A'} = R(\langle x, x' \rangle, x')1_{A'} \) for any \( x' \). The representation
\[
\pi(x)1_{A'} = \max_{x' \in E'} R(\langle x, x' \rangle, x')1_{A'}
\]
trivially holds true on \( A' \). The thesis follows pasting together equations (24) and (25)

Step 3: we now suppose that there exists \( \varepsilon \in L^0_{+} \) such that \( C_\varepsilon \neq \emptyset \). The definition of \( Y_\varepsilon \) implies that \( C_\varepsilon1_A = E1_A \) and \( A \) is the maximal element i.e. \( A = A_{C_\varepsilon} \) (given by Definition 4). Moreover this set is conditionally evenly convex and \( x \) is outside \( C_\varepsilon \). The definition of evenly convex set guarantees that there exists \( x_\varepsilon \in E' \) such that
\[
\langle x, x_\varepsilon \rangle > \langle \xi, x_\varepsilon \rangle \quad \text{on } D_{C_\varepsilon} = A^+, \forall \xi \in C_\varepsilon.
\]

Claim:
\[
\{ \xi \in E | \langle x, x_\varepsilon \rangle1_{A^+} \leq \langle \xi, x_\varepsilon \rangle1_{A^+} \} \subseteq \{ \xi \in E | \pi(\xi) > (\pi(x) - \varepsilon)1_{G} + \varepsilon1_{G \cap A^c} \text{ on } A^+ \}.
\]
In order to prove the claim take \( \xi \in E \) such that \( \langle x, x_\varepsilon \rangle1_{A^+} \leq \langle \xi, x_\varepsilon \rangle1_{A^+} \). By contra we suppose that there exists a \( F \subseteq A^+, F \in \mathcal{G} \) and \( P(F) > 0 \) such that \( \pi(\xi)1_F \leq (\pi(x) - \varepsilon)1_{G \cap F} + \varepsilon1_{G \cap F} \). Take \( \eta \in C_\varepsilon \) and define \( \bar{\xi} = \eta1_{F \cap C_\varepsilon} + \xi1_{F \cap C_\varepsilon} \in C_\varepsilon \) so that we conclude that \( \langle x, x_\varepsilon \rangle > \langle \bar{\xi}, x_\varepsilon \rangle \) on \( A^+ \).

Since \( \langle \bar{\xi}, x_\varepsilon \rangle = \langle \xi, x'_\varepsilon \rangle \) on \( F \) we reach a contradiction.

Once the claim is proved we end the argument observing that
\[
\begin{align*}
\pi(x)1_{A^+} &\geq \sup_{x' \in E'} R(\langle x, x' \rangle, x')1_{A^+} = R(\langle x, x'_\varepsilon \rangle, x'_\varepsilon)1_{A^+} \\
&= \inf_{\xi \in E} \{ \pi(\xi)1_{A^+} | \langle x, x'_\varepsilon \rangle1_A \leq \langle \xi, x'_\varepsilon \rangle1_{A^+} \} \\
&\geq \inf_{\xi \in E} \{ \pi(\xi)1_{A^+} \mid \pi(\xi) > (\pi(x) - \varepsilon)1_{G} + \varepsilon1_{G \cap A^c} \text{ on } A^+ \} \\
&\geq (\pi(x) - \varepsilon)1_{G \cap A^+} + \varepsilon1_{G \cap A^c},
\end{align*}
\]
The representation (7) follows by taking \( \varepsilon \) arbitrary small on \( G \cap A^c \) and arbitrary big on \( G^c \cap A^c \) and pasting together the result with equation (24). ■

**Proof of Proposition 27.** Fix \( X \in E \) and consider the classes of sets

\[
A := \{ B \in G \mid \forall \xi \in E \; \pi(\xi) \geq \pi(X) \text{ on } B \}, \\
A^c := \{ B \in G \mid \exists \xi \in E \; \pi(\xi) < \pi(X) \text{ on } B \}.
\]

Then \( A = \{ B \in G \mid \forall Y \in F \; Y \geq Y_0 \text{ on } B \} \), where \( F := \{ \pi(\xi) \mid \xi \in E \} \) and \( Y_0 = \pi(X) \). Applying Lemma 36, there exist two maximal elements \( A \in A \) and \( A^c \in A^c \) so that: \( P(A \cup A^c) = 1, \)

\[
P(A \cap A^c) = 0,
\]

\[
\pi(\xi) \geq \pi(X) \text{ on } A \text{ for every } \xi \in E \quad \text{and} \quad \exists \xi \in E \; \pi(\xi) < \pi(X) \text{ on } A^c.
\]

Clearly for every \( \mu \in E^c \),

\[
\pi(X)1_A \geq R(\mu(X), \mu)1_A \geq \pi(X)1_A.
\]  \hfill (30)

Consider \( \delta \in L^0_+ (G) \). The set

\[
\mathcal{O} := \{ \xi \in E \mid \pi(\xi)1_{T^c} < \pi(X)1_{A^c} + (\pi(X) + \delta)1_A \}
\]

is open, \( L^0(G) \)-convex (from Remark 16 ii) and not empty. Clearly \( X \notin \mathcal{O} \) and \( \mathcal{O} \) satisfies (CSet). We thus can apply Theorem 3.15 in [Gu10] and find \( \mu_* \in E^c \) so that

\[
\mu_* < \mu_*(X) \quad \text{on } H(\{X\}, \mathcal{O}) \cap \mathcal{O}, \forall \xi \in \mathcal{O}.
\]

Notice that \( P(H(\{X\}, \mathcal{O}) \setminus A^c) = 0 \). We apply the argument in Step 3 of the proof of Theorem 17 to find that

\[
\{ \xi \in E \mid \mu_*(X)1_{A^c} \leq \mu_*(\xi)1_{A^c} \} \subseteq \{ \xi \in E \mid \pi(\xi)1_{A^c} = \pi(X)1_{A^c} \}.
\]

From (19)-(20) we derive

\[
\pi(X)1_{A^c} \geq R(\mu_*(X), \mu_*)1_{A^c} = \inf_{\xi \in E} \pi(\xi)1_{A^c} = \pi(X)1_{A^c} \geq \pi(X)1_{A^c}.
\]

The thesis then follows from (30). ■

7 Appendix

**Lemma 35** For any sets \( C \subseteq E \) and \( D \subseteq E \) set:

\[
A = \{ B \in G \mid \forall y \in D^{cc} \exists \xi \in C^{cc} \text{ s.t. } 1_{By} = 1_B \xi \}, \\
A^c = \{ B \in G \mid \exists y \in D^{cc} \text{ s.t. } y \text{ is outside } |B| C^{cc} \}.
\]

Then there exist the maximal set \( A_M \in A \) of \( A \) and the maximal set \( A^c_M \in A^c \) of \( A^c \), one of which may have zero probability, that satisfy

\[
\forall y \in D^{cc} \exists \xi \in C^{cc} \text{ s.t. } 1_{A_M} y = 1_{A_M} \xi,
\]

\[
\exists y \in D^{cc} \text{ s.t. } y \text{ is outside } |A^c_M| C^{cc},
\]

and \( A^c_M \) is the \( \mathbb{P} \)-a.s unique complement of \( A_M \).

Suppose in addition that \( D = E \) and \( C = C^{cc} \). Then the class \( A \) coincides with the class \( A(C) = \{ B \in G \mid 1_A E = 1_A C \} \) introduced in the Notation 4. Henceforth: the maximal set of \( A(C) \) is \( A_C = A_M; D_C = A^c_M; 1_{A_C} E = 1_{A_C} C \); and there exists \( y \in E \text{ s.t. } y \text{ is outside } C \). If \( x \notin C \) then \( P(D_C) > 0 \) and \( \chi = \{ y \in E \mid y \text{ is outside } C \} \neq \emptyset \).
Proof. The two classes \( A \) and \( A^\circ \) are closed with respect to countable union. Indeed, for the family \( A^\circ \), suppose that \( B_i \in A^\circ \), \( y_i \in D^\circ \) s.t. \( y_i \) is outside \( B_i \). Define \( B_1 := B_1 \), \( \tilde{B}_i := B_i \setminus B_{i-1} \), \( B := \bigcup_{i=1}^{\infty} \tilde{B}_i = \bigcup_{i=1}^{\infty} B_i \). Then \( y_i \) is outside \( \tilde{B}_i \). \( \tilde{B}_i \) are disjoint elements of \( A^\circ \) and \( y^* := \sum_{i=1}^{\infty} y_i \tilde{B}_i \in D^\circ \). Since \( y_1 \tilde{B}_i = y^* \tilde{B}_i \), \( y \) is outside \( \tilde{B}_i \) for all \( i \) and so \( y \) is outside \( B \). Thus \( B \in A^\circ \). Similarly for the class \( A \).

The Remark 30 guarantees the existence of the two maximal sets \( A_M \in A \) and \( A_M^\circ \in A^\circ \), so that \( B \in A \) implies \( B \subseteq A_M \); \( B^\circ \in A^\circ \) implies \( B^\circ \subseteq A_M^\circ \).

Obviously, \( P(A_M \cap A_M^\circ) = 0 \), as \( A_M \in A \) and \( A_M^\circ \in A^\circ \). We claim that

\[
P(A_M \cup A_M^\circ) = 1. \tag{31}
\]

To show (31) let \( D := \Omega \setminus \{A_M \cup A_M^\circ\} \in \mathcal{G} \). By contradiction suppose that \( P(D) > 0 \). From \( D \subseteq (A_M \cap A_M^\circ) \cap \Omega \) the maximality of \( A_M \) we get \( D \nsubseteq A \). This implies that there exists \( y \in D^\circ \) such that

\[
1_D \{y\} \cap 1_{D^\circ} \neq \emptyset \tag{32}
\]

and obviously \( y \notin D^\circ \), as \( P(D) > 0 \). By the Lemma 31 there exists a set \( H_{D^\circ,y} := H \in \mathcal{G} \) satisfying \( P(H) > 0 \), (14) and (15) with \( \mathcal{C} \) replaced by \( \mathcal{D}^\circ \).

Condition (15) implies that \( H \in A^\circ \) and then \( H \subseteq A_M^\circ \). From (14) we deduce that there exists \( \xi \in C^\circ \) s.t. \( 1_A \xi = 1_A \xi^\circ \) for all \( A \subseteq \Omega \setminus H \). Then (32) implies that \( D \) is not contained in \( \Omega \setminus H \), so that \( P(D \cap H) > 0 \). This is a contradiction since \( D \cap H \subseteq D \subseteq (A_M^\circ)^\circ \) and \( D \cap H \subseteq H \subseteq A_M^\circ \).

Item 1 is a trivial consequence of the definitions. \( \square \)

Lemma 36 With the symbol \( \sqsupseteq, \sqsubset \), \( \sqsupset \), \( \sqsubset \), \( \sqsupset \) and \( \sqsubset \) its negation. Consider a class \( F \subseteq L^0(\mathcal{G}) \) of random variables, \( Y_0 \in L^0(\mathcal{G}) \) and the classes of sets

\[
A := \{ A \in \mathcal{G} \mid \forall Y \in F \ Y \sqsupseteq Y_0 \text{ on } A \},
\]

\[
A^\circ := \{ A^\circ \in \mathcal{G} \mid \exists Y \in F \ s.t. \ Y \sqsubset Y_0 \text{ on } A^\circ \}.
\]

Suppose that for any sequence of disjoint sets \( A^i \in A^\circ \) and the associated r.v. \( Y_i \in F \) we have \( \sum_{i=1}^{\infty} Y_i A^i_1 \in F \). Then there exist two maximal sets \( A_M \in A \) and \( A_M^\circ \in A^\circ \) such that

\[
P(A_M \cap A_M^\circ) = 0, \ P(A_M \cup A_M^\circ) = 1 \quad \text{and}
\]

\[
Y \sqsupseteq Y_0 \text{ on } A_M, \forall Y \in F \quad \text{and} \quad Y \sqsubset Y_0 \text{ on } A_M^\circ, \text{ for some } Y \in F.
\]

Proof. Notice that \( A \) and \( A^\circ \) are closed with respect to countable union. This claim is obvious for \( A \). For \( A^\circ \), suppose that \( A^i \in A^\circ \) and that \( Y_i \in F \) satisfies \( P(Y_i \sqsubseteq Y_0) = P(A^i) \). Defining \( B_1 := A^i_1 \), \( B_i := A^i_1 \setminus B_{i-1} \), \( A^i_\infty := \bigcup_{i=1}^{\infty} A^i_1 = \bigcup_{i=1}^{\infty} B_i \) we see that \( B_i \) are disjoint elements of \( A^\circ \) and that \( Y^* := \sum_{i=1}^{\infty} Y_i 1_{B_i} \in F \) satisfies \( P(\{Y^* \sqsubseteq Y_0\} \cap A^i_\infty) = P(A^i_\infty) \) and so \( A^i_\infty \in A^\circ \).

The Remark 30 guarantees the existence of two sets \( A_M \in A \) and \( A_M^\circ \in A^\circ \) such that:

(a) \( P(A \cap (A_M^\circ)) = 0 \) for all \( A \in A \).
(b) \( P(A^\circ \cap (A_M^\circ)) = 0 \) for all \( A^\circ \in A^\circ \).

Obviously, \( P(A_M \cap A_M^\circ) = 0 \), as \( A_M \in A \) and \( A_M^\circ \in A^\circ \). To show that \( P(A_M \cup A_M^\circ) = 1 \), let \( D := \Omega \setminus \{A_M \cup A_M^\circ\} \in \mathcal{G} \). By contradiction suppose that \( P(D) > 0 \). As \( D \subseteq (A_M^\circ)^\circ \), from condition (a) we get \( D \notin A \). Therefore, \( \exists Y \in F \) s.t. \( P(\{Y \sqsupset Y_0\} \cap D) < P(D) \), i.e. \( P(\{Y \sqsubset Y_0\} \cap D) > 0 \). If we set \( B := \{Y \sqsubset Y_0\} \cap D \) then it satisfies \( P(\{Y \sqsubset Y_0\} \cap D) = P(B) > 0 \) and, by definition of \( A^\circ \), \( B \) belongs to \( A^\circ \). On the other hand, as \( B \subseteq D \subseteq (A_M^\circ)^\circ \), \( P(B) = P(B \cap (A_M^\circ)^\circ) \), and from condition (b) \( P(B \cap (A_M^\circ)^\circ) = 0 \), which contradicts \( P(B) > 0 \). \( \square \)
References


