On the existence of minimax martingale measures*

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Abstract

Embedding asset pricing in a utility maximization framework leads naturally to the concept of minimax martingale measures. We consider a market model where the price process is assumed to be a $\mathbb{R}^d$– semimartingale $X$ and the set of trading strategies consists of all predictable, $X$– integrable, $\mathbb{R}^d$ valued processes $H$ for which the stochastic integral $(H.X)$ is uniformly bounded from below. When the market is free of arbitrage, we show that a sufficient condition for the existence of the minimax measure is that the utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ is concave and nondecreasing. We also show the equivalence between the No Free Lunch with Vanishing Risk condition, the existence of a separating measure, and a properly defined notion of viability.

Keywords: Utility maximization, martingale measures, incomplete markets, asset pricing, viability, duality, relative entropy.

1 Introduction

A recurrent theme in the mathematical finance literature is the selection of a pricing functional in incomplete markets. One main stream of research is based on the consideration that the valuation of unattainable claims is no longer independent of the preference structure of investors. This suggests considering asset pricing in a utility maximization context, as for example in Davis [7], and using a marginal rate of substitution argument to define the fair price of the claims.

In a continuous time diffusion incomplete model, Karatzas, Lehoczky, Shreve and Xu [24] showed that one way to handle the utility maximization problem

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from terminal wealth is to consider the dual problem, to apply the martingale technology in order to find its solution, and then to recover the solution of the original problem. Similar techniques were developed by He and Pearson [20] to analyze the dynamic consumption portfolio problem when asset prices follow Ito processes.

In these models the dual problem can be formulated as a minimization problem over the set $\mathcal{M}$ of martingale measures, and one central issue is to guarantee that the minimum is indeed attained by an element in $\mathcal{M}$. He and Pearson introduced the notion of a minimax martingale measure and showed its relevance for the analysis of the consumption portfolio problem, both in the finite dimensional case [19] and in the continuous time case [20].

One aim of this paper is to provide sufficient conditions for the existence of minimax measures in a general setting, where asset price processes are only assumed to be semimartingales.

Another key issue is to show that most existing criteria for selecting a martingale measure in incomplete markets (as the optimal variance measure, Schweizer [30]; the minimal entropy measure, Frittelli [13]; or the Kakutani Hellinger measure, Grandits [17]) are dual to a utility maximization problem. In this way we provide a powerful economic justification of these criteria. Examples of these minimal measures are collected in Section 5.

In this paper we formulate sufficient conditions on the utility functions which guarantee that the infimum of the above mentioned dual problem is attained by a martingale measure.

In section 1.1 we present our market model and the main assumption of the paper. We introduce the concept of separating measures that replaces the notion of martingale measures in our general setting (for bounded processes the two coincide). Next, in section 1.2 we clarify the utility maximization problem, and in section 1.3 we give the definition of minimax measure and state the main result of the paper (Theorem A). After Theorem A we further discuss the literature related to this subject.

In section 2 we present some results (Proposition 1 and Theorem 1) from duality theory. Proposition 1 and Theorem 1 are formulated in a general setting where an arbitrary subset $N_1$ of the unit ball of $L^1$ takes the place of the set of martingale measures. Theorem A will be deduced, in Section 2.2, from Theorem 1 and its Corollary.

The proofs of Proposition 1 and Theorem 1 are collected in section 3 and are based on the characterization of the conjugate of the integral functional on $L^\infty$, as provided by Rockafellar [26].

Viability, No Arbitrage and existence of separating measures are the content of section 4. Here we show that these three conditions are all equivalent. This is not a surprise given the well known results of Kreps [25]. Even though our definition of viability is clearly inspired by Kreps, it is not equivalent to his definition, and his results cannot immediately be recast in our context.
We collected in the appendix some of the results needed in the proof of Theorem 1.

1.1 The market model \((\mathbb{P}, X, \mathcal{H})\)

(I) Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(\mathbb{P}\) be the class of probability measures equivalent to \(P\).

As a preliminary observation note that all definitions in this subsection (of \(X, K, C, M, M_1\), of separating measures and of NFLVR) remain unchanged if the probability measure \(P \in \mathbb{P}\) is replaced by any other probability in \(\mathbb{P}\).

(II) \(X = (X_t)_{t \in [0,T]}\) is an \(\mathbb{R}^d\)-valued cadlag semi-martingale on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\), where we assume that the filtration \((\mathcal{F}_t)_{t \in [0,T]}\) satisfies the usual assumptions of right continuity and completeness.

The process \(X\) represents the deflated price process of \(d\) tradeable assets. An \(\mathbb{R}^d\)-valued predictable process \(H = (H_t)_{t \in [0,T]}\) is called admissible if \(H\) is \(X\)-integrable and the stochastic process \((H.X)\), defined by

\[
(H.X)_t \triangleq \int_0^t H_s \cdot dX_s, \quad t \in [0,T],
\]

is \(P\)-a.s. uniformly bounded from below.

(III) The class \(\mathcal{H}\) of trading strategies consists of all admissible processes \(H\).

We denote with \(L^\infty = L^\infty(\Omega, \mathcal{F}, P; \mathbb{R})\) the space of essentially bounded random variables on \((\Omega, \mathcal{F}, P)\) having values in \(\mathbb{R}\), with \(L^0 = L^0(\Omega, \mathcal{F}, P; \mathbb{R})\) the space of \(P\)-a.s. finite random variables having values in \(\mathbb{R}\), with \(L^- = \{ w \in L^0 : \|w^-\|_\infty < \infty \}\) the cone of bounded from below random variables. The positive cone of a vector lattice \((V, \geq)\) is denoted by \(V_+ = \{ v \in V : v \geq 0 \}\), similarly for \(V_-\). Set:

\[
\chi \triangleq \{(H.X) : H \in \mathcal{H}\},
\]

\[
K \triangleq \{(H.X)_T : H \in \mathcal{H}\},
\]

\[
C \triangleq (K - L^0_+) \cap L^\infty.
\]

\((H.X)_T\) is the time \(T\) deflated financial gain from the admissible trading strategy \(H\). \(K\) is the cone of bounded from below claims that are attainable, at zero initial cost, from trading in the \(d\) assets in a self-financing way. The convex cone \(C\) satisfies \(L^\infty_0 \subseteq C \subseteq L^\infty\), and consists of all essentially bounded random variables that are dominated by some element of \(K\). It is the cone of bounded claims that can be super-replicated by attainable claims at zero initial cost.

We now introduce the set of “pricing measures”. 

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The norm dual space of $L^\infty$ is $ba = ba(\Omega, \mathcal{F}, P)$, the set of bounded additive set functions on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $P$. Let $C^0$ be the polar cone of $C$ with respect to the dual system $(L^\infty, ba)$:

$$C^0 \triangleq \{ \xi \in ba : \xi(w) \leq 0 \forall w \in C \},$$

and set:

$$M \triangleq C^0 \cap L^1(P) = \{ z \in L^1_+(P) : E_P[wz] \leq 0 \forall w \in C \}, \quad P \in \mathbb{P},$$

$$M_1 \triangleq \{ z \in M : E_P[z] = 1 \}.$$

The set $M$ consists of those elements $z \in C^0$ that are countably additive, and hence doesn’t depend on which $P \in \mathbb{P}$ is chosen. For all $P \in \mathbb{P}$, $M \subseteq L^1_+(P)$ (since $L^\infty \subseteq C$), $M$ is closed in $L^1(P)$ and, if $M_1 \neq \emptyset$, the convex cone $M$ is generated by the convex set $M_1$.

Without further mentioning, we will frequently identify a $P$–absolute continuous probability measure with its Radon Nikodym density in $L^1(P)$. Therefore, we have:

$$M_1 = \{ Q \ll P : E_Q[w] \leq 0 \forall w \in C \}.$$

**Definition 1** A $P$–absolutely continuous probability measure $Q$ is called a separating measure if $K \subseteq L^1(Q)$ and $E_Q[k] \leq 0 \forall k \in K$. It is called an equivalent separating measure if in addition $Q \in \mathbb{P}$. The set of $P$–absolutely continuous probability measures $Q$ such that $X$ is a $(Q, (\mathcal{F}_t)_{t \in [0, T]})$ martingale (resp. local martingale, $\sigma$-martingale) is denoted by $\mathcal{M}$ (resp. $\mathcal{M}_{\text{loc}}, \mathcal{M}_\sigma$).

**Lemma 2** $Q$ is a separating measure if and only if $Q \in M_1$. Moreover:

a) If $X$ is bounded then $M_1 = \mathcal{M}$.

b) If $X$ is locally bounded then $M_1 = \mathcal{M}_{\text{loc}}$.

c) In general (for possibly unbounded $X$), if $M_1 \neq \emptyset$ then $M_\sigma \neq \emptyset$ and $\mathcal{M}_\sigma = M_1$, where the closure is taken in the $L^1$–norm topology.

d) In general (for possibly unbounded $X$) $M_1$ is the set of $P$–absolutely continuous probability measures such that all processes in $X$ are supermartingales.

**Proof.** Since $E_Q[k] \leq 0 \forall k \in K$, then $E_Q[w] \leq 0 \forall w \in C$ and $Q \in M_1$.

Conversely, suppose that $Q \in M_1$. Let $k \in K$ and set $k_n = \min(k, n)$. Then, $k_n = k - (k - k_n) \in C$, and $k_n \uparrow k P$–a.s, hence also $Q$ – a.s.. By monotone convergence we get $E_Q[k] = \lim E_Q[k_n] \leq 0$. Hence $Q$ is a separating measure.

a) follows easily from definitions. Indeed, if $X$ is bounded and $f = 1_A(X_t - X_s), A \in \mathcal{F}_s, 0 \leq s < t \leq T$, then $f \in K$ and $-f \in K$, so that $Q \in M_1$ implies $E_Q[1_A(X_t - X_s)] = 0$ and $Q$ is a martingale measure.

b) follows from the same argument, using the localizing sequence of stopping times of the locally bounded process $X$.

c) is the remarkable achievement of Delbaen-Schachermayer [9] and Kabanov [22].

d) Define:

$$S = \{ Q \ll P \text{ such that all processes in } X \text{ are supermartingales under } Q \}.$$
It is clear from the definition of $\mathcal{S}$ and $M_1$ that $\mathcal{S} \subseteq M_1$. We’ll prove that $\mathcal{S} = M_1$.

First we show that $M_{\sigma} \subseteq \mathcal{S}$. Let $Q \in M_{\sigma}$. By the definition of a $\sigma-$martingale, the integral $(\varphi, X)$ is a local martingale under $Q$ for some predictable $(0, 1]-$valued process $\varphi$. Then it follows from Corollary 3.5 of Ansel and Stricker [1] that $(H, X) = ((H \varphi^{-1})(\varphi, X))$ is a local martingale (and hence a supermartingale) under $Q$, for each admissible $H$. Hence $Q \in \mathcal{S}$.

Therefore, $\overline{M_{\sigma}} \subseteq \overline{\mathcal{S}} \subseteq \overline{M_1}$, where the closure is taken in the $L^1(P)-$norm topology. However, $M_1$ is already closed in $L^1(P)$ and it is easy to check, using the property that the processes are bounded from below, that also $\mathcal{S}$ is closed in $L^1(P)$. Then the thesis follows from item c).

**Remark 3** The results on minimax measures and on the duality that we will provide in this paper hold for a semimartingale $X$ and for the set $M_1$ of separating measures. It is clear from the above Lemma that whenever the process $X$ is bounded (resp. locally bounded) then the same results hold for the set of $P-$absolutely continuous martingale measures (resp. local martingale measures).

The fundamental theorem of asset pricing essentially states that the existence of an equivalent separating measure is equivalent to a properly defined condition of No Arbitrage or No Free Lunch (see [6], [9], [14], [18], [22] and [25] for various formulations). The following condition of No Free Lunch with Vanishing Risk was introduced by Delbaen and Schachermayer [8]:

$$\textbf{NFLVR:} \quad \mathcal{C} \cap L^\infty_+ = \{0\}, \text{ where } \mathcal{C} \text{ is the } L^\infty_- \text{ norm closure of } C.$$  

Delbaen and Schachermayer [9] proved the following remarkable Theorem:

$$\textbf{NFLVR} \Leftrightarrow M_1 \cap \mathcal{P} \neq \emptyset. \quad (2)$$

This justifies the following

**Assumption 1:** $M_1 \cap \mathcal{P} \neq \emptyset$.

In I, II and III we have defined the market model $(\mathcal{P}, X, \mathcal{H})$ that we will always consider in the sequel. Assumption 1 will guarantee that the model is “free of arbitrage opportunities”.

### 1.2 The utility maximization problem

If the market is incomplete, the selection of a pricing measure in $M_1$ becomes a non trivial issue and one way to tackle this problem is to take into consideration the preferences of investors.

Assumption U will hold true throughout the paper.
**Assumption U**: The utility function \( u : \mathbb{R} \to \mathbb{R} \cup \{ -\infty \} \) is an upper semicontinuous concave function on \( \mathbb{R} \), its effective domain \( \mathcal{D}(u) \) is an interval of \( \mathbb{R} \) and \( u \) is nondecreasing on \( \mathcal{D}(u) \).

Note that \( u \) nowhere assumes the value \(+\infty\). It is understood that a concave function \( u : \mathbb{R} \to \mathbb{R} \cup \{ -\infty \} \) satisfies:

\[ \lambda u(x_1) + (1 - \lambda)u(x_2) \leq u(\lambda x_1 + (1 - \lambda)x_2), \quad \text{for all } 0 < \lambda < 1, \; x_1, \; x_2 \in \mathbb{R}, \]

the **effective domain** of \( u \) is the interval

\[ \mathcal{D}(u) = \{ x \in \mathbb{R} : u(x) > -\infty \}, \]

and \( u \) is continuous on \( \text{int}(\mathcal{D}(u)) \), the (assumed non empty) interior of \( \mathcal{D}(u) \).

The class of functions satisfying Assumption **U** is quite large. It comprehends for example the functions: \( u(x) = \ln(x) \), with \( \mathcal{D}(u) = (0, +\infty) \); \( u(x) = x^p \), with \( 0 < p < 1 \) and \( \mathcal{D}(u) = [0, +\infty) \); \( u(x) = -\frac{1}{x^p} \), with \( p > 0 \) and \( \mathcal{D}(u) = (0, +\infty) \).

Consider also the following stronger

**Assumption U2**: The utility function \( u : \mathbb{R} \to \mathbb{R} \) is concave and non decreasing, i.e. Assumption **U** is satisfied and \( \mathcal{D}(u) = \mathbb{R} \).

The quadratic-flat utility, defined by

\[
\begin{align*}
    u(x) &= \begin{cases} 
        -x^2 & \text{if } x < 0 \\
        0 & \text{if } x \geq 0
    \end{cases}, \quad (3)
\end{align*}
\]

and the exponential utility \( u(x) = -e^{-\alpha x}, \; \alpha > 0 \), are two examples of functions satisfying Assumption **U2**.

Note that neither in Assumption **U** nor in **U2** we require that the utility is strictly increasing and/or differentiable. Thus the class of utility functions satisfying Assumption **U2** comprehends concave and nondecreasing functions of the form

\[
\begin{align*}
    u_d(x) &= \begin{cases} 
        \tilde{u}(x) & x < d \\
        \tilde{u}(d) & x \geq d
    \end{cases}, \quad d \in (-\infty, +\infty),
\end{align*}
\]

which are constant for all \( x \) greater than some real \( d \).

In the sequel we will explicitly declare whether the use of Assumption **U2** is needed.

We denote with \( L^\infty(\mathcal{D}) \) the convex set of random variables having \( P \)-a.s. values in a compact set contained in the interior of the interval \( \mathcal{D} \subseteq \mathbb{R} \). This notation will be useful when we will deal with the integral functional \( w \to E_P[u(w)] \). Note that \( E_P[u(w)] \) is finite at each \( w \in L^\infty(\mathcal{D}(u)) \). It is clear that
$L^\infty(\mathcal{D}) \subseteq L^\infty$. If $\mathcal{D}(u) = \mathbb{R}$, as in the case of Assumption U2, then $L^\infty(\mathbb{R}) = L^\infty$.

Let $w_0 \in L^\infty(\mathcal{D}(u))$ be a fixed random variable, $P \in \mathbb{P}$ be a given probability measure and $G \subseteq L^0$ be a non empty convex cone in $L^0$. We formally define:

$$U_{P,G}(w_0) \triangleq \sup_{w \in G} E_P[u(w_0 + w)] \leq +\infty,$$  \hspace{1cm} (4)

where it is assumed that the supremum is taken over the set of random variables $w \in G$ for which $E_P[u(w_0 + w)]$ is finite. When $w_0$ is a constant, we interpret $w_0$ as the initial endowment of the agent. Since $w_0 \in L^\infty(\mathcal{D}(u))$ and $0 \in G$, we always have $U_{P,G}(w_0) > -\infty$.

**Definition 4** The maximum attainable utility from $w_0 \in L^\infty(\mathcal{D}(u))$ is given by:

$$U_{P,K}(w_0) = \sup_{w \in K} E_P[u(w_0 + w)] = \sup_{H \subseteq \mathcal{H}} E_P[u(w_0 + (H.X)_T)].$$  \hspace{1cm} (5)

In order to apply a duality result based on the dual system $(L^\infty, ba)$ we show, in the next Lemma, when we may replace in equation (5) the set $K$ with the set $C \subseteq L^\infty$. It is clear that, even though the value in equation (5) will not change with this substitution, the supremum will not in general be attained by an element in $L^\infty$.

Since $M_1 \not= \emptyset$, the polar cone $M^0$ of $M$ can be represented as:

$$M^0 \triangleq \{w \in L^\infty : E_Q[w] \leq 0 \text{ } \forall Q \in M_1\}.$$

**Lemma 5** Under Assumption 1 we have: $C = M^0$. Under Assumptions 1 and U we have:

$$U_{P,K}(w_0) \leq U_{P,C}(w_0) = U_{P,M^0}(w_0), \text{ } w_0 \in L^\infty(\mathcal{D}(u)).$$  \hspace{1cm} (6)

Under Assumptions 1 and U2 we have:

$$U_{P,K}(w_0) = U_{P,C}(w_0) = U_{P,M^0}(w_0), \text{ } w_0 \in L^\infty(\mathcal{D}(u)).$$  \hspace{1cm} (7)

**Proof.** The polar of $C$, with respect to the dual system $(L^\infty, L^1(P))$, is $M$. Delbaen and Schachermayer (see [8] & [9]) proved the following remarkable implication, true also for unbounded processes: NFLVR $\Rightarrow$ $C$ is weak* closed. When considering the dual system $(L^\infty, L^1(P))$, the bipolar theorem assures that $M^0 = C^{\text{bipolar}} = C$. The first assertion, the equality in (6) and the second equality in (7) follow.

To show (6) note that if $k \in K$ and $k_n \triangleq \min(k, n)$, then $k_n \uparrow k$ $P$-a.s, $k_n \in L^\infty$ and $k_n = k - (k - k_n) \in K - L^0_\uparrow$. Hence $k_n \in C$. By monotone convergence $Eu(x+k_n) \uparrow Eu(x+k)$, thus $\sup \{Eu(x+k)|k \in K\} \leq \sup \{Eu(x+f)|f \in C\}$.

Under Assumption U2, $u$ is nondecreasing on $\mathbb{R}$ and so

$$\sup \{Eu(x+f)|f \in C\} \leq \sup \{Eu(x+f)|f \in K - L^0_\uparrow\} = \sup \{Eu(x+f)|f \in K\},$$

and (7) follows.■
1.3 The existence of minimax measures

In this subsection we assume that \( w_0 \) is a constant and set \( w_0 = x \in \text{int}(\mathcal{D}(u)) \). By definition:

\[
U_{P,M^0}(x) = \{ \sup E_P[u(x + w)] \mid w \in L^\infty : E_Q[w] \leq 0 \ \forall Q \in M_1 \} = \{ \sup E_P[u(w)] \mid w \in L^\infty : E_Q[w] \leq x \ \forall Q \in M_1 \}. 
\]

Then, given the interpretation of \( M_1 \) as the set of pricing measures, \( U_{P,M^0}(x) \) represents the maximum expected utility that an agent can reach with the requirement that the maximum “No Arbitrage price” of the claim \( w \) is not bigger than the initial wealth \( x \).

For \( x \in \text{int}(\mathcal{D}(u)) \) and \( Q \ll P \) define:

\[
U(x; Q, P) \triangleq \{ \sup E_P[u(x + w)] \mid w \in L^\infty : E_Q[w] \leq 0 \} \quad (8) = \{ \sup E_P[u(w)] \mid w \in L^\infty : E_Q[w] \leq x \}.
\]

\( U(x; Q, P) \) is the maximum expected utility that an agent can reach with an initial endowment \( x \in \mathbb{R} \) when the pricing measure \( Q \) is known and hence the budget constraint set is given by: \( \{ w \in L^\infty : E_Q[w] \leq x \} \).

In the definition of \( U(x; Q, P) \) we are taking the supremum over bounded elements. But, as shown in the next Lemma, this is only an apparent restriction, since the supremum can be computed over the set of bounded from below random variables.

**Lemma 6** Under Assumption \( U \) we have

\[
U(x; Q, P) = \{ \sup E_P[u(x + w)] \mid w \in L^- : E_Q[w] \leq 0 \}, \ x \in \text{int}(\mathcal{D}(u)).
\]

The proof is similar to the second part of the proof of Lemma 5 and is omitted.

Notice that, for each \( Q \in M_1 \), \( U_{P,M^0}(x) \leq U(x; Q, P) \) and so \( U_{P,M^0}(x) \leq \inf_{Q \in M_1} U(x; Q, P) \).

**Definition 7** A probability measure \( \tilde{Q}_x \in M_1 \) is called a minimax measure if it satisfies:

\[
U_{P,M^0}(x) = \min_{Q \in M_1} U(x; Q, P) = U(x; \tilde{Q}_x, P), \ x \in \text{int}(\mathcal{D}(u)).
\]

The minimax measure \( \tilde{Q}_x \) has the property that the maximum expected utility \( U(x; \tilde{Q}_x, P) \), when the pricing measure is \( \tilde{Q}_x \), is equal to the maximum expected utility \( U_{P,M^0}(x) \) from all claims having maximum “no arbitrage price” not bigger than the initial wealth \( x \).

Why is the minimax measure a good selection for a pricing measure? Suppose that an agent has preferences represented by a utility function \( u \) satisfying
Assumption \(U\) and that she/he adopts a pricing functional which is determined by a measure \(Q^* \in M_1\) different from the minimax measure \(\tilde{Q}_x\). Then \(U_{P,M^0}(x)\) is greater or equal to the maximum attainable utility \(U_{P,K}(x)\) from initial endowment \(x \in \text{int}(D(u))\), and

\[
\sup_{w \in L^2 : E_{Q^*}[w] \leq x} E_P[u(w)] > \sup_{H \in \mathcal{H}} E_P[u(x + (H X)_T)],
\]

which doesn’t seem economically reasonable.

For the existence of a minimax measure one has first to exclude a “duality gap” between the two optimization problems (\(\sup E_P[u(x+w)]\) and \(\inf U(x; Q, P)\)), and then guarantee that the infimum is attained. Convex duality is obviously the main tool for this analysis.

In the primal problem the value function is \(U_{P,C}(x)\), and the maximum is taken on the set \(C\) of super replicable claims which is a subset of \(L^\infty\). Hence, the “natural” dual minimization problem is over \(ba\), and not over \(L^1(P)\). For this reason, in general, one may not expect to find a solution in the set of martingale measures.

We note that even the existence of a martingale measure is not guaranteed by the existence of an optimal solution to the primal problem, as it was clearly shown in a counterexample by Back and Pliska [2], where the pricing operator is in \(ba\), but lacks the countably additive property.

The main result of the paper is the following theorem which will be derived in Section 2.2 from a slightly more general result (Theorem 1). It guarantees the existence of the minimax measure for concave nondecreasing utility functions that are finite valued on \(\mathbb{R}\), as for example the exponential and quadratic-flat utility functions (see corollaries 3 and 4 for further details).

**Theorem (A)** Suppose that Assumptions 1 and \(U2\) hold true and assume that \(x \in \mathbb{R}\) and \(P \in \mathbb{P}\) satisfy \(U_{P,K}(x) < \sup_{y \in \mathbb{R}} u(y)\). Then

\[
U_{P,M^0}(x) = \min_{Q \in M_1} U(x; Q, P).
\]

If in addition \(\sup_{y \in \mathbb{R}} u(y) = +\infty\) then

\[
U_{P,M^0}(x) = \min_{Q \in M_1 \cap \mathbb{P}} U(x; Q, P).
\]

If \(X\) is bounded (resp. locally bounded) then the above statements are true if we replace \(M_1\) with \(\mathcal{M}\) (resp. \(\mathcal{M}_{loc}\)).

**Remark 8** By Lemma 5, we may replace, in the statement of Theorem A, \(U_{P,K}(x)\) with \(U_{P,C}(x)\) or \(U_{P,M^0}(x)\). Note that under Assumptions 1 and \(U2\), the condition \(U_{P,K}(x) = U_{P,M^0}(x) < \sup_{y \in \mathbb{R}} u(y)\) is equivalent to

\[
\exists Q \in M_1 : U(x; Q, P) < \sup_{y \in \mathbb{R}} u(y).
\]

In Section 4 we will discuss the boundedness assumption on \(U_{P,K}(x)\).
The relevance of the question about the existence of minimax measures was pointed out in the previous cited works ([24], [19] and [20]) where the central issue is the solution of the consumption portfolio problem, and also in Bellini-Frittelli [3], where \( \{u^{-1}(U(x; Q, P)) - x\} \) is interpreted as a generalized distance between probability measures.

Following the latter interpretation, we will see that many known criteria for choosing a pricing functional, that are based on the martingale approach, are special cases of the minimization of \( U(x; Q, P) \). Depending on which utility function is selected, the minimax measure that minimizes \( U(x; Q, P) \) will coincide with the optimal variance measure, as defined by Schweizer [30] (see remark 16 for the precise statement), or the minimal entropy measure, as defined in Frittelli [13], or the minimal Kakutani Hellinger measure. All these examples are given in Section 5.

General results on the existence of probability measures that minimizes divergence type of distances are provided in Csiszar [5] and Rüschendorf [27].

In a recent paper Kramkov and Schachermayer [23] studied the problem of the maximization of the utility of terminal wealth in a general setting, under minimal assumptions on utility functions defined on \((0, +\infty)\). They showed that, without restriction on the utility, the dual problem has an optimal solution only in a properly defined set of supermartingale measures. They also provided an example ([23], Example 5.1 bis) in which: \( u(x) = \log(x) \); the set of martingale measures is closed in variation (the price process is bounded); there is no duality gap; but, nevertheless, the infimum is not attained by a martingale measure.

We briefly mention some other related results that have appeared in literature after the first version of this paper was submitted.

In Cvitanic, Schachermayer and Wang [4] some of the results of Kramkov and Schachermayer [23] are extended to cover the case of utility maximization with random endowment, when the utility functions are defined on \((0, +\infty)\).

Schachermayer [28] treats the case of utility functions that are continuously differentiable, increasing and strictly concave on \((-\infty, +\infty)\). A sufficient (and “almost” necessary) condition for the existence of the optimal solution to the utility maximization problem is that the utility has reasonable asymptotic elasticity. As shown in Theorem A and also in Schachermayer [28], this condition is not necessary for the existence of the optimal solution to the dual minimization problem.

Goll and Rüschendorf [16] examine the relations between minimax measures and minimal distance measures in a fashion similar to ours, and characterize optimal utility based portfolios in terms of minimax measures.

The value of a claim, as introduced in Frittelli [15], is also based on utility maximization. The duality results stated in the present paper are used to compute pricing functionals \( \pi_u \) coherent with the no arbitrage principle. These functionals need not to be linear and, depending on which utility \( u \) is chosen,
admit representations of the forms

\[ \pi_u(w) = \inf_{Q \in \mathcal{M}_1} F_u(Q)E_Q[w], \]

where \( F_u : M_1 \to [1, +\infty) \) satisfies appropriate conditions, or

\[ \pi_u(w) = \inf_{Q \in \mathcal{M}_1} \{ E_Q[w] + H(Q, P) \} - \inf_{Q \in \mathcal{M}_1} H(Q, P), \]

where \( H(Q, P) \) is the relative entropy and \( u(x) = -e^{-x}. \)

Following an approach similar to the one proposed in Frittelli [15], El Karoui and Rouge [12] consider specifically the case of the exponential utility function, and compute the price of a claim exploiting the duality relation between utility maximization and relative entropy minimization. Further results in a dynamic context are provided using dynamic programming methods.

Also Delbaen et al. [11] treat specifically the exponential utility function and analyze the duality between utility maximization and entropy minimization. Under suitable assumptions on the class of trading strategies they provide sufficient conditions for the optimality of both problems.

2 Duality results

2.1 The general formulation

In this subsection we will not consider the market model presented in Section 1.1, since our results depend only on simple geometric properties of the set \( C. \)

To avoid any sort of confusion with the preceding section, we will denote with \( G \) the subset of \( L^\infty \) in which we are interested. The reader should keep in mind that the set \( G \) takes the place of the set \( C \) defined in equation (1). Analogously, in this section we replace \( M \) with the set \( N = G^0 \cap L^1(P). \) We will see that in the setting of Section 1.1, Corollary 13 will provide two sufficient conditions, one based on Proposition 9 and the other based on Theorem 11, for the existence of minimax measures.

The concave conjugate \( u^* : \mathbb{R} \to \mathbb{R} \cup \{ -\infty \} \) of \( u \) is given by:

\[ u^*(x^*) = \inf_{x \in \mathbb{R}} \left\{ x x^* - u(x) \right\}, \quad x^* \in \mathbb{R}. \]

Recall the definition of \( U_{P,G}(w_0) \) given in equation (4). An application of Fenchel duality theorem will allow us to prove, in Section 3, the next

**Proposition 9** Suppose that Assumption U holds true, \( N \subseteq L^1_+(P) \) is a not empty convex cone \( \sigma(\mathfrak{h}a, L^\infty) \) closed, \( G = N^0 = \{ w \in L^\infty : E_P[zw] \leq 0 \ \forall z \in N \} \) and that \( P \in \mathbb{P} \) and \( w_0 \in L^\infty(D(u)) \) satisfy \( U_{P,G}(w_0) < +\infty. \) Then \( N = G^0 \) and

\[ U_{P,G}(w_0) = \min_{z \in N} E_P[z w_0 - u^*(z)]. \quad (9) \]
Recall the definition of $U(x; Q, P)$ given in equation (8).

**Corollary 10** Suppose that Assumption U holds true, that $Q \ll P$ and $x \in \text{int}(\mathcal{D}(u))$ satisfy $U(x; Q, P) < \sup_{y \in \mathbb{R}} u(y)$. Then

$$U(x; Q, P) = \min_{\lambda \in (0, +\infty)} \left\{ \lambda x - E_P \left[ u^* \left( \frac{dQ}{dP} \right) \right] \right\}, \quad (10)$$

**Proof.** Let $Q$ and $x$ be given, set $N = \left\{ z \in L^1_+(P) : z = \lambda \frac{dQ}{dP}, \; \lambda \geq 0 \right\}$ and $w_0 = x$. Then $G = N^0 = \{ w \in L^\infty : E_Q[w] \leq 0 \}$, $U(x; Q, P) = U_{P,G}(x)$ and the hypotheses of Proposition 9 are satisfied. From Proposition 9 we get:

$$U(x; Q, P) = U_{P,G}(x) = \min_{z \in N} E_P[z x - u^*(z)]$$

$$= \min_{\lambda \in (0, +\infty)} \left\{ \lambda x - E_P[u^*(\lambda \frac{dQ}{dP})] \right\}.$$  

We may exclude $\lambda = 0$, since otherwise $U(x; Q, P) = E_P[-u^*(0)] = \sup_{y \in \mathbb{R}} u(y)$.

Using formula (10), in Section 5 we compute $U(x; Q, P)$ for many different functions $u$.

Our main result (Theorem A) will follow from Theorem 1. We state and prove Theorem 1 under an assumption which is slightly weaker than Assumption U2.

**Assumption U1:** Assumption U is satisfied and $u(x)$ is finite for all $x < d$, for some $d \in \mathbb{R}$.

The functions satisfying Assumption U1 are finite valued for all $x < d$, but may jump to $-\infty$ for $x \geq d$. It is clear that:

$$\text{U2} \Rightarrow \text{U1} \Rightarrow \text{U}.$$  

We are now ready to state the main result of this section:

**Theorem 11** Suppose that Assumption U1 holds true, that $G$ is a convex cone with $L^\infty \subseteq G \subseteq L^\infty$ and that $P \in \mathcal{P}$ and $w_0 \in L^\infty(\mathcal{D}(u))$ satisfy $U_{P,G}(w_0) < +\infty$. Then:

$$U_{P,G}(w_0) = \min_{z \in N = G \cap L^1(P)} E_P[z w_0 - u^*(z)]. \quad (11)$$

**Remark 12** Equations 9 and 11 are the same and in both equations $N \subseteq L^1_+(P)$. In Theorem 11 the set $N$ is closed in $L^1(P)$ but we require the stronger Assumption U1. On the other hand, in Proposition 9 we only need the weaker Assumption U but the set $N$ has to be $\mathcal{M}(ba, L^\infty)$-closed.

If $\mathcal{D}(u) = (0, +\infty)$ and $N$ is not $\mathcal{M}(ba, L^\infty)$-closed, the minimum may not be attained by an element in $N$. This is shown in counterexample 5.1 bis of [23] where $u(x) = \log(x)$.  

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Corollary 13  Under either the assumptions of Proposition 1 or of Theorem 1 we have:

a) If \( U_{P,G}(w_0) < \sup_{x \in \mathbb{R}} u(x) \), then \( N_1 \triangleq \{ z \in N : E_P[z] = 1 \} \neq \emptyset \) and
\[
U_{P,G}(w_0) = \min_{Q \in N_1} U(E_Q[w_0]; Q, P).
\]

b) If \( \sup_{x \in \mathbb{R}} u(x) = +\infty \), then \( N_1 \cap P \neq \emptyset \) and
\[
U_{P,G}(w_0) = \min_{Q \in N_1 \cap P} U(E_Q[w_0]; Q, P).
\]

Proof. We can assume that either equation (9) or (11) holds true. We may exclude that \( z = 0 \) attains the minimum, otherwise \( U_{P,G}(w_0) = E_P[-u^*(0)] = \sup_{x \in \mathbb{R}} u(x) \). Hence \( N_1 \subseteq L_+(P) \) is not empty. We have
\[
U_{P,G}(w_0) = \min_{z \in N, z \neq 0} \{ E_P[z w_0] - E_P[u^*(z)] \}
= \min_{Q \in N_1} \min_{\lambda \in (0, +\infty)} \left\{ \lambda E_Q[w_0] - E_P \left[ u^* \left( \frac{dQ}{dP} \right) \right] \right\}
= \min_{Q \in N_1} U(E_Q[w_0], Q, P),
\]
where the last equality follows from Corollary 10. Item a) is proved. If \( u \) is unbounded from above, then \( u^*(z^*) = \inf_{x \in \mathbb{R}} \{ x z^* - u(x) \} = -\infty \) on the set \( \{ z^* = 0 \} \). If \( z^* \) attains the minimum and if \( P(\{ z^* = 0 \}) > 0 \), then \( U_{P,G}(w_0) = E_P[z^* w_0 - u^*(z^*)] = +\infty \), which is impossible. Therefore \( P(\{ z^* = 0 \}) = 0 \) and item b) follows. ■

2.2 Formulation in the setting of Section 1.1

We apply the above results to the market model presented in Section 1.1. Let \( G = C, N = M, N_1 = M_1 \) and suppose that Assumption 1 holds true.

Proof of Theorem A

Recall that the set \( C \) is defined in equation (1) and it is a convex cone satisfying \( L^\infty \subseteq C \subseteq L^\infty \). Since \( U2 \Rightarrow U1 \), we may apply Theorem 11 to \( G = C, N = M \) and \( w_0 = x \). Since \( U_{P,M^0}(x) = U_{P,C}(x) \), from conditions a) and b) of Corollary 13 we immediately deduce Theorem A. ■

Let
\[ u(x) = -e^{-x}, \quad x \in \mathbb{R}, \]
be the exponential utility function. Then \( \mathcal{D}(u) = \mathbb{R}, U_{P,K}(w_0) = U_{P,C}(w_0) = U_{P,M^0}(w_0), w_0 \in L^\infty, \) and \( U(x, Q, P) = -e^{-H(Q,P)-x} \), where \( H(Q, P) \) is the relative entropy (see Section 5.1). When \( X \) is bounded (resp. locally bounded) the minimax measure coincides with the minimal entropy martingale (resp. local martingale) measure, as defined in Frittelli [13].

In this case from Corollary 13 we immediately deduce
Corollary 14 Suppose that Assumption 1 holds true and that \( w_0 \in L^\infty \). If there exists \( Q \in M_1 \) such that \( H(Q,P) < +\infty \) then
\[
\sup_{H \in \mathcal{H}} E_P[-e^{-w_0 - (H,X)_T}] = \min_{Q \in M_1} \left\{ -e^{-E_Q[w_0]} - H(Q,P) \right\}.
\]

Proof. Let \( u(x) = -e^{-x} \), \( w_0 \in L^\infty \), \( G = C \) and \( N_1 = M_1 \). The thesis follows from item a) of Corollary 13 by noting that the following conditions are all equivalent: i) \( U_{PC}(w_0) < \sup_{y \in R} u(y) \); ii) \( \exists Q \in M_1 : U(E_Q[w_0], Q, P) < \sup_{y \in R} u(y) \); iii) \( \exists Q \in M_1 : -e^{-H(Q,P) - E_Q[w_0]} < 0 \); iv) \( \exists Q \in M_1 : H(Q,P) < +\infty \).

This means that when the market is “free of arbitrage”, the duality relation between utility maximization and entropy minimization holds under the only assumption of the existence of a separating measure with finite relative entropy.

Let \( u \) be the quadratic-flat utility function defined in equation (3). Then \( \mathcal{D}(u) = \mathbb{R} \), \( U_{PK}(w_0) = U_{PC}(w_0) = U_{PM}(w_0) \), \( w_0 \in L^\infty \), and
\[
U(x, Q, P) = \frac{u(x)}{E_P[(\frac{dQ}{dP})^2]}, \text{ for all } x \in \mathbb{R}.
\]
As in the case of the exponential utility, from Corollary 13 we immediately deduce

Corollary 15 Suppose that Assumption 1 holds true and that \( w_0 \in L^\infty(-\infty, 0) \). If there exists \( Q \in M_1 \) such that \( E_P[(\frac{dQ}{dP})^2] < +\infty \) then
\[
\sup_{H \in \mathcal{H}} E_P[u(w_0 + (H,X)_T)] = \min_{Q \in M_1} \frac{u(E_Q[w_0])}{E_P[(\frac{dQ}{dP})^2]} = - \max_{Q \in M_1} \frac{(E_Q[w_0])^2}{E_P[(\frac{dQ}{dP})^2]}.
\]

where \( u \) is given in equation (3).

Remark 16 Under the same assumptions of the previous corollary and if \( w_0 = x < 0 \), we have \( u(x) < 0 \) and
\[
\sup_{H \in \mathcal{H}} E_P[u(x + (H,X)_T)] = \frac{-x^2}{\min_{Q \in M_1} E_P[(\frac{dQ}{dP})^2]}.
\]

This implies the existence of the minimax measure \( \hat{Q} \in M_1 \) that minimizes the variance of \( \frac{dQ}{dP} \) over the set \( M_1 \), i.e.
\[
\min_{Q \in M_1} Var_P[\frac{dQ}{dP}] = Var_P[\frac{d\hat{Q}}{dP}] = 0.
\]
Since all elements in \( M_1 \) are probability measures, \( \hat{Q} \) coincides with the optimal variance signed local martingale measure \( Q^{var} \), as defined in Schweizer [30], whenever \( M_1 = M_{loc} \) and \( Q^{var} \) is a probability measure - not only a signed measure. Both conditions are satisfied, for example, if the process \( X \) is continuous (see Lemma 2 item b, and Delbaen & Schachermayer [10]).
As already noticed, the class of utility functions satisfying Assumption $U$ is quite large. In Section 5 we will provide some examples of minimax measures that arise also from these utility functions. However, under Assumption $U$, we are able to prove (Proposition 1 and Corollary 13) the existence of the minimax measure only under the additional assumption that the set $M$ is $\sigma(\mathcal{B}, \mathcal{L}^\infty)$-closed.

This assumption is satisfied in the simple case of a finite dimensional market or in the case of a complete market where $M_1$ is a singleton.

When the utility functions are finite valued on $(0, +\infty)$, other sufficient conditions for the existence of a minimax measure are provided in Krankov and Schachermayer [23].

3 Proofs

We start with some preliminary facts from convex duality.

We denote the indicator functional of a convex set $F \subseteq \mathcal{L}^\infty$ with

$$\delta(w|F) \triangleq \begin{cases} 0 & w \in F \\ +\infty & w \notin F \end{cases},$$

and the convex conjugate of the indicator functional $\delta(\cdot|F)$ with $\delta_F^*: ba \to \mathbb{R}$

$$\delta_F^*(\xi) \triangleq \sup_{w \in F} \xi(w).$$

Define the concave integral functional

$$I_u : \mathcal{L}^\infty \to \mathbb{R} \cup \{-\infty\}, \quad I_u(w) \triangleq E\mathbb{P}[u(w)],$$

and its effective domain

$$\mathcal{D}(I_u) = \{w \in \mathcal{L}^\infty : I_u(w) > -\infty\}.$$

By definition, the concave conjugate $I_u^*: ba \to \mathbb{R} \cup \{-\infty\}$ of the integral functional $I_u$ is

$$I_u^*(\xi) = \inf_{w \in \mathcal{L}^\infty} \{\xi(w) - I_u(w)\}, \quad \xi \in ba.$$

The next Lemma follows from the characterization of the conjugate of the integral functional provided by Rockafellar (in Appendix we prove Lemma 17 and state Rockafellar’s Theorem).

**Lemma 17** Suppose that $u$ satisfies Assumption $U$ and let $u^*$ be the concave conjugate of $u$. Then:

a) $I_u$ is continuous in the norm topology at any $w_0 \in \mathcal{L}^\infty(\mathcal{D}(u))$;
b) $I_u^*: L^1(P) \to \mathbb{R} \cup \{-\infty\}, \quad I_u^*(z) \triangleq E\mathbb{P}[u^*(z)]$ is well defined on $L^1(P)$ and satisfies

$$I_u^*(z) = I_u^*(z) \quad \forall z \in L^1(P);$$

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c) the concave functional $I_u^* : ba \rightarrow \mathbb{R} \cup \{-\infty\}$ conjugate to $I_u$ is given by

$$I_u^*(\xi) = I_u(\xi_a) - \delta_{D(I_u)}(-\xi_s), \quad \xi \in ba,$$

where $\xi = \xi_a + \xi_s$ is the Yosida-Hewitt decomposition of $\xi$ (see Appendix).

To simplify the notations we denote with $f$ the indicator functional of the set $w_0 + G$:

$$f : L^\infty \rightarrow \mathbb{R} \cup \{+\infty\}, \quad f(w) \overset{\Delta}{=} \delta(w|w_0 + G).$$

The proof of both Proposition 1 and Theorem 1 is based on the following application of Fenchel duality theorem:

**Lemma 18** Suppose that Assumption $U$ holds true, that $G$ is a convex cone with $L^\infty \subseteq G \subseteq L^\infty$, and that $P \in \mathbb{P}$ and $w_0 \in L^\infty(D(u))$ satisfy $U_{P,G}(w_0) < +\infty$. Then

$$U_{P,G}(w_0) = \min_{\xi \in G^0} \{f^*(\xi) - I_u^*(\xi)\}.$$  

**Proof.** Let $w_0 \in L^\infty(D(u))$. Then $I_u$ is continuous at $w_0$ (Lemma 17 item a)), $w_0$ is an interior point of $D(I_u)$ and the epigraph of $I_u$ has non empty interior in the product topology of $L^\infty \times \mathbb{R}$.

We have:

$$f^*(\xi) = \sup_{w \in \mathbb{R} \cup G} \xi(w) = \xi(w_0) + \sup_{w \in G} \xi(w) = \begin{cases} \xi(w_0) & \xi \in G^0 \subseteq ba \\ +\infty & \text{otherwise} \end{cases}. $$

Since $L^\infty \subseteq G$, then $\{w \in L^\infty : w \leq w_0, P - a.s.\} \subseteq w_0 + G = D(f)$, the effective domain of $f$. Therefore $w_0 \in D(f) \cap D(I_u)$ and the set $D(f) \cap D(I_u)$ has non empty interior in the $L^\infty$-norm topology.

Since $U_{P,G}(w_0)$ is finite, from Fenchel Theorem in Appendix, we deduce:

$$U_{P,G}(w_0) = \sup_{w \in \mathbb{R} \cup G} \xi(w) = \sup_{w \in \mathbb{R} \cup G} \{I_u(w) - \delta(w|w_0 + G)\}$$

$$= \min_{\xi \in ba} \{f^*(\xi) - I_u^*(\xi)\}$$

$$= \min_{\xi \in G^0} \{f^*(\xi) - I_u^*(\xi)\},$$

where the last equality follows from the previous computation of $f^*(\xi)$. ■

**Proof of Proposition 1**

Since $N \subseteq L^\infty_+(P)$ is a convex cone $\sigma(ba, L^\infty)$-closed, the bipolar theorem guarantees that $G^0 = N^{00} = N$. If $z \in G^0 \subseteq L^\infty_+(P)$ then $f^*(z) = E_P[zw_0]$. $G = N^0$ is a convex cone satisfying $L^\infty \subseteq G \subseteq L^\infty$, hence from Lemma 18 and Lemma 17 item b) we get

$$U_{P,G}(w_0) = \min_{\xi \in G^0} \{f^*(\xi) - I_u^*(\xi)\}$$

$$= \min_{z \in N} \{E_P[zw_0] - E_P[u^*(z)]\}. \blacksquare$$
Lemma 19 Suppose that $\xi = \xi_u + \xi_a \in ba$ satisfies $I^*_u(\xi) > -\infty$. If $(-\infty, \varepsilon) \subseteq D(u)$, $\varepsilon > 0$, then $\xi_a \in (ba)_-$. 

Proof. First note that, if $(-\infty, \varepsilon) \subseteq D(u)$, then $L^\infty \subseteq D(I_u)$, the polar of $D(I_u)$ satisfies:
$$D(I_u)^0 = \{ \xi \in ba : \xi(w) \leq 0 \ \forall w \in D(I_u) \} \subseteq (ba)_+,$$
and we have:
$$\delta^*_D(I_u)(\xi) = \begin{cases} 0 & \text{if } \xi \in D(I_u)^0 \\ +\infty & \text{otherwise} \end{cases}.$$ 

Therefore, from Lemma 17 item c), $I^*_u(\xi) > -\infty$ implies $\delta^*_D(I_u)(-\xi_a) < +\infty$ and $-\xi_a \in D(I_u)^0 \subseteq (ba)_+$ (note that when $D(u) = \mathbb{R}$, we simply have $D(I_u)^0 = \{0\}$, and so $I^*_u(\xi) > -\infty$ implies $\xi_a = 0$). 

Proof of Theorem 1

We first assume that $(-\infty, \varepsilon) \subseteq D(u)$, $\varepsilon > 0$.

From Lemma 18 we have:
$$U_{P,G}(w_0) = \min_{\xi \in G^0} \{ f^*(\xi) - I^*_u(\xi) \} \quad (12)$$
$$= \min_{z \in G^0 \cap L^1_+} \{ EP[z w_0] - EP[u^*(z)] \}. \quad (13)$$

To show the last equality suppose that $\xi^* \in G^0$ attains the minimum in equation (12). Then we have $I^*_u(\xi^*) > -\infty$, which implies $\xi^*_a \in (ba)_-$, by Lemma 19. From $L^\infty \subseteq G$ we get $G^0 \subseteq (ba)_+$. Then $\xi^*_a \in G^0$ implies $\xi^*_a = 0$ and $\xi^* \in L^1_+$. But then, by Lemma 17 item b), $I^*_u(\xi^*) = EP[u^*(\xi^*)]$ and equation (13) follows.

We now end the proof of Theorem 1 by showing that the condition $(-\infty, \varepsilon) \subseteq D(u)$ can be easily avoided. Under Assumption U1 we have that $(-\infty, d) \subseteq D(u), \ \text{where } d \in \mathbb{R}$ is arbitrary. Let $U_{P,G}(w_0) = \sup_{w \in G} EP[u(w_0 + w)] < +\infty$ and $w_0 \in L^\infty(D(u))$. Let $\varepsilon > 0$, $\alpha \triangleq d - \varepsilon$, $u_a(x) \triangleq u(\alpha + x)$, $w_a \triangleq w_0 - \alpha$ and $U_{P,G}(w_0) = \sup_{w \in G} EP[u_a(w_0 + w)]$. Then $D(u_a) = \{ D(u) - \alpha \} = \{ x - \alpha, x \in D(u) \}$, $(-\infty, \varepsilon) \subseteq D(u_a)$ and $w_a \in L^\infty(D(u_a))$. $U^*_a(w_a) = U_{P,G}(w_a) < +\infty$ and $u^*_a(x^*) = -\alpha x^* + u^*(x^*)$. Since $(-\infty, \varepsilon) \subseteq D(u_a)$, then we may apply Theorem 11 to the function $u_a$ and get:
$$U_{P,G}(w_0) = U^*_a(w_a) = \min_{z \in G^0 \cap L^1_+} EP[z w_a - u^*_a(z)] = \min_{z \in G^0 \cap L^1_+} EP[z w_0 - u^*(z)].$$

4 Viability, separating measures and No Arbitrage

We now consider the market model presented in Section 1.1, but we don’t suppose a priori that Assumption 1 holds true, because we want to analyze the
relationship between the condition $U_{P,K}(x) < +\infty$ and Assumption 1. In this section we suppose that the utility functions satisfy the following

**Assumption U3:** The utility function $u : \mathbb{R} \to \mathbb{R}$ is concave, nondecreasing and unbounded from above (i.e. Assumption U2 is satisfied and $u(+\infty) = +\infty$).

We say that the market $(\mathbb{P}, X, \mathcal{H})$ is viable if there exists at least an agent with initial endowment $x \in \mathbb{R}$ that cannot increase arbitrarily his expected utility from terminal wealth by trading in claims $w \in K$ i.e. claims which are attainable by admissible trading strategies $H \in \mathcal{H}$.

**Definition 20** The market $(\mathbb{P}, X, \mathcal{H})$ is viable if there exist $P \in \mathbb{P}$, $x \in \mathbb{R}$ and a utility function $u$ (satisfying Assumption U3) such that:

$$U_{P,K}(x) = \sup_{w \in K} E_P[u(x + w)] < +\infty.$$

This definition can be compared with the one given in Kreps [25]. In our context the “commodity set” is $L^-$ and the set of “marketed bundles” (having zero initial price) is $K$, which are only convex cones and not linear spaces, as in Kreps. We require the preference relations to be representable by expected utility, to be nondecreasing and concave. However, we do not require, as in Kreps, the existence of an optimal solution $w^* \in K$. But we do require that the supremum of expected utility is finite, even though the utility function is unbounded from above (and from below).

In his setting, Kreps showed that viability is equivalent to the existence of an “extension operator”. If the extension operator is representable by an element of $L^1$, then it corresponds to an equivalent martingale measure (if the process $X$ is bounded).

In the next Proposition we are able to prove the existence of an equivalent separating measure without requiring a priori the existence of an optimal solution to the primal problem. Other sufficient conditions, for the existence of a pricing operator in $L^1$, are provided in Back and Pliska [2], again under the additional assumption of the existence of an optimal demand.

**Proposition 21** The market $(\mathbb{P}, X, \mathcal{H})$ is viable if and only if there exists an equivalent separating measure.

**Proof.** Since $u$ is finite valued on $\mathbb{R}$, the definitions of $C$ and $K$ imply $U_{P,C}(x) \leq U_{P,K}(x)$. Hence, if the market is viable, $U_{P,C}(x) < +\infty$. Since U3 implies U1, we may apply Theorem 11 and Corollary 2 to the sets $G = C$ and $N_1 = M_1$, and deduce the existence of an equivalent separating measure.

Viceversa, if $Q$ is an equivalent separating measure then $Q \in \mathbb{P}$ and $E_Q[w] \leq 0$ for all $w \in K$. Then for all concave, nondecreasing utility $u$ and $x \in \mathbb{R}$ we get:

$$\sup_{w \in K} E_Q[u(x + w)] \leq \sup_{w \in K} u(x + E_Q[w]) \leq u(x) < +\infty,$$
and indeed $U_{Q,K}(x) = u(x)$. ■

As an immediate consequence of Proposition 21 and the Fundamental Theorem of Asset Pricing stated in equation (2) we get:

**Theorem 22** The following conditions are equivalent:
1) There exists an equivalent separating measure,
2) The market $(\mathbb{P}, X, \mathcal{H})$ is viable,
3) NFLVR holds.

## 5 Examples of minimax measures

When appropriate regularity conditions are satisfied, it is known that the expression for $U(x; Q, P)$ in equation (10) can be rewritten as:

$$U(x; Q, P) = E_P \left[ u \left( I \left( \lambda_{x, Q} \frac{dQ}{dP} \right) \right) \right],$$

where $I \triangleq (u')^{-1}$ and $\lambda_{x, Q} > 0$ is the unique solution of the equation $E_Q \left[ I \left( \lambda_{x, Q} \frac{dQ}{dP} \right) \right] = x$. In the following examples we will set $\varphi = \frac{dQ}{dP}$.

### 5.1 Exponential utility - Relative entropy $H(Q, P)$

Let $u(x) = -e^{-x}$, $\mathcal{D}(u) = (-\infty, +\infty)$. Then

$$U(x; Q, P) = -e^{-H(Q, P) - x}, \quad x \in (-\infty, +\infty),$$

where $H(Q, P) = E_P[\varphi \ln \varphi]$ is the relative entropy of $Q$ with respect to $P$.

This means that the relative entropy

$$H(Q, P) = \sup_{u \in L^\infty : \mathbb{E}_Q[u] \leq 0} u^{-1}(E_P[u(x + w)]) - x = \sup_{u \in L^\infty : \mathbb{E}_Q[u] = 0} u^{-1}(E_P[u(w)])$$

is the maximum certainty equivalent of all contingent claims having zero prices under $Q$.

The minimax measure minimizes the relative entropy $H(Q, P)$.

### 5.2 Log utility - Relative entropy $H(P, Q)$

Let $u(x) = \ln x$, $\mathcal{D}(u) = (0, +\infty)$. Then

$$U(x; Q, P) = u(x) + H(P, Q), \quad x \in (0, +\infty), \quad Q \sim P.$$

The minimax measure minimizes the relative entropy $H(P, Q)$, for $Q \sim P$. 

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5.3 Power class of utility

Let

\[ u(x) = \frac{p}{1-p} |x|^p, \text{ where } \begin{cases} D(u) = (0, +\infty) & \text{if } p < 0 \\ D(u) = [0, +\infty) & \text{if } 0 < p < 1 \\ D(u) = (-\infty, 0] & \text{if } p > 1 \end{cases} \]

Then, letting \( q \in \mathbb{R} : 1/p + 1/q = 1 \),

\[ U(x; Q, P) = u(x) (E_P[\varphi^q])^{1-p}, \text{ x \in \text{int}(D(u))}. \]

5.3.1 Kakutani-Hellinger measure

1. For \( p < 0 \) we have \( 0 < q = \frac{p}{p-1} < 1 \), \( u(x) < 0 \) and

\[ U(x; Q, P) = u(x) (H(q; Q, P))^{1-p}, \text{ x \in (0, +\infty)}, \]

where \( H(q; Q, P) = E_P[\varphi^q] \) is the Hellinger integral of order \( q \). The minimax measure maximizes the Hellinger integral of order \( q \).

2. For \( p = -1 \) we get \( u(x) = -\frac{1}{2} x^2 \) and

\[ U(x; Q, P) = u(x) \left( H\left(\frac{1}{2}; Q, P\right)\right)^2, \text{ x \in (0, +\infty)}. \]

The minimax measure minimizes the Kakutani-Hellinger distance \( \rho(Q, P) = \sqrt{1 - H\left(\frac{1}{2}; Q, P\right)} \).

5.3.2 Optimal variance measure

1. For \( p = 2 \) we get \( u(x) = -2x^2 \) and

\[ U(x; Q, P) = \frac{u(x)}{\text{Var}_P[\varphi^2]} = \frac{u(x)}{\text{Var}_P[\varphi] + 1}, \text{ x \in (-\infty, 0)}. \]

The minimax measure minimizes the variance of \( \frac{dQ}{dP} \).

2. For \( p = \frac{2}{3} \) we get \( u(x) = 2 \sqrt[3]{x^2} \) and

\[ U(x; Q, P) = \frac{u(x)}{\sqrt[3]{E_P\left[\frac{1}{\varphi^2}\right]}}, \text{ x \in (0, +\infty)}. \]

The minimax measure minimizes the variance of \( \frac{dP}{dQ} \), for \( Q \sim P \).

5.3.3 Measure with minimal expectation

For \( p = \frac{1}{2} \) we get \( u(x) = \sqrt{x} \) and

\[ U(x; Q, P) = \frac{u(x)}{\sqrt{E_P\left[\frac{1}{\varphi}\right]}}, \text{ x \in (0, +\infty)}. \]

The minimax measure minimizes the expected value of \( \frac{dP}{dQ} \), for \( Q \sim P \).
6 Appendix

A relevant application of Hahn Banach theorem is the well known Fenchel duality theorem, which we restate for the dual system \((L^\infty, ba)\).

**Theorem 23** (Fenchel duality theorem. See Luenberger [21], Th.1, Ch.7.12)

Suppose that \(f : L^\infty \to \mathbb{R} \cup \{+\infty\} \) is convex, \(g : L^\infty \to \mathbb{R} \cup \{-\infty\} \) is concave, \(D(f) \cap D(g) \) contains points in the relative interior of \(D(f) \) and \(D(g) \), and either the epigraph of \(f \) or of \(g \) has non empty interior in the product topology of \(L^\infty \times \mathbb{R} \) where \(L^\infty \) is endowed with the norm topology. If \(\sup_{w \in L^\infty} \{g(w) - f(w)\} \) is finite then

\[
\sup_{w \in L^\infty} \{g(w) - f(w)\} = \min_{\xi \in ba} \{f^*(\xi) - g^*(\xi)\}
\]

and the minimum is attained by an element in \(ba\).

Let \(\pi \in ba(\Omega, \mathcal{F}, P), \pi \geq 0\). Then \(\pi\) is said to be purely finitely additive if the only countably additive non negative set function \(\xi \in ba(\Omega, \mathcal{F}, P)\) such that \(\xi \leq \pi\) is \(\xi = 0\).

**Theorem 24** (Decomposition of ba. See Yosida-Hewitt [31], Th. 1.23)

If \(\xi \geq 0\) is in \(ba(\Omega, \mathcal{F}, P)\), then there exists unique \(\xi_a \geq 0, \xi_s \geq 0\) in \(ba(\Omega, \mathcal{F}, P)\) such that \(\xi_a\) is countably additive, \(\xi_s\) is purely finitely additive and \(\xi = \xi_a + \xi_s\).

The next theorem is a reformulation of theorems 1 and 2 of Rockafellar [26] in a form convenient for our purposes.

**Theorem 25** (Characterization of \(I^*_u\). Rockafellar [26], Th. 1 and 2)

Suppose that \(u : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \) is a lower semicontinuous convex function and that \(\overline{\omega} \in L^\infty\) and \(r \in \mathbb{R}_+\) are such that \(u(x + \overline{\omega})\) is integrable whenever \(|x| < r\). Let \(u^*\) be the convex conjugate of \(u\). Then:

1) \(I^*_u : L^\infty \to \mathbb{R} \cup \{+\infty\}\) is well defined on \(L^\infty\) and is continuous in the \(L^\infty\) - norm topology at \(w\) whenever \(|w - \overline{\omega}| < r\);

2) \(I^*_u : L^1(P) \to \mathbb{R} \cup \{+\infty\}, I^*_u(z) = E[u^*(z)]\) is well defined on \(L^1(P)\) and satisfies

\[
I^*_u(z) = I^*(z) \quad \forall z \in L^1(P);
\]

3) the convex functional \(I^*_u : ba \to \mathbb{R} \cup \{+\infty\}\) conjugate to \(I_u\) is given by

\[
I^*_u(\xi) = I^*_u(\xi_a) + \delta_D(\xi_s), \quad \xi \in ba
\]

where \(D = \{w \in L^\infty : I_u(w) < +\infty\}\) and \(\xi = \xi_a + \xi_s\) is the Yosida-Hewitt decomposition of \(\xi \in ba\).

We note that item 2) follows from lines 1-3 in the proof of Rockafellar [26], Theorem 1.

**Proof. of Lemma 17.**
a) follows from Theorem 25 item 1 and the definition of $L^\infty(D)$.
b) follows from Theorem 25 item 2.
c) is a direct application of Theorem 25 item 3. Just recall that $g^*(\xi) = -f^*(-\xi)$ if $f = -g$ is convex and “*” denotes the appropriate concave/convex conjugate. Then:

\[ I^*_a(\xi) = - I^*_u(-\xi) = - \{ I_{(-u)}(-\xi_u) + \delta_D(-\xi_u) \} = I_{-u}(\xi_u) - \delta_D(I_{-u})(-\xi_u) \]

and $D = \{ w \in L^\infty : I_{-u}(w) < +\infty \} = \{ w \in L^\infty : I_u(w) > -\infty \} = D(I_u)$. ■

References


