Complete duality for quasiconvex dynamic risk measures on modules of the $L^p$-type

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Abstract
We provide a dual representation of quasiconvex conditional risk measures defined on $L^0$ modules of the $L^p$ type. This is a consequence of more general result which extend the usual Penot-Volle representation for quasiconvex real valued maps. We establish, in the conditional setting and in the module framework, a complete duality between quasiconvex risk measures and the appropriate class of dual functions.

Keywords: quasiconvex functions, dual representation, complete duality, $L^0$-modules, dynamic risk measures, quasiconvex risk measures.

1 Introduction
The already-twenty-years-old theory of risk measures is still originating many questions and springing out lots of new problems which trigger off the interest of researchers. Recently Kupper and Schachermayer [KS10] showed that in a dynamic framework only the entropic risk measure is in agreement with all the usual assumptions such as cash additivity, monotonicity, convexity, law invariance and time consistency. It’s thus natural to question if these assumptions are too restrictive: cash additivity was the first to be doubted and weakened to cash subadditivity, in El Karoui and Raveneau [ER09].

Currently a debate between convexity and quasiconvexity is trying to give a better explanation to the concept of diversification, see Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio [CV10]. In this latter paper, the representation of a quasiconvex cash subadditive (real valued) risk measure $\rho$ is written in terms of the dual function $R$. In the conditional setting, where the maps take values in a set of random variables - for example, $\rho : L^p(\Omega, \mathcal{F}_T, \mathbb{P}) \to L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, $t < T$ - the representation of dynamic quasiconvex maps is obtained by Frittelli and Maggis [FM09] and the particular case of the Conditional Certainty

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Equivalent is treated in Frittelli Maggis [FM10]. We stress that the conditional setting is very relevant in all applications involving dynamic features.

In [CV09], Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio provide a complete duality for real valued quasiconvex functions: the idea is to prove a one to one relationship between quasiconvex monotone functionals \( \rho \) and the function \( R \) in the dual representation. Obviously \( R \) will be unique only in an opportune class of maps satisfying certain properties. In Decision Theory this uniqueness is a very important feature (see [CV09]). A similar result has been recently obtained also by Drapeau and Kupper in [DK10].

In all these papers the quasiconvex maps are defined on vector spaces. This classical approach has recently been argued by Filipovic, Kupper and Vogelpoth [FKV10], where they showed some of the advantages of working in a module framework. The intuition behind the use of modules is simple and natural: given a probability space \( (\Omega, \mathcal{F}_T, \mathbb{P}) \) and a filtration \( \mathcal{F} = \{ \mathcal{F}_t \}_{0 \leq t \leq T} \), suppose that a set \( L \) of time-\( T \) maturity contingent claims is fixed (for concreteness let \( L = L^p(\mathcal{F}_T) \)) and an agent is computing the risk of a portfolio at an intermediate time \( t < T \). All the \( \mathcal{F}_t \)-measurable random variables are going to be known at time \( t \), thus the \( \mathcal{F}_t \)-measurable random variables will act as scalars in the process of diversification of our portfolio, forcing to consider the new set

\[
L^p_{\mathcal{F}_t}(\mathcal{F}_T) := L^0(\Omega, \mathcal{F}_t, \mathbb{P}) \cdot L^p(\Omega, \mathcal{F}_T, \mathbb{P}) = \{ YX \mid Y \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}), X \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}) \}
\]

as the domain of the risk measures. This product structure is exactly the one that constitutes the \( L^0 \)-modules.

The concept of module over a ring of functions is not new: it already appeared in the Fifties, [Gh50], [Ha64], [Ha65] and [Or69]. Hahn Banach type extension theorems were firstly provided for particular classes of rings and finally at the end of seventies (see for instance [BS77]) general ordered rings were considered, so that the case of \( L^0 \) was included. However, until [FKV09], no Hyperplane Separation Theorems were obtained. It is well known that many fundamental results in Mathematical Finance rely on it: for instance Arbitrage Theory and the duality results on risk measure or utility maximization.

In the series of three papers [FKV10], [FKV09] and [KV09] the authors brilliantly succeed in the task of giving a relevant and useful topological structure to \( L^0 \)-modules and to extend those theorems from functional analysis, which are relevant for financial applications. Once this rigorous analytical background has been carefully built up, it is possible to develop it further and obtain other interesting results and applications.

The overall aim of this paper is twofold:

(a) the extension of the results in [FM09] to \( L^0 \)-modules (Theorem 16 and Corollaries 17 and 18);
(b) the establishment of a complete duality for quasiconvex conditional maps (Theorem 22).

Our findings may be applied in a dynamic framework in several areas, such as Decision Theory (see [CV09]) and Risk Measures (see [CV10]).

a) On the dual representation of quasiconvex maps. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space and $\mathcal{G} \subseteq \mathcal{F}$ be any sigma algebra contained in $\mathcal{F}$. We prove that a quasiconvex regular map $\pi : E \to L^0(\mathcal{G})$ can be represented as

$$\pi(X) = \sup_{\mu \in \mathcal{L}(E, L^0(\mathcal{G}))} \mathcal{R}(\mu(X), \mu),$$

where

$$\mathcal{R}(Y, \mu) := \inf_{\xi \in E} \{ \pi(\xi) \mid \mu(\xi) \geq Y \}, \, Y \in L^0(\mathcal{G}),$$

$E$ is a $L^0$-module and $\mathcal{L}(E, L^0(\mathcal{G}))$ is the module of continuous $L^0$-linear functionals over $E$. A posteriori, adding the assumption of monotonicity, we can restrict the optimization problem over the set of positive and normalized functional, as we show in Theorem 22.

The proof of this result is based on a version of the hyperplane separation theorem and not on some approximation or scalarization arguments, as it happened in the vector space setting. By carefully analyzing the proof one may appreciate many similarities with the original demonstration in the static setting by Penot and Volle [PV90]. One key difference with [PV90], in addition to the conditional setting, is the continuity assumption needed to obtain the representation (1). We choose to work, as in [CV09], with evenly quasiconvex functions, i.e. function having evenly convex lower level sets. This is an assumption weaker than quasiconvexity and lower (or upper) semicontinuity.

To this end we introduce the concept of $L^0(\mathcal{G})$–evenly convex set that is tailor made for the conditional setting and show that the same implications true in the static framework (any convex open or closed set is evenly convex) hold also in the conditional one.

As a corollary, we obtain the representation (1) under the lower (or upper) semicontinuity and quasiconvexity assumptions.

b) On the complete duality of evenly quasiconvex conditional maps. In the conditional case, uniqueness of the representation is a more delicate issue and, to the best of our knowledge, this is the only result available in literature.

The complete duality for conditional risk measures (see Theorem 22 for the precise statement), perfectly matches what had been obtained in [CV09] for the static case and provide great evidence of the power of the module approach.

Under suitable conditions, $\rho : L^p_\mathbb{E}(\mathcal{F}) \to L^0(\mathcal{G})$ is an evenly quasiconvex conditional risk measure if and only if

$$\rho(X) = \sup_{Q \in \mathbb{P}^\mathbb{E}} R \left( E \left[ \frac{dQ}{d\mathbb{P}} X \mid \mathcal{G} \right], Q \right)$$

(2)
where $\mathcal{P}^q$ is a subset of probabilities having densities in $L^q_0(\mathcal{F})$ and $R$ is unique in the class $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$ (see the Definition 21).

In the present paper we limit ourselves to consider conditional maps of the form $\rho_t : L^p_T(\mathcal{F}_T) \to L^0(\mathcal{F}_t)$, $t < T$, and we defer to a forthcoming paper the study of the temporal consistency of the family of maps $(\rho_t)_{t \in [0,T]}$.

The paper is organized as follows. In Section 2 we provide some preliminary notions and facts: a short review about $L^0$-module; the concept and the properties of $L^0(\mathcal{G})$-evenly convex set; the regularity, quasiconvexity and continuity assumptions of the maps $\pi : E \to L^0(\mathcal{G})$.

Our main findings regarding the dual representation are collected in Section 3. In Section 4 we work on modules of the $L^p$-type and we state the complete duality for quasiconvex conditional risk measures. We include in Section 4.2 some complementary results. Section 5 is devoted to the proofs of the main contributions of the paper stated in Sections 3 and 4.1. Two more technical lemmas are deferred to the appendix.

2 Notations, setting and preliminary results

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed throughout this chapter and $\mathcal{G} \subseteq \mathcal{F}$ is any sigma algebra contained in $\mathcal{F}$. We denote with $L^0(\Omega, \mathcal{F}, \mathbb{P}) = L^0(\mathcal{F})$ (resp. $L^0(\mathcal{G})$ ) the space of $\mathcal{F}$ (resp. $\mathcal{G}$) measurable random variables that are $\mathbb{P}$ a.s. finite, whereas by $\overline{L}^0(\mathcal{F})$ the space of extended random variables which may take values in $\mathbb{R} \cup \{\infty\}$. In general since $(\Omega, \mathbb{P})$ are fixed we will always omit them. We define $L^+_0(\mathcal{F}) = \{X \in L^0(\mathcal{F}) \mid X \geq 0\}$ and $L^{++}_0(\mathcal{F}) = \{X \in L^0(\mathcal{F}) \mid X > 0\}$.

We remind that all equalities/inequalities among random variables are meant to hold $\mathbb{P}$-a.s. As the expected value $E_{\mathbb{P}}[\cdot]$ is mostly computed w.r.t. the reference probability $\mathbb{P}$, we will often omit $\mathbb{P}$ in the notation.

Moreover the essential (almost surely) supremum $\sup_{\mathbb{P}}(X_\lambda)$ of an arbitrary family of random variables $X_\lambda \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ will be simply denoted by $\sup_{\mathbb{P}}(X_\lambda)$, and similarly for the essential infimum. The symbol $\lor$ (resp. $\land$) denotes the essential (almost surely) maximum (resp. the essential minimum) between two random variables, which are the usual lattice operations.

2.1 A short review on $L^0$-modules

This subsection is based on Filipovic et al. [FKV09]. To help the reader in finding further details we use the same notations as in [FKV09] and [KV09].

$L^0(\mathcal{G})$ equipped with the order of the almost sure dominance is a lattice ordered ring. For every $\varepsilon \in L^0_{++}(\mathcal{G})$, the ball $B_\varepsilon := \{Y \in L^0(\mathcal{G}) \mid |Y| \leq \varepsilon\}$ centered in 0 in $L^0(\mathcal{G})$ gives the neighborhood basis of 0. A set $V \subset L^0(\mathcal{G})$ is a neighborhood of $Y \in L^0(\mathcal{G})$ if there exists $\varepsilon \in L^0_{++}(\mathcal{G})$ such that $Y + B_\varepsilon \subset V$. A set $V$ is open if it is a neighborhood of all $Y \in V$. $(L^0(\mathcal{G}), \|\cdot\|)$ stands for $L^0(\mathcal{G})$ endowed with this topology: in this case the space looses the property of
Definition 1 A topological $L^0(G)$-module $(E, \tau)$ is an algebraic module $E$ on the ring $L^0(G)$, endowed with a topology $\tau$ such that the operations

(i) $(E, \tau) \times (E, \tau) \rightarrow (E, \tau)$, $(X_1, X_2) \mapsto X_1 + X_2$,
(ii) $(L^0(G), \| \cdot \|) \times (E, \tau) \rightarrow (E, \tau)$, $(\Gamma, X_2) \mapsto \Gamma X_2$

are continuous w.r.t. the corresponding product topology.

A set $C$ is said to be $L^0(G)$-convex if for every $X_1, X_2 \in C$ and $\Lambda \in L^1(G)$, $0 \leq \Lambda \leq 1$, we have $\Lambda X_1 + (1 - \Lambda)X_2 \in C$.

A topology $\tau$ on $E$ is locally $L^0(G)$-convex if $(E, \tau)$ is a topological $L^0(G)$-module and there is a neighborhood base $\mathcal{U}$ of $0 \in E$ for which each $U \in \mathcal{U}$ is $L^0(G)$-convex, $L^0(G)$-absorvent and $L^0(G)$-balanced. In this case $(E, \tau)$ is a locally $L^0(G)$-convex module.

Definition 2 A function $\| \cdot \| : E \rightarrow L^0_+(G)$ is a $L^0(G)$-seminorm on $E$ if

(i) $\|\Gamma X\| = |\Gamma| \|X\|$ for all $\Gamma \in L^0(G)$ and $X \in E$,
(ii) $\|X_1 + X_2\| \leq \|X_1\| + \|X_2\|$ for all $X_1, X_2 \in E$.

The function $\| \cdot \|$ becomes a $L^0(G)$-norm if in addition

(iii) $\|X\| = 0$ implies $X = 0$.

Any family $\mathcal{S}$ of $L^0(G)$-seminorms on $E$ induces a topology in the following way. For any finite $\mathcal{S} \subset \mathcal{S}$ and $\varepsilon \in L^0_+(G)$ we define

$$U_{\mathcal{S}, \varepsilon} := \{ X \in E | \sup_{\| \cdot \| \in \mathcal{S}} \|X\| \leq \varepsilon \}$$

$$\mathcal{U} := \{ U_{\mathcal{S}, \varepsilon} | \mathcal{S} \subset \mathcal{S} \text{ finite and } \varepsilon \in L^0_+(G) \}.$$ 

$\mathcal{U}$ gives a $L^0(G)$-convex neighborhood base of 0 and it induces a topology on $E$ so that it become a locally $L^0(G)$-convex module. In fact Filipovic et al. proved (Theorem 2.4 [FKV09]) that a topological $L^0(G)$-convex module $(E, \tau)$ is locally $L^0(G)$-convex if and only if $\tau$ is induced by a family of $L^0(G)$-seminorms. When $\| \cdot \|$ is a norm we will always endow $E$ with the topology induced by $\| \cdot \|$ and say that $(E, \| \cdot \|)$ is a $L^0(G)$-normed module.

Remark 3 Notice that if $O_i \subseteq E$, $i = 1, 2$, are open and non empty and $A \in G$, then the set $O_11_A + O_21_{AC}$ is open. To show this claim let $\alpha := \alpha_11_A + \alpha_21_{AC}$ with $\alpha_i \in O_i$ and let $U_0$ be a neighborhood of 0 satisfying $\alpha_1 + U_0 \subseteq O_1$. Then the set $U := (\alpha_1 + U_0)1_A + (\alpha_2 + U_0)1_{AC} = \alpha + U_01_A + U_01_{AC}$ is contained in $O_11_A + O_21_{AC}$ and it is a neighborhood of $\alpha$, since $U_01_A + U_01_{AC}$ contains $U_0$ and is therefore a neighborhood of $0$.

The previous property generally fails when we paste together a countable sequence of open sets. For this reason we need a stronger assumption, namely the concatenation property, which is fully discussed in [FKV09] and always satisfied by $L^0(G)$-normed modules.
Definition 4 A topological \(L^0(\mathcal{G})\)-module has the countable concatenation property if for every countable collection \( \{ U_n \}_n \) of neighborhoods of 0 ∈ \( E \) and for every countable partition \( \{ A_n \}_n \subseteq \mathcal{G} \) the set \( \sum_n 1_{A_n} U_n \) is again a neighborhood of 0 ∈ \( E \).

When \( E \subseteq L^0(F) \) is a locally \( L^0(\mathcal{G}) \)-convex module, we denote by \( \mathcal{L}(E, L^0(\mathcal{G})) \) the \( L^0(\mathcal{G}) \)-module of continuous \( L^0(\mathcal{G}) \)-linear maps, which is not empty. Recall that \( \mu : E \to L^0(\mathcal{G}) \) is \( L^0(\mathcal{G}) \)-linear if for all \( \alpha, \beta \in L^0(\mathcal{G}) \) and \( X_1, X_2 \in E \)

\[
\mu(\alpha X_1 + \beta X_2) = \alpha \mu(X_1) + \beta \mu(X_2).
\]

In particular this implies

\[
\mu(X_1 1_A + X_2 1_{A^c}) = \mu(X_1) 1_A + \mu(X_2) 1_{A^c}
\]

which corresponds to the property (REG) in Definition 10. On the other hand \( \mu : E \to L^0(\mathcal{G}) \) is continuous if the counterimage of any open set (in the topology of almost sure dominance provided on \( L^0(\mathcal{G}) \)) is an open set in \( \tau \).

We will assume that the locally \( L^0(\mathcal{G}) \)-convex module \( E \) satisfies:

there exists \( \mu \in \mathcal{L}(E, L^0(\mathcal{G})) \) such that \( \mu(1_\Omega) > 1 \). (3)

Remark 5 Condition (3) holds true under very weak assumptions on the locally \( L^0(\mathcal{G}) \)-convex module \( E \), as for instance:

- There exists a seminorm \( \| \cdot \| \in \mathcal{Z} \) with \( \|1_\Omega\| > 0 \). This is always satisfied by normed \( L^0(\mathcal{G}) \) modules;
- \( (E, \tau) \) is a Hausdorff topological space.

Without further mention, in the sequel of the paper the following assumption holds true:

Assumption (H): \( E \) is a locally \( L^0(\mathcal{G}) \)-convex module contained in \( L^0(F) \) satisfying the countable concatenation property and condition (3).

2.1.1 On the \( L^0(\mathcal{G}) \)-normed module \( L^p_0(\mathcal{F}) \)

We now give an important class of \( L^0(\mathcal{G}) \)-normed modules (satisfying H) which play a key role in the financial applications and are studied in detail in [KV09] Section 4.2.

As before, let \( \mathcal{G} \subseteq \mathcal{F} \) be any sigma algebra contained in \( \mathcal{F} \) and consider the generalized conditional expectation of \( \mathcal{F} \)-measurable non negative random variables: \( E[|\cdot|; \mathcal{G}] : L^0_+(\mathcal{F}) \to L^0_+(\mathcal{G}) \)

\[
E[X|\mathcal{G}] =: \lim_{n \to +\infty} E[X \wedge n|\mathcal{G}].
\]

The basic properties of conditional expectation still hold true. In particular for every \( X, X_1, X_2 \in L^0_+(\mathcal{F}) \) and \( Y \in L^0(\mathcal{G}) \)
i) $YE[X|\mathcal{G}] = E[YX|\mathcal{G}]$;

ii) $E[X_1 + X_2|\mathcal{G}] = E[X_1|\mathcal{G}] + E[X_2|\mathcal{G}]$;

iii) $E[X] = E[E[X|\mathcal{G}]]$.

Let $p \in [1, \infty]$ and consider the algebraic $L^0$-module defined as

$$L^p_0(\mathcal{F}) = \{ X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \|X\|_p \in L^0(\Omega, \mathcal{G}, \mathbb{P}) \}$$

where $\| \cdot \|_p$ is the $L^0(\mathcal{G})$-norm assigned by

$$\|X\|_p = \begin{cases} 
\frac{E[|X|^p|\mathcal{G}]^{\frac{1}{p}}}{p} & \text{if } p < +\infty \\
\inf \{ Y \in L^0(\mathcal{G}) \mid Y \geq |X| \} & \text{if } p = +\infty
\end{cases} \quad (4)$$

We remind that $L^p_0(\mathcal{F})$, endowed with the $L^0$-module topology induced by (4), becomes a $L^0(\mathcal{G})$-normed module having the product structure i.e.

$$L^p_0(\mathcal{F}) = L^0(\mathcal{G})L^p(\mathcal{F}) = \{ YX \mid Y \in L^0(\mathcal{G}), X \in L^p(\mathcal{F}) \}.$$

This last property allows the conditional expectation to be well defined for every $X \in L^p_0(\mathcal{F})$; indeed, if $X = YX$ with $Y \in L^0(\mathcal{G})$ and $X \in L^p(\mathcal{F})$, then $E[X|\mathcal{G}] = YE[X|\mathcal{G}]$ is a finite valued random variable.

For $p < \infty$, any $L^0(\mathcal{G})$-linear continuous functional $\mu : L^p_0(\mathcal{F}) \to L^0(\mathcal{G})$ can be identified with a random variable $Z \in L^p_0(\mathcal{F})$ as $\mu(\cdot) = E[Z\cdot|\mathcal{G}]$ where $\frac{1}{p} + \frac{1}{q} = 1$.

### 2.2 On Evenly Convex Sets in the Conditional Setting

We recall that a subset $V$ of a locally convex TVS is even convex if it is the intersection of a family of open half spaces, or equivalently, if every $X \notin V$ can be separated from $V$ by a continuous linear functional. In this section we provide a generalization of the concept of an evenly convex set which is tailor made for the conditional setting.

**Definition 6** For $D \in \mathcal{G}$, $X \in E$ and a subset $\mathcal{V}$ of $E$, we say that $X$ is outside$_D \mathcal{V}$ if $1_A\{X\} \cap 1_A\mathcal{V} = \emptyset$ for every $A \in \mathcal{G}$ with $A \subseteq D$ and $\mathbb{P}(A) > 0$.

This is of course a much stronger requirement than $X \notin \mathcal{V}$ and it strongly depends on the selection of $D \in \mathcal{G}$, as it may happen that $X$ is not outside$_D \mathcal{V}$ but is outside$_{D'} \mathcal{V}$ if $D' \subset D$ (take for example $X = 1_{D'}$ with $\mathbb{P}(D') > 0$ and $\mathcal{V} = \{0\}$).

An arbitrary set $\mathcal{V}$ may present some components which degenerate to the entire module. Basically it might occur that for some $A \in \mathcal{G}$

$$\forall 1_A = E1_A,$$

i.e., for each $\xi \in E$ there exists $\eta \in \mathcal{V}$ such that $\eta 1_A = \xi 1_A$. In this case there are no chances of finding an $X$ which is outside$_A \mathcal{V}$ and consequently no separation results of the [FKV09] type can be applied. Thus we need to determine the maximal $\mathcal{G}$-measurable set on which $\mathcal{V}$ reduces to $E$. To this end, we recall the following simple fact that will be applied many times in the paper.
**Remark 7** Let $A$ be any class of subsets of a sigma algebra $\mathcal{G}$ that is closed with respect to finite union (i.e. $A_1, A_2 \in A \Rightarrow A_1 \cup A_2 \in A$) and consider the set $\chi := \{1_A \mid A \in A\}$. By definition of the essential supremum, $\text{ess.sup}\{\chi\}$ is a $\mathcal{G}$ measurable random variable. As $1_{A_1} \lor 1_{A_2} = 1_{A_1 \cup A_2} \in \chi$, if $A_1, A_2 \in A$, the set $\chi$ is upward directed and we can find a sequence $A_n \in A$ such that $1_{A_n} \uparrow \text{ess.sup}\{\chi\}$. As $1_{A_n} \uparrow 1_{A_M}$, where $A_M := \bigcup_{n \in \mathbb{N}} A_n$ is $\mathcal{G}$-measurable, we conclude that $\text{ess.sup}\{\chi\} = 1_{A_M}$, $A_M = \sup A$ and $A_M$ satisfies:

$$A \in A \text{ implies } A \subseteq A_M$$

(more precisely, $P(A \cap A_M) = P(A)$ for all $A \in A$). If in addition, $A$ is closed with respect to countable union then $A_M \in A$ is the maximal element in $A$.

Based on this remark the following definition is well posed.

**Definition 8** For every $V \subseteq E$ we denote with $A_V$ the ($\mathcal{G}$-measurable) maximal element of the class $\{A \in \mathcal{G} \mid \forall 1_A = E1_A\}$ and with $D_V$ the “complement” of $A_V$, i.e. a set $D_V \in \mathcal{G}$ satisfying $P(A_V \cup D_V) = 1$ and $P(A_V \cap D_V) = 0$. A set $V \subset E$ is said to be even $L^1(\mathcal{G})$-convex if for every $X$ outside $|D_V, V$, there exists a $\mu \in L(E, L^0(\mathcal{G}))$ such that

$$\mu(X) > \mu(\xi) \text{ on } D_V, \forall \xi \in V.$$  

The following proposition shows that the above definition of evenly $L^0(\mathcal{G})$-convex sets is the appropriate generalization, in the context of locally $L^0(\mathcal{G})$-convex module, of the notion of an evenly convex subsets of a topological vector space, as in both setting convex sets that are either closed or open are evenly convex.

**Proposition 9**

(i) If a non empty subset of $E$ is closed and $L^1(\mathcal{G})$-convex then it is evenly $L^1(\mathcal{G})$-convex.

(ii) If a non empty subset of $E$ is open and $L^1(\mathcal{G})$-convex then it is evenly $L^1(\mathcal{G})$-convex.

**Proof.** (i) First observe that the topology $\tau$ of any locally $L^1(\mathcal{G})$-convex module $E$ is generated by seminorms satisfying $\|1_A(\xi - \xi)\| = 1_A\|\xi - \xi\| \leq \|\xi - \xi\|$, for $\xi, \eta \in E$, $\xi, \eta \in G$, and therefore:

$$\xi \sim \Rightarrow (\xi_1 + \eta 1_{A_1}) \sim \Rightarrow (\xi_1 + \eta 1_{A_2})$$

for any net $\{\xi_\alpha\} \subseteq E$, $\xi \in E$, $A \in G$, and $\eta \in E$. In particular, $\xi \sim \Rightarrow \xi \Rightarrow (\xi_1A_1) \sim \Rightarrow (\xi_1 A)$. 

Let $C \subset E$ be closed, $L^1(\mathcal{G})$-convex, $C \neq \emptyset$ and suppose that $X \in E$ satisfies $X \cap A_C = \emptyset$ for every $A \in G$, $A \subseteq D_C$, $P(A) > 0$. Recall that $D_C$ is the complement of the maximal set $A_C \in G$ such that $C1_{A_C} = E1_{A_C}$. 


Let $C_0 = C_1_{D_C} + \{X + \varepsilon\}1_{A_C}$ where $\varepsilon \in L^0_+ (\mathcal{G})$. Clearly $C_0$ is $L^0(\mathcal{G})$-convex. In order to prove that $C_0$ is closed consider any net $\xi_{\alpha} \rightarrow \xi$, $\{\xi_{\alpha}\} \subset C_0$. Then $\xi_{\alpha} = Z_{\alpha}1_{D_C} + \{X + \varepsilon\}1_{A_C}$, with $Z_{\alpha} \in C$, and $(X + \varepsilon)1_{A_C} = \xi_{\alpha}1_{A_C}$.

Take any $\eta \in C$. As $C$ is $L^0(\mathcal{G})$-convex, $\xi_{\alpha}1_{D_C} + \eta 1_{A_C} = Z_{\alpha}1_{D_C} + \eta 1_{A_C} \in C$ and, as $C$ is closed, $\xi_{\alpha}1_{D_C} + \eta 1_{A_C} \rightarrow \xi1_{D_C} + \eta 1_{A_C} \Rightarrow Z \in C$. Therefore, $\xi = Z1_{D_C} + \{X + \varepsilon\}1_{A_C} \in C_0$.

Since $C_0$ is closed and $L^0(\mathcal{G})$-convex and $\{X\}1_A \cap C_0 1_A = \emptyset$ for every $A \in \mathcal{G}$, we apply Hahn Banach (see [FKV09] Theorem 2.8) and deduce the existence of $D$ being $L^0(\mathcal{G})$-convex. In order to prove that $\pi$ is regularity, i.e. $\pi(X) \rightarrow \pi(\xi)$ for every $\xi \in O_0$, we use the conditional setting of the Penot-Volle dual representation of quasiconvex functions. The additional basic property that is needed in this general setting is being even.

(ii) First recall that by Remark 3 if $O_i \subseteq E$, $i = 1, 2$, are open and non empty and $A \in G$, then the set $O_11_A + O_21_{A^c}$ is open. Let $O \subseteq E$ be open, $L^0(\mathcal{G})$-convex, $O \neq \emptyset$ and let $A_O \in \mathcal{G}$ be the maximal set given in Definition 8, being $D_O$ its complement. Suppose that $X \in E$ satisfies $\{X\}1_A \cap O1_A = \emptyset$ for every $A \in \mathcal{G}$, $A \subseteq D_O$, $P(A) > 0$.

Take a non trivial $\nu \in L(E, L^0(\mathcal{G}))$, i.e. an element satisfying (3), define the open, non empty and $L^0(\mathcal{G})$-convex set $E := \{\xi \in E \mid \nu(\xi) \leq \nu(X)\}$, and notice that $\{X\}1_A \cap E1_A = \emptyset$ for every $A \in \mathcal{G}$. Then the set $O_0 = O_11_{D_O} + E1_{A O}$ is $L^0(\mathcal{G})$-convex, open and satisfies $\{X\}1_A \cap O_01_A = \emptyset$ for every $A \in \mathcal{G}$ s.t. $P(A) > 0$.

Applying Hahn Banach separation Theorem (see [FKV09] Theorem 2.6) we separate $O_0$ from $X$ by the mean of $\mu \in L(E, L^0(\mathcal{G}))$ so that $\mu(X) > \mu(\xi)$ $\forall \xi \in O_0$, which implies $\mu(X) > \mu(\xi)$ on $D_O$, $\forall \xi \in O$, i.e. $O$ is evenly $L^0(\mathcal{G})$-convex.

2.3 On regularity of maps $\pi : E \rightarrow \bar{L}^0(\mathcal{G})$

We are interested in giving a counterpart in the module framework and in the conditional setting of the Penot-Volle dual representation of quasiconvex functions. The additional basic property that is needed in this general setting is regularity.

Definition 10 A map $\pi : E \rightarrow \bar{L}^0(\mathcal{G})$ is

(REG) regular if for every $X_1, X_2 \in E$ and $A \in \mathcal{G}$, $\pi(X_11_A + X_21_{A^c}) = \pi(X_1)1_A + \pi(X_2)1_{A^c}$.

Remark 11 (On REG) It is well known that (REG) is equivalent to:

$$\pi(X1_A)1_A = \pi(X)1_A, \forall A \in \mathcal{G}, \forall X \in E.$$ 

In the module setting it is even true that (REG) is equivalent to countably regularity, i.e.

$$\pi\left(\sum_{i=1}^{\infty} X_i1_{A_i}\right) = \sum_{i=1}^{\infty} \pi(X_i)1_{A_i}, \text{ on } \cup_{i=1}^{\infty} A_i$$
if \( X_i \in E \) and \( \{ A_i \}_i \) is a sequence of disjoint \( \mathcal{G} \) measurable sets. Indeed, from the module properties, \( X := \sum_{i=1}^{\infty} X_i 1_{A_i} \in E \) and \( \sum_{i=1}^{\infty} \pi(X_i) 1_{A_i} \in L^0(\mathcal{G}) \); (REG) then implies \( \pi(X) 1_{A_i} = \pi(X_1) 1_{A_i} = \pi(X_i) 1_{A_i} \).

Let \( \pi : E \to L^0(\mathcal{G}) \) be (REG). There might exist a set \( A \in \mathcal{G} \) on which the map \( \pi \) is constant, in the sense that \( \pi(\xi) 1_A = \pi(\eta) 1_A \) for every \( \xi, \eta \in E \). For this reason we introduce

\[ A := \{ A \in \mathcal{G} \mid \pi(\xi) 1_A = \pi(\eta) 1_A \ \forall \xi, \eta \in E \}. \]

Applying Lemma 33 in Appendix with \( F := \{ \pi(\xi) - \pi(\eta) \mid \xi, \eta \in E \} \) (we consider the convention \(+\infty - \infty = 0\)) and \( Y_0 = 0 \) we can deduce the existence of two maximal sets \( T_\pi \in \mathcal{G} \) and \( \Upsilon_\pi \in \mathcal{G} \) for which \( P(T_\pi \cap \Upsilon_\pi) = 0 \), \( P(T_\pi \cup \Upsilon_\pi) = 1 \) and

\[
\begin{align*}
\pi(\xi) &= \pi(\eta) \quad \text{on } \Upsilon_\pi \text{ for every } \xi, \eta \in E, \\
\pi(\zeta_1) &< \pi(\zeta_2) \quad \text{on } T_\pi \text{ for some } \zeta_1, \zeta_2 \in E. \quad (5)
\end{align*}
\]

Suppose that a map \( \pi \) satisfies: \( P(\Upsilon_\pi) > 0 \) and \( \pi(\xi)1_{\Upsilon_\pi} = +\infty 1_{\Upsilon_\pi} \) for every \( \xi \in E \). Then its lower level sets \( \{ X \in E \mid \pi(X) \leq \eta \}, \eta \in L^0(\mathcal{G}) \), are all empty (and so convex and closed). This would imply that any such map is quasiconvex and lower semicontinuous, regardless of its behavior on (the relevant set) \( T_\pi \). This explains the need of introducing the set \( T_\pi \) in the following definition of (QCO), (LSC) and (EVQ).

### 2.4 Quasiconvexity and Continuity Assumptions

Hereafter we state the conditional version of some relevant properties of the maps under investigation.

**Definition 12** Let \( \pi : E \to L^0(\mathcal{G}) \), \( Y \in L^0(\mathcal{G}) \) and \( U^Y_\pi := \{ \xi \in E \mid \pi(\xi)1_{T_\pi} \leq Y \} \). The map \( \pi \) is:

- **(QCO)** quasiconvex if the sets \( U^Y_\pi \) are \( L^0(\mathcal{G}) \)-convex \( \forall Y \in L^0(\mathcal{G}) \).
- **(EVQ)** evenly quasiconvex if the sets \( U^Y_\pi \) are evenly \( L^0(\mathcal{G}) \)-convex \( \forall Y \in L^0(\mathcal{G}) \).
- **(LSC)** lower semicontinuous if the sets \( U^Y_\pi \) are closed \( \forall Y \in L^0(\mathcal{G}) \).
- **(USC)** strongly upper semicontinuous if the sets \( \{ \xi \in E \mid \pi(\xi)1_{T_\pi} < Y \} \) are open \( \forall Y \in L^0(\mathcal{G}) \).

**Remark 13** Let \( \pi : E \to L^0(\mathcal{G}) \).

(i) The quasiconvexity of \( \pi \) is equivalent to the condition

\[
\pi(\Lambda X_1 + (1 - \Lambda)X_2) \leq \pi(X_1) \vee \pi(X_2), \quad (6)
\]

for every \( X_1, X_2 \in E \), \( \Lambda \in L^0(\mathcal{G}) \) and \( 0 \leq \Lambda \leq 1 \).

Indeed, let \( \pi(X_i)1_{T_\pi} \leq Y, \ i = 1, 2 \). If (6) holds then:

\[
\pi(\Lambda X_1 + (1 - \Lambda)X_2)1_{T_\pi} \leq \max\{\pi(X)1_{T_\pi}, \pi(Y)1_{T_\pi}\} \leq Y
\]
Viceversa, set $Y := \max \{ \pi(X_1), \pi(X_2) \}$. Then $X_1, X_2 \in \mathcal{U}(Y)$ implies, by the convexity of $\mathcal{U}(Y)$, that $\Delta X_1 + (1 - \Delta) X_2 \in \mathcal{U}(Y)$ and that $\pi(\Delta X_1 + (1 - \Delta) X_2) \leq Y$. On the other hand on the set $\mathcal{T}_\pi$ the map $\pi$ is constant so that

$$\pi(\Delta X_1 + (1 - \Delta) X_2) \mathcal{T}_\pi = \max \{ \pi(X_1), \pi(X_2) \}.$$

(ii) If $\pi$ is (QCO) then the sets $\{ \xi \in E \mid \pi(\xi) \mathcal{T}_\pi < Y \}$ are $L^0(\mathcal{G})$-convex for every $Y \in \mathcal{U}(\mathcal{G})$ and $D \in \mathcal{G}$. This follows immediately from (6).

(iii) (USC)* implies that $\{ \xi \in E \mid \pi(\xi) \mathcal{T}_\pi < Y \}$ are open for every $Y \in \mathcal{U}(\mathcal{G})$. Notice that in the condition (USC)* we do not ask that $U^Y_\pi$ is closed, as the complement of $\{ \xi \in E \mid \pi(\xi) \mathcal{T}_\pi < Y \}$, is not $U^Y_\pi$. This explains the label: (USC)* “strong upper semicontinuity”.

Regularity also guarantees that evenly quasiconvex maps are indeed quasiconvex:

**Lemma 14** Let $\pi : E \to \mathcal{U}(\mathcal{G})$ be (REG). If $\pi$ is (EVQ) then it is (QCO).

**Proof.** We need to prove that if the set

$$\mathcal{V} := \{ \xi \in E \mid \pi(\xi) \mathcal{T}_\pi \leq Y \}, \ Y \in L^0(\mathcal{G}),$$

is evenly $L^0(\mathcal{G})$-convex then it is also $L^0(\mathcal{G})$-convex. We may and do assume that $\mathcal{V} \neq \emptyset$ and therefore that $Y \mathcal{T}_\pi \geq 0$. Notice that

$$A \in \mathcal{G} \text{ and } \mathcal{V}A = E \mathcal{A}_A \text{ iff } \pi(\xi) \mathcal{T}_\pi A \leq \mathcal{Y} \mathcal{T}_\pi \forall \xi \in E \quad (7)$$

The "only if" condition is trivial, while the "if" condition holds as $\pi(\xi) \mathcal{T}_\pi A \leq \mathcal{Y} \mathcal{T}_\pi$ and (REG) imply $\pi(\eta) \mathcal{T}_\pi \leq Y$ where $\eta = \xi A + \theta_A \mathcal{A}_A \mathcal{A}$, $\theta_A \in \mathcal{V}$.

From $\pi(\xi) \mathcal{T}_\pi \mathcal{T}_\pi = 0 \leq \mathcal{Y} \mathcal{T}_\pi \forall \xi \in E$, the equivalence (7) and the maximality property of $A$, we deduce that $\mathcal{T}_\pi \subseteq \mathcal{A}$ and so $D \subseteq \mathcal{T}_\pi \subseteq \mathcal{A}$. Set

$$\chi := \{ X \in E \mid \mathcal{A}_A \cap \mathcal{V} \mathcal{V} = \emptyset \ \forall A \in \mathcal{G} \text{ with } A \subseteq D \text{ and } \mathbb{P}(A) > 0 \}.$$ 

We may assume that $\mathbb{P}(D \mathcal{V}) > 0$, otherwise $\mathcal{V} = E$. We first show that $\chi \neq \emptyset$. From (7) we see that (up to null sets) $A$ is the maximal element of $A$. Applying Lemma 33 in Appendix, with $F = \{ \pi(\xi) \mathcal{T}_\pi \mid \xi \in E \}$, we conclude that $D \mathcal{V}$ is the maximal element of $A$ and there exists a $\xi_0 \in E$ such that $\pi(\xi_0) \mathcal{T}_\pi > Y$ on $D \mathcal{V}$, i.e. $\xi_0 \in \chi$.

For all $X \in \chi$ there exists $\mu_X \in \mathcal{L}(E, L^0(\mathcal{G}))$ such that $\mu_X(X) > \mu_X(\xi)$ on $D \mathcal{V}$, $\forall \xi \in \mathcal{V}$. Consider the $L^0(\mathcal{G})$-convex set

$$\mathcal{H}_{\mu_X}^X := \{ \xi \in E \mid \mu_X(X) > \mu_X(\xi) \text{ on } D \mathcal{V} \}.$$
exists

Therefore $V \subseteq H$. Then $\pi$ Notice that for which $\mu(X) \in L^0(G)$ such that $\mu(X_0) > \mu(X_0(\xi))_\mu$ on $D_V$. If $\xi \in H^{\mu}_\mu$ then $\mu(X_0) > \mu(X_0(\xi))$ on $D_V$ and $\mu(X_0) > \mu(X_0(\xi))$ on $B$. Hence $\eta \notin H^{\mu}_\mu$ and so $\eta \notin H$. Therefore $H \subseteq V$. It then follows that $V = H$, which is $L^0(G)$-convex. ■

The next Lemma shows that the property (EVQ) is weaker than (QCO) plus either (LSC) or (USC)*.

**Lemma 15**

(i) Every (QCO) and (LSC) map $\pi : E \to L^0(G)$ is (EVQ).

(ii) Every (REG), (QCO) and (USC)* map $\pi : E \to L^0(G)$ is (EVQ).

**Proof.** (i) follows directly from Proposition 9 (i).

(ii) Step 1: assume that $P(\Upsilon_\pi) = 0$. Fix any $Y \in L^0(G)$, define $V = \{ \xi \in E \mid \pi(\xi) \leq Y \}$. Suppose that $X \in E$ satisfies $1_A \{ X \} \cap 1_A V = \emptyset$ for every $A \in G$ s.t. $P(A) > 0$ and $A \subseteq D_V$. We need to show that there exists $\mu \in \mathcal{L}(E, L^0(G))$ for which

$$\mu(X) > \mu(\xi), \quad D_V \forall \xi \in V.$$ 

Notice that $\pi(X) > Y$ on $D_V$, otherwise if $\pi(X) 1_B \leq Y 1_B$, with $B \subseteq D_V$ and $P(B) > 0$, then $X 1_B \in V 1_B$. Hence, if $\xi \in V$ then $\pi(\xi) \leq Y < \pi(X)$ on $D_V$. In particular if $\xi \in V$ then $\pi(\xi) < \pi(X)$ $1_D + (\pi(X) + \varepsilon) 1_A$, where $\varepsilon \in L^0(G)$. Define:

$$\mathcal{O} := \{ \xi \in E \mid \pi(\xi) < \pi(X) 1_D(\pi(X) + \varepsilon) 1_A \}.$$ 

Then $\mathcal{O}$ is open (see Remark 13 (iii)), $L^0(G)$-convex (by Remark 13 (iii)) and $V \subseteq \mathcal{O}$. As $1_A \{ X \} \cap 1_A \mathcal{O} = \emptyset$ for every $A \in G$, we apply the Hanh Banach Separation Theorem (see FKV09, Theorem 2.6) and deduce that there exists $\mu \in \mathcal{L}(E, L^0(G))$ s.t.

$$\mu(X) > \mu(\xi), \forall \xi \in \mathcal{O}.$$ 

As $V \subseteq \mathcal{O}$ we have the thesis.

Step 2: Suppose that $P(\Upsilon_\pi) > 0$. Then the map $\bar{\pi}$ build in Lemma 34 in Appendix satisfies (REG), (QCO), (USC*) and $P(\Upsilon_\pi) = 0$. Hence, by Step 1, $\bar{\pi}$ is (EVQ) and Lemma 34 (iii) guarantees that $\pi$ is (EVQ). ■

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3 Quasiconvex duality on general $L^0$-modules

Theorem 16 matches the representation by Maccheroni et al. in [CV09], which was obtained in topological vector spaces and in the static setting. The interesting feature here is that in the module and conditional framework we are able to provide a dual representation for evenly quasiconvex maps, which is a condition weaker than (QCO) plus (LSC) (resp. (USC)$^*$) and is an important starting point to obtain a complete quasiconvex duality. The proofs of this section are postponed to Section 5.2. Recall that we work under Assumption H.

Theorem 16 If $\pi : E \to \bar{L}^0(G)$ is (REG) and (EVQ) then

$$\pi(X) = \sup_{\mu \in \mathcal{L}(E, L^0(G))} \mathcal{R}(\mu(X), \mu),$$

(9)

where

$$\mathcal{R}(Y, \mu) := \inf_{\xi \in E} \{\pi(\xi) \mid \mu(\xi) \geq Y\},$$

(10)

with $Y \in L^0(G)$ and $\mu \in \mathcal{L}(E, L^0(G))$.

As an immediate consequence of Lemma 15 and Theorem 16 we then deduce:

Corollary 17 If $\pi : E \to L^0(G)$ is (REG), (QCO) and either (LSC) or (USC)$^*$ then (9) holds true.

In the upper semicontinuous case we can say more:

Proposition 18 If $\pi : E \to \bar{L}^0(G)$ is (REG), (QCO) and (USC)$^*$ then

$$\pi(X) = \max_{\mu \in \mathcal{L}(E, L^0(G))} \mathcal{R}(\mu(X), \mu).$$

(11)

In Theorem 16, $\pi$ can be represented as a supremum but not as a maximum. The following corollary shows that nevertheless we can find a $\mathcal{R}(\mu(X), \mu)$ arbitrary close to $\pi(X)$.

Corollary 19 Under the same assumption of Theorem 16 or Corollary 17, for every $\varepsilon > 0$ there exists $\mu_\varepsilon \in \mathcal{L}(E, L^0(G))$ such that

$$\pi(X) - \mathcal{R}(\mu_\varepsilon(X), \mu_\varepsilon) < \varepsilon \text{ on the set } \{\pi(X) < +\infty\}.$$

(12)

4 Conditional Risk Measures defined on $L^p_G(\mathcal{F})$.

From now on we consider $E = L^p_G(\mathcal{F})$, the $L^0(\mathcal{G})$-normed module satisfying the Assumption H and described in Section 2.1.1, and assume: $1 \leq p < \infty$.

Definition 20 We say that a map $\rho : L^p_G(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ is:

(↓ MON) monotone decreasing if $X_1 \geq X_2 \Rightarrow \pi(X_1) \leq \pi(X_2)$. 

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A quasiconvex conditional risk measure is a map \( \rho : L^p_0(\mathcal{F}) \to \bar{L}^0(\mathcal{G}) \) satisfying (REG), (\text{JMON}) and (QCO).

Recall that the principle of diversification states that “diversification should not increase the risk”, i.e. the diversified position \( \lambda X + (1-\lambda)Y \) is less risky than either the positions \( X \) or \( Y \). Thus the mathematical formulation of this principle is exactly quasiconvexity, i.e. the property (6). Under the cash additivity axiom (\text{CAS}) \( \rho(X + \Lambda) = \rho(X) - \Lambda \), for any \( \Lambda \in L^0(\mathcal{G}) \) and \( X \in L^p_0(\mathcal{F}) \), convexity and quasiconvexity are equivalent, so that they both provide the right interpretation of this principle. As vividly discussed by El Karoui and Ravanelli \[ER09\] the lack of liquidity of zero coupon bonds is the primary reason of the failure of cash additivity. In addition, in the time consistent case, the cash subadditivity property (\text{CSA}) \( \rho(X + \Lambda) \geq \rho(X) - \Lambda \), for any \( \Lambda \in L^0_+(\mathcal{G}) \) and \( X \in L^p_0(\mathcal{F}) \), is the adequate property of a conditional risk measure for processes (see the discussion in Section 5, \[FP06\]).

Thus it is unavoidable in the dynamic setting to relax the convexity axiom to quasiconvexity (and (CAS) to (CSA)) in order to regain the best modeling of diversification.

### 4.1 Complete Duality

This section is devoted to one of the most interesting result: a complete quasiconvex duality between the risk measure \( \rho \) and the dual map \( R \). We restrict the discussion to the particular case of \( L^0(\mathcal{G}) \)-normed modules of the \( L^p_0(\mathcal{F}) \) type, \( p < \infty \), as in this case there is a full knowledge of the dual module \( L(E, L^0(\mathcal{G})) \) which can be identified with \( L^q_0(\mathcal{F}) \) (see Section 2.1.1). The following proof may be adapted to other modules (for example of the Orlicz type) for which the dual module is known and contained in \( L^1_0(\mathcal{F}) \).

As discussed in the previous section, the duality concerning conditional quasiconvex risk measures holds only under an additional continuity assumption (either EVQ, LSC or USC*). For the analysis of the complete duality in this section, we chose the weakest assumption, i.e. EVQ, and we leave the other two cases for further investigation. We remark that, in virtue of Lemma 14, any map satisfying the assumptions of the following Theorem 22 is a conditional quasiconvex risk measure.

Due to the assumption that \( \rho \) is monotone decreasing, we modify, with just a difference in the sign, the definition of the dual function and rename it as:

\[
R(Y, Z) := \inf_{\xi \in L^p_0(\mathcal{F})} \{ \rho(\xi) \mid E[-\xi Z|\mathcal{G}] \geq Y \}. \tag{13}
\]
The function $R$ is well defined on the domain

$$\Sigma = \{(Y, Z) \in L^0(G) \times L^q_0(\mathcal{F}) \mid \exists \xi \in L^p_0(\mathcal{F}) \text{ s.t. } E[-Z\xi|G] \geq Y\}. \quad (14)$$

Let also introduce the following set:

$$P^q := \{Z \in L^q_0(G) \mid Z \geq 0, E[Z|G] = 1\}$$

and with a slight abuse of notation we will write that the probability $Q \in \mathcal{P}^q$ instead of $Z = \frac{dQ}{dP} \in \mathcal{P}^q$ and $R(Y, Q)$ instead of $R\left(Y, \frac{dQ}{dP}\right)$.

**Definition 21** The class $\mathcal{M}(L^0(G) \times \mathcal{P}^q)$ is composed by maps $K : L^0(G) \times \mathcal{P}^q \to L^0(G)$ s.t.

i) $K$ is increasing in the first component.

ii) $K(Y_1 A, Q_1) = K(Y, Q) 1_A$ for every $A \in \mathcal{G}$ and $(Y, \frac{dQ}{dP}) \in \Sigma$.

iii) $\inf_{Y \in L^0(\mathcal{G})} K(Y, Q) = \inf_{Y \in L^0(\mathcal{G})} K(Y, Q')$ for every $Q, Q' \in \mathcal{P}^q$.

iv) $K$ is \(\diamond\)-evenly $L^0(\mathcal{G})$-quasiconcave: for every $(Y^*, Q^*) \in L^0(\mathcal{G}) \times \mathcal{P}^q$, $A \in \mathcal{G}$ and $\alpha \in L^0(\mathcal{G})$ such that $K(Y^*, Q^*) < \alpha$ on $A$, there exists $(S^*, X^*) \in L^0_+(\mathcal{G}) \times L^p_0(\mathcal{F})$ with

$$Y^*S^* + E\left[X^* \frac{dQ^*}{dP} \mid G\right] < YS^* + E\left[X^* \frac{dQ}{dP} \mid G\right] \text{ on } A$$

for every $(Y, Q)$ such that $K(Y, Q) \geq \alpha$ on $A$.

v) the set $\mathcal{K}(X) = \left\{K(E[X \frac{dQ}{dP}|G], Q) \mid Q \in \mathcal{P}^q\right\}$ is upward directed for every $X \in L^p_0(\mathcal{F})$.

vi) $K(Y_1 1_A) = K(Y_2 1_A)$, if $\frac{dQ_1}{dP} 1_A = \frac{dQ_2}{dP} 1_A$, $Q_i \in \mathcal{P}^q$, and $A \in \mathcal{G}$.

We will show in Lemma 30 that the class $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$ is not empty.

**Theorem 22** The map $\rho : L^p_0(\mathcal{F}) \to L^0(\mathcal{G})$ satisfies (REG), (↓MON), (EVQ) if and only if

$$\rho(X) = \sup_{Q \in \mathcal{P}^q} R\left(E\left[\frac{dQ}{dP} X\mid G\right], Q\right) \quad (15)$$

where

$$R(Y, Q) = \inf_{\xi \in L^p_0(\mathcal{F})} \left\{\rho(\xi) \mid E\left[-\xi \frac{dQ}{dP} \mid G\right] = Y\right\}$$

is unique in the class $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$.

**Proof.** In Section 5.3.
4.2 Complements

From Theorem 22 we can deduce the following proposition which confirm what was obtained in [FM09].

**Proposition 23** Suppose that \( \rho \) satisfies the same assumptions of Theorem 22. Then the restriction \( \hat{\rho} := \rho 1_{L^p(F)} \) defined by \( \hat{\rho}(X) = \rho(X) \) for every \( X \in L^p(F) \) can be represented as

\[
\hat{\rho}(X) = \sup_{Q \in P^q, \xi \in L^p(F)} \left\{ \hat{\rho}(\xi) \mid E[-\xi \frac{dQ}{dP}|G] = E[-X \frac{dQ}{dP}|G] \right\}.
\]

**Proof.** For every \( X \in L^p(F), Q \in P^q \) we have

\[
\hat{\rho}(X) \geq \inf_{\xi \in L^p(F)} \left\{ \hat{\rho}(\xi) \mid E[-\xi \frac{dQ}{dP}|G] = E[-X \frac{dQ}{dP}|G] \right\}
\]

and hence the thesis. \( \square \)

The following result is meant to confirm that the dual representation chosen for quasiconvex maps is indeed a good generalization of the convex case.

**Corollary 24** Let \( \rho : L^0_0(F) \to L^0(G) \).

(i) If \( Q \in P^q \) and if \( \rho \) is (MON), (REG) and (CAS) then

\[
R(E_Q(-X|G), Q) = E_Q(-X|G) - \rho^*(-Q)
\]

where

\[
\rho^*(-Q) = \sup_{\xi \in L^0_0(F)} \{ E_Q[-\xi|G] - \rho(\xi) \}.
\]

(ii) Under the same assumptions of Theorem 22 and if \( \rho \) satisfies in addition (CAS) then

\[
\rho(X) = \sup_{Q \in P^q} \{ E_Q(-X|G) - \rho^*(-Q) \}.
\]

**Proof.** Denote by \( \mu(\cdot) := E \left[ \frac{dQ}{dP} \mid G \right] \); by definition of \( R \)

\[
R(E_Q(-X|G), Q) = \inf_{\xi \in L^0_0(F)} \{ \rho(\xi) \mid \mu(-\xi) = \mu(-X) \}
\]

\[
= \mu(-X) + \inf_{\xi \in L^0_0(F)} \{ \rho(\xi) - \mu(-X) \mid \mu(-\xi) = \mu(-X) \}
\]

\[
= \mu(-X) + \inf_{\xi \in L^0_0(F)} \{ \rho(\xi) - \mu(-\xi) \mid \mu(-\xi) = \mu(-X) \}
\]

\[
= \mu(-X) - \sup_{\xi \in L^0_0(F)} \{ \rho(\xi) - \mu(-X) \mid \mu(-\xi) = \mu(-X) \}
\]

\[
= \mu(-X) - \rho^*(-Q).
\]
where the last equality follows from

\[
\rho^*(-Q) = \sup_{\xi \in L^p_0(F)} \{\mu(-\xi - \mu(X - \xi)) - \rho(\xi + \mu(X - \xi))\} = \sup_{\eta \in L^p_0(F)} \{\mu(-\eta) - \rho(\eta) \mid \eta = \xi + \mu(X - \xi)\} \leq \sup_{\eta \in L^p_0(F)} \{\mu(-\eta) - \rho(\eta) \mid \mu(-\eta) = \mu(-X)\} \leq \rho^*(-Q).
\]

(ii) It is a consequence of (i) and Theorem 22. ■

Remark 25 It’s not hard to show that for every \(X \in L^p_0(F), Q \in \mathcal{P}^q\) and any map \(\rho : L^p_0(F) \to L^0(G)\) we have:

\[
R \left( E \left[ -\frac{dQ}{d\mu}X \big| G \right], Q \right) \geq E \left[ -\frac{dQ}{d\mu}X \big| G \right] - \rho^*(-Q). \tag{16}
\]

From equation (15) we deduce that whenever the preferences of an agent are described by a quasiconvex - not convex - risk measure we cannot recover the risk only taking a supremum of the Fenchel conjugate, i.e. of the RHS of (16), over all the possible probabilistic scenarios. We shall need a more cautious approach represented by the new penalty function \(R \left( E \left[ -\frac{dQ}{d\mu}X \big| G \right], Q \right)\). The quantity \(R(Y,Q)\) is therefore the reserve amount required at the intermediate time \(t (F_t = G)\) under the scenario \(Q\), to cover an expected loss \(Y \in L^0(G)\) in the future.

4.2.1 A characterization via the risk acceptance family

In this subsection we assume for the sake of simplicity that \(\rho(0) \in L^0(G)\). In this way we do not loose any generality imposing \(\rho(0) = 0\) (if not, just define \(\tilde{\rho}(\cdot) := \rho(\cdot) - \rho(0)\)). We remind that if \(\rho(0) = 0\) then (REG) is equivalent to the condition

\[
\rho(X1_A) = \rho(X)1_A, \ A \in L^0(G).
\]

Given a risk measure one can always define for every \(Y \in L^0(G)\) the risk acceptance set of level \(Y\) as

\[
A^Y_\rho = \{X \in L^p_0(F) \mid \rho(X) \leq Y\}.
\]

This set represents the collection of financial positions whose risk is smaller of the fixed level \(Y\) and are strictly related to the Acceptability Indices [CM09].

Given a risk measure \(\rho\) we can associate a family of risk acceptance sets, namely \(\{A^Y_\rho \mid Y \in L^0(G)\}\), as it was suggested in the static case in [DK10]. In general

**Definition 26** A family \(\mathcal{A} = \{A^Y_\rho \mid Y \in L^0(G)\}\) of subsets \(A^Y_\rho \subset L^p_0(F)\) is called risk acceptance family if the following properties hold:

(i) convexity: \(A^Y_\rho\) is \(L^0(G)\)-convex for every \(Y \in L^0(G)\);
Moreover, that for every risk acceptance family \( \mathcal{V} \) vice versa is a well defined quasiconvex conditional risk measure \( \rho \) with \( \mathcal{A} \).

\( \rho \) is \( \mathcal{L} \)-monotone and \( \mathcal{L} \)-coquadratic.

Proposition 27: For any quasiconvex conditional risk measure \( \rho : L^0_G(\mathcal{F}) \rightarrow \mathbb{L}^0(\mathcal{G}) \) the family
\[
\mathcal{A}_\rho = \{ \mathcal{A}_\rho^Y | Y \in L^0(\mathcal{G}) \}
\]
with \( \mathcal{A}_\rho^Y = \{ X \in L^0_G(\mathcal{F}) | \rho(X) \leq Y \} \) is a risk acceptance family.

Vice versa for every risk acceptance family \( \mathcal{A} \) the map
\[
\rho_\mathcal{A}(X) = \inf \{ Y | Y \in L^0(\mathcal{G}) \text{ s.t. } X \in \mathcal{A} \}
\]
is a well defined quasiconvex conditional risk measure \( \rho_\mathcal{A} : L^0_G(\mathcal{F}) \rightarrow \mathbb{L}^0(\mathcal{G}) \) such that \( \rho(0) = 0 \).

Moreover, \( \rho_\mathcal{A} = \rho \) and if \( \mathcal{A}^Y = \bigcap_{Y' \geq Y} \mathcal{A}^{Y'} \) for every \( Y \in L^0(\mathcal{G}) \) then \( \mathcal{A}_{\rho_\mathcal{A}} = \mathcal{A} \).

Proof. The proof is an extension from the static case provided in [CM09] and [DK10].

(\( \mathcal{L} \)-MON) and (QCO) of \( \rho \) imply that \( \mathcal{A}_\rho^Y \) is convex and monotone. Also notice that
\[
\inf \{ Y | Y \in L^0(\mathcal{G}) \text{ s.t. } X1_G \in \mathcal{A}_\rho^Y \} = \inf \{ Y | \rho(X1_G) \leq Y \text{ for } Y \in L^0(\mathcal{G}) \}
\]
\[
= \rho(X1_G) = \rho(X)1_G = \inf \{ Y1_G | Y \in L^0(\mathcal{G}) \text{ s.t. } \rho(X) \leq Y \}
\]
\[
= \inf \{ Y1_G | Y \in L^0(\mathcal{G}) \text{ s.t. } X \in \mathcal{A}_\rho^Y \},
\]
i.e. \( \mathcal{A}_\rho^Y \) is regular.

Vice versa: we first prove that \( \rho_\mathcal{A} \) is (REG). For every \( G \in \mathcal{G} \)
\[
\rho_\mathcal{A}(X1_G) = \inf \{ Y | Y \in L^0(\mathcal{G}) \text{ s.t. } X1_G \in \mathcal{A}^Y \}
\]
\[
= \inf \{ Y1_G | Y \in L^0(\mathcal{G}) \text{ s.t. } X \in \mathcal{A}^Y \} = \rho_\mathcal{A}(X)1_G
\]
Now consider \( X_1, X_2 \in L^0_G(\mathcal{F}), X_1 \leq X_2 \). Let \( \mathcal{G}^C = \{ \rho_\mathcal{A}(X_1) = +\infty \} \) so that
\[
\rho_\mathcal{A}(X_11_C) \geq \rho_\mathcal{A}(X_21_C).
\]
Otherwise consider the collection of \( Y \)'s such that \( X_11_G \in \mathcal{A}^Y \). Since \( \mathcal{A}^Y \) is monotone we have that \( X_21_G \in \mathcal{A}^Y \) if \( X_11_G \in \mathcal{A}^Y \) and this implies that
\[
\rho_\mathcal{A}(X_1)1_G = \inf \{ Y1_G | Y \in L^0(\mathcal{G}) \text{ s.t. } X_1 \in \mathcal{A}^Y \}
\]
\[
= \inf \{ Y | Y \in L^0(\mathcal{G}) \text{ s.t. } X_11_G \in \mathcal{A}^Y \}
\]
\[
\geq \inf \{ Y | Y \in L^0(\mathcal{G}) \text{ s.t. } X_21_G \in \mathcal{A}^Y \}
\]
\[
= \inf \{ Y1_G | Y \in L^0(\mathcal{G}) \text{ s.t. } X_2 \in \mathcal{A}^Y \} = \rho_\mathcal{A}(X_2)1_G.
\]
Let $X_1, X_2 \in L^0_\rho(F)$ and take any $\Lambda \in L^0(\mathcal{G})$, $0 \leq \Lambda \leq 1$. Define the set $B = \{ \rho_h(X_1) \leq \rho_h(X_2) \}$. If $X_11_B + X_21_B \in \mathcal{A}Y$ for some $Y' \in L^0(\mathcal{G})$ then for sure $Y' \geq \rho_h(X_1) \lor \rho_h(X_2) \geq \rho(X_i)$ for $i = 1, 2$. Hence also $\rho(X_i) \in \mathcal{A}Y$ for $i = 1, 2$ and by convexity we have that $\Lambda X_1 + (1-\Lambda)X_2 \in \mathcal{A}Y$. Then $\rho_h(\Lambda X_1 + (1-\Lambda)X_2) \leq \rho_h(X_1) \lor \rho_h(X_2)$.

If $X_11_B + X_21_B \notin \mathcal{A}Y$ for every $Y' \in L^0(\mathcal{G})$ then from property (iii) we deduce that $\rho_h(X_1) = \rho_h(X_2) = +\infty$ and the thesis is trivial.

Now consider $B = \{ \rho(X) = +\infty \}$: $\rho_h(X) = \rho(X)$ follows from

\[
\rho_h(X)1_B = \inf \{ Y1_B \mid Y \in L^0(\mathcal{G}) \text{ s.t. } \rho(X) \leq Y \} = +\infty1_B
\]

\[
\rho_h(X)1_{BC} = \inf \{ Y1_{BC} \mid Y \in L^0(\mathcal{G}) \text{ s.t. } \rho(X) \leq Y \}
\]

\[
= \inf \{ Y \mid Y \in L^0(\mathcal{G}) \text{ s.t. } \rho(X)1_{BC} \leq Y \} = \rho(X)1_{BC}
\]

For the second claim notice that if $X \in \mathcal{A}Y$ then $\rho_h(X)$ $\leq Y$ which means that $X \in \mathcal{A}_{\rho_h}Y$. Conversely if $X \in \mathcal{A}_{\rho_h}Y$ then $\rho_h(X)$ $\leq Y$ and by monotonicity this implies that $X \in \mathcal{A}Y$ for every $Y' > Y$. From the right continuity we take the intersection and get that $X \in \mathcal{A}Y$. $\blacksquare$

5 Proofs

5.1 General properties of $\mathcal{R}(Y, \mu)$

Following the path traced in [FM09], we adapt to the module framework the proofs of the foremost properties holding for the function $\mathcal{R} : L^0(\mathcal{G}) \times L(E, L^0(\mathcal{G})) \rightarrow L^0(\mathcal{G})$ defined in (10). Let the effective domain of the function $\mathcal{R}$ be:

$$\Sigma_{\mathcal{R}} := \{ (Y, \mu) \in L^0(\mathcal{G}) \times L(E, L^0(\mathcal{G})) \mid \exists \xi \in E \text{ s.t. } \mu(\xi) \geq Y \}. \quad (17)$$

**Lemma 28** Let $\mu \in \mathcal{L}(E, L^0(\mathcal{G}))$, $X \in E$ and $\pi : E \rightarrow \bar{L}^0(\mathcal{G})$ satisfy (REG).

i) $\mathcal{R}(\cdot, \mu)$ is monotone non decreasing.

ii) $\mathcal{R}(\Lambda \mu(X), \Lambda \mu) = \mathcal{R}(\mu(X), \mu)$ for every $\Lambda \in L^0(\mathcal{G})$.

iii) For every $Y \in L^0(\mathcal{G})$ and $\mu \in \mathcal{L}(E, L^0(\mathcal{G}))$, the set

$$\mathcal{A}_\mu(Y) \doteq \{ \pi(\xi) \mid \xi \in E, \mu(\xi) \geq Y \}$$

is downward directed in the sense that for every $\pi(\xi_1), \pi(\xi_2) \in \mathcal{A}_\mu(Y)$ there exists $\pi(\xi^*) \in \mathcal{A}_\mu(Y)$ such that $\pi(\xi^*) \leq \min\{ \pi(\xi_1), \pi(\xi_2) \}$.

In addition, if $\mathcal{R}(Y, \mu) < \alpha$ for some $\alpha \in L^0(\mathcal{G})$ then there exists $\xi$ such that $\mu(\xi) \geq Y$ and $\pi(\xi) < \alpha$.

iv) For every $A \in \mathcal{G}$, $(Y, \mu) \in \Sigma_{\mathcal{R}}$

\[
\mathcal{R}(Y, \mu)1_A = \inf_{\xi \in E} \{ \pi(\xi)1_A \mid Y \geq \mu(X) \}
\]

\[
= \inf_{\xi \in E} \{ \pi(\xi)1_A \mid Y1_A \geq \mu(X1_A) \} = \mathcal{R}(Y1_A, \mu)1_A \quad (18)
\]
v) For every $X_1, X_2 \in E$
   
   (a) $\mathcal{R}(\mu(X_1), \mu) \land \mathcal{R}(\mu(X_2), \mu) = \mathcal{R}(\mu(X_1) \land \mu(X_2), \mu)$
   
   (b) $\mathcal{R}(\mu(X_1), \mu) \lor \mathcal{R}(\mu(X_2), \mu) = \mathcal{R}(\mu(X_1) \lor \mu(X_2), \mu)$
   
   vi) The map $\mathcal{R}(\mu(X), \mu)$ is quasi-affine with respect to $X$ in the sense that for every $X_1, X_2 \in E$, $\Lambda \in L^0(\mathcal{G})$ and $0 \leq \Lambda \leq 1$, we have
   
   $\mathcal{R}(\mu(\Lambda X_1 + (1 - \Lambda)X_2), \mu) \geq \mathcal{R}(\mu(X_1), \mu) \lor \mathcal{R}(\mu(X_2), \mu)$ (quasiconcavity)
   
   $\mathcal{R}(\mu(\Lambda X_1 + (1 - \Lambda)X_2), \mu) \leq \mathcal{R}(\mu(X_1), \mu) \land \mathcal{R}(\mu(X_2), \mu)$ (quasiconvexity).
   
   vii) $\inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu_1) = \inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu_2)$ for every $\mu_1, \mu_2 \in \mathcal{L}(E, L^0(\mathcal{G}))$.
   
   Proof. i) and ii) follow trivially from the definition.
   
   iii) The set $\{\pi(\xi) \mid \xi \in E, \mu(\xi) \geq Y\}$ is clearly downward directed. Thus there exists a sequence $\{\xi_m^\mu\}_{m=1}^\infty \in E$ such that
   
   $\mu(\xi_m^\mu) \geq Y \ \forall \ m \geq 1, \ \pi(\xi_m^\mu) \downarrow \mathcal{R}(Y, \mu) \ \text{as} \ m \uparrow \infty.$
   
   Now let $\mathcal{R}(Y, \mu) < \alpha$: consider the sets $F_m = \{\pi(\xi_m^\mu) < \alpha\}$ and the partition of $\Omega$ given by $G_1 = F_1$ and $G_m = F_m \setminus G_{m-1}$. From the properties of the module $E$ and (REG) we get:
   
   $\xi = \sum_{m=1}^\infty \xi_m^\mu 1_{G_m} \in E, \ \mu(\xi) \geq Y \ \text{and} \ \pi(\xi) < \alpha.$
   
   iv), v) and vi) follow as in [FM09].
   
   (vii) Notice that $\mathcal{R}(Y, \mu) \geq \inf_{\xi \in E} \pi(\xi), \forall Y \in L^0(\mathcal{G})$, implies: $\inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu) \geq \inf_{\xi \in E} \pi(\xi)$. On the other hand, $\pi(\xi) \geq \mathcal{R}(\mu(\xi), \mu) \geq \inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu), \forall \xi \in E$, implies: $\inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu) \leq \inf_{\xi \in E} \pi(\xi)$. □
   
   Remark 29 Corresponding properties hold also for the function $R$ defined in (13).
   
   5.2 Proofs of the results of Section 3
   
   Proof of Theorem 16.
   
   Step 1. Assume that
   
   $P(Y_\pi) = 0.$ \hspace{1cm} (20)
   
   Fix $X \in E$ and denote $G = \{\pi(X) < +\infty\}$. For every $\varepsilon \in L^0_+ (\mathcal{G})$ we set
   
   $Y_\varepsilon := (\pi(X) - \varepsilon) 1_G + \varepsilon 1_{G^c} \in L^0(\mathcal{G})$ and
   
   $\mathcal{A} := \{A \in \mathcal{G} \mid \forall \xi \in E \ \pi(\xi) > Y_\varepsilon \ \text{on} \ A\},$
   
   $\mathcal{A}_+ := \{A \in \mathcal{G} \mid \exists \xi \in E \ \text{s.t.} \ \pi(\xi) \leq Y_\varepsilon \ \text{on} \ A\}.$
   
   Then $A = \{A \in \mathcal{G} \mid \forall Y \in F \ Y > Y_\varepsilon \ \text{on} \ A\}$, where $F := \{\pi(\xi) \mid \xi \in E\}$. Then (REG) guarantees (recall Remark 11) that we may apply Lemma 33 which assure the existence of two maximal elements $A_\varepsilon \in \mathcal{A}$ and $A_\varepsilon^+ \in \mathcal{A}^+$ so that: $P(A_\varepsilon \cup A_\varepsilon^+) = 1, P(A_\varepsilon \cap A_\varepsilon^+) = 0,$
   
   $\pi(\xi) > Y_\varepsilon \ \text{on} \ A_\varepsilon \ \text{for every} \ \xi \in E$
and 
\[ \exists \xi \in E \text{ s.t. } \pi(\xi) \leq Y_\varepsilon \text{ on } A^*_\varepsilon. \]
In particular it follows that \( R(\mu(X), \mu) \geq Y_\varepsilon \) on \( A_\varepsilon \) for every \( \mu \in \mathcal{L}(E, L^0(\mathcal{G})) \) and thus
\[
\pi(X)1_{A_\varepsilon} \geq \sup_{\mu \in \mathcal{L}(E, L^0(\mathcal{G}))} R(\mu(X), \mu)1_{A_\varepsilon} \\
\geq (\pi(X) - \varepsilon)1_{G \cap A_\varepsilon} + \varepsilon 1_{G^c \cap A_\varepsilon},
\]
(21)
Since \( \mathbb{P}(Y_\varepsilon) = 0 \), we know from (5) that there exist \( \zeta_1, \zeta_2 \in E \) such that \( \pi(\zeta_1) < \pi(\zeta_2) \). The set
\[ C_\varepsilon := \{ \xi \in E \mid \pi(\xi) \leq Y_\varepsilon 1_{A^*_\varepsilon} + \pi(\zeta_1)1_{A_\varepsilon} \} \]
is not empty, as \( 1_{A^*_\varepsilon} + \zeta_1 1_{A_\varepsilon} \in C_\varepsilon \), and the random variable
\[ \tilde{X} := X1_{A^*_\varepsilon} + \zeta_2 1_{A_\varepsilon} \]
satisfies
\[ 1_B \{ \tilde{X} \} \cap 1_B C_\varepsilon = \emptyset \]
for every \( B \in \mathcal{G} \) with \( \mathbb{P}(B) > 0 \) (this means that the maximal set \( A_{C_\varepsilon} \) in Definition 8 has null measure).

As \( \pi \) is (EVQ) and \( \pi(\zeta_1) \in L^0(\mathcal{G}) \), the set \( C_\varepsilon \) is evenly \( L^0(\mathcal{G}) \)-convex and therefore we can find \( \mu_\varepsilon \in \mathcal{L}(E, L^0(\mathcal{G})) \) such that
\[ \mu_\varepsilon(\tilde{X}) > \mu_\varepsilon(\xi) \quad \forall \xi \in C_\varepsilon. \]
(22)

Step 2. We claim that
\[ \{ \xi \in E \mid \mu_\varepsilon(\tilde{X})1_{A^*_\varepsilon} \leq \mu_\varepsilon(\xi)1_{A^*_\varepsilon} \} \subseteq \{ \xi \in E \mid \pi(\xi) > (\pi(\tilde{X}) - \varepsilon)1_{G} + \varepsilon 1_{G^c} \text{ on } A^*_\varepsilon \}. \]
(23)
To this end, define for each \( A \in A^* \) such that \( \mathbb{P}(A) > 0 \)
\[ C_{\varepsilon}^A := \{ \xi \in E \mid \pi(\xi)1_A \leq (\pi(\tilde{X}) - \varepsilon)1_{A \cap G} + \varepsilon 1_{A \cap G^c} \}, \]
\[ D_{\varepsilon}^A := \{ \xi \in E \mid \mu_\varepsilon(\tilde{X}) > \mu_\varepsilon(\xi) \text{ on } A \}. \]
Notice that
\[ C_{\varepsilon}^A \subseteq D_{\varepsilon}^A. \]
(24)
Indeed, let \( \xi \in C_{\varepsilon}^A, \eta \in C_\varepsilon \) and define \( \tilde{\xi} = \xi 1_A + \eta 1_{A^c} \), which belong to \( C_\varepsilon \). Hence \( \mu_\varepsilon(\tilde{X}) > \mu_\varepsilon(\xi) \) and, as \( \mu_\varepsilon(\tilde{\xi})1_A = \mu_\varepsilon(\xi 1_A)1_A, \mu_\varepsilon(\tilde{\xi}) > \mu_\varepsilon(\xi) \) on \( A \).

The inclusion (24) implies:
\[ \bigcap_{A \in A^*} (D_{\varepsilon}^A)^C \subseteq \bigcap_{A \in A^*} (C_{\varepsilon}^A)^C. \]
(25)
We show that (25) is exactly (23). By definition
\[ (C_{\varepsilon}^A)^C = \{ \xi \in E \mid \exists B \subseteq A, \mathbb{P}(B) > 0 \text{ and } [*] \}) \]
where

\[
[\star] \leftrightarrow \begin{cases} 
\pi(\xi)(\omega) > \pi(\bar{X})(\omega) - \varepsilon(\omega) & \text{for a.e. } \omega \in B \cap G \\
\pi(\xi)(\omega) > \varepsilon(\omega) & \text{for a.e. } \omega \in B \cap G^C .
\end{cases}
\]

Then:

\[
\bigcap_{A \in A^r} (C^\varepsilon_A)^C = \{ \xi \in E \mid \forall A \in A^r, \exists B \subseteq A, \mathbb{P}(B) > 0 \text{ and } [\star] \}
\]

Indeed, if \( \xi \in E \) and \( \pi(\xi) > (\pi(\bar{X}) - \varepsilon)1_G + \varepsilon 1_{G^C} \) on \( A^r_\varepsilon \), then \( \xi \in \bigcap_{A \in A^r} (C^\varepsilon_A)^C \).

Vice versa let \( \xi \in \bigcap_{A \in A^r} (C^\varepsilon_A)^C \) and suppose by contradiction that there exists \( D \in A^r_\varepsilon, \mathbb{P}(D) > 0 \), such that \( \pi(\xi)1_D \leq \left[ (\pi(\bar{X}) - \varepsilon)1_G + \varepsilon 1_{G^C} \right] 1_D \). By definition of \( (C^D)^C \) we can find \( B \in \mathcal{G} \) and \( B \subseteq D \) such that \( \pi(\xi) > \pi(\bar{X}) - \varepsilon \) on \( B \cap G \) or \( \pi(\xi) > \varepsilon \) on \( B \cap G^C \) and this is clearly a contradiction. By the previous argument one can prove that

\[
\bigcap_{A \in A^r} (D^\varepsilon_A)^C = \{ \xi \in E \mid \mu_\varepsilon(\bar{X})1_{A^r_\varepsilon} \leq \mu_\varepsilon(\xi)1_{A^r_\varepsilon} \}.
\]

**Step 3.** From (18)-(19) and \( \mu(\bar{X})1_{A^r_\varepsilon} = \mu(X)1_{A^r_\varepsilon} \) we have \( R(\mu(\bar{X}), \mu)1_{A^r_\varepsilon} = \mathcal{R}(\mu(X), \mu)1_{A^r_\varepsilon} \) and then we obtain:

\[
\pi(X)1_{A^r_\varepsilon} \geq \sup_{\mu \in \mathcal{L}(E, L^0(\mathcal{G}))} \mathcal{R}(\mu(X), \mu)1_{A^r_\varepsilon} = \sup_{\mu \in \mathcal{L}(E, L^0(\mathcal{G}))} \mathcal{R}(\mu(\bar{X}), \mu)1_{A^r_\varepsilon} \tag{26}
\]

\[
\geq \mathcal{R}(\mu_\varepsilon(\bar{X}), \mu_\varepsilon)1_{A^r_\varepsilon} = \inf_{\xi \in E} \{ \pi(\xi)1_{A^r_\varepsilon} \mid \mu_\varepsilon(\bar{X})1_{A^r_\varepsilon} \leq \mu_\varepsilon(\xi)1_{A^r_\varepsilon} \}
\]

\[
\geq \inf_{\xi \in E} \left\{ \pi(\xi)1_{A^r_\varepsilon} \mid \pi(\xi) > (\pi(\bar{X}) - \varepsilon)1_G + \varepsilon 1_{G^C} \right\} \text{ on } A^r_\varepsilon
\]

\[
\geq (\pi(X) - \varepsilon)1_{G \cap A^r_\varepsilon} + \varepsilon 1_{G^C \cap A^r_\varepsilon}, \tag{27}
\]

where we used (23) and \( \pi(\bar{X})1_{A^r_\varepsilon} = \pi(X)1_{A^r_\varepsilon} \). By (21) and this last sequence of inequalities we can assure that, for any \( \varepsilon \in L^0_{+,+}(\mathcal{G}) \),

\[
\pi(X) \geq \sup_{\mu \in \mathcal{L}(E, L^0(\mathcal{G}))} \mathcal{R}(\mu(X), \mu) \geq (\pi(X) - \varepsilon)1_G + \varepsilon 1_{G^C}.
\]

The representation (9) follows by taking \( \varepsilon \) arbitrary small on \( G \) and arbitrary big on \( G^C \).

**Step 4.** If \( \mathbb{P}(\Upsilon_\pi) > 0 \) then \( \pi(X) = \mathcal{R}(\mu(X), \mu) \) on \( \Upsilon_\pi \), for any \( \mu \), and the representation

\[
\pi(X)1_{\Upsilon_\pi} = \max_{\mu \in \mathcal{L}(E, L^0(\mathcal{G}))} \mathcal{R}(\mu(X), \mu)1_{\Upsilon_\pi} \tag{28}
\]
trivially holds true on $\Upsilon_\pi$. Moreover, the map $\tilde{\pi}$ build in Lemma 34 satisfies (REG), (EVQ) and $\mathbb{P}(\Upsilon_\pi) = 0$. Denoting with $\mathcal{R}$ the dual function associated to $\tilde{\pi}$, we deduce from the previous steps:

$$\pi(X)1_{T_a} = \sup_{\mu \in \mathcal{L}(E, L^0(\mathcal{G})))} \mathcal{R}(\mu(X), \mu)1_{T_a}$$

as (18) implies $\mathcal{R}(Y, \mu)1_{T_a} = \mathcal{R}(Y, \mu)1_{T_a}$. □

**Proof of Proposition 18.** Lemma 34 (ii) and the same argument used in Step 4 of the proof of Theorem 16, allows us to assume w.l.o.g. that $\mathbb{P}(\Upsilon_x) = 0$. Fix $X \in E$ and consider, similarly to the proof of Theorem 16, the classes of sets

$$\mathcal{A} := \{ A \in \mathcal{G} \mid \forall \xi \in E \pi(\xi) \geq \pi(X) \text{ on } A \},$$

$$\mathcal{A}^+ := \{ A \in \mathcal{G} \mid \exists \xi \in E \text{ s.t. } \pi(\xi) < \pi(X) \text{ on } A \}.$$  

Then $\mathcal{A} = \{ A \in \mathcal{G} \mid \forall \xi \in F Y \geq Y_0 \text{ on } A \}$, where $F := \{ \pi(\xi) \mid \xi \in E \}$ and $Y_0 = \pi(X)$. Applying Lemma 33, there exist two maximal elements $A_M \in \mathcal{A}$ and $A_M^+ \in \mathcal{A}^+$ so that: $P(A_M \cup A_M^+) = 1$, $P(A_M \cap A_M^+) = 0$,

$$\pi(\xi) \geq \pi(X) \text{ on } A_M \text{ for every } \xi \in E$$

and

$$\exists \xi \in E \text{ s.t. } \pi(\xi) < \pi(X) \text{ on } A_M^+.$$  

Clearly

$$\pi(X)1_{A_M} \geq \mathcal{R}(\mu(X), \mu)1_{A_M} \geq \pi(X)1_{A_M}$$

for every $\mu \in \mathcal{L}(E, L^0(\mathcal{G}))$.

As $\mathbb{P}(\Upsilon_x) = 0$, we know from (5) that there exist $\zeta_1, \zeta_2 \in E$ such that $\pi(\zeta_1) < \pi(\zeta_2)$. The set

$$\mathcal{O} := \{ \xi \in E \mid \pi(\xi) < \pi(X)1_{A_M^+} + \pi(\zeta_2)1_{A_M} \}$$

is open, $L^0(\mathcal{G})$-convex (from Remark 13 ii) and not empty, as $\tilde{\xi}1_{A_M^+} + \zeta_11_{A_M} \in \mathcal{O}$. The random variable

$$\tilde{X} = X1_{A_M^+} + \zeta_21_{A_M}$$

satisfies $1_{B}\{\tilde{X}\} \cap 1_{B}\mathcal{O} = \emptyset$, for every $B \in \mathcal{G}$ s.t. $P(B) > 0$. We thus can apply the generalized Hahn Banach Separation Theorem (see [FKV09] Theorem 2.6) and find $\mu_\ast \in \mathcal{L}(E, L^0(\mathcal{G}))$ so that

$$\mu_\ast(\tilde{X}) > \mu_\ast(\xi) \quad \forall \xi \in \mathcal{O}.$$  

For each $A \in \mathcal{A}^+$ such that $\mathbb{P}(A) > 0$ let:

$$\mathcal{C}^A := \{ \xi \in E \mid \pi(\xi) < \pi(X) \text{ on } A \},$$

$$\mathcal{D}^A := \{ \xi \in E \mid \mu_\ast(\tilde{X}1_A) > \mu_\ast(\xi1_A) \text{ on } A \},$$

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and apply the argument in Step 2 of the proof of Theorem 16 to find that
\[\{\xi \in E \mid \mu_*(\bar{X})1_{A_m^*} \leq \mu_*(\xi)1_{A_m^*}\} \subseteq \{\xi \in E \mid \pi(\xi)1_{A_m^*} \geq \pi(\bar{X})1_{A_m^*}\}.\]

From \(\bar{X}1_{A_m^*} = X1_{A_m^*}\), (18)-(19) and \(R(\mu_*(\bar{X}),\mu_*)1_{A_m^*} = R(\mu_*(X),\mu_*)1_{A_m^*}\) we derive
\[\pi(X)1_{A_m^*} \geq R(\mu_*(X),\mu_*)1_{A_m^*} = R(\mu_*(\bar{X}),\mu_*)1_{A_m^*}\]
\[= \inf_{\xi \in E}\{\pi(\xi)1_{A_m^*} \mid \mu_*(\bar{X})1_{A_m^*} \leq \mu_*(\xi)1_{A_m^*}\}\]
\[\geq \inf_{\xi \in E}\{\pi(\xi)1_{A_m^*} \mid \pi(\xi)1_{A_m^*} \geq \pi(\bar{X})1_{A_m^*}\} \geq \pi(\bar{X})1_{A_m^*} = \pi(X)1_{A_m^*}.\]

The thesis then follows from (29).

\[\textbf{Proof of Corollary 19.} \quad \text{The statement is a direct consequence of the inequalities (26) through (27) of Step 3 in the proof of Theorem 16.} \]

5.3 Proofs of the complete duality stated in Section 4

We need some preliminary results

\textbf{Lemma 30} Let \(\rho\) be \((\text{REG})\). The function \(R\) defined in (13) belongs to \(\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)\)

\[\textbf{Proof.} \quad \text{We check the items in Definition 21.}\]

i) Is trivial.

ii) It can be shown as in Lemma 28 iv).

iii) Observe that \(R(Y,Q) \geq \inf_{\xi \in L_0^0(\mathcal{F})} \rho(\xi)\), for all \((Y,Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q\), so that
\[\inf_{Y \in L^0(\mathcal{G})} R(Y,Q) \geq \inf_{\xi \in L_0^0(\mathcal{F})} \rho(\xi).\]

Conversely notice that the set \(\{\rho(\xi) \mid \xi \in L_0^0(\mathcal{F})\}\) is downward directed and then there exists \(\rho(\xi_n) \downarrow \inf_{\xi \in L_0^0(\mathcal{F})} \rho(\xi)\). For every \(Q \in \mathcal{P}^q\) we have
\[\rho(\xi_n) \geq R\left(E\left[-\xi_n \frac{dQ}{d\mathbb{P}}\mid \mathcal{G}\right], Q\right) \geq \inf_{Y \in L^0(\mathcal{G})} R(Y,Q)\]
and therefore
\[\inf_{Y \in L^0(\mathcal{G})} R(Y,Q) \leq \inf_{\xi \in L_0^0(\mathcal{F})} \rho(\xi).\]

iv) For \(\alpha \in L^0(\mathcal{G})\) and \(A \in \mathcal{G}\) define \(U^A_\alpha = \{(Y,Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q \mid R(Y,Q) \geq \alpha \text{ on } A\}\), and suppose \(\emptyset \neq U^A_\alpha \neq L^0(\mathcal{G}) \times \mathcal{P}^q\). Let \((Y^*,Q^*) \in L^0(\mathcal{G}) \times \mathcal{P}^q\) such that \(R(Y^*,Q^*) < \alpha\) on \(A\).

As in Lemma 28 (iii) there exists \(X^* \in L^0_\alpha(\mathcal{F})\) such that \(E[-X^* \frac{dQ}{d\mathbb{P}}\mid \mathcal{G}] \geq Y^*\) and \(\rho(X^*) \leq \alpha\) on \(A\). Since \(R(Y,Q) \geq \alpha\) on \(A\) for every \((Y,Q) \in U^A_\alpha\), we deduce that \(E[-X^* \frac{dQ}{d\mathbb{P}}\mid \mathcal{G}] \leq Y\) on \(A\), for every \((Y,Q) \in U^A_\alpha\). Otherwise we could define
Finally we can conclude that for every \((Y,Q)\) \(\in U^A_{\alpha}\)
\[
Y^* + E \left[ X^* \frac{dQ^*}{dP} | G \right] \leq 0 < Y + E \left[ X^* \frac{dQ}{dP} | G \right]
\] on \(A\).

v) Take \(Q_1,Q_2 \in \mathcal{P}^q\) and define \(F = \{ R(E[X \frac{dQ}{dP} | G],Q_1) \geq R(E[X \frac{dQ_2}{dP} | G],Q_2) \}\) and let \(\hat{Q}\) given by
\[
\frac{d\hat{Q}}{dP} := 1_F \frac{dQ_1}{dP} + 1_{F^c} \frac{dQ_2}{dP} \in \mathcal{P}^q.
\]
It is easy to show, using an argument similar to the one in [FM09], Lemma 3.5 v) that
\[
R \left( E \left[ X \frac{d\hat{Q}}{dP} | G \right], \hat{Q} \right) = R \left( E \left[ X \frac{dQ_1}{dP} | G \right], Q_1 \right) \lor R \left( E \left[ X \frac{dQ_2}{dP} | G \right], Q_2 \right),
\]
which shows that the set \(\{ R(E[X \frac{dQ}{dP} | G],Q) \mid Q \in \mathcal{P}^q \}\) is upward directed.

vi) It follows from the same argument used in [FM09], Lemma 3.5 iv).

Lemma 31 If \(Q \in \mathcal{P}^q\) and if \(\rho\) is \((\downarrow \text{MON})\) and \((\text{REG})\) then
\[
R(Y,Q) = \inf_{\xi \in L^p_G(\mathcal{F})} \left\{ \rho(\xi) \mid E \left[ -\xi \frac{dQ}{dP} | G \right] = Y \right\}. \tag{30}
\]

**Proof.** For sake of simplicity we denote by \(\mu(\cdot) = E[\frac{dQ}{dP} | G]\) and \(r(Y,\mu)\) the right hand side of equation (30). Notice that \(R(Y,\mu) \leq r(Y,\mu)\). By contradiction, suppose that \(\mathbb{P}(\textbf{A}) > 0\) where \(\textbf{A} = \{ R(Y,\mu) < r(Y,\mu) \}\). From Lemma 28, there exists a r.v. \(\xi \in L^p_G(\mathcal{F})\) satisfying the following conditions

- \(\mu(-\xi) \geq Y\) and \(\mathbb{P}(\mu(-\xi) > Y) > 0\).
- \(R(Y,\mu)(\omega) \leq \rho(\xi)(\omega) < r(Y,\mu)(\omega)\) for \(\mathbb{P}\)-almost every \(\omega \in A\).

Then \(Z := \mu(-\xi) - Y \in L^0(G) \subseteq L^p_G(\mathcal{F})\) satisfies \(Z \geq 0\), \(\mathbb{P}(Z > 0) > 0\) and, thanks to \((\downarrow \text{MON})\), \(\rho(\xi) \geq \rho(\xi + Z)\). From \(\mu(-(\xi + Z)) = Y\) we deduce:
\[
R(Y,\mu)(\omega) \leq \rho(\xi)(\omega) < r(Y,\mu)(\omega) \leq \rho(\xi + Z)(\omega)\text{ for }\mathbb{P}\text{-a.e. }\omega \in A,
\]
which is a contradiction. \(\blacksquare\)

5.3.1 **Proof of Theorem 22**

During the whole proof we fix an arbitrary \(X \in L^p_G(\mathcal{F})\).
ONLY IF. Lemma 34 and the same argument used in Step 4 of the proof of Theorem 16, allows us to assume w.l.o.g. that \( \mathbb{P}(Y_{\ast}) = 0 \). Therefore, there exist \( \zeta_1, \zeta_2 \in L^0_0(\mathcal{F}) \) such that \( \rho(\zeta_1) < \rho(\zeta_2) \).

We refer to the proof of Theorem 16 for the following definitions and notations. We recall that the evenly convex set

\[
C_\varepsilon = \{ \xi \in L^0_0(\mathcal{F}) \mid \rho(\xi) \leq Y_\varepsilon 1_{A_\varepsilon} + \rho(\zeta_1)1_{A_\varepsilon} \} \neq \emptyset
\]

may be separated from \( \bar{X} = X1_{A_\varepsilon} + \zeta_21_{A_\varepsilon} \) by \( \mu_\varepsilon \in \mathcal{L}(L^0_0(\mathcal{F}), L^0(\mathcal{G})) \), i.e.

\[
\mu_\varepsilon(\bar{X}) > \mu_\varepsilon(\xi) \quad \forall \xi \in C_\varepsilon.
\]

(31)

Let \( \eta \in L^0_0(\mathcal{F}), \eta \geq 0 \). If \( \xi \in C_\varepsilon \) then (\( \downarrow \) MON) implies \( \xi + n\eta \in C_\varepsilon \) for every \( n \in \mathbb{N} \). Since \( \mu_\varepsilon(\cdot) = E[Z_\varepsilon \cdot | \mathcal{G}] \) for some \( Z_\varepsilon \in L^0_0(\mathcal{F}) \), from (31) we deduce:

\[
E[Z_\varepsilon(\xi + n\eta) | \mathcal{G}] < E[Z_\varepsilon \bar{X} | \mathcal{G}] \implies \frac{E[Z_\varepsilon(\xi - \bar{X}) | \mathcal{G}]}{n}, \quad \forall n \in \mathbb{N}
\]

i.e. \( E[Z_\varepsilon \eta | \mathcal{G}] \leq 0 \) for every \( \eta \in L^0_0(\mathcal{F}), \eta \geq 0 \). This implies, as \( 1_{\{Z_\varepsilon > 0\}} \in L^0_0(\mathcal{F}), \) that \( Z_\varepsilon \leq 0 \).

We now show that \( Z_\varepsilon < 0 \). Suppose there existed a \( \mathcal{G} \)-measurable set \( G, \mathbb{P}(G) > 0 \), on which \( Z_\varepsilon \leq 0 \) and fix \( \xi \in C_\varepsilon \). From \( E[Z_\varepsilon \xi | \mathcal{G}] < E[Z_\varepsilon \bar{X} | \mathcal{G}] \) we can find a \( \delta_\varepsilon \in L^{+\infty}(\mathcal{G}) \) such that

\[
E[Z_\varepsilon \xi | \mathcal{G}] + \delta_\varepsilon < E[Z_\varepsilon \bar{X} | \mathcal{G}] \implies \delta_\varepsilon 1_G = E[Z_\varepsilon 1_G \xi | \mathcal{G}] + \delta_\varepsilon 1_G \leq E[Z_\varepsilon 1_G \bar{X} | \mathcal{G}] = 0.
\]

which is a contradiction since \( \mathbb{P}(\delta_\varepsilon 1_G > 0) > 0 \).

We deduce that \( E[Z_\varepsilon 1_B] = E[E[Z_\varepsilon | \mathcal{G}]1_B] < 0 \) for every \( B \in \mathcal{G} \) and then \( \mathbb{P}(E[Z_\varepsilon | \mathcal{G}] < 0) = 1 \). Hence we may normalize \( Z_\varepsilon \) to \( Z_\varepsilon E[Z_\varepsilon | \mathcal{G}] = \frac{dQ_{\varepsilon}}{d\mathbb{P}} \in L^1(\mathcal{F}) \).

From equations (21) and (27) in the proof of Theorem 16 we can deduce that

\[
\rho(X) \geq \inf_{\xi \in L^0_0(\mathcal{F})} \{ \rho(\xi) \mid \mu_\varepsilon(\xi) \geq \mu_\varepsilon(X) \}
\]

\[
= \inf_{\xi \in L^0_0(\mathcal{F})} \left\{ \rho(\xi) \mid E\left[ -\xi \frac{dQ_{\varepsilon}}{d\mathbb{P}} | \mathcal{G} \right] \geq E\left[ -X \frac{dQ_{\varepsilon}}{d\mathbb{P}} | \mathcal{G} \right] \right\}
\]

\[
\geq (\pi(X) - \varepsilon)1_G + \varepsilon 1_{C^c} \quad (32)
\]

and hence

\[
\rho(X) = \sup_{Q \in \mathcal{P}, \xi \in L^0_0(\mathcal{F})} \left\{ \rho(\xi) \mid E\left[ -\xi \frac{dQ}{d\mathbb{P}} | \mathcal{G} \right] \geq E\left[ -X \frac{dQ}{d\mathbb{P}} | \mathcal{G} \right] \right\}.
\]

Applying Lemma 31 we can substitute \( = \) in the constraint.

IF. We assume that \( \rho(X) = \sup_{Q \in \mathcal{P}, \xi \in L^0_0(\mathcal{F})} R(E[-X \frac{dQ}{d\mathbb{P}} | \mathcal{G}], Q) \) holds for some \( R \in \mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}) \). Since \( R \) is monotone in the first component and \( R(Y1_A, Q)1_A = R(Y, Q)1_A \) for every \( A \in \mathcal{G} \) we easily deduce that \( \rho \) is (MON) and (REG). We
need to show that \( \rho \) is (EVQ).

Let \( V_\alpha = \{ \xi \in L^p_0(\mathcal{F}) \mid \rho(\xi)1_{T_\alpha} \leq \alpha \} \) where \( \alpha \in L^0(\mathcal{G}) \) and recall that \( D_{V_\alpha} \) is the complementary of the set provided in Definition 8. Notice that \( D_{V_\alpha} \subseteq T_\alpha \).

Take \( X^* \in L^0_0(\mathcal{F}) \) satisfying \( X^*1_A \cap V_\alpha 1_A = \emptyset \) for every \( A \in \mathcal{G} \), \( A \subseteq D_{V_\alpha} \), \( P(A) > 0 \). Hence

\[
\rho(X^*) = \sup_{Q \in \mathcal{P}^q} R(E[-X^*dQ/d\mu] \mid \mathcal{G}, Q) > \alpha
\]
on the set \( D_{V_\alpha} \). Since the set \( \{ R(E[-X^*dQ/d\mu] \mid \mathcal{G}, Q) \mid Q \in \mathcal{P}^q \} \) is upward directed there exists \( \{ Q_m \} \subset \mathcal{P}^q \) s.t.

\[
R\left(E\left[-X^*\frac{dQ_m}{d\mu}\mid \mathcal{G}\right], Q_m\right) \uparrow \rho(X^*) \quad \text{as} \ m \uparrow +\infty.
\]

Let \( \delta \in L^0_{++}(\mathcal{G}) \) satisfies \( \delta < \rho(X) - \alpha \) and consider the sets

\[
F_m = \left\{ R(E\left[-X^*\frac{dQ_m}{d\mu}\mid \mathcal{G}\right], Q_m) > \rho(X) - \delta \right\}
\]

and the partition of \( \Omega \) given by \( G_1 = F_1 \) and \( G_m = F_m \setminus G_{m-1} \). We have from the properties of the module \( L^q(\mathcal{F}) \) that

\[
\frac{dQ^*}{d\mu} = \sum_{m=1}^{\infty} \frac{dQ_m}{d\mu}1_{G_m} \in L^q(\mathcal{F})
\]

and then \( Q^* \subset \mathcal{P}^q \) with \( R(E[-X^*dQ^*/d\mu] \mid \mathcal{G}, Q^*) > \alpha \) on the set \( D_{V_\alpha} \).

Let \( \xi \in V_\alpha \). It remains to show that this \( Q^* \) separates \( X^* \) from \( V_\alpha \) on the set \( D_{V_\alpha} \). If there existed \( A \subseteq D_{V_\alpha} \in \mathcal{G} \) such that \( E[\xi dQ^*/d\mu]1_A | \mathcal{G} \leq E[X^*dQ^*/d\mu]1_A | \mathcal{G} \)
on \( A \) then \( \rho(\xi 1_A) \geq R(E[-\xi dQ^*/d\mu]1_A | \mathcal{G}, Q^*) \geq R(E[-X^*dQ^*/d\mu]1_A | \mathcal{G}, Q^*) > \alpha \) on \( A \). This implies \( \rho(\xi) > \alpha \) on \( A \) which is a contradiction unless \( \mathbb{P}(A) = 0 \). Hence \( E[\xi dQ^*/d\mu] \geq E[X^*dQ^*/d\mu] \) on \( D_{V_\alpha} \) for every \( \xi \in V_\alpha \).

**UNIQUENESS.** First we need the following preliminary result. Define the set

\[
\mathcal{A}(Y, Q) = \left\{ \xi \in L^p_0(\mathcal{F}) \mid E\left[-\xi \frac{dQ}{d\mu}\mid \mathcal{G}\right] \geq Y \right\}.
\]

**Lemma 32** If \( K \in \mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q) \), then for each \( (Y^*, Q^*) \in L^0(\mathcal{G}) \times \mathcal{P}^q \)

\[
K(Y^*, Q^*) = \sup_{Q \in \mathcal{P}^q} \inf_{X \in \mathcal{A}(Y^*, Q^*)} K \left(E\left[-X \frac{dQ}{d\mu}\mid \mathcal{G}\right], Q\right) \quad (33)
\]

**Proof.** Consider

\[
\psi(Q, Q^*, Y^*) = \inf_{X \in \mathcal{A}(Y^*, Q^*)} K \left(E\left[-X \frac{dQ}{d\mu}\mid \mathcal{G}\right], Q\right)
\]

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Notice that $E[-X \frac{dQ^*}{dP} | G] \geq Y^*$ for every $X \in \mathcal{A}(Y^*, Q^*)$ implies
\[
\psi(Q^*, Q^*, Y^*) = \inf_{X \in \mathcal{A}(Y^*, Q^*)} K\left( E\left[ -X \frac{dQ^*}{dP} | G \right], Q^* \right) \geq K(Y^*, Q^*)
\]

On the other hand $E[Y^* \frac{dQ^*}{dP} | G] = Y^*$ so that $-Y^* \in \mathcal{A}(Y^*, Q^*)$ and the second inequality is actually an equality
\[
\psi(Q^*, Q^*, Y^*) \leq K\left( E\left[ -(Y^*) \frac{dQ^*}{dP} | G \right], Q^* \right) = K(Y^*, Q^*)
\]

If we show that $\psi(Q, Q^*, Y^*) \leq \psi(Q^*, Q^*, Y^*)$ for every $Q \in \mathcal{P}^q$ then (33) is proved. To this aim we define
\[
\mathcal{A} := \left\{ A \in \mathcal{G} \mid E\left[ X \frac{dQ^*}{dP} | G \right] 1_A = E\left[ X \frac{dQ}{dP} | G \right] 1_A, \ \forall X \in L^p_G(F) \right\}
\]

For every $A \in \mathcal{A}$ and every $X \in L^p_G(F)$
\[
K\left( E\left[ -X \frac{dQ^*}{dP} | G \right], Q \right) 1_A = K\left( E\left[ -X \frac{dQ}{dP} | G \right] 1_A, Q \right) 1_A = K\left( E\left[ -X \frac{dQ^*}{dP} | G \right], Q^* \right) 1_A
\]
which implies
\[
\psi(Q, Q^*, Y^*) 1_A = \psi(Q^*, Q^*, Y^*) 1_A.
\] (34)

Notice that $\mathcal{A} = \{ A \in \mathcal{G} \mid Y = 0 \text{ on } A, \forall Y \in F \}$, where
\[
F := \left\{ E\left[ X \frac{dQ^*}{dP} | G \right] - E\left[ X \frac{dQ}{dP} | G \right] \mid X \in L^p_G(F) \right\}
\]

As the conditional expectation is (REG), we may apply Lemma 33 and deduce the existence of two maximal sets $A_M \in \mathcal{A}$ and $A_M^c \in \mathcal{A}^c$ such that: $P(A_M \cap A_M^c) = 0, P(A_M \cup A_M^c) = 1$; $E\left[ X \frac{dQ^*}{dP} | G \right] 1_{A_M} = E\left[ X \frac{dQ}{dP} | G \right] 1_{A_M}, \ \forall X \in L^p_G(F)$; and $E\left[ -X \frac{dQ^*}{dP} | G \right] \neq E\left[ -X \frac{dQ}{dP} | G \right]$ on $A_M^c$, for some $X^* \in L^p_G(F)$.

Considering $A_M \in \mathcal{A}$ we then deduce from (34)
\[
\psi(Q, Q^*, Y^*) 1_{A_M} = \psi(Q^*, Q^*, Y^*) 1_{A_M}.
\]

Now we consider $A_M^c \in \mathcal{A}^c$ and define $Z^* := X^* - E\left[-X^* \frac{dQ^*}{dP} | G \right]$. Surely $E\left[ Z^* \frac{dQ^*}{dP} | G \right] = 0$ and $E\left[ Z^* \frac{dQ}{dP} | G \right] \neq 0$ on $A_M^c$. We deduce that for every $\alpha \in L^p(\mathcal{G})$, $-Y^* + \alpha Z^* \in \mathcal{A}(Y^*, Q^*)$. Also notice that any $Y \in L^p(\mathcal{G})$ can be written as $Y 1_{A_M^c} = E[(-Y^* + \alpha Y^* Z^*) \frac{dQ^*}{dP} | G] 1_{A_M^c}$, with $\alpha Y \in L^p(\mathcal{G})$.

Finally
\[
\psi(Q, Q^*, Y^*) 1_{A_M^c} \leq \inf_{\alpha \in L^p(\mathcal{G})} K\left( E\left[ -(Y^* + \alpha Z^*) \frac{dQ^*}{dP} | G \right], Q \right) 1_{A_M^c}
\]
\[
= \inf_{Y \in L^p(\mathcal{G})} K\left( Y 1_{A_M^c}, Q \right) 1_{A_M^c} = \inf_{Y \in L^p(\mathcal{G})} K\left( Y 1_{A_M^c}, Q^* \right) 1_{A_M^c}
\]
\[
\leq K(Y^*, Q^*) 1_{A_M^c}
\]

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As \( P(A_M \cup A_M^*) = 1 \), we conclude that \( \psi(Q, Q^*, Y^*) \leq \psi(Q^*, Q^*, Y^*) = K(Y^*, Q^*) \) and the claim is proved. 

To prove the uniqueness we show that for every \( K \in \mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q) \) such that

\[
\rho(X) = \sup_{Q \in \mathcal{P}^q} K \left( E \left[ -X \frac{dQ}{dP} | \mathcal{G} \right], Q \right),
\]

\( K \) must satisfy

\[
K(Y, Q) = \inf_{\xi \in \mathcal{L}_0^\infty(\mathcal{F})} \left\{ \rho(\xi) \mid E \left[ -\xi \frac{dQ}{dP} | \mathcal{G} \right] \geq Y \right\}.
\]

By the Lemma 32

\[
K(Y^*, Q^*) = \sup_{Q \in \mathcal{P}^q} \inf_{X \in \mathcal{A}(Y^*, Q^*)} K \left( E \left[ -X \frac{dQ}{dP} | \mathcal{G} \right], Q \right)
\leq \inf_{X \in \mathcal{A}(Y^*, Q^*)} \sup_{Q \in \mathcal{P}^q} K \left( E \left[ -X \frac{dQ}{dP} | \mathcal{G} \right], Q \right) = \inf_{X \in \mathcal{A}(Y^*, Q^*)} \rho(X).
\]

It remains to prove the reverse inequality, i.e.

\[
K(Y^*, Q^*) \leq \inf_{X \in \mathcal{A}(Y^*, Q^*)} \rho(X).
\] (35)

Consider the class:

\[
\mathcal{A} := \{ A \in \mathcal{G} \mid K(Y, Q)1_A \leq K(Y^*, Q^*)1_A \ \forall (Y, Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q \}.
\]

Notice that \( \mathcal{A} = \{ A \in \mathcal{G} \mid Z \leq Z_0 \ \forall Z \in \mathcal{A}, \ \forall Z \in \mathcal{F} \} \)

where

\[
\mathcal{F} := \{ K(Y, Q) \mid (Y, Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q \}.
\]

and \( Z_0 = K(Y^*, Q^*) \). In order to apply Lemma 33, let \( A_i \in \mathcal{A}^c \) be a sequence of disjoint sets and \( Z_i = K(Y_i, Q_i) \) be the corresponding element in \( \mathcal{F} \). From \( K(Y1_{A_i}, Q)1_A = K(Y, Q)1_A \) we deduce (as in Remark 13 i) that

\[
K(\sum_{i=1}^\infty Y_i1_{A_i}, \sum_{j=1}^\infty Q_j1_{A_j})1_{A_{\infty}} = \sum_{i=1}^\infty K(Y_i, \sum_{j=1}^\infty Q_j1_{A_j})1_{A_i},
\]

\[
= \sum_{i=1}^\infty K(Y_i, Q_i)1_{A_i} = \sum_{i=1}^\infty Z_i1_{A_i}
\]

showing that \( \sum_{i=1}^\infty Z_i1_{A_i} \in \mathcal{F} \). From Lemma 33 we may deduce the existence of two maximal sets \( A_M \in \mathcal{A} \) and \( A_M^* \in \mathcal{A}^c \) such that: \( P(A_M \cap A_M^*) = 0 \), \( P(A_M \cup A_M^*) = 1 \); \( K(Y, Q)1_{A_M} \leq K(Y^*, Q^*)1_{A_M} \ \forall (Y, Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q \); and

\[
K(Y^*, Q^*) < K(Y, Q) \ \text{on} \ A_M^*.
\] (36)
for some \((\bar{Y}, \bar{Q}) \in L^0(\mathcal{G}) \times \mathcal{P}^n\). On \(A_M \in \mathcal{A}\) the inequality (35) is obviously true and we need only to show it on the set \(A^*_M\).

From (36) we can easily build a \(\beta \in L^0(\mathcal{G})\) such that \(K(Y^*, Q^*) < \beta \leq K(\bar{Y}, \bar{Q})\) on \(A^*_M\) and \(\beta\) is arbitrarily close to \(K(Y^*, Q^*)\) on \(A^*_M\). An example of such \(\beta\) is obtained by taking \(\lambda \downarrow 0\) in the family:

\[
\beta_\lambda := 1_{A^*_M} \left[ \frac{\lambda K(\bar{Y}, \bar{Q})}{\lambda} + (1 - \lambda) K(Y^*, Q^*) \right] 1_{\{K(\bar{Y}, \bar{Q}) < \infty\} \cap \{K(Y^*, Q^*) > -\infty\}} + 1_{A^*_M} 1_{\{K(\bar{Y}, \bar{Q}) = \infty\}} \left[ (K(Y^*, Q^*) + \lambda) 1_{\{K(Y^*, Q^*) > -\infty\}} - 1_{\lambda} 1_{\{K(Y^*, Q^*) = -\infty\}} \right].
\]

Since the set \(U_\beta := \{(Y, Q) \in L^0(\mathcal{G}) \times \mathcal{P}^n | K(Y, Q) > \beta\} \cap A^*_M\) is not empty, the assumption that \(K\) is \(\sigma\)-evenly \(L^0(\mathcal{G})\)-quasiconcave implies the existence of \((S^*, X^*) \in L^0(\mathcal{G}) \times L^0(\mathcal{F})\) with

\[
Y^* S^* + E \left[ X^* \frac{dQ^*}{dP} \mid \mathcal{G} \right] < Y S^* + E \left[ X^* \frac{dQ}{dP} \mid \mathcal{G} \right] \text{ on } A^*_M
\]

for every \((Y, Q) \in U_\beta\).

We claim that for every \((Y, Q) \in U_\beta\)

\[
Y + E \left[ \hat{X} \frac{dQ}{dP} \mid \mathcal{G} \right] > 0 \text{ on } A^*_M,
\]

where \(\hat{X} := \frac{X^*}{S^*} + \Lambda\) and \(\Lambda := -Y^* - E[\frac{X^*}{S^*} \frac{dQ^*}{dP} \mid \mathcal{G}]\). Indeed, for every \((Y, Q) \in U_\beta\)

\[
Y^* S^* + E \left[ X^* \frac{dQ^*}{dP} \mid \mathcal{G} \right] < Y S^* + E \left[ X^* \frac{dQ}{dP} \mid \mathcal{G} \right] \text{ on } A^*_M,
\]

implies \(
Y^* + E \left[ \hat{X} \frac{dQ^*}{dP} \mid \mathcal{G} \right] < Y + E \left[ \hat{X} \frac{dQ}{dP} \mid \mathcal{G} \right] \text{ on } A^*_M,
\]

implies \(
Y^* + E \left[ \hat{X} \frac{dQ^*}{dP} \mid \mathcal{G} \right] < Y + E \left[ \hat{X} \frac{dQ}{dP} \mid \mathcal{G} \right] \text{ on } A^*_M,
\]

i.e. the claim holds, as \(E[\hat{X} \frac{dQ}{dP} \mid \mathcal{G}] = -Y^*\).

For every \(Q \in \mathcal{P}^n\) define \(Y_Q := E \left[ -\hat{X} \frac{dQ}{dP} \mid \mathcal{G} \right].\) We show that

\[
K(Y_Q, Q) < \beta \text{ on } A^*_M.
\]

Suppose by contradiction that there exists \(B \subseteq A^*_M, B \in \mathcal{G}, P(B) > 0\), such that \(K(Y_Q, Q) \geq \beta\) on \(B\). Take \((Y_1, Q_1) \in U_\beta\) and define \(\bar{Y} := Y_Q 1_B + Y_1 1_{B^C}\) and \(\bar{Q} \in \mathcal{P}^n\) by

\[
\frac{d\bar{Q}}{dP} = \frac{dQ}{dP} 1_B + \frac{dQ_1}{dP} 1_{B^C}.
\]

Thus \(K(\bar{Y}, \bar{Q}) \geq \beta\) on \(A^*_M\) and \(\bar{Y} + E \left[ \hat{X} \frac{dQ}{dP} \mid \mathcal{G} \right] > 0\) on \(A^*_M\), which implies \(Y_Q + E \left[ \hat{X} \frac{dQ}{dP} \mid \mathcal{G} \right] > 0\) on \(B\) and this is impossible and (37) is proven.

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Since $\hat{X} \in \mathcal{A}(Y^*, Q^*)$ we can conclude that

$$K(Y^*, Q^*)\mathbf{1}_{A^*_M} \leq \inf_{X \in \mathcal{A}(Y^*, Q^*), \mathbb{P}} \sup_{Q \in \mathbb{P}} K \left( E \left[ -\hat{X} \frac{dQ}{dP} | \mathcal{G} \right], Q \right) \mathbf{1}_{A^*_M}$$

$$\leq \sup_{Q \in \mathbb{P}} K \left( E \left[ -\hat{X} \frac{dQ}{dP} | \mathcal{G} \right], Q \right) \mathbf{1}_{A^*_M} \leq \beta \mathbf{1}_{A^*_M}.$$  

As $\beta$ is arbitrarily close to $K(Y^*, Q^*)$, the equality must hold and then we obtain:

$$K(Y^*, Q^*) = \inf_{X \in \mathcal{A}(Y^*, Q^*)} \rho(X) \text{ on } A^*_M.$$  

This concludes the proof of Theorem 22.

6 Appendix

The next Lemma is used several times in the proofs of the paper. It says that for any subset $F \subseteq L^0(\mathcal{G})$ that is “closed w.r. to pasting” it is possible to determine a maximal set $A_M \in \mathcal{G}$ (which may have zero probability) such that $Y 1_{A_M} \geq 0 \ \forall Y \in F$ and one element $\overline{Y} \in F$ for which $\overline{Y} < 0$ on the complement of $A_M$.

Lemma 33 With the symbol $\triangleright$ denote any one of the binary relations $\geq, \leq, =, >, <$ and with $\prec$ its negation. Consider a class $F \subseteq \overline{L}^0(\mathcal{G})$ of random variables, $Y_0 \in \overline{L}^0(\mathcal{G})$ and the classes of sets

$$\mathcal{A} := \{ A \in \mathcal{G} | \forall Y \in F \ \forall Y \triangleright Y_0 \text{ on } A \},$$

$$\mathcal{A}^\triangleright := \{ A^\triangleright \in \mathcal{G} | \exists Y \in F \text{ s.t. } Y \prec Y_0 \text{ on } A^\triangleright \}.$$  

Suppose that for any sequence of disjoint sets $A_i^\triangleright \in \mathcal{A}^\triangleright$ and the associated r.v. $Y_i \in F$ we have $\sum_{i=1}^{\infty} Y_i 1_{A_i^\triangleright} \in F$. Then there exist two maximal sets $A_M \in \mathcal{A}$ and $A_M^\triangleright \in \mathcal{A}^\triangleright$ such that $P(A_M \cap A_M^\triangleright) = 0$, $P(A_M \cup A_M^\triangleright) = 1$ and

$$Y \triangleright Y_0 \text{ on } A_M, \forall Y \in F$$

$$\overline{Y} < Y_0 \text{ on } A_M^\triangleright, \text{ for some } \overline{Y} \in F.$$  

**Proof.** Notice that $\mathcal{A}$ and $\mathcal{A}^\triangleright$ are closed with respect to countable union. This claim is obvious for $\mathcal{A}$. For $\mathcal{A}^\triangleright$, suppose that $A_i^\triangleright \in \mathcal{A}^\triangleright$ and that $Y_i \in F$ satisfies $P(\{ Y_i \prec Y_0 \} \cap A_i^\triangleright) = P(A_i^\triangleright)$. Defining $B_1 := A_1^\triangleright$, $B_i := A_i^\triangleright \setminus B_{i-1}$, $A_i := \bigcup_{i=1}^{\infty} A_i$, we see that $B_i$ are disjoint elements of $\mathcal{A}^\triangleright$ and that $Y_* := \sum_{i=1}^{\infty} Y_i 1_{B_i} \in F$ satisfies $P(\{ Y_* \prec Y_0 \} \cap A_*^\triangleright) = P(A_*^\triangleright)$ and so $A_*^\triangleright \in \mathcal{A}^\triangleright$.

The Remark 7 guarantees the existence of two sets $A_M \in \mathcal{A}$ and $A_M^\triangleright \in \mathcal{A}^\triangleright$ such that:

(a) $P(\mathcal{A} \cap (A_M)^C) = 0$ for all $A \in \mathcal{A}$,

(b) $P(A \cap (A_M^\triangleright)^C) = 0$ for all $A^\triangleright \in \mathcal{A}^\triangleright$.

Obviously, $P(A_M \cap A_M^\triangleright) = 0$, as $A_M \in \mathcal{A}$ and $A_M^\triangleright \in \mathcal{A}^\triangleright$. To show that $P(A_M \cup A_M^\triangleright) = 1$, let $D := \Omega \setminus \{ A_M \cup A_M^\triangleright \} \in \mathcal{G}$. By contradiction suppose
that $P(D) > 0$. As $D \subseteq (A_M)^C$, from condition (a) we get $D \notin A$. Therefore, 
\[ \exists Y \in F \text{ s.t. } P(\{Y \supset Y_0\} \cap D) < P(D), \text{ i.e. } P(\{Y \subset Y_0\} \cap D) > 0. \]
If we set $B := \{Y \subset Y_0\} \cap D$ then it satisfies $P(\{Y \subset Y_0\} \cap B) = P(B) > 0$ and, by
definition of $A^*$, $B$ belongs to $A^*$. On the other hand, as $B \subseteq D \subseteq (A_M)^C$, $P(B) = P(B \cap (A_M)^C)$, and from condition (b) $P(B \cap (A_M)^C) = 0$, which
contradicts $P(B) > 0$. ■

The following Lemma allows us to simplify many proofs in the paper.

**Lemma 34** Suppose that $\pi : E \rightarrow \bar{L}^0(\mathcal{G})$ is (REG). Consider the map

\[ \bar{\pi}(\cdot) := \pi(\cdot)1_{T_{\pi}} + \mu(\cdot)1_{T_{\bar{\pi}}}, \]

where $\mu \in \mathcal{L}(E, \bar{L}^0(\mathcal{G}))$ satisfies $\mu(1_{\Omega}) > 0$. Then $\bar{\pi}$ satisfies (REG) and
$\mathbb{P}(\bar{\pi}) = 0$. Moreover, (i) if $\pi$ is (QCO) then $\bar{\pi}$ is (QCO); (ii) if $\pi$ is (USC*)
then $\bar{\pi}$ is (USC*); (iii) $\pi$ is (EVQ) iff $\bar{\pi}$ is (EVQ).

**Proof.** Clearly, $\bar{\pi}$ satisfies (REG) and $\mathbb{P}(\bar{\pi}) = 0$.
(i) Applying $(\cdot \lor \cdot)1_D = (\cdot)1_D \lor (\cdot)1_D$
\[ \bar{\pi}(\lambda X_1 + (1 - \lambda)X_2) = \pi(X_1 + (1 - \lambda)X_2)1_{T_{\pi}} + \mu(X_1 + (1 - \lambda)X_2)1_{T_{\bar{\pi}}}, \]
\[ \leq [\pi(X_1) \lor \pi(X_2)]1_{T_{\pi}} + [\mu(X_1) \lor \mu(X_2)]1_{T_{\bar{\pi}}} \]
\[ = \pi(X_1)1_{T_{\pi}} \lor \pi(X_2)1_{T_{\bar{\pi}}} + \mu(X_1)1_{T_{\pi}} \lor \mu(X_2)1_{T_{\bar{\pi}}}. \]

(ii) Suppose that $\pi$ is (USC*) and let $Z \in \bar{L}^0(\mathcal{G})$. We need to prove that the
set $\bar{\mathcal{O}}^\pi := \{\xi \in E \mid \bar{\pi}(\xi) < Z\}$ is open. Let $\xi \in \bar{\mathcal{O}}^\pi$ and define the open sets
$\mathcal{O}^\pi := \{\xi \in E \mid \pi(\xi) < Z1_{T_{\pi}}\}$ and $\mathcal{O}^\mu := \{\xi \in E \mid \mu(\xi) < Z1_{T_{\bar{\pi}}}\}$. As $\xi 1_{T_{\pi}} \in \mathcal{O}^\pi$
and $\xi 1_{T_{\bar{\pi}}} \in \mathcal{O}^\mu$ we may find a neighborhood $U_0$ of 0 satisfying $\xi 1_{T_{\pi}} + U_0 \subseteq \mathcal{O}^\pi$
and $\xi 1_{T_{\bar{\pi}}} + U_0 \subseteq \mathcal{O}^\mu$. Then the set $U := (\xi 1_{T_{\pi}} + U_0)1_{T_{\pi}} + (\xi 1_{T_{\bar{\pi}}} + U_0)1_{T_{\bar{\pi}}} = \xi + U_01_{T_{\pi}} + U_01_{T_{\bar{\pi}}}$ is contained in $\mathcal{O}^\pi 1_{T_{\pi}} + \mathcal{O}^\mu 1_{T_{\bar{\pi}}}$ and is a neighborhood of $\xi$,
since $U_01_{T_{\pi}} + U_01_{T_{\bar{\pi}}}$ contains $U_0$ and is therefore a neighborhood of 0. Thanks
to (REG) $\mathcal{O}^\pi 1_{T_{\pi}} + \mathcal{O}^\mu 1_{T_{\bar{\pi}}} \subseteq \bar{\mathcal{O}}^\pi$, so that $\bar{\mathcal{O}}^\pi$ is open.
(iii) Suppose that $\pi$ is (EVQ). To prove that $\bar{\pi}$ is (EVQ) let $Z \in \bar{L}^0(\mathcal{G})$ and
suppose that
\[ \mathcal{C}^\pi_{\bar{\pi}} := \{\xi \in E \mid \bar{\pi}(\xi) \leq Z\}, \]
is not empty. Define
\[ \mathcal{C}_{Z_1} := \{\xi \in E \mid \pi(\xi)1_{T_{\pi}} \leq Z_1\} \]
\[ \mathcal{C}_{Z_2} := \{\xi \in E \mid \mu(\xi) \leq Z_2\}, \]

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where \( Z_1 := Z \mathbf{1}_{\Upsilon_*} \) and \( Z_2 := Z \mathbf{1}_{\Upsilon_*} + \mu(X) \mathbf{1}_{\Upsilon_*}, X \in E \). In this way, if \( \xi \in C^\pi_Z \) then \( \xi_1 := \xi \mathbf{1}_{\Upsilon_*} \in C^\pi_{Z_1} \neq \emptyset \) and \( \xi_2 := \xi \mathbf{1}_{\Upsilon_*} + X \mathbf{1}_{\Upsilon_*} \in C^\mu_{Z_2} \neq \emptyset \).

We set \( \tilde{\mathcal{V}} := C^\pi_{Z_2} \) and show that \( \tilde{\mathcal{V}} \) is evenly \( L^0(\mathcal{G}) \)-convex. Recalling Definition 8, we find the maximal set \( A_{\tilde{\mathcal{V}}} \) and its complement set \( D_{\tilde{\mathcal{V}}} \) which will depend on the map \( \tilde{\pi} \) and not on \( \pi \). Since \( \mu(1_\Omega) > 0 \), then \( \tilde{\mathcal{V}} \) has no trivial component on \( \Upsilon_{\pi} \) and hence \( A_{\tilde{\mathcal{V}}} \subseteq T_* \). Let \( \eta \in E \) such that for every \( A \in \mathcal{G}, \ A \subseteq D_{\tilde{\mathcal{V}}}, \ P(A) > 0 \),

\[
1_A(\eta) \cap 1_A C^\pi_{Z_2} = \emptyset \text{ on } D_{\tilde{\mathcal{V}}}. \quad (38)
\]

We need to construct a \( \mu \in \mathcal{L}(E, L^0(\mathcal{G})) \) such that \( \mu(\eta) > \mu(\xi) \) on \( D_{\tilde{\mathcal{V}}}, \ \forall \xi \in C^\pi_{Z_2} \). Condition (38) implies:

\[
1_A(\eta) \cap 1_A C^\mu_{Z_1} = \emptyset \text{ for all } A \subseteq (T_* \setminus A_{\tilde{\mathcal{V}}}), \ A \in \mathcal{G}, \ P(A) > 0.
\]

Notice also that if \( \zeta_2 \in C^\mu_{Z_2} \) and \( \xi \in C^\pi_{Z_2} \) then \( \zeta := \zeta_2 \mathbf{1}_{\Upsilon_*} + \xi \mathbf{1}_{\Upsilon_*} \in C^\pi_{Z_2} \) and therefore (38) implies

\[
1_A(\eta) \cap 1_A C^\mu_{Z_2} = \emptyset \text{ for all } A \subseteq \Upsilon_{\pi}, \ A \in \mathcal{G}, \ P(A) > 0.
\]

Since both \( \pi, \mu \) are (EVQ) then we may find \( \mu_1, \mu_2 \in \mathcal{L}(E, L^0(\mathcal{G})) \) such that

\[
\begin{align*}
\mu_1(\eta) &> \mu_1(\xi) \quad \text{on } (T_* \setminus A_{\tilde{\mathcal{V}}}) \forall \xi \in C^\pi_{Z_1}, \\
\mu_2(\eta) &> \mu_2(\xi) \quad \text{on } \Upsilon_{\pi} \forall \xi \in C^\mu_{Z_2}.
\end{align*}
\]

Setting \( \mu := \mu_1 \mathbf{1}_{\Upsilon_*} + \mu_2 \mathbf{1}_{T_*} \) we deduce \( \mu(\eta) > \mu(\xi), \ \forall \xi \in C^\pi_{Z_2} \) on \( D_{\mathcal{V}} = \Upsilon_{\pi} \cup (T_* \setminus A_{\tilde{\mathcal{V}}}) \).

Viceversa, suppose that \( \tilde{\pi} \) is (EVQ) and let \( \eta \in L^0(\mathcal{G}) \). We need to show that the set \( C_{\tilde{\pi}} := \{ \xi \in E \mid \pi(\xi) \mathbf{1}_{T_*} \leq \eta \} \) is evenly \( L^0(\mathcal{G}) \)-convex. Again using Definition 8 with \( \mathcal{V} = C_{\tilde{\pi}} \) we find the maximal set \( A_{\tilde{\mathcal{V}}} \) and its complement set \( D_{\tilde{\mathcal{V}}} \). We may assume that \( \eta \mathbf{1}_{T_*} \geq 0 \), otherwise \( C_{\tilde{\pi}} = \emptyset \). Then we have that \( \Upsilon_{\pi} \subseteq A_{\mathcal{V}} \) and consequently \( D_{\mathcal{V}} \subseteq T_* \). We will have to find a separating functional over the set \( D_{\mathcal{V}} \). Notice that \( \xi \in C_{\pi} \) if and only if \( \pi(\xi) \leq \eta \) on \( T_* \).

Let \( X \in E \) s.t. \( 1_A(\eta) \cap 1_A C_{\tilde{\pi}} = \emptyset \) for all \( D_{\mathcal{V}}, \ A \in \mathcal{G}, \ P(A) > 0 \). As \( \tilde{\pi} \) is (EVQ) and \( \mathcal{P}(\Upsilon_{\tilde{\pi}}) = 0 \), the set

\[
C_{\tilde{\pi}} := \{ \xi \in E \mid \tilde{\pi}(\xi) \leq \eta \mathbf{1}_{T_*} + \mu(X - 1_\Omega) \mathbf{1}_{T_*} \}
\]

is evenly \( L^0(\mathcal{G}) \)-convex. Since \( \mu(1_\Omega) > 0 \), \( 1_A(\eta) \cap 1_A C_{\tilde{\pi}} = \emptyset \) for every \( A \in \mathcal{G} \), \( P(A) > 0 \). This means that the maximal set provided by Definition 8 for \( C_{\tilde{\pi}} \) is given by \( T_* \cap A_{\mathcal{V}} \). Then there exists \( \phi \in \mathcal{L}(E, L^0(\mathcal{G})) \) such that \( \phi(X) > \phi(\xi) \) on \( \Omega \setminus (T_* \cap A_{\mathcal{V}}), \ \forall \xi \in C_{\tilde{\pi}} \). Notice that if \( \xi \in C_{\pi} \) then \( \xi := \xi \mathbf{1}_{T_*} + (X - 1_\Omega) \mathbf{1}_{T_*} \in C_{\tilde{\pi}} \) and \( \tilde{\xi} \mathbf{1}_{T_*} = \xi \mathbf{1}_{T_*} \). This implies \( \phi(X) > \phi(\xi) \) on \( T_* \cap D_{\mathcal{V}}, \ \forall \xi \in C_{\pi} \) and therefore \( C_{\pi} \) is evenly \( L^0(\mathcal{G}) \)-convex. ■
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