A note on the super replication price of unbounded claims

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Abstract

We prove an extension of the theorem that provides the dual representation of the weak super replication price of a claim. With this generalization we admit unbounded utility functions and unbounded stochastic integrals.

The aim of this note is the proof of Theorem 1, which is a generalization of Theorem 5 in Biagini Frittelli [1]. We defer to [1] for the interpretation of the results, but in order to be self-contained, we state from [1] all the definitions that are needed to comprehend the subject.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $L^{bb}(P) \triangleq \{f \in L^0(P) \,|\, \exists c \in \mathbb{R} \text{ s.t. } f \geq c \; P\text{−a.s.}\}$ be the set of bounded from below random variables. We will work with a fixed non empty convex cone $\mathcal{K} \subseteq L^0(P)$, which is *not necessarily contained in* $L^{bb}(P)$. When the price processes are modeled by locally bounded semimartingales the set K of terminal values from the classical class of admissible trading strategies is contained in $L^{bb}(P)$. However, when the price processes are semimartingales that may be unbounded, the set $\tilde{K}$ of terminal values from the appropriate class of "admissible" trading strategies *isn't any more contained in* $L^{bb}(P)$. Therefore, if we select $\mathcal{K} = \tilde{K}$, the Theorems 1 and 3 below turn out to be very useful not only in the context of the super replication price treated in this note, but also in the utility maximization problem with unbounded processes. In this case, equation (3) will provide the dual representation of the domain of the optimization problem. The precise formulations of the results concerning the utility maximization problem with unbounded semimartingales can be found in Biagini Frittelli [2].

Recall that $\Phi : (0, +\infty) \to \mathbb{R}$ is a strictly convex, differentiable function satisfying the growth condition $G(\Phi)$: For each compact interval $[\lambda_0, \lambda_1]$ contained in $(0, +\infty)$ there exist constants $\alpha > 0$ and $\beta > 0$ such that:

$$\Phi(\lambda y) \leq \alpha \Phi(y) + \beta(y + 1), \text{ for } y > 0 \text{ and } \lambda \in [\lambda_0, \lambda_1].$$

For any subset $G \subseteq L^1(P)$ we set: $G_1 \triangleq \{z \in G : E[z] = 1\}$. Define

$$\begin{aligned} M_1 &\triangleq \left\{Q \ll P : k \in L^1(Q) \text{ and } E_Q[k] \leq 0 \text{ for all } k \in \mathcal{K}\right\}, \\ \mathcal{P}_\Phi &\triangleq \{Q \ll P \mid E[\Phi(\frac{dQ}{dP})] < +\infty\}, \\ M_\Phi &\triangleq M_1 \cap \mathcal{P}_\Phi, \\ C_\Phi &\triangleq \bigcap_{Q \in M_\Phi} \overline{\mathcal{K} - L^1_+(Q)}^Q, \end{aligned}$$

where $\overline{C}^Q$ denotes the $L^1(Q)-$closure of a set C. In the financial interpretation M_1 is the set of pricing measures and Φ is the conjugate of a utility function. The following theorem was stated and proved in Biagini Frittelli, Theorem 5 [1], under the additional assumptions that $\mathcal{K} \subseteq L^{bb}(P)$, that M_Φ contained at least one probability measure equivalent to P and that $\Phi(0^+) = \lim_{y \downarrow 0} \Phi(y) < +\infty$. Note that when Φ is the convex conjugate of an utility function u, the condition $\Phi(0^+) < +\infty$ is equivalent to the requirement that u is bounded from above, which may therefore constitute a non trivial assumption. When $\mathcal{K} = K \subseteq L^{bb}(P)$ (see [1] for the exact definition of K) equation (1) is exactly equation (11) in [1], and provides the dual representation of the *weak super replication price* of a claim f.

Theorem 1 *If Φ satisfies $G(\Phi)$, $M_\Phi \neq \varnothing$ and $f \in \bigcap_{Q \in M_\Phi} L^1(Q)$ then*

$$\inf\{x \in \mathbb{R} \mid f - x \in C_\Phi\} = \sup_{Q \in M_\Phi} E_Q[f]. \tag{1}$$

This same extension, but under the hypothesis $\mathcal{K} \subseteq L^{bb}(P)$, of Theorem 5 [1] is provided also by Owen, Theorem 6.6 [4]. However, the proof of Owen relies on unnecessary auxiliary definitions and results that hide the simplicity of this generalization. Here, we provide a direct, much shorter proof based on duality theory and on Theorem 3 below. As in [1], the key device is the appropriate selection of the topology that allow for the duality to work easily. When $M_\Phi \neq \varnothing$, let's define the linear spaces

$$L = \bigcap_{Q \in M_\Phi} L^1(Q) \text{ and } L' = Lin\{M_\Phi\} \subseteq L^1(P).$$

For all $z \in L$ and $z' \in L'$, we have that $(zz') \in L^1(P)$ and the bilinear form: $z \times z' \to z'(z) \triangleq E[zz']$ is well defined. We denote with τ a locally convex topology on L compatible with the duality (L, L'); call ν a compatible topology on L' and all polars are defined with respect to this dual system (see Remark 22 in [1] for details on this dual system). We remark that $\mathcal{K} \subseteq L$ by construction.

As shown in [1], if $\Phi(0^+)$ is infinite, then M_Φ may not be closed and $co(M_\Phi) \subseteq (C_\Phi)^0$ with possibly strict inclusion. However, we will show (see equation (5)) that $(C_\Phi)^0_1 = \overline{M_\Phi}^\nu$ always holds true, and then Theorem 1 will easily follow from

Theorem 2 *(Theorem 10 [1]) Let $G \subseteq L$ be a convex cone satisfying $G^{00} = G$ and assume that $-\mathbf{1} \in G$. If $G_1^0 \neq \varnothing$ then for all $f \in L$ we have:*

$$\inf\{x \in \mathbb{R} \mid f - x\mathbf{1} \in G\} = \sup\left\{z(f) \mid z \in G_1^0\right\}. \tag{2}$$

Equation (3) in the next theorem is an essential step in the proof of Theorem 1. It was first proved in Frittelli, Theorem 20 [3], with two additional assumptions: $\mathcal{K} \subseteq L^{bb}(P)$ and M_Φ contains at least one measure equivalent to P.

Theorem 3 *If Φ satisfy $G(\Phi)$ and $M_\Phi \neq \varnothing$ then*

$$C_\Phi = \bigcap_{Q \in M_\Phi} \overline{\mathcal{K} - L_+^1(Q)}^Q = \{f \in L \mid E_Q[f] \leq 0 \text{ for all } Q \in M_\Phi\} = (co(M_\Phi))^0. \tag{3}$$

In general, the convex cone generated by a closed convex set N may not be closed and so it may happen that $co(\overline{N}) \subseteq \overline{co(N)}$ with strict inclusion (if N is not bounded this may even happen in $\mathbb{R}^n$). However, as shown in the next lemma, it is always true that $(\overline{co(N)})_1 = \overline{N}$. The simple proof, based on the Hanh Banach theorem is omitted.

Lemma 4 *Let X be a locally convex topological vector space, X' its topological dual space and $H = \{x \in X \mid x_0'(x) = a\}$, $a \neq 0$, $x_0' \in X'$, be a closed hyperplane. If N is a non empty and convex subset of H then $\overline{co(N)} \cap H = \overline{N}$.*

We have now collected all the tools for the proof of the main result of this note.

Proof of Theorem 1. As an immediate consequence of (3) we have:

$$C_\Phi \text{ is } \tau - \text{closed}, \ (C_\Phi)^{00} = C_\Phi, \ (C_\Phi)^0 = \overline{co(M_\Phi)}^\nu. \tag{4}$$

Lemma 4 applied to $X = L'$, $N = M_\Phi$, $x_0' = 1 \in L$ and $H = \{z \in L' \mid E[z] = 1\}$ guarantees that

$$(C_\Phi)_1^0 = (\overline{co(M_\Phi)}^\nu)_1 = \overline{M_\Phi}^\nu. \tag{5}$$

From (2), where $G = C_\Phi$, we get for all $f \in L$:

$$\begin{aligned} \inf\{x \in \mathbb{R} \mid f - x \in C_\Phi\} &= \sup\left\{E_Q[f] \mid Q \in (C_\Phi)_1^0\right\} = \sup\left\{E_Q[f] \mid Q \in \overline{M_\Phi}^\nu\right\} \\ &= \sup\{E_Q[f] \mid Q \in M_\Phi\}, \end{aligned}$$

since $E_\bullet[f] : L' \to \mathbb{R}$ is $\nu-$continuous. ■

Remark 5 *(i) If the assumptions of Theorem 20 [3] are satisfied, then we already know that equation (3) is true. Thus, in addition to Theorem 10 [1], only the simple consequences (4) and (5) are needed to prove Theorem 1.*

(ii) It is possible to characterize $(C_\Phi)_1^0$ as the set of probability measures $Q \in M_1$ such that $E[\widehat{\Phi}(\frac{dQ}{dP})] < +\infty$, where $\widehat{\Phi}$ is an auxiliary function obtained by appropriately modifying Φ. This was first shown by Owen in [4].

Proof of Theorem 3. For all $Q \in M_\Phi$ we have: $\mathcal{K} \subseteq L^1(Q)$. So, if $f \in \overline{\mathcal{K} - L^1_+(Q)}^Q$ then $E_Q[f] \leq 0$. Therefore $C_\Phi \subseteq (co(M_\Phi))^0$. To show the opposite inclusion let $k \in (co(M_\Phi))^0$ and suppose by contradiction that there exists $Q_0 \in M_\Phi$ such that $k \notin \overline{\mathcal{K} - L^1_+(Q_0)}^{Q_0}$.

By the Hahn-Banach Theorem, there exists $\widetilde{\xi} \in L^\infty(Q_0)$ such that:

$$\sup_{f \in \mathcal{K} - L^1_+(Q_0)} E_{Q_0}[\widetilde{\xi} f] \leq 0 < E_{Q_0}[\widetilde{\xi} k]. \tag{6}$$

Since $-I_{\{\widetilde{\xi}<0\}} \in (\mathcal{K} - L^1_+(Q_0))$, we deduce that $\widetilde{\xi} \geq 0$ $Q_0 - a.s.$ Since $\widetilde{\xi}\frac{dQ_0}{dP} \geq 0$ $P - a.s.$, we normalize $\widetilde{\xi}$, call it ξ, and define a probability $Q_1 \ll P$ by setting: $\frac{dQ_1}{dP} = \xi \frac{dQ_0}{dP}$. From equation (6) we then derive $Q_1 \in M_1$ and $E_{Q_1}[k] > 0$.

First consider the case $\Phi(0^+) < +\infty$. We claim that $Q_1 \in P_\Phi$, so that $Q_1 \in M_\Phi$, which is in contradiction with $k \in (co(M_\Phi))^0$. The case of a decreasing Φ on $(0, +\infty)$ is obvious. Then suppose that $\Phi(+\infty) = +\infty$. Fix $\lambda_0 > 0$ and let $c^* = \inf\{y \geq 0 : \Phi \text{ is increasing on } (y, +\infty)\}$. We have:

$$E\left[\Phi(\frac{dQ_1}{dP})\right] \leq E\left[\Phi(\xi\frac{dQ_0}{dP})I_{\{\xi>\lambda_0\}}\right] + \Phi(0) + E\left[\Phi(\lambda_0\frac{dQ_0}{dP})I_{\{\xi\frac{dQ_0}{dP}>c^*\}}I_{\{\xi\leq\lambda_0\}}\right]$$

and the rhs is finite since $Q_0 \in M_\Phi = M_1 \cap P_\Phi$ and we may apply $G(\Phi)$ to $E\left[\Phi(\lambda_0\frac{dQ_0}{dP})\right]$ and to $E\left[\Phi(\xi\frac{dQ_0}{dP})I_{\{\xi>\lambda_0\}}\right]$ because ξ is bounded $P-$a.s on $\left\{\frac{dQ_0}{dP} > 0\right\}$.

Now suppose that $\Phi(0^+) = +\infty$. Then the condition $Q_0 \in P_\Phi$ implies $Q_0 \sim P$ and therefore $\frac{dQ_1}{dQ_0} \in L^\infty(P)$. Consider the convex combination $Q^\lambda = \lambda Q^1 + (1-\lambda)Q^0 \in M_1$. If $\lambda \in [0,1)$ then $0 < 1-\lambda \leq \frac{dQ^\lambda}{dQ^0} \leq \lambda\frac{dQ_1}{dQ_0} + 1 - \lambda$ holds $P - a.s$ and therefore $G(\Phi)$ and $Q^0 \in P_\Phi$ imply again $\Phi(\frac{dQ^\lambda}{dP}) \in L^1(P)$. So $\forall\lambda \in [0,1)$ $Q_\lambda \in M_\Phi$. However, since $E_{Q_\lambda}[k] = \lambda E_{Q_1}[k] + (1-\lambda)E_{Q_0}[k]$ and $E_{Q_1}[k] > 0$, there exists $\lambda^* \in [0,1)$ such that $E_{Q_{\lambda^*}}[k] > 0$, which contradicts $k \in (co(M_\Phi))^0$. ■

References

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