Abstract

In order to evaluate the quality of the scientific research, we introduce a new family of scientific performance measures, called Scientific Research Measures (SRM). Our proposal originates from the more recent developments in the theory of risk measures and is an attempt to resolve the many problems of the existing bibliometric indices.

The SRM that we introduce are based on the whole scientist’s citation record and are: coherent, as they share the same structural properties; flexible to fit peculiarities of different areas and seniorities; granular, as they allow a more precise comparison between scientists, and inclusive, as they comprehend several popular indices.

Another key feature of our SRM is that they are planned to be calibrated to the particular scientific community. We also propose a dual formulation of this problem and explain its relevance in this context.

Keywords: Bibliometric Indices, Citations, Risk Measures, Scientific Impact Measures, Calibration, Duality.

1 Introduction

In the recent years the evaluation of the scientist’s performance has become increasingly important. In fact, most crucial decisions regarding faculty recruitment, research projects, research time, academic promotion, travel money, award of grants depend on great extent upon the scientific merits of the involved researchers.

The scope of the valuation of the scientific research is twofold:

• Provide an updated picture of the existing research activity, in order to allocate financial resources in relation to the scientific quality and scientific production;

• Determine an increase in the quality of the scientific research (of the structures).
Even though both aims seem quite obvious, it is worthwhile to emphasize that the selection of erroneous valuation criteria (one trivial example would be "the number of the publications") could have important negative impact on the future research quality. The methodologies for the valuation can be divided into two categories:

- content valuation, based on internal judgments committee and external reviews of peer panels.
- context valuation, based on bibliometrics (i.e. statistics derived from citation data) and the characteristics of the Journals associated to the publications.

Economic considerations strongly depone of using the context method on a systematic (yearly) base, while peer review is more plausible on a multiple year base and should also be finalized to check, harmonize, and tune the outcomes based on bibliometric indices.

Thanks also to the major availability of the online database (i.e. Google Scholar, ISI Web of Science, MathSciNet, Scopus) several different bibliometric measures have been recently introduced and applied.

There are several critics, as those clearly underlined by the Citation Statistics Report of the International Mathematical Union (2008) [CIT], to the use of the citations as a key factor in the assessment of the quality of the research. However, many of these critics can be satisfactorily addressed and our proposal is one reasonable way to achieve this task.

We agree that the quality of the scientific research can not be reduced to citations, but we also believe that the information embedded in citations should be properly quantified and should be one component of the valuation of the quality of the scientific research.

We emphasize that the output of the valuation is the classification of authors (and structures) into few merit classes of homogeneous research quality: it is not intended to provide a fine ranking. In the Appendix we listed a brief summary of the pros and cons of bibliometric indices and of the peer review process.

In 2005 Hirsch [H05] proposed the $h$-index, that is now the most popular and used citation-based metric. The $h$-index of an author is defined as the largest number $h \in \mathbb{N}$ satisfying the condition that $h$ distinct publications of the author have (each one) $h$ citations. The $h$-index is a vague attempt to measure at the same time the production in terms of number of publications and the research quality in terms of citations per publication. Our approach aims exactly to take better in consideration the balance between these two components.

After its introduction, the $h$-index received wide attention from the scientific community and it has been extended by many authors who proposed other indices (for an overview see Alonso et al., 2009 [ACHH]) in order to overcome some of the drawbacks of it (see Bornmann and Daniel, 2007 [BD07]).

In this paper we introduce three novel features in the methodology regarding the measurement of the quality of scientific research:
1. The coherency of the research measures
2. The calibration technique
3. The dual setting.

1. Differently from any existing approach, our formulation is clearly germinated from the Theory of Risk Measures. The axiomatic approach developed in the seminal paper by Artzner et al. [ADEH99] turned out to be, in this last decade, very influential for the theory of risk measures: instead of focusing on some particular measurement of the risk carried by financial positions (the variance, the $V@R$, etc. etc.), [ADEH99] proposed a class of measures satisfying some reasonable properties (the “coherent” axioms). Ideally, each institution could select its own risk measure, provided it obeyed the structural coherent properties. This approach added flexibility in the selection of the risk measure and, at the same time, established a unified framework.

We propose the same approach in order to determine a good class of scientific performance measures, that we call Scientific Research Measures (SRM).

The theory of coherent risk measures was later extended to the class of convex risk measures (Follmer and Schied [FS02], Frittelli and Rosazza [FR02]). The origin of our proposal can be traced in the more recent development of this theory, leading to the notion of quasi-convex risk measures introduced by Cerreia-Vioglio et al. [CMMM] and further developed in the dynamic framework by Frittelli and Maggis [FM11]. Additional papers in this area include: Cherny and Madan [CM09], that introduced the concept of an Acceptability Index having the property of quasi-concavity; Drapeau and Kupper [DK10], where the correspondence between a quasi-convex risk measure and the associated family of acceptance sets - already present in [CM09] - is fully analyzed.

The representation of quasi-convex monotone maps in terms of family of acceptance sets, as well as their dual formulations, are the key mathematical tools underlying our definition of SRM.

2. A second feature of our approach is that our SRM are planned to be "calibrated from the market data", a typical feature of modeling in finance. As explained in Section 4, we calibrate the SRM from the historic data that are available for one particular scientific area and seniority. In this way, each SRM will fit appropriately the characteristics of the research field and seniority under consideration.

3. Our third innovation in this area, is provided by the dual approach to the valuation of the quality of the scientific research. As explained in Section 3.2, we establish a duality between the primal space, the space of random variables (representing the citations records) defined on the set of Journals and its dual space, the space of the "Arrow-Debreu price" of each Journal, which could be
given by the impact factor of the Journal. In section 3, we discuss this duality and show that our SRM fits very well in this framework.

We finally report some empirical results obtained by calibrating the performance curves to a specific data set.

To summarize, we propose a family of SRMs that are:

- **coherent**, as they share the same structural properties - based on an axiomatic approach;
- **calibrated** to the particular scientific community;
- **flexible** in order to fit peculiarities of different areas and ages;
- **robust**, as they can be defined, via duality, through a set of probabilities representing the “value” of each Journal;
- **granular**, as they allow a more precise comparison between scientists;
- **inclusive**, as they comprehends several popular indices.

2 On a class of Scientific Research Measure

We represent each author by a vector \( X \) of citations, where the \( i \)-th component of \( X \) represents the number of citations of the \( i \)-th publication and the components of \( X \) are ranked in decreasing order. We consider the whole citation curve of an author as a decreasing bounded step functions \( X \) (see Fig.1) in the convex cone:

\[
\mathcal{X}^+ = \left\{ X : \mathbb{R} \to \mathbb{R}_+ \mid X \text{ is bounded, with only a finite numbers of values, decreasing on } \mathbb{R}_+ \text{ and such that } X(x) = 0 \text{ for } x \leq 0. \right\}
\]

![Fig. 1. Author’s Citation Curve](image-url)
We compare the citation curve $X$ of an author with a theoretical citation curve $f_q$ representing the desiderata citations at a fix performance level $q$. For this purpose we introduce the following class of curves. Let $\mathcal{I} \subseteq \mathbb{R}$ be the index set of the performance level. For any $q \in \mathcal{I}$ we define the theoretical performance curve of level $q$ as a function $f_q : \mathbb{R} \to \mathbb{R}_+$ that associates to each publication $x \in \mathbb{R}$ the corresponding number of citations $f_q(x) \in \mathbb{R}_+$.

**Definition 1 (Performance curves)** Given a index set $\mathcal{I} \subseteq \mathbb{R}$ of performance levels $q \in \mathcal{I}$, a class $\mathcal{F} := \{ f_q \}_{q \in \mathcal{I}}$ of functions $f_q : \mathbb{R} \to \mathbb{R}_+$ is a family of performance curves if

- i) $\{ f_q \}$ is increasing in $q$, i.e. if $q \geq p$ then $f_q(x) \geq f_p(x)$ for all $x$;
- ii) for each $q$, $f_q(x)$ is left continuous in $x$;
- iii) $f_q(x) = 0$ for all $x \leq 0$ and all $q$.

The main feature of these curves is that a higher performance level implies a higher number of citations. This family of curves is crucial for our objective to build a SRM able to comprehend many of the popular indices and to be calibrated to the scientific area and the seniority of the authors.

**Definition 2 (Performance sets and SRM)** Given a family of performance curves $\mathcal{F} = \{ f_q \}$, we define the family of performance sets $\mathcal{A}_\mathcal{F} := \{ A_q \}_{q \in \mathcal{I}}$ by

$A_q := \{ X \in \mathcal{X}^+ \mid X(x) \geq f_q(x) \text{ for all } x \in \mathbb{R} \}$.

The Scientific Research Measure (SRM) is the map $\phi_\mathcal{F} : \mathcal{X}^+ \to \mathbb{R}$ associated to $\mathcal{F}$ and $\mathcal{A}_\mathcal{F}$ given by

$$\phi_\mathcal{F}(X) := \sup \{ q \in \mathcal{I} \mid X \in A_q \} = \sup \{ q \in \mathcal{I} \mid X(x) \geq f_q(x) \text{ for all } x \in \mathbb{R} \}.$$ (1)

The SRM $\phi_\mathcal{F}$ is obtained by the comparison between the real citation curve of an author $X$ (the red line in Fig.2) and the family $\mathcal{F}$ of performance curves (the blue line in Fig.2): the $\phi_\mathcal{F}(X)$ is the greatest level $q$ of the performance...
curve $f_q$ below the author’s citation curve $X$.

![Diagram of citation curve and performance curves]

**Fig. 2.** Determination of a particular SRM, the $h$-index (that in this example is equal to 4).

### 2.1 Some examples of existing SRMs

The previous definition points out the importance of the family of theoretical performance curves for the determination of the SRM. It is clear that different choices of $F := \{f_q\}_q$ lead to different SRM $\phi_F$. The following examples show that some well known indices of scientific performance are particular cases of our SRM. In the following examples, if $X$ has $p \geq 1$ publications that received at least one citation, we set: $X = \sum_{i=1}^{p} x_i 1_{(i-1,i]}$, with $x_i \geq x_{i+1} \geq 1$ for all $i$, and $p$ satisfies $X = X \uparrow_{(0,p]}$.

**Example 3 (max # of citations)** The maximum number of citations of the most cited author’s paper is the SRM $\phi_{c_{\text{max}}}$ defined by (1), where the performance curves are: $f_q = q \uparrow_{(0,1]}$.

![Diagram of citation curve and performance curves with Example 3 example]

(2)
Example 4 (total number of publications) The total number of publications with at least one citation is the SRM $\phi_{q_p}$ defined by (1), where the performance curves are: $f_q = 1_{[0,q]}$.

Example 5 (h-index) The h-index defined by Hirsch [H05] may be rewritten in our setting. Indeed, the h-index is the SRM $\phi_{q_h}$ defined by (1), where the performance curves are: $f_q = q^{1_{[0,q]}}$.

Example 6 (h$^2$-index) According to Kosmulski, 2006 [K06] a scientist has h$^2$-index $q$ if $q$ of his $n$ papers have at least $q^2$ citations each and the other $n-q$ papers have fewer than $q^2$ citations each. This index is the SRM $\phi_{q_{h^2}}$ defined by (1), where the performance curves are: $f_q = q^{2_{1_{[0,q]}}}$.

Example 7 (h$_\alpha$-index) Eck and Waltman, 2008 [EW06] proposed the h$_\alpha$-index as a generalization of the h-index defined as: "a scientist has h$_\alpha$-index h$_\alpha$ if $h_\alpha$ of his $n$ papers have at least $\alpha$-h$_\alpha$ citations each and the other $n-\alpha$ papers have fewer than $\alpha$-h$_\alpha$ citations each". Hence, the h$_\alpha$-index is the SRM $\phi_{h_{h_\alpha}}$ defined by (1), where the performance curves are: $f_q = q^{\alpha_{1_{[0,q]}}}, \alpha \in (0, \infty)$.

Example 8 (w-index) Woeginger, 2008 [W0308] introduced the w-index defined as: "a w-index of at least $k$ means that there are $k$ distinct publications that have at least 1, 2, 3, 4, ..., $k$ citations, respectively". It is the SRM $\phi_{x_w}$ defined by (1), where the performance curves are: $f_q(x) = (-x + q + 1)^{1_{[0,q]}}$.

Example 9 (h$_{rat}$-index & h$_r$-index) The rational and the real h-index, h$_{rat}$-index and h$_r$-index, introduced respectively by Ruane and Tol, 2008 [RT08] and Guns and Rousseau, 2009 [GR09] are SRMs, indeed they could be defined as the h-index but taking respectively $q \in \mathbb{I} \subseteq \mathbb{Q}$ and $q \in \mathbb{I} \subseteq \mathbb{R}$. 

7
2.2 Key properties of the SRM

**Proposition 10** Let $\mathcal{F}$ be a family of performance curves, $\mathcal{A}_q = \{A_q\}_q$ be the associated family of performance sets and $\phi_q$ be the associated SRM. Let $X_1, X_2 \in \mathcal{X}^+$. Then:

1. i) $\{A_q\}$ is decreasing monotone: $A_q \subseteq A_p$ for any level $q \geq p$;
   ii) for any $q$, $A_q$ is monotone: $X_1 \in A_q$ and $X_2 \geq X_1$ implies $X_2 \in A_q$;
   iii) for any $q$, $A_q$ is convex: if $X_1, X_2 \in A_q$ then $\lambda X_1 + (1-\lambda)X_2 \in A_q$, $\lambda \in [0,1]$.

   i) $\phi_q$ is monotone increasing: if $X_1 \leq X_2 \Rightarrow \phi_q(X_1) \leq \phi_q(X_2)$;
   ii) $\phi_q$ is quasi-concave: $\phi_q(\lambda X_1 + (1-\lambda)X_2) \geq \min(\phi_q(X_1), \phi_q(X_2))$,
       $\lambda \in [0,1]$.

**Proof.**

1) The proof of the monotonicity and convexity of $\mathcal{A}_q$ follows from the definition.

2.i) If $X_1 \leq X_2$, then $X_1 \geq f_q$ implies $X_2 \geq f_q$. Hence $\{q \in \mathcal{I} \mid X_1 \geq f_q\} \subseteq \{q \in \mathcal{I} \mid X_2 \geq f_q\}$.

2.ii) Let $\phi_q(X_1) \geq m$ and $\phi_q(X_2) \geq m$. By definition of $\phi_q$, $\forall \varepsilon > 0 \exists q_i$ s.t. $X_i \geq f_{q_i}$ and $q_i \geq \phi_q(X_i) - \varepsilon \geq m - \varepsilon$. Then $X_i \geq f_{q_i} \geq f_{m-\varepsilon}$, as $\{f_q\}_q$ is an increasing family, and therefore $\lambda X_1 + (1-\lambda)X_2 \geq f_{m-\varepsilon}$. As this holds for any $\varepsilon > 0$, we conclude that $\phi_q(\lambda X_1 + (1-\lambda)X_2) \geq m$ and $\phi_q$ is quasi-concave.

It is obviously reasonable that a SRM should be increasing: if the citations of a researcher $X_2$ dominate the citations of another researcher $X_1$, publication by publication, then $X_2$ has a performance greater than $X_1$.

**Example 11** We show that a SRM is not in general quasi-convex, that is $\phi_q(\lambda X_1 + (1-\lambda)X_2) \leq \max(\phi_q(X_1), \phi_q(X_2))$ for all $\lambda \in [0,1]$. Consider two vectors, $X_1 = [8, 6, 4, 2]$ and $X_2 = [4, 2, 2, 2, 2]$, where $X_2$ has more publications than $X_1$ but less cited. Computing, for example, the $w$-index we obtain that $\phi_{w}(X_1) = 4$ and $\phi_{w}(X_2) = 3$, while for the convex combination $X = \frac{1}{2}X_1 + \frac{1}{2}X_2 = [6, 4, 3, 2, 1]$ we have: $\phi_{w}(X) = 5$. 

8
2.3 Additional properties of SRMs

We have seen that all the SRMs $\phi_q$ share the same structural properties of monotonicity and quasiconcavity. Of course other relevant properties of $\phi_q$ could be considered, which could also be built in from the corresponding features of the family of performance curves. In this section we show that this is the case for the behavior of $\phi_q$ with respect to the addition of citations (C-additivity) to existing papers.

**Definition 12** A SRM $\phi_q : X^+ \to \mathbb{R}$ is:

a) C-superadditive if $\phi_q(X + m) \geq \phi_q(X) + m$ for all $m \in \mathbb{R}_+$;

b) C-subadditive if $\phi_q(X + m) \leq \phi_q(X) + m$ for all $m \in \mathbb{R}_+$;

c) C-additive if $\phi_q(X + m) = \phi_q(X) + m$ for all $m \in \mathbb{R}_+$.

**Definition 13** A family $\mathcal{F}$ of performance curves is:

a) slowly increasing in $q$ if $f_{q+m} - f_q \leq m$ for all $m \in \mathbb{R}_+$;

b) fast increasing in $q$ if $f_{q+m} - f_q \geq m$ for all $m \in \mathbb{R}_+$;

c) linear increasing in $q$ if $f_{q+m} - f_q = m$ for all $m \in \mathbb{R}_+$.

These properties of the family of performance curves can be express in terms of corresponding properties of the family $A_q$ of performance sets.

**Lemma 14** The family $\mathcal{F}$ of performance curve is slowly (resp. fast, linear) increasing in $q$, if and only if

$$A_q + m \subseteq A_{q+m} \quad (\text{resp. } A_{q+m} \subseteq A_q + m, A_{q+m} = A_q + m) \quad (5)$$

for all $m \in \mathbb{R}_+$ and $q \in I$.

**Proof.** In order to show that $A_q + m \subseteq A_{q+m}$ we observe that:

$$A_{q+m} := \{X \mid X \geq f_{q+m}\}$$

$$A_q + m = \{X + m \mid X \geq f_q\}$$

$$= \{X \mid X \geq f_q + m\}.$$

From $f_q + m \geq f_{q+m}$, we deduce that $X \geq f_q + m$ implies $X \geq f_{q+m}$. Hence $X \in A_q + m \implies X \in A_{q+m}$. Hence $X \in A_q + m \implies X \in A_{q+m}$.

Regarding the other implication, we know that if $X \in A_q + m$ then $X \in A_{q+m}$, that is $X \geq f_q + m$ implies $X \geq f_{q+m}$. This implies that $f_q + m \geq f_{q+m}$. Similarly for the other cases. ■

**Lemma 15** If a family $\mathcal{F}$ of performance curves is slowly (resp. fast, linear) increasing in $q$, then $\phi_q$ is C-superadditive (resp. C-subadditive, C-additive).
**Proof.** In order to show that \( \phi_p(X + m) - m \geq \phi_p(X) \) for all \( m \in \mathbb{R}_+ \) we use the definition in (1) and we observe that
\[
\phi_p(X + m) - m = \sup \{ q \mid X + m \geq f_q \} - m \\
= \sup \{ q - m \mid X \geq f_q - m \} \\
= \sup \{ q \mid X \geq f_{q+m} - m \}.
\]
Hence it’s sufficient to show that \( \{ q \mid X \geq f_q \} \subseteq \{ q \mid X \geq f_{q+m} - m \} \) and this is true since \( f_q \geq f_{q+m} - m \). Similarly for the other cases. 

As shown in the following examples, the reverse implication in the above Lemma does not hold true.

**Example 16**.

- The \( h \)-index in the example (5) is a \( C \)-subadditive SRM, but the associated family \( F \) of performance curves defined in (4) is not fast increasing in \( q \), nor slowly increasing. Indeed the family is linear increasing on the Hirsch core \([0, h]\), but not outside it.

- The same considerations hold for the \( h^2 \)- and \( h_{\alpha} \)-index (see examples (6) and (7)).

- The family \( F \) defined in Example 8, associated to the \( w \)-index, is slowly increasing in \( q \). This condition is sufficient to say that the \( w \)-index is a \( C \)-superadditive SRM.

- The maximum number of citations of an article (see example 3) is a \( C \)-additive SRM, even if the family \( F \) of performance curves defined in 2 is not linear increasing in \( q \). This property holds only on \([0, 1]\), since the performance curves are equal to zero outside.

- The total number of publications (see example 4) is a \( C \)-superadditive SRM since the family \( F \) of performance curves defined in 3 is slowly increasing in \( q \).

A further property concerns the addition of a single publication to the author’s citation record.

**Definition 17** Let \( p \) be the maximum number of publications with at least one citation, so that \( p \) satisfies: \( X = X_{1(0,p)} \). A SRM \( \phi_p : X^+ \to \mathbb{R} \) is

a) \( P \)-superadditive if \( \phi_p(X + 1\{p+1\}) \geq \phi_p(X) + 1 \);

b) \( P \)-subadditive if \( \phi_p(X + 1\{p+1\}) \leq \phi_p(X) + 1 \);

c) \( P \)-additive if \( \phi_p(X + 1\{p+1\}) = \phi_p(X) + 1 \);

c) \( P \)-invariance if \( \phi_p(X + 1\{p+1\}) = \phi_p(X) \).
A SRM is P-superadditive if the addition of a new publication with one citation leads to an increase of the measure more than linear. Many known SRMs are P-invariance (i.e. the $c_{\text{max}}$, $h_-$, $h^2$- and $h_n$-index in the examples (3) (5), (6) and (7)) as the addition of one citation to a new publication leaves the SRM invariant. The $w$-index (in the example (8)) is P-subadditive as the addition of one citation to a new publication makes it greater at most of 1 unit. While the total number of publications $p$ with at least one citation (in the example (4)) is clearly P-additive.

3 On the Dual Representation of the SRM

The goal of this section is to provide a dual representation of the SRM. To this scope, we need some topological structure. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ be a probability space, where $\mathcal{B}(\mathbb{R})$ is the $\sigma$-algebra of the Borel sets, $\mu$ is a probability measure on $\mathcal{B}(\mathbb{R})$. Since the citation curve of an author $X$ is a bounded function, it appears natural to take $X \in L^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$, where $L^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ is the space of $\mathcal{B}(\mathbb{R})$-measurable functions that are $\mu$ almost surely bounded. If we endow $L^\infty$ with the weak topology $\sigma(L^\infty, L^1)$ then $L^1 = (L^\infty, \sigma(L^\infty, L^1))'$ is its topological dual. In the dual pairing $(L^\infty, L^1, \langle \cdot, \cdot \rangle)$ the bilinear form $\langle \cdot, \cdot \rangle : L^\infty \times L^1 \to \mathbb{R}$ is given by $\langle X, Z \rangle = E[ZX]$, the linear function $X \mapsto E[ZX]$, with $Z \in L^1$, is $\sigma(L^\infty, L^1)$ continuous and $(L^\infty, \sigma(L^\infty, L^1))$ is a locally convex topological vector space. In this framework, each element of a performance family $\mathcal{F} = \{f_q\}_q$ is a $\mathcal{B}(\mathbb{R})$-measurable function, the inequalities between random variables are meant to hold $\mu$-a.s., and we set:

$$\mathcal{A}_q : = \{X \in L^\infty \mid X \geq f_q\},$$
$$\phi_p(X) : = \sup \{q \in \mathcal{I} \mid X \in \mathcal{A}_q\}. \quad (6)$$

We have seen in the Section 1 that the SRM is a quasi-concave and monotone map. Under appropriate continuity assumptions, the dual representation of these type of maps can be found in [PV90],[Vo98], [CMMM].

**Definition 18** A map $\phi : L^\infty(\mathbb{R}) \to \mathbb{R}$ is $\sigma(L^\infty, L^1)$-upper-semicontinuous if the upper level sets

$$\{X \in L^\infty(\mathbb{R}) \mid \phi(X) \geq q\}$$

are $\sigma(L^\infty, L^1)$-closed for all $q \in \mathbb{R}$.

**Lemma 19** If $\mathcal{A}_\mathcal{F} = \{\mathcal{A}_q\}_q$ is a family of performance sets then $\mathcal{A}_q$ is $\sigma(L^\infty, L^1)$-closed for any $q$.

**Proof.** To prove that $\mathcal{A}_q$ is $\sigma(L^\infty, L^1)$-closed let $Y_\alpha \in \mathcal{A}_q$ be a net satisfying $Y_\alpha \xrightarrow{\sigma(L^\infty, L^1)} Y \in L^\infty$. By contradiction, suppose that $\mu(B) > 0$ where $B := \{Y < f_q\} \in \mathcal{B}(\mathbb{R})$. Taking as a continuous linear functional $Z = 1_B \in L^1$, from $Y_\alpha \xrightarrow{\sigma(L^\infty, L^1)} Y$ we deduce: $E[1_B f_q] \leq E[1_B Y_\alpha] \to E[1_B Y] < E[1_B f_q]$. □
The following proposition shows the relation between the continuity property of the family $\mathcal{F}$ of performance curves, those of the family $\mathcal{A}_F$ of performance sets and those of the SRM $\phi_E$.

**Proposition 20** Let $\mathcal{F}$ be a family of performance curves. If $\mathcal{F}$ is left continuous in $q$, that is
\[ f_{q-\varepsilon}(x) \uparrow f_q(x) \text{ for } \varepsilon \downarrow 0, \mu-\text{a.s.,} \]
then:

1. $\mathcal{A}_F$ is left-continuous in $q$, that is
   \[ \mathcal{A}_q = \bigcap_{\varepsilon > 0} \mathcal{A}_{q-\varepsilon}. \]

2. \[ \mathcal{A}_q = \{ X \in L^\infty \mid \phi_E(X) \geq q \}, \text{ for all } q \in \mathcal{I}. \] (7)

3. $\phi_E$ is $\sigma(L^\infty,L^1)$-upper-semicontinuous.

**Proof.**

1. By assumption we have that $f_{q-\varepsilon}(x) \uparrow f_q(x)$ for $\varepsilon \to 0$, $\mu$-a.s.. We have proved in Proposition (10) that $\{ \mathcal{A}_q \}$ is decreasing monotone, hence we know that $\mathcal{A}_q \subseteq \bigcap_{\varepsilon > 0} \mathcal{A}_{q-\varepsilon}$. We need to prove that $\bigcap_{\varepsilon > 0} \mathcal{A}_{q-\varepsilon} \subseteq \mathcal{A}_q$. By contradiction we suppose that there exist $X \in L^\infty$ such that $X \geq f_{q-\varepsilon}$ for every $\varepsilon > 0$ but $X(x) < f_q(x)$ on a set of positive measure $\mu$. Then there exist a $\delta > 0$ such that $f_q(x) > X(x) + \delta$ on a measurable set $B$ with $b := \mu(B) \in (0,1]$. Since $f_{q-\varepsilon} \uparrow f_q$ we may find $\varepsilon > 0$ such that $f_{q-\varepsilon}(x) > f_q(x) - \frac{\delta}{2}$ on a measurable set $C$ with $\mu(C) > 1 - b$. Thus $\mu(B \cap C) > 0$ and $X(x) \geq f_{q-\varepsilon}(x)$ for all $\varepsilon > 0$, therefore $X \in \mathcal{A}_{q-\varepsilon}$. By item 1 and the left continuity in $q$ of the family $\mathcal{F}$ we know that $\{ \mathcal{A}_q \}$ is left-continuous in $q$ and so: $X \in \bigcap_{\varepsilon > 0} \mathcal{A}_{q-\varepsilon} = \mathcal{A}_q$.

2. Now let
   \[ B_q := \{ X \in L^\infty \mid \phi_E(X) \geq q \}. \]
   $\mathcal{A}_q \subseteq B_q$ follows directly from the definition of $\phi_E$. We have to show that $B_q \subseteq \mathcal{A}_q$. Let $X \in B_q$. Hence $\phi_E(X) \geq q$ and, for all $\varepsilon > 0$, there exists $\overline{q}$ such that $\overline{q} > \phi_E(X) - \varepsilon \geq q - \varepsilon$ and $X \geq f_{\overline{q}}$. Since $\{ f_q \}_q$ is increasing in $q$ we have that $X \geq f_{q-\varepsilon}$ for all $\varepsilon > 0$, therefore $X \in \mathcal{A}_{q-\varepsilon}$. By item 1 and the left continuity in $q$ of the family $\mathcal{F}$ we know that $\{ \mathcal{A}_q \}$ is left-continuous in $q$ and so: $X \in \bigcap_{\varepsilon > 0} \mathcal{A}_{q-\varepsilon} = \mathcal{A}_q$.

3. By Lemma (19) we know that $\mathcal{A}_q$ is $\sigma(L^\infty,L^1)$-closed for any $q$ and therefore the upper level sets $B_q = \mathcal{A}_q$ are $\sigma(L^\infty,L^1)$-closed and $\phi_E$ is $\sigma(L^\infty,L^1)$ upper semicontinuous.

The next lemma will be applied in the proof of theorem 22. It can be proved in a way similar to the convex case (see for example [FS04])
Lemma 21 Let $\phi_q : L^\infty \to \mathbb{R}$ be a SRM. Then the following are equivalent:

1. $\phi_q$ is $\sigma(L^\infty, L^1)$-upper semicontinuous;
2. $\phi_q$ is continuous from above: $X_m, X \in L^\infty$ and $X_m \downarrow X$ imply $\phi_q(X_m) \downarrow \phi_q(X)$

Proof. Let $\phi_q$ be $\sigma(L^\infty, L^1)$-upper semicontinuous and suppose that $X_m \downarrow X$. The monotonicity of $\phi_q$ implies $\phi_q(X_m) \geq \phi_q(X)$ and therefore $q := \lim_m \phi_q(X_m) \geq \phi_q(X)$. Hence $\phi_q(X_m) \geq q$ and $X_m \in B_q := \{Y \in L^\infty \mid \phi_q(Y) \geq q\}$ which is $\sigma(L^\infty, L^1)$-closed by assumption. As the elements in $L^1$ are order continuous, from $X_m \downarrow X$ we get $X \sigma(L^\infty, L^1) X$ and therefore $X \in B_q$. This implies that $\phi_q(X) = q$ and that $\phi_q$ is continuous from above.

Conversely, suppose that $\phi_q$ is continuous from above. We have to show that the convex set $B_q$ is $\sigma(L^\infty, L^1)$-closed for any $q$. By the Krein Smulian Theorem it is sufficient to prove that $C := B_q \cap \{X \in L^\infty \mid \|X\| \leq r\}$ is $\sigma(L^\infty, L^1)$-closed for any fixed $r > 0$. As $C \subseteq L^\infty \subseteq L^1$ and as the embedding

$$(L^\infty, \sigma(L^\infty, L^1)) \hookrightarrow (L^1, \sigma(L^1, L^\infty))$$

is continuous it is sufficient to show that $C$ is $\sigma(L^1, L^\infty)$-closed. Since the $\sigma(L^1, L^\infty)$ topology and the $L^1$ norm topology are compatible, and $C$ is convex, it is sufficient to prove that $C$ is closed in $L^1$. Take $X_n \in C$ such that $X_n \rightharpoonup X$ in $L^1$. Then there exists a subsequence $\{Y_n\}_n \subseteq \{X_n\}_n$ such that $Y_n \rightharpoonup X$ a.s. and $\phi_q(Y_n) \geq q$ for all $n$. Set $Z_m := \sup_{n \geq m} Y_n \vee X$. Then $Z_m \in L^\infty$, since $\{Y_n\}_n$ is uniformly bounded, and $Z_m \geq Y_m$, $\phi_q(Z_m) \geq \phi_q(Y_m)$ and $Z_m \downarrow X$. From the continuity from above we conclude: $\phi_q(X) = \lim_m \phi_q(Z_m) \geq \limsup_m \phi_q(Y_m) \geq q$. Thus $X \in B_q$ and consequently $X \in C$. \qed

When the family of performance curves $F$ is left continuous, Proposition (20) shows that the SRM is $\sigma(L^\infty, L^1)$-upper semicontinuous. Hence we can provide a dual representation for the SRM in the same spirit of [Vo98], [CMMM] and [DK10]. In the following theorem we first provide the representation of $\phi_q$ in terms of the dual function $H$ defined in (9) and then we show that $\phi_q$ can also be represented in terms of the right continuous version of $H$, which can be written in a different way as in (10). This dual representation will provide an interesting interpretation of the SRM (see section 3.2).

Denote

$$\mathcal{P} := \{Q \ll P\} \text{ and } \mathcal{Z} := \left\{Z = \frac{dQ}{dP} \mid Q \in \mathcal{P}\right\} = \left\{Z \in L^1_+ \mid E[Z] = 1\right\}.$$

Theorem 22 Suppose that the family of performance curves $F$ is left continuous. Each SRM $\phi_q : L^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu) \to \mathbb{R}$ defined in (6) can be represented as

$$\phi_q(X) = \inf_{Z \in \mathcal{Z}} H(Z, E[ZX]) = \inf_{Z \in \mathcal{Z}} H^+(Z, E[ZX]) \quad (8)$$

$$= \inf_{Q \in \mathcal{P}} H^+(Q, E_Q[X]) \quad \text{for all } X \in L^\infty$$
where $H : L^1 \times \mathbb{R} \to \overline{\mathbb{R}}$ is defined by
\[ H(Z,t) := \sup_{\xi \in L^1} \{ \phi_{\xi}(t) | E[Z\xi] \leq t \} , \]
(9)

$H^+(Z,\cdot)$ is its right continuous version:
\[ H^+(Z,t) := \inf_{s \geq t} H(Z,s) \]
\[ = \sup \{ q \in \mathbb{R} | t \geq \gamma(Z,q) \} , \]
(10)

and $\gamma : L^1 \times \mathbb{R} \to \overline{\mathbb{R}}$ is defined by:
\[ \gamma(Z,q) := \inf_{X \in L^1} \{ E[X] | \phi_{\xi}(X) \geq q \} . \]
(11)

**Proof.** Step 1: $\phi_{\xi}(X) = \inf_{Z \in Z} H(Z,E[ZX])$.
Fix $X \in L^1$. As $X \in \{ \xi \in L^\infty \mid E[Z\xi] \leq E[Z\xi] \}$, by the definition of $H(Z,E[Z\xi])$ we deduce that, for all $Z \in L^1$,
\[ H(Z,E[Z\xi]) \geq \phi_{\xi}(X) \]

hence
\[ \inf_{Z \in L^1} H(Z,E[Z\xi]) \geq \phi_{\xi}(X) . \]
(12)

We prove the opposite inequality. Let $\varepsilon > 0$ and define the set
\[ C_{\varepsilon} := \{ \xi \in L^\infty \mid \phi_{\xi}(\xi) \geq \phi_{\xi}(X) + \varepsilon \} \]

As $\phi_{\xi}$ is quasi-concave and $\sigma(L^\infty,L^1)$-upper semicontinuous (Propositions 10 and 20), $C$ is convex and $\sigma(L^\infty,L^1)$-closed. Since $X \notin C_{\varepsilon}$, if $\phi_{\xi}(X) = -\infty$, we may take $C_M := \{ \xi \in L^\infty \mid \phi_{\xi}(\xi) \geq -M \}$ and the following argument would hold as well) the Hahn Banach theorem implies the existence of a continuous linear functional that strongly separates $X$ and $C_{\varepsilon}$, that is there exist $Z_{\varepsilon} \in L^1$ such that
\[ E[Z_{\varepsilon}\xi] > E[Z_{\varepsilon}X] \text{ for all } \xi \in C_{\varepsilon} . \]
(13)

Hence
\[ \{ \xi \in L^\infty \mid E[Z\xi] \leq E[Z_{\varepsilon}X] \} \subseteq C_{\varepsilon}^c := \{ \xi \in L^\infty \mid \phi_{\xi}(\xi) < \phi_{\xi}(X) + \varepsilon \} \]

and from (12)
\[ \phi_{\xi}(X) \leq \inf_{Z \in L^1} H(Z,E[Z\xi]) \leq H(Z_{\varepsilon},E[Z_{\varepsilon}X]) \]
\[ = \sup \{ \phi_{\xi}(\xi) \mid \xi \in L^\infty \text{ and } E[Z_{\varepsilon}\xi] \leq E[Z_{\varepsilon}X] \} \]
\[ \leq \sup \{ \phi_{\xi}(\xi) \mid \xi \in L^\infty \text{ and } \phi_{\xi}(\xi) < \phi_{\xi}(X) + \varepsilon \} \leq \phi_{\xi}(X) + \varepsilon . \]

Therefore, $\phi_{\xi}(X) = \inf_{Z \in L^1} H(Z,E[Z\xi])$. To show that the $\inf$ can be taken over the positive cone $L^1_+$, it is sufficient to prove that $Z_{\varepsilon} \subseteq L^1_+$. Let $Y \in L^1_+$
and \( \xi \in C_\varepsilon \). Given that \( \phi_\varepsilon \) is monotone increasing, \( \xi + nY \in C_\varepsilon \) for every \( n \in \mathbb{N} \) and, from (13), we have:

\[
E[Z_\varepsilon (\xi + nY)] > E[Z_\varepsilon X] \Rightarrow E[Z_\varepsilon Y] > \frac{E[Z_\varepsilon (X - \xi)]}{n} \to 0, \text{ as } n \to \infty.
\]

As this holds for any \( Y \in L_1^\infty \) we deduce that \( Z_\varepsilon \subseteq L_1^1 \). Therefore, \( \phi_\varepsilon (X) = \inf_{Z \in L_1^1} H(Z, E[ZX]). \)

By definition of \( H(Z, t) \),

\[
H(Z, E[ZX]) = H(\lambda Z, E[(\lambda Z)X]) \quad \forall Z \in L_1^1, Z \neq 0, \lambda \in (0, \infty).
\]

Hence we deduce

\[
\phi_\varepsilon (X) = \inf_{Z \in L_1^1(\mathbb{R})} H(Z, E[ZX]) = \inf_{Z \in Z} H(Z, E[ZX]) = \inf_{Q \in \mathcal{P}} H(Q, E[QX]).
\]

Step 2: \( \phi_\varepsilon (X) = \inf_{Z \in Z} H^+(Z, E[ZX]). \)

Since \( H(Z, \cdot) \) is increasing and \( Z \in L_1^1 \), we obtain

\[
H^+(Z, E[ZX]) := \inf_{s > E[X]} H(Z, s) \leq \lim_{X \to X} H(Z, E[ZX_m]),
\]

where in the last equality we applied Lemma 21 that guarantees the continuity from above of \( \phi_\varepsilon \).

Step 3: \( H^+(Z, t) := \inf_{s > t} H(Z, s) = \sup \{ q \in \mathbb{R} \mid \gamma(Z, q) \leq t \} \) where \( \gamma \) is defined in (11).

Denote

\[
S(Z, t) := \sup \{ q \in \mathbb{R} \mid \gamma(Z, q) \leq t \}, \quad (Z, t) \in L_1^1 \times \mathbb{R},
\]

and note that \( S(Z, \cdot) \) is the right inverse of the increasing function \( \gamma(Z, \cdot) \) and therefore \( S(Z, \cdot) \) is right continuous.

To prove that \( H^+(Z, t) \leq S(Z, t) \) it is sufficient to show that for all \( p > t \) we have:

\[
H(Z, p) \leq S(Z, p), \tag{14}
\]

Indeed, if (14) is true

\[
H^+(Z, t) = \inf_{p > t} H(Z, p) \leq \inf_{p > t} S(Z, p) = S(Z, t),
\]

as both \( H^+ \) and \( S \) are right continuous in the second argument.

Writing explicitly the inequality (14)

\[
\sup_{\xi \in L_1^\infty} \{ \phi_\varepsilon (\xi) \mid E[Z\xi] \leq p \} \leq \sup \{ q \in \mathbb{R} \mid \gamma(Z, q) \leq p \}
\]

15
and letting $\xi \in L^\infty$ satisfying $E[Z\xi] \leq p$, we see that it is sufficient to show the existence of $q \in \mathbb{R}$ such that $\gamma(Z,q) \leq p$ and $q \geq \phi_q(\xi)$. If $\phi_q(\xi) = \infty$ then $\gamma(Z,q) \leq p$ for any $q$ and therefore $S(Z,p) = H(Z,p) = \infty$.

Suppose now that $\infty > \phi_q(\xi) > -\infty$ and define $q := \phi_q(\xi)$. As $E[\xi] \leq p$ we have:

$$\gamma(Z,q) := \inf \{ E[Z\xi] \mid \phi_q(\xi) \geq q \} \leq p.$$  

Then $q \in \mathbb{R}$ satisfies the required conditions.

To obtain $H^+(Z,t) := \inf_{p>t} H(Z,p) \geq S(Z,t)$ it is sufficient to prove that, for all $p > t$, $H(Z,p) \geq S(Z,t)$, that is:

$$\sup_{\xi \in L^\infty} \{ \phi_q(\xi) \mid E[Z\xi] \leq p \} \geq \sup \{ q \in \mathbb{R} \mid \gamma(Z,q) \leq t \}.$$  

(15)

Fix any $p > t$ and consider any $q \in \mathbb{R}$ such that $\gamma(Z,q) \leq t$. By the definition of $\gamma$, for all $\varepsilon > 0$ there exists $\xi_\varepsilon \in L^\infty$ such that $\phi_q(\xi_\varepsilon) \geq q$ and $E[Z\xi_\varepsilon] \leq t + \varepsilon$. Take $\varepsilon$ such that $0 < \varepsilon < p - t$. Then $E[Z\xi_\varepsilon] \leq p$ and $\phi_q(\xi_\varepsilon) \geq q$ and (15) follows.

**Remark 23** (Interpretation of formulas 10 and 11) Let $Q$ be the 'weight' that we can assign to the author’s publications (for example, the impact factor of the Journal where the article is published). For a fixed $Q$, the term $\gamma(Q,q) := \inf \{ E_Q[\xi] \mid \phi_q(\xi) \geq q \}$ represents the smallest $Q$-average of citations that a generic author needs in order to have the SRM at least of $q$. We observe that this term is independent from the citations of the author $X$.

On the light of these considerations we can interpret the term $H^+(Q,E_Q[X]) := \sup \{ q \in \mathbb{R} \mid E_Q[X] \geq \gamma(Q,q) \}$ as the greatest performance level that the author $X$ can reach, in the case that we attribute the weight $Q$ to the publications. Namely, we compare the $Q$-average of the author $X$, $E_Q[X]$, with the minimum $Q$-average necessary to reach each level $q$, that is $\gamma(Q,q)$.

### 3.1 Examples

In the following examples we find the dual representation of some existing indices. In all these examples the family $\mathbb{F}$ of performance curves is left continuous hence, by Proposition (20), the associated SRM $\phi_q$ is $\sigma(L^\infty,L^1)$-upper semicontinuous and, from (7), $X$ satisfies:

$$\phi_q(X) \geq q \text{ iff } X \in A_q \text{ iff } X \geq f_q.$$  

Therefore, we find the dual representation computing $\gamma$, $H^+$ and $\phi_q$ applying the formulas: (11),(10) and (8). Let $X \in L^\infty_+$, $Z \in L^1_+$, $q \in \mathbb{R}_+$. Then:

$$\gamma(Z,q) := \inf_{\phi_q(X) \geq q} E[ZX] = \inf_{X \in A_q} E[ZX] = E[Zf_q].$$  

Recall that $X = \sum_{i=1}^p x_i 1_{(i-1,i]}$, with $x_i \geq x_{i+1} > 0$ for all $i$, and that $p$ satisfies $X = X 1_{(0,p]} \in L^\infty_+$. 

16
Example 24 (max # of citations) Consider the example (3), where \( f_q = q1_{(0,1)} \). Then:

\[
\gamma(Z, q) = qE[Z1_{(0,1)}]
\]

and we obtain

\[
H^+(Z, E[ZX]) := \sup \{ q \in \mathbb{R} \mid E[ZX] \geq qE[Z1_{(0,1)}] \} = \frac{E[ZX]}{E[Z1_{(0,1)}]}
\]

(we may assume that \( E[Z1_{(0,1)}] \neq 0 \), otherwise \( H^+(Z, E[ZX]) = +\infty \) and it does not contribute to \( \phi_{E_{\text{c,max}}} \)). In our application, any non zero citation vector \( X \) always satisfies \( X \geq x1_{(0,1]} \) and, since \( E[X1_{(0,1)}] = x1E[1_{(0,1)}] \), we also have:

\[
\frac{X}{E[X1_{(0,1)}]} \geq \frac{1_{(0,1)}}{E[1_{(0,1)}]}. \text{ Therefore,}
\]

\[
E \left[ \frac{Z}{E[1_{(0,1)}]} \right] \leq E \left[ \frac{Z}{E[X1_{(0,1)}]} \right] \forall Z \in L^+ \left( \mathbb{R} \right)
\]

and

\[
\frac{E[ZX]}{E[Z1_{(0,1)}]} \geq \frac{E[1_{(0,1)}]}{E[1_{(0,1)}]} \forall Z \in L^+ \left( \mathbb{R} \right).
\]

Hence:

\[
\phi_{E_{\text{c,max}}} (X) = \inf_{Z \in L^+ \left( \mathbb{R} \right)} \ H^+(Z, E[ZX]) = \inf_{Z \in L^+ \left( \mathbb{R} \right)} \frac{E[ZX]}{E[Z1_{(0,1)}]}
\]

\[
= \frac{E[1_{(0,1)}]}{E[1_{(0,1)}]} = x1,
\]

i.e. the infimum is attained at \( Z = 1_{(0,1]} \in L^+ \), which is of course natural as this SRM weights only the first publication.

Example 25 (total # of publications) Consider the example (4), where \( f_q = 1_{(0,q)} \). Then

\[
\gamma(Z, q) = E[Z1_{(0,q)}]
\]

and we obtain

\[
H^+(Z, E[ZX]) := \sup \{ q \in \mathbb{R} \mid E[ZX] \geq E[Z1_{(0,q)}] \}.
\]

Hence the dual representation of the total number of publications with at least one citation is

\[
\phi_{E_{\text{p}}} (X) = \inf_{Z \in L^+ \left( \mathbb{R} \right)} \left\{ \sup_{E[ZX] \geq E[Z1_{(0,q)}]} q \right\}
\]

We show indeed that \( \phi_{E_{\text{p}}} (X) = p \), where \( p \) is characterized by \( X = X1_{(0,p]} \). For all \( Z \in L^+ \), and \( q \leq p \) we have

\[
E[ZX] = E[ZX1_{(0,q]} \geq E[1_{(0,q]}Z]
\]

17
and therefore

\[ \sup_{E[ZX] \geq E[Z1_{(0,q)}]} q \geq p \quad \forall Z \in L^1_+, \]

and \( \phi_{\phi_p}(X) \geq p \). Regarding the \( \leq \) inequality, it is enough to take \( Z = 1_{(p,p+\delta)} \), with \( \delta > 0 \). In this case, the condition \( E[ZX] \geq E[Z1_{(0,q)}] \) becomes

\[ 0 = E[1_{(p,p+\delta)}X] \geq E[1_{(p,p+\delta)}1_{(0,q)}] \]

that holds only for \( q \leq p \), hence

\[ H^+(Z, E[ZX]) = \sup_{E[1_{(p,p+\delta)}X] \geq E[1_{(p,p+\delta)}1_{(0,q)}]} q = p \]

and \( \phi_{\phi_p}(X) \leq p \).

**Example 26 (h-index)** Consider the example (5), where \( f_q = q1_{(0,q)} \). Then

\[ \gamma(Z, q) = E[Zq1_{(0,q)}] \]

and we obtain

\[ H^+(Z, E[ZX]) := \sup \{ q \in \mathbb{R} \mid E[ZX] \geq E[Zq1_{(0,q)}] \}. \]

Hence the dual representation of the h-index is

\[ \phi_{\phi_p}(X) = \inf_{Z \in L^1_+(\mathbb{R}^+)} \sup_{E[ZX] \geq E[Zq1_{(0,q)}]} q \]

We indeed show that \( \phi_{\phi_p}(X) = h \), where \( h \) is characterized by \( X1_{(0,h]} \geq h1_{(0,h]} \) and \( X1_{(h,+,\infty]} \leq h1_{(h,+,\infty]} \). First we check that \( \phi_{\phi_p}(X) \geq h \). For all \( Z \in L^1_+ \), and \( q \leq h \) we have

\[ E[ZX] \geq E[ZX1_{(0,h]}] \geq E[Zh1_{(0,h]}] \geq E[Zq1_{(0,q)}], \]

hence

\[ \sup_{E[ZX] \geq E[Zq1_{(0,q)}]} q \geq h \quad \forall Z \in L^1_+ \]

and \( \phi_{\phi_p}(X) \geq h \).

Regarding the \( \leq \) side, take \( Z = 1_{(h,h+\delta]} \) with \( \delta > 0 \). For all \( q > h \) there exists \( \delta > 0 \) such that \( h + \delta < q \) and then

\[ E[1_{(h,h+\delta)}X] \leq E[1_{(h,h+\delta)}h] < E[1_{(h,h+\delta)}q1_{(0,q)}] \]

hence

\[ \sup_{E[1_{(h,h+\delta)}X] \geq E[1_{(h,h+\delta)}q1_{(0,q)}]} q \leq h \]

and \( \phi_{\phi_p}(X) \leq h \).
3.2 On the dual approach to SRM

The dual representation in Theorem 22 and the Remark 23 suggest us another approach for the definition of a class of SRMs.

In other words, which is the interpretation of the duality that we are discovering?

The primal space is given by the set of all the possible author’s citation records, i.e. by all the random variables $X(w)$ defined on the events $w \in \Omega$, where each event now corresponds to the journal in which the paper appeared.

The dual space is then represented by all possible linear valuation (the "Arrow-Debreu price") of the journals.

We may fix a plausible family of probabilities $P \subseteq \{Q \ll P\}$ where each $Q(w)$ then represents the 'value' attributed to the journal $w \in \Omega$. The valuation criterion for journals (i.e. the selection of the family $P$) has to be determined a priori and could be based on the 'impact factor' or other criterion. A specific $Q$ could attribute more importance to the journals with a large number of citations (a large impact factor); another particular $Q$ to the journals having a "high quality". A priori there will be no consensus on the selection of the family $P$, hence a robust approach is needed.

As suggested from the dual representation results and in particular from the equations (8) and (10) we consider, independently to the particular scientist $X$, a family $\{\gamma_\beta\}_{\beta \in \mathbb{R}}$ of functions $\gamma_\beta : P \to \mathbb{R}$ that associate to each $Q$ the value $\gamma_\beta(Q)$, that represents the smallest $Q$-average of citations in order to reach a quality index at least of $\beta$.

So given a particular value $Q(w_i)$ for each $i^{th}$-journal and the average citations $\gamma_\beta(Q)$ necessary to have an index level greater than $\beta$, we build the SRM in the following way. We define the function $H^+ : \mathcal{P} \times \mathbb{R} \to \mathbb{R}$ that associates to each pair $(Q, E_Q(X))$ the number

$$H^+(Q, E_Q(X)) := \sup \{\beta \in \mathbb{R} \mid E_Q(X) \geq \gamma_\beta(Q)\},$$

which represents the greatest quality index that the author $X$ can reach when $Q$ is fixed, and we build the SRM as follows:

$$\phi(X) := \inf_{Q \in \mathcal{P}} H^+(Q, E_Q(X))$$

which represents a prudential and robust approach with respect to $\mathcal{P}$, the plausible different selections of the evaluation of the Journals. This SRM is by construction quasi-concave and monotone increasing. Theorem 22 exhibits the relationship between the performance curve approach and this dual approach.

4 Example of the calibration of a SRM

Since the SRM introduced in Section 2 depends on the particular family $\mathbb{F}$ of performance curves, in this section we provide an example showing how to
calibrate the family $\mathcal{F}$ from the historic data available for one particular scientific area and seniority. In this way, the SRM will fit appropriately the characteristics of the research field and seniority under consideration. We recall that the SRM should be used only in relative terms (to compare the author quality with respect to the other researchers in the same area) in order to classify the authors (and structures) into few classes of homogeneous research quality.

### 4.1 Determination of the family $\{f_q\}_q$ and of the SRM

The first step consists in the selection of a representative sample of $M$ authors in the same scientific area and with the same seniority and then from this sample of authors we need to extrapolate the family of curves $\{f_q\}_q$ that better represents the citation curve of the area and seniority. The analysis of the citation vectors of each author (see Fig.4.1) shows that the theoretical model may be described (for this particular scientific area) by the formula

$$f_q(x) = \frac{q}{x^\beta}$$

(16)

![Fig. 4.1 Citation curves of 20 senior authors in Math Finance area.](image)

with $q, \beta \in \mathbb{R}_+$. Setting $\ln f_q = Y$, $\ln(q) = \tilde{q}$, $\ln x = X$, $\beta = \tilde{\beta}$ we obtain the linearized model

$$Y = \tilde{q} - \tilde{\beta}X.$$  

(17)

For each $i$-th author of the sample we determine $\hat{\beta}_i$ that minimizes the sum of the square distances of the points from the line (17). Then, we compute $\bar{\beta}$ as the average of the $\hat{\beta}_i$:

$$\bar{\beta} = \frac{1}{M} \sum_{i=1}^{M} \hat{\beta}_i.$$
Once the parameter $\overline{\beta}$ is fixed, we obtain the family of performance curves $f_q(x) = \frac{x^q}{x^q}$ and then the associated SRM (hereafter called the $\phi$-index) is:

$$\phi(X) = \sup \left\{ q \in \mathbb{R} \mid X(x) \geq \frac{q}{x^{\overline{\beta}}} \quad \forall x \right\}$$

(18)

### 4.2 The empirical results

We have chosen a group of 20 well established researchers in the mathematical finance area. We have computed the $\beta_i$ for each author and we have found that $\overline{\beta} = 1.62$.

In the following table (Fig.4.2.a) we report the results and the respective ranking obtained calculating the $\phi$-index as in (18) and the $h$-index for each author. Fig.4.2.b shows that the hyperbole-type curve (red line) corresponding to the author’s $\phi$-index is always below his citation curve (blue line), in the domain $(0, p)$.

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![Fig. 4.2.a](image1.png)  
![Fig. 4.2.b](image2.png)

Notice that the author $F$ increases his index, from the 14th position in the $h$-index to the 5th one of the $\phi$-index. If we compare this author with the author $I$, we note that both have almost the same $h$-index but the $\phi$-index of $F$ is much greater than the $\phi$-index of $I$. Analyzing their citation curves we observe that they have the same number of publications, but $F$ has in general a lot more citations for any publication than $I$, especially those in the Hirsch-core. The same reasons explain the different ranking of the authors $H$ and $D$.

The conclusion is that the calibrated family of performance curves $\mathcal{F}$ takes in the correct account the balance between the number of publications and
the citations per publication, a characteristic indeed of the specific area under consideration.

5 Appendix

We provide a brief summary of the positive and negative features of the peer review process and of the bibliometric indices. We are well aware that these remarks are incomplete and represent a subjective, not scientifically based, view of this complex and controversial theme. In the references, as outlined in the introduction, more detailed arguments can be found. The following remarks however shows that many possible drawbacks of bibliometrics indices may be smoothened and reduced by an appropriate use of them and by the selection of a more convenient class of indices of the type we presented in the previous sections.

5.1 Summary of the pros and cons of context evaluation (bibliometric indices)

Pros:

- **Easily accessible**, from the online databases (Google Scholar, ISI Web, MathSciNet, Scopus...);
- **Not expensive**: can be used systematically, especially if tested - every $n$ years - with peer review.
- **Quick to compute**
- **“Objective”**, in the reductive meaning of being independent from individual judgements.

Cons:

- **Subjective interpretation of citations**, as it can be more subjective than the judgment of experts - see Citation Statistics Report of the International Mathematical Union (2008) [CIT].
  - The new metric must be validated against other (possibly non metric) criterion already validated.
  - It has been pointed out - see the discussion in the American Scientist Open Access Forum, 2008 [ASOAF]- that citation metrics are extremely correlated with peer reviews.
- **Improper comparison** of papers belonging to different fields.
– The SRM should be used to rank each author inside his scientific community (e.g.: top 10% - top 30% - average...). It provides relative - to fields - values, not absolute values. However, this allows also for a coarse comparison of authors belonging to different areas, in the sense that it is possible to easily recognize the authors that are in the same (top/lower/ ...) merit class in each area.

• **Improper comparison of papers having different ages.**
  – Our SRM may be calibrated to different ages (as well as different areas).

• **Different databases provide different citations.**
  – Many areas (naturally) share the same database.
  – The outcome of the scientific measure is in relative terms: the ranking of one author is compared with the ranking of all researchers in the same area (hence using the same database).
  – Different databases (Google Scholar, MathSciNet,...) provide different numbers (in terms of citation of each paper), but only via a scaling factor: the overall ranking of the papers, with respect to the number of citations received, remains essentially the same, see [ASOAF].

• **Co-authors**
  – It is possible to normalize the citation numbers per each single author. For some fields (where papers have typically many co-authors) this may be problematic.

• **Incorrect citations** attributed to an author and self citations
  – Both problems can be easily addressed by the systematic use of Author Codes (a code that identify the author).

• **A single number is insufficient for the evaluation of a complex feature,** such as scientific research.
  – We agree: It is necessary to find multiple metrics (including time-based metrics). We propose one of them.
  – This argument should not lead to abandon the search of appropriate multiple metrics.

• **Quality** of the scientific research **can not be reduced to citations**
  – Agree: it is only one component that however should be properly quantified.
• **Negative credit**: citations may be attributed *not as reward citations* (to give credit to the work of the cited author) but as negative credit (or “rhetorical credit” due to the prestige of the cited author).

  – True. Many are the motivations of citations and they varies among authors: they do not always reflects reward, but certainly a large percentage of citations are credit ones. Indeed:

  – The fact that citation based statistics often agree with other validated form of valuation (peer review), see [ASOAF], suggests that, to some degree, these metrics indeed reflects the impact of the author’s research.

  – The periodical peer review valuation should point out the macroscopic exceptions to reward citations (papers mostly cited for their fallacy).

• **Disincentive** for young researcher to study subjects more innovative but less popular

  – True, even though this could be compensated by the consideration that innovative paper (in a new field) typically receive many citations.

• **Negative Implications**: The use of citation based metrics will increase the number of citations (and improper ones).

  – The abuse of citations is comparable with intentional misjudgment by referee: unfortunately this is always possible.

  – When citations number are high (in the order of hundreds) it is difficult to modify the citation records with self or friendly citations.

  – It is not completely unfair that a strong scientific group (capable to produce a large number of published papers) receives additional credit (due to potential additional citations from the group).

5.2 **Summary of the pros and cons of content valuation (peer review)**

**Pros:**

• **effective assessment** of the quality of the research;

**Cons:**

• **expensive**, in term of time and people involved: *It can not be used systematically.*

• **subjective**, since the result depends on the referees: do they operate properly, are they competent and reliable? The choice of the referees is a very delicate issue.
• **non-uniformity of the judgment**, as each evaluator has a personal scaling preferences leading to different ranking (specially in different areas).

**References**


