Abstract

We propose a novel concept of a Systemic Optimal Risk Transfer Equilibrium (SORT), which is inspired by the Bühlmann's classical notion of an Equilibrium Risk Exchange. We provide sufficient general assumptions that guarantee existence, uniqueness, and Pareto optimality of such a SORTE. In both the Bühlmann and the SORTE definition, each agent is behaving rationally by maximizing his/her expected utility given a budget constraint. The two approaches differ by the budget constraints. In Bühlmann's definition the vector that assigns the budget constraint is given a priori. On the contrary, in the SORTE approach, the vector that assigns the budget constraint is endogenously determined by solving a systemic utility maximization. SORTE gives priority to the systemic aspects of the problem, in order to optimize the overall systemic performance, rather than to individual rationality.

Keywords: Equilibrium, Systemic Utility Maximization, Optimal Risk Sharing, Systemic Risk.
Mathematics Subject Classification (2010): 91G99; 91B30; 60A99; 91B50; 90B50.

1 Introduction

We introduce the concept of Systemic Optimal Risk Transfer Equilibrium, denoted with SORTE, that conjugates the classical Bühlmann’s notion of an equilibrium risk exchange with capital allocation based on systemic expected utility optimization.

The capital allocation and risk sharing equilibrium that we consider can be applied to many contexts, such as: equilibrium among financial institutions, agents, or countries; insurance and...
In this paper we will refer to a participant in these problems (financial institution or firms or countries) as an **agent**; the class consisting of these $N$ agents as the **system**; the individual risk of the agents (or the random endowment or future profit and loss) as the **risk vector** $X := (X^1, ..., X^N)$; the amount $Y := (Y^1, ..., Y^N)$ that can be exchanged among the agents as random **allocation**. We will generically refer to a central regulator authority, or CCP, or executive manager as a **central bank** (CB).

We now present the main concepts of our approach and leave the details and the mathematical rigorous presentation to the next sections. In a one period framework, we consider $N$ agents, each one characterized by a concave, strictly monotone utility function $u_n : \mathbb{R} \rightarrow \mathbb{R}$ and by the original risk $X_n \in L^0(\Omega, \mathcal{F}, P)$, for $n = 1, ..., N$. Here, $(\Omega, \mathcal{F}, P)$ is a probability space and $L^0(\Omega, \mathcal{F}, P)$ is the vector space of real valued $\mathcal{F}$-measurable random variables. The sigma algebra $\mathcal{F}$ represents all possible measurable events at the final time $T$. $E[\cdot]$ denotes the expectation under $P$. Given another probability measure $Q$, $E_Q[\cdot]$ denotes the expectation under $Q$. For the sake of simplicity and w.l.o.g., we are assuming zero interest rate. We will use the bold notation to denote vectors.

1. **Bühlmann’s risk exchange equilibrium**

We recall Bühlmann’s definition of a risk exchange equilibrium in a pure exchange economy (or in a reinsurance market). The initial wealth of agent $n$ is denoted by $x^n \in \mathbb{R}$ and the variable $X^n$ represents the original risk of this agent. In this economy each agent is allowed to exchange risk with the other agents. Each agent has to agree to receive (if positive) or to provide (if negative) the amount $\tilde{Y}^n(\omega)$ at the final time in exchange of the amount $E_Q[\tilde{Y}^n]$ paid (if positive) or received (if negative) at the initial time, where $Q$ is some pricing probability measure. Hence $\tilde{Y}^n$ is a time $T$ measurable random variable. In order that at the final time this risk sharing procedure is indeed possible, the exchange variables $\tilde{Y}^n$ have to satisfy the **clearing condition**

$$\sum_{n=1}^{N} \tilde{Y}^n = 0 \quad P\text{-a.s. .}$$

As in [11] and [12], we say that a pair $(\tilde{Y}_X, Q_X)$ is an **risk exchange equilibrium** if:

(a) for each $n$, $\tilde{Y}_X^n$ maximizes: $E\left[u_n(x^n + X^n + \tilde{Y}^n - E_Q[\tilde{Y}^n])\right]$ among all variables $\tilde{Y}^n$;

(b) $\sum_{n=1}^{N} \tilde{Y}_X^n = 0 \quad P\text{-a.s. .}$

It is clear that only for some particular choice of the equilibrium pricing measure $Q_X$, the optimal solutions $\tilde{Y}_X^n$ to the problems in (a) will also satisfy the condition in (b). In addition it is evident that this risk exchange equilibrium presupposes a cooperative attitude among all agents at the final time, as the clearing condition implies that some agent will provide capital $\tilde{Y}_X^n(\omega)$ to some other agent. Define

$$C_R := \left\{ Y \in (L^0(\Omega, \mathcal{F}, P))^N \mid \sum_{n=1}^{N} Y^n \in \mathbb{R} \right\}$$ (1)
that is, \( C_\mathbb{R} \) is the set of random vectors such that the sum of the components is \( P \)-a.s. a deterministic number.

Observe that with the change of notations \( Y^n := x^n + \tilde{Y}^n - E_{Q_X}[\tilde{Y}^n] \), we obtain variables with \( E_{Q_X}[Y^n] = x^n \) for each \( n \), and an optimal solution \( Y^n_\mathbb{X} \) still belonging to \( C_\mathbb{R} \) and satisfying

\[
\sum_{n=1}^N Y^n_\mathbb{X} = \sum_{n=1}^N x^n \quad P\text{-a.s.}.
\] (2)

As also shown in Appendix A.4, Item 1, observe that

\[
\sup_{\tilde{Y}^n} \mathbb{E}\left[u_n(x^n + X^n + \tilde{Y}^n - E_{Q_X}[\tilde{Y}^n])\right] = \sup_{Y^n} \{\mathbb{E}[u_n(X^n + Y^n)] \mid E_{Q_X}[Y^n] \leq x^n\}.
\]

Hence the two above conditions in the definition of a risk exchange equilibrium may be equivalently reformulated as

(a') for each \( n \), \( Y^n_\mathbb{X} \) maximizes: \( \mathbb{E}[u_n(X^n + Y^n)] \) among all variables satisfying \( E_{Q_X}[Y^n] \leq x^n \);

(b') \( Y_\mathbb{X} \in C_\mathbb{R} \) and \( \sum_{n=1}^N Y^n_\mathbb{X} = \sum_{n=1}^N x^n \quad P\text{-a.s.} \)

We remark that here the quantity \( x^n \in \mathbb{R} \) is preassigned to each agent.

2. Systemic Optimal (deterministic) Allocation

To simplify the presentation, we now suppose that the initial wealth of each agent is already absorbed in the notation \( X^n \), so that \( X^n \) represents the initial wealth plus the original risk of agent \( n \). We assume that the system has at disposal a total amount of capital \( A \in \mathbb{R} \) to be used at a later time in case of necessity. This amount could have been assigned by the Central Bank, or could have been the result of the previous trading in the system, or could have been collected ad hoc by the agents. In any case, we consider the quantity \( A \) as exogenously determined. This amount is allocated among the agents in order to optimize the overall systemic satisfaction. If we denote with \( a^n \in \mathbb{R} \) the cash received (if positive) or provided (if negative) by agent \( n \), then the time \( T \) wealth at disposal of agent \( n \) will be \( (X^n + a^n) \). The optimal vector \( a_\mathbb{X} \in \mathbb{R}^N \) could be determined according to the following aggregate time-\( T \) criterion

\[
\sup \left\{ \sum_{n=1}^N \mathbb{E}[u_n(X^n + a^n)] \mid a_\in \mathbb{R}^N \text{ s.t. } \sum_{n=1}^N a^n = A \right\}.
\] (3)

This criterion presupposes a cooperative attitude among all the agents, as each agent is not optimizing his own utility function. As the vector \( a_\in \mathbb{R}^N \) is deterministic, it is known at time \( t = 0 \) and therefore the cooperation among the agents is required only at such initial time.

However, under the assumption that also at the final time the agents will continue to be cooperative and have confidence in the overall reliability of the other agents, one can combine the two approaches outlined in Items 1 and 2 above to further increase the optimal total expected systemic utility and simultaneously guarantee that each agent will optimize his/her own single expected utility, taking into consideration an aggregated budget constraint assigned by the system. Of course an alternative assumption to cooperation and trustworthiness could be that the rules are enforced by the CB.
We denote with $L^n \subseteq L^0(\Omega,\mathcal{F},P)$ a space of admissible random variables and assume that $L^n + \mathbb{R} = L^n$. We will consider maps $p^n : L^n \to \mathbb{R}$ that represent the pricing or cost functionals, one for each agent $n$. As we shall see, in some relevant cases, all agents will adopt the same functional $p^n = \ldots = p^N$, which will then be interpreted as the equilibrium pricing functional, as in Bühlmann’s setting above, where $p^n(\cdot) := E_{Q^n}[\cdot]$ for all $n$. However, we do not have to assume this a priori. Instead we require that the maps $p^n$ satisfy for all $n = 1,\ldots,N$:

i) $p^n$ is monotone increasing;

ii) $p^n(0) = 0$;

iii) $p^n(Y + c) = p^n(Y) + c$ for all $c \in \mathbb{R}$ and $Y \in L^n$.

Such assumptions in particular imply $p^n(c) = c$ for all constants $c \in \mathbb{R}$. A relevant example of such functionals are

$$p^n(\cdot) := E_{Q^n}[\cdot],$$

where $Q^n$ are probability measures for $n = 1,\ldots,N$.

Now we will apply both approaches, outlined in Items 1 and 2 above, to describe the concept of a Systemic Optimal Risk Transfer Equilibrium.


As explained in Item 1, given some amount $a^n$ assigned to agent $n$, this agent may buy $\tilde{Y}^n$ at the price $p^n(\tilde{Y}^n)$ in order to optimize

$$E \left[ u_n(a^n + X^n + \tilde{Y}^n - p^n(\tilde{Y}^n)) \right].$$

The pricing functionals $p^n$, $n = 1,\ldots,N$ have to be selected so that the optimal solution verifies the clearing condition

$$\sum_{n=1}^N \tilde{Y}^n = 0 \quad P\text{-a.s.}$$

However, as in Item 2, $a^n$ is not exogenously assigned to each agent, but only the total amount $A$ is at disposal of the whole system. Thus the optimal way to allocate $A$ among the agents is given by the solution $(\tilde{Y}^n_X, p^n_X, a^n_X)$ of the following problem:

$$\sup_{a \in \mathbb{R}^N} \left\{ \sum_{n=1}^N \sup_{\tilde{Y}^n} \left\{ E \left[ u_n(a^n + X^n + \tilde{Y}^n - p^n_X(\tilde{Y}^n)) \right] \right\} \right\} \Bigg| \sum_{n=1}^N a^n = A, \quad \sum_{n=1}^N \tilde{Y}^n = 0 \quad P\text{-a.s.} .$$

As shown in Appendix A.4, Item 2, from (5) and (6) it follows that an optimal solution $(\tilde{Y}^n_X, p^n_X, a^n_X)$ fulfills

$$\sum_{n=1}^N p^n_X(\tilde{Y}^n_X) = 0 .$$

Further, letting $Y^n := a^n + \tilde{Y}^n - p^n_X(\tilde{Y}^n)$, from the cash additivity of $p^n_X$ we deduce $p^n_X(\tilde{Y}^n) = a^n + p^n_X(Y^n) = a^n + \sum_{n=1}^N Y^n_X = \sum_{n=1}^N a^n + \sum_{n=1}^N \tilde{Y}^n_X - \sum_{n=1}^N p^n_X(Y^n) = \sum_{n=1}^N a^n$.
and, similar as before, the above optimization problem can be reformulated as

\[
\sup_{a \in \mathbb{R}^N} \left\{ \sum_{n=1}^{N} \sup_{Y_n \in \mathbb{R}} \left\{ E\left[u_n(X^n + Y^n) \right] \mid p^n_X(Y^n) \leq a^n \right\} \right\},
\]

where analogously to (7) we have that a solution \((Y^n_X, p^n_X, a^n_X)\) satisfies \(\sum_{n=1}^{N} a^n = A\), by (8) and (9).

The two optimal values in (5) and (8) coincide. We see that while each agent is behaving optimally according to his preferences, the budget constraint \(p^n_X(Y^n) \leq a^n\) are not a priori assigned, but are endogenously determined through an aggregate optimization problem. The optimal value \(a^n_X\) determines the optimal risk allocation of each agent. It will turn out that \(a^n_X = p^n_X(Y^n_X)\). Obviously, the optimal value in (5) is greater than (or equal to) the optimal value in (3), which can be economically translated in the statement that an increase in the cooperation (not only at initial time but also at the end of the period) increases the systemic performance.

In addition to the condition in (9), we introduce further possible constraints on the optimal solution, by requiring that

\[
Y_X \in \mathcal{B},
\]

where \(\mathcal{B} \subseteq \mathcal{C}_\mathbb{R}\).

In the paper, see Section 3.4, we formalize the above discussion and show the existence of the solution \((Y^n_X, p^n_X, a^n_X)\) to (8), (9) and (10), which we call Systemic Optimal Risk Transfer Equilibrium (SORTE). We show that \(p^n_X\) can be chosen to be of the particular form \(p^n_X(\cdot) := E_{Q^n_X}[\cdot]\), for a probability vector \(Q^n_X = (Q^n_1, ..., Q^n_N)\). The crucial step, Theorem 4.7, is the proof of the dual representation and the existence of the optimizer of the associated problem (23). The optimizer of the dual formulation provides the optimal probability vector \(Q^n_X\) that determines the functional \(p^n_X(\cdot) := E_{Q^n_X}[\cdot]\). The characteristics of the optimal \(Q^n_X\) depend on the feasible allocation set \(\mathcal{B}\). When no constraints are enforced, i.e., when \(\mathcal{B} = \mathcal{C}_\mathbb{R}\), then all the components of \(Q^n_X\) turn out to be equal. Hence we find that the implicit assumption of one single equilibrium pricing measure, made in the Bühlmann’s framework, is in our theory a consequence of the particular selection \(\mathcal{B} = \mathcal{C}_\mathbb{R}\), but for general \(\mathcal{B}\) this is not always the case.

Bühlmann’s equilibrium \((Y_X)\) satisfies two relevant properties: Pareto optimality (there are no feasible allocation \(Y\) such that all agents are equal or better off - compared with \(Y^n_X\) - and at least one of them is better off) and Individual Rationality (each agent is better off with \(Y_X^n\) than without it). Any feasible allocation satisfying these two properties is called an optimal risk sharing rule, see [2] or [25].

We show that a SORTE is unique (once the class of pricing functionals is restricted to those in the form \(p^n(\cdot) = E_{Q^n}[\cdot]\)). We also prove Pareto optimality, see the Definition 3.1 and the exact formulation in Theorem 4.19.
However, a SORTE lacks Individual Rationality. This is shown in the toy example of Section 4.7, but it is also evident from the expression in equation (8). As already mentioned, each agent is performing rationally, maximizing her expected utility, but under a budget constraint \( p(X^n) \leq a^n \) that is determined globally via an additional systemic maximization problem (\( \sup_{a \in \mathbb{R}^N} \{ \ldots | \sum_{n=1}^N a^n = A \} \)) that assigns priority to the systemic performance, rather than to each individual agent. In the SORTE we replace individual rationality with such a systemic induced individual rationality, which also shows the difference between the concepts of SORTE and of an optimal risk sharing rule. We also point out that the participation in the risk sharing mechanism may be appropriately mitigated or enforced by the use of adequate sets \( B \), see e.g. Example 4.22 for risk sharing restricted to subsystems. From the technical point of view, we will not rely on any of the methods and results related to the notion of inf-convolution, which is a common tool to prove existence of optimal risk sharing rules (see for example [2] or [25]) in the case of monetary utility functions, as we do not require the utility functions to be cash additive. Our proofs are based on the dual approach to (systemic) utility maximization. This is summarized in Section 4.2.

**Review of literature:** This paper originates from the systemic risk approach developed in [4] and [5]. For an exhaustive overview on the literature on systemic risk, see [24] and of [22]. Risk sharing equilibria have been studied in different forms starting from the seminal papers of [10], where Pareto-optimal allocations are proved to be comonotonic for concave utility functions, and [13]. In [11] and [12] existence of risk equilibria is proved in a pure exchange economy. In [2] inf-convolution of convex risk measures has been introduced as a fundamental tool for studying risk sharing. Existence of optimal risk sharing for law-determined monetary utility functions is obtained in [25] and then generalized to the case of non-monotone risk measures by [1] and [21], to multivariate risks by [14] and [15], to cash-subadditive and quasi-convex measures by [28]. Further works on risk sharing are also [17], [23], [32], [33]. Risk sharing problems with quantile-based risk measures are studied in [19] by explicit construction, and in [18] for heterogeneous beliefs. In [20] Capital and Risk Transfer is modelled as (deterministically determined) redistribution of capital and risk by means of a finite set of non deterministic financial instruments. Existence issues are studied and related concepts of equilibrium are introduced. Recent further extensions have been obtained in [27].

**2 Notations**

Let \((\Omega,\mathcal{F},P)\) be a probability space and consider the following set of probability vectors on \((\Omega,\mathcal{F})\)

\[ \mathcal{P}^N := \{ Q = (Q^1,\ldots,Q^N) \mid \text{such that } Q^j \ll P \text{ for all } j = 1,\ldots,N \} . \]

For a vector of probability measures \( Q \) we write \( Q \ll P \) to denote \( Q^1 \ll P,\ldots,Q^N \ll P \). Similarly for \( Q \sim P \). Set \( L^0(\Omega,\mathcal{F},P;\mathbb{R}^N) = (L^0(P))^N \). For \( Q \in \mathcal{P}^1 \) let \( L^1(Q) := L^1(\Omega,\mathcal{F},Q;\mathbb{R}) \) be the vector space of \( Q \)-integrable random variables and \( L^\infty(Q) := L^\infty(\Omega,\mathcal{F},Q;\mathbb{R}) \) be the space of \( Q \)-essentially bounded random variables. Set \( L^1_+(Q) = \{ Z \in L^1(Q) \mid Z \geq 0 \text{ } Q \text{-a.s.} \} \) and \( L^\infty_+(Q) = \{ Z \in L^\infty(Q) \mid Z \geq 0 \text{ } Q \text{-a.s.} \} \). For \( Q \in \mathcal{P}^N \) let

\[ L^1(Q) := L^1(Q^1) \times \ldots \times L^1(Q^N), \quad L^1_+(Q) := L^1_+(Q^1) \times \ldots \times L^1_+(Q^N), \]
\[ L^\infty(Q) := L^\infty(Q^1) \times \cdots \times L^\infty(Q^N), \quad L^\infty_+(Q) := L^\infty_+(Q^1) \times \cdots \times L^\infty_+(Q^N). \]

For each \( j = 1, \ldots, N \) consider a vector subspace \( L^j \) with \( \mathbb{R} \subseteq L^j \subseteq L^0(\Omega, \mathcal{F}, P; \mathbb{R}) \) and set
\[ L := L^1 \times \cdots \times L^N \subseteq (L^0(P))^N. \]

Consider now a subset \( \mathcal{Q} \subseteq \mathcal{P}^N \) and assume that the pair \((L, \mathcal{Q})\) satisfies that for every \( Q \in \mathcal{Q} \)
\[ L \subseteq L^1(Q). \]

One could take as \( L^j \), for example, \( L^\infty \) or some Orlicz space. Our optimization problems will be defined on the vector space \( L \) to be specified later.

For each \( n = 1, \ldots, N \), let \( u_n : \mathbb{R} \to \mathbb{R} \) be concave and strictly increasing. Fix \( X = (X^1, \ldots, X^N) \in \mathcal{L} \).

For \( (Q, a, A) \in \mathcal{Q} \times \mathbb{R}^N \times \mathbb{R} \) define
\[ U_n^Q(a^n) := \sup \{ E[u_n(X^n + Y)] \mid Y \in L^n, E_{Q^n}[Y] \leq a^n \}, \quad (11) \]
\[ S^Q(A) := \sup \left\{ \sum_{n=1}^N U_n^Q(a^n) \mid a \in \mathbb{R}^N \text{ s.t. } \sum_{n=1}^N a^n \leq A \right\}, \quad (12) \]
\[ \Pi^Q(A) := \sup \left\{ E \left[ \sum_{n=1}^N u_n(X^n + Y^n) \right] \mid Y \in \mathcal{L}, \sum_{n=1}^N E_{Q^n}[Y^n] \leq A \right\}. \quad (13) \]

Obviously, such quantities depend also on \( X \), but as \( X \) will be kept fixed throughout most of the analysis, we may avoid to explicitly specify this dependence in the notations. As \( u_n \) is increasing we can replace, in the definitions of \( U_n^Q(a^n) \), \( S^Q(A) \) and \( \Pi^Q(A) \) the inequality in the budget constraint with an equality.

When a vector \( Q \in \mathcal{Q} \) is assigned, we can consider two problems. First, for each \( n \), \( U_n^Q(a^n) \) is the optimal value of the classical one dimensional expected utility maximization problem with random endowment \( X^n \) under the budget constraint \( E_{Q^n}[Y] \leq a^n \), determined by the real number \( a^n \) and the valuation operator \( E_{Q^n}[\cdot] \) associated to \( Q^n \). Second, if we interpret the quantity \( \sum_{n=1}^N u_n(\cdot) \) as the aggregated utility of the system, then \( \Pi^Q(A) \) is the maximal expected utility of the whole system \( X \), among all \( Y \in \mathcal{L} \) satisfying the overall budget constraint \( \sum_{n=1}^N E_{Q^n}[Y^n] \leq A \). Notice that in these problems the vector \( Y \) is not required to belong to \( \mathcal{C}_R \), but only to the vector space \( \mathcal{L} \). We will show in Lemma 4.13 the quite obvious equality \( S^Q(A) = \Pi^Q(A) \).

### 3 On several notions of Equilibrium

#### 3.1 Pareto Allocation

**Definition 3.1.** Given a set of feasible allocations \( \mathcal{V} \subseteq \mathcal{L} \) and a vector \( X \in \mathcal{L}, \hat{Y} \in \mathcal{V} \) is a *Pareto allocation for \( \mathcal{V} \)* if
\[ Y \in \mathcal{V} \quad \text{and} \quad E[u_n(X^n + Y^n)] \geq E[u_n(X^n + \hat{Y}^n)] \quad \text{for all} \quad n \]
\[ \text{imply} \quad E[u_n(X^n + Y^n)] = E[u_n(X^n + \hat{Y}^n)] \quad \text{for all} \quad n. \]

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In general Pareto allocations are not unique and, not surprisingly, the following version of the First Welfare Theorem holds true. Define the optimization problem

\[
\Pi(\mathcal{Y}) := \sup_{Y \in \mathcal{Y}} \sum_{n=1}^{N} \mathbb{E} \left[ u_n(X^n + Y^n) \right].
\]

(15)

**Proposition 3.2.** Whenever \( \hat{Y} \in \mathcal{Y} \) is the unique optimal solution of \( \Pi(\mathcal{Y}) \), then it is a Pareto allocation for \( \mathcal{Y} \).

**Proof.** Let \( \hat{Y} \) be optimal for \( \Pi(\mathcal{Y}) \), so that \( \mathbb{E} \left[ \sum_{n=1}^{N} u_n(X^n + \hat{Y}^n) \right] = \Pi(\mathcal{Y}) \). Suppose that there exists \( Y \) such that (14) holds true. As \( Y \in \mathcal{Y} \) we have:

\[
\mathbb{E} \left[ \sum_{n=1}^{N} u_n(X^n + \hat{Y}^n) \right] = \Pi(\mathcal{Y}) \geq \mathbb{E} \left[ \sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq \mathbb{E} \left[ \sum_{n=1}^{N} u_n(X^n + \hat{Y}^n) \right],
\]

by (14). Hence also \( Y \) is an optimal solution to \( \Pi(\mathcal{Y}) \). Uniqueness of the optimal solution implies \( Y = \hat{Y} \).

\( \square \)

### 3.2 Systemic utility maximization

The next definition is the utility maximization problem, in the case of a system of \( N \) agents.

**Definition 3.3.** Fix \( Q \in \mathcal{Q} \). The pair \((Y_X, a_X) \in \mathcal{L} \times \mathbb{R}^N\) is a **Q–Optimal Allocation** with budget \( A \in \mathbb{R} \) if

1) for each \( n \), \( Y^n_X \) is optimal for \( U^n_Q(a^n_X) \),
2) \( a_X \) is optimal for \( S^Q(A) \),
3) \( Y_X \in \mathcal{L} \).

Note that in the above definition the vector \( Q \in \mathcal{Q} \) is exogenously assigned. Given a total budget \( A \in \mathbb{R} \), the vector \( a_X \in \mathbb{R}^N \) maximizes the systemic utility \( \sum_{n=1}^{N} U^n_Q(a^n) \) among all feasible \( a \in \mathbb{R}^N (\sum_{n=1}^{N} a^n \leq A) \) and \( Y^n_X \) maximizes the single agent expected utility \( \mathbb{E} [u_n(X^n + Y^n)] \) among all feasible allocations \( Y \in \mathcal{L} \) s.t. \( E_Q^n[Y] \leq a^n_X \). Since \( Q \in \mathcal{Q} \) is given, the budget constraint \( E_Q^n[Y] \leq a^n_X \) is well defined for all \( Y \in \mathcal{L} \) and we do not need additional conditions of the form \( Y \in \mathcal{C}_R \). A generalization of the classical single agent utility maximization yields the following existence result.

**Proposition 3.4.** Under Assumption 4.1.(a) select \( \mathcal{Q} = \{Q\} \) for some \( Q \in \mathcal{Q} \), (see (20)) with \( Q \sim P \). Set \( \mathcal{L} = L^1(Q^1) \times \cdots \times L^1(Q^N) \) and let \( X \in M^F \) (see (62)). Then a Q–Optimal Allocation exists.

**Proof.** The proof can be found in Section 4.2 [4], but for completeness we provide a sketch of the proof in Section A.3 in Appendix.

\( \square \)

Let \((Y_X, a_X) \in \mathcal{L} \times \mathbb{R}^N\) be a Q–Optimal Allocation. Due to Lemma 4.13, \( \Pi^Q(A) = S^Q(A) \) and

\[
\Pi^Q(A) = S^Q(A) = \sup_{a \in \mathbb{R}^N, \sum_{n=1}^{N} a^n = A} \sum_{n=1}^{N} \sup_{Y^n \in \mathcal{L}^n} \left\{ \mathbb{E} [u_n(X^n + Y^n)] \mid E_Q^n[Y^n] = a^n \right\}
\]

\[
= \sum_{n=1}^{N} \sup_{Y^n \in \mathcal{L}^n} \left\{ \mathbb{E} [u_n(X^n + Y^n)] \mid E_Q^n[Y^n] = a^n_X \right\}.
\]

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Hence the systemic utility maximization problem $\Pi^Q(A)$ with overall budget constraint $A$ reduces to the sum of $n$ single agent maximization problems, where, however, the budget constraint of each agents is assigned by $a^n_X = E_Q^n[Y^n_X]$ and the vector $a_X$ maximizes the overall performance of the system. We will also recover this feature in the notion of a SORTE, where the probability vector $Q$ will be endogenously determined, instead of being a priori assigned, as in this case.

### 3.3 Risk Exchange Equilibrium

We here formalize Bühlmann’s risk exchange equilibrium in a pure exchange economy, [11] and [12], already mentioned in conditions (a’) and (b’), Item 1 of the Introduction. To be consistent with Definition 3.3 we keep the same numbering for the corresponding conditions. Let $Q^1$ be the set of vectors of probability measures having all components equal:

$$Q^1 := \{ Q \in \mathcal{P}^N \mid Q^1 = \ldots = Q^N \}.$$ 

**Definition 3.5.** Fix $A \in \mathbb{R}$, $a \in \mathbb{R}^N$ such that $\sum_{n=1}^{N} a^n = A$. The pair $(Y_X, Q_X) \in \mathcal{L} \times Q^1$ is a risk exchange equilibrium (with budget $A$ and allocation $a \in \mathbb{R}^N$) if:

1) for each $n$, $Y^n_X$ is optimal for $U^n_{Q_X}(a^n)$,

2) $a_X$ is optimal for $S^{Q_X}(A)$,

3) $Y_X \in C_\mathbb{R}$, $\sum_{n=1}^{N} Y^n_X = A \text{ P-a.s.}$

**Theorem 3.6** (Bühlmann, [12]). For twice differentiable, concave, strictly increasing utilities $u_1, \ldots, u_n : \mathbb{R} \to \mathbb{R}$ such that their risk aversions are positive Lipschitz and for $\mathcal{L} = (L^\infty(P))^N$, $\mathcal{Q} = Q^1$ and $X \in \mathcal{L}$, there exists a unique risk exchange equilibrium that is Pareto optimal.

**Proof.** See [12].

In a risk exchange equilibrium with budget $A$, the vector $a \in \mathbb{R}^N$ such that $\sum_{n=1}^{N} a^n = A$ is exogenously assigned, while both the optimal exchange variable $Y_X$ and the equilibrium price measure $Q_X$ are endogenously determined. On the contrary, in a Q–Optimal Allocation the pricing measure is assigned a priori, while the optimal allocation $Y_X$ and optimal budget $a_X$ are endogenously determined. We shall now introduce a notion which requires to endogenously recover the triple $(Y_X, Q_X, a_X)$ from the systemic budget $A$.

### 3.4 Systemic Optimal Risk Transfer Equilibrium (SORTE)

The novel equilibrium concept presented in equations (8) (9) and (10) can now be formalized as follows. To this end, recall from (1) the definition of $C_\mathbb{R}$ and fix a convex cone

$$B \subseteq C_\mathbb{R}$$

of admissible allocations such that $\mathbb{R}^N + B = B$.

**Definition 3.7** (SORTE). The triple $(Y_X, Q_X, a_X) \in \mathcal{L} \times \mathcal{Q} \times \mathbb{R}^N$ is a Systemic Optimal Risk Transfer Equilibrium with budget $A \in \mathbb{R}$ if:

1) for each $n$, $Y^n_X$ is optimal for $U^n_{Q_X}(a^n_X)$,

2) $a_X$ is optimal for $S^{Q_X}(A)$,

3) $Y_X \in B \subseteq C_\mathbb{R}$ and $\sum_{n=1}^{N} Y^n_X = A \text{ P-a.s.}$
 Remark 3.8. It follows from the monotonicity of each \( u_n \) that \( \sum_{n=1}^{N} a^n_X = A \) and \( E_{Q^n_X}[Y^n_X] = a^n_X \). Hence
\[
\sum_{n=1}^{N} E_{Q^n_X}[Y^n_X] = \sum_{n=1}^{N} a^n_X = A,
\]
and
\[
\sum_{n=1}^{N} Y^n_X = \sum_{n=1}^{N} E_{Q^n_X}[Y^n_X] \text{ P-a.s.}. \tag{16}
\]

The main aim of the paper is to provide sufficient general assumptions that guarantee existence and uniqueness as well as good properties of a SORTE.

Remark 3.9. We will show the existence of a triple \((Y_X, Q_X, a_X) \in \mathcal{L} \times \mathcal{P} \times \mathbb{R}^N\) verifying the three conditions in Definition 3.7. Hence, we also obtain the existence of the SORTE in the formulations given in (5), (6), (10) or in (8), (9), (10), for generic functional \( p^n \) verifying the conditions (i), (ii) and (iii) stated in the Introduction (see also Remark 4.5).

Theorem 3.10. In the setup of Section 4.1 a Systemic Optimal Risk Transfer Equilibrium \((Y_X, Q_X, a_X)\) exists, with \( Q^n_X, \ldots, Q^N_X \) equivalent to \( P \).

Theorem 3.11. In the setup of Section 4.1 and under the additional Assumption that \( \mathcal{B} \) is closed under truncation (Definition 4.15) the Systemic Optimal Risk Transfer Equilibrium is unique and is a Pareto optimal allocation.

The formal statements and proofs are postponed to Section 4, Theorem 4.14 and Theorem 4.19.

Remark 3.12. A priori there are no reasons why a \( Q \)-optimal allocation \( Y_X \) in Definition 3.3 would also satisfy the constraint \( \sum_{n=1}^{N} Y^n_X \in \mathbb{R} \). The existence of a SORTE is indeed the consequence of the existence of a probability measure \( Q_X \) such that the \( Q_X \)-optimal allocation \( Y_X \) in Definition 3.3 satisfies also the additional risk transfer constraint \( \sum_{n=1}^{N} Y^n_X = A \text{ P-a.s.} \).

Remark 3.13. Without the additional feature expressed by 2) in the Definition 3.7, for all choices of \( a_X \) satisfying \( \sum_{n=1}^{N} a^n_X = A \) there exists an equilibrium \((Y_X, Q_X)\) in the sense of Definition 3.5 (see Theorem 3.6). The uniqueness of a SORTE is then a consequence of the uniqueness of the optimal solution in condition 2).

Remark 3.14. Depending on which one of the three objects \((Y, Q, a) \in \mathcal{L} \times \mathcal{P} \times \mathbb{R}^N\) we keep a priori fixed, we get a different notion of equilibrium (see the various definitions above). The characteristic features of the risk exchange equilibriums and of a SORTE, compared with the more classical utility optimization problem in the systemic framework of Section 3.2, are the condition \( \sum_{n=1}^{N} Y^n_X = A \text{ P-a.s.} \) and the existence of the equilibrium pricing vector \( Q_X \).

For both concepts of equilibrium (Definitions 3.5 and SORTE), each agent is behaving rationally by maximizing his expected utility given a budget constraint. The two approaches differ by the budget constraints. In Bühlmann’s definition the vector \( a \in \mathbb{R}^N \) that assigns the budget constraint \( E_{Q^n_X}[Y^n] \leq a_n \) is prescribed a priori. On the contrary, in the SORTE approach, the vector \( a \in \mathbb{R}^N \), with \( \sum_{n=1}^{N} a_n = A \), that assigns the budget constraint \( E_{Q^n_X}[Y^n] \leq a_n \) is determined by optimizing the problem in condition 2), hence by taking into account the optimal systemic utility \( S_{Q_X}(A) \), which is (by definition) larger than the systemic utility \( \sum_{n=1}^{N} U^n_{Q_X}(a^n) \) in Bühlmann’s equilibrium.

The SORTE gives priority to the systemic aspects of the problem in order to optimize the overall
systemic performance. A toy example showing the difference between a B"uhlmann’s equilibrium
and a SORTE is provided in Section 4.7.

Example 3.15. We now consider the example of a cluster of agents, already introduced in [4]. For
\( h \in \{1, \cdots, N\} \), let \( I := (I_m)_{m=1,\ldots,h} \) be some partition of \( \{1, \cdots, N\} \). We introduce the following
family
\[
B^{(I)} = \left\{ Y \in L^0(\mathbb{R}^N) \mid \exists d = (d_1, \cdots, d_h) \in \mathbb{R}^h : \sum_{i \in I_m} Y^i = d_m \text{ for } m = 1, \cdots, h \right\} \subseteq \mathcal{C}_R. \tag{17}
\]
For a given \( I \), the values \( (d_1, \cdots, d_h) \) may change, but the elements in each of the \( h \) groups \( I_m \) is
fixed by the partition \( I \). It is then easily seen that \( B^{(I)} \) is a linear space containing \( \mathbb{R}^N \) and closed
with respect to convergence in probability. We point out that the family \( B^{(I)} \) admits two extreme
cases:

(i) the strongest restriction occurs when \( h = N \), i.e., we consider exactly \( N \) groups, and in this
  case \( B^{(I)} = \mathbb{R}^N \) corresponds to no risk sharing;

(ii) on the opposite side, we have only one group \( h = 1 \) and \( B^{(I)} = \mathcal{C}_R \) is the largest possible
  class, corresponding to risk sharing among all agents in the system. This is the only case
  considered in B"uhlmann’s definition of equilibrium.

Remark 3.16. One additional feature of a SORTE, compared with the B"uhlmann’s notion, is the
possibility to require, in addition to \( \sum_{n=1}^N Y^n = A \) that the optimal solution belongs to a pre-
assigned set \( B \) of admissible allocations, satisfying Assumption 4.1 (b). In particular, we allow for
the selection of the sets \( B = \mathbb{R}^N \) or \( B = \mathcal{C}_R \). Note also that, differently from Definition 3.5, in
Definition 3.7 a SORTE includes a vector of probability measures \( Q_X = (Q^1_X, \ldots, Q^N_X) \) and not just
a \( Q \in \mathcal{Q}^I \) having all the same components. Indeed, the characteristics of the optimal probability
\( Q_X \) depend on the admissible set \( B \). For \( B = \mathcal{C}_R \), all the components of \( Q_X \) turn out to be equal.
For \( B = B^{(I)} \) all the components \( Q^i_X \) of \( Q_X \) are equal for all \( i \in I_m \), for each group \( I_m \). Additional
examples of sets \( B \) are provided in Section 4.6.

4 Proof of Theorem 3.10 and Theorem 3.11

4.1 Setup

In the sequel we will work under the following Assumption 4.1.

Assumption 4.1.

(a) Utilities: \( u_1, \ldots, u_N : \mathbb{R} \to \mathbb{R} \) are concave, strictly increasing differentiable functions with
\[
\lim_{x \to -\infty} \frac{u_n(x)}{x} = +\infty \quad \lim_{x \to +\infty} \frac{u_n(x)}{x} = 0.
\]
Moreover we assume that the following property holds: for any \( n \in \{1, \ldots, N\} \) and \( Q^n \ll P \)
\[
\mathbb{E} \left[ v_n \left( \lambda \frac{dQ^n}{dP} \right) \right] < +\infty \text{ for some } \lambda > 0 \iff \mathbb{E} \left[ v_n \left( \lambda \frac{dQ^n}{dP} \right) \right] < +\infty \text{ for all } \lambda > 0, \tag{18}
\]
where \( v_n(y) := \sup_{x \in \mathbb{R}} \{u_n(x) - xy\} \) denotes the convex conjugate of \( u_n \).
(b) **Constraints**: $\mathcal{B} \subseteq \mathcal{C}_R$ is a convex cone, closed in probability, such that $\mathbb{R}^N + \mathcal{B} = \mathcal{B}$.

**Remark 4.2.** In particular, Assumptions 4.1 (b) implies that all constant vectors belong to $\mathcal{B}$. The condition (18) is related to the Reasonable Asymptotic Elasticity condition on utility functions, which was introduced in [31]. This assumption, even though quite weak (see [6] Section 2.2), is fundamental to guarantee the existence of the optimal solution to classical utility maximization problems (see [6] and [31]).

We need the following concepts and notations:

1. The utility functions in Assumption 4.1 induce an Orlicz Space structure: see Appendix A.1 for the details and the definitions of the functions $\Phi$ and $\Phi^*$, the Orlicz space $L^\Phi$ and the Orlicz Heart $M^\Phi$. Here we just recall the following inclusions among the Banach Spaces $L^\infty(P) \subseteq M^\Phi \subseteq L^\Phi \subseteq L^1(P)$ and that $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^{\Phi^*}$ implies $L^\Phi \subseteq L^1(\mathbb{Q})$. From now on we assume that $X \in M^\Phi$.

2. For any $A \in \mathbb{R}$ we set $B_A := \mathcal{B} \cap \{ Y \in (L^0(P))^N \mid \sum_{n=1}^N Y^n \leq A \text{ P-a.s.} \}$.

Observe that $B_0 \cap M^\Phi$ is a convex cone.

3. We introduce the following problem for $X \in M^\Phi$ and for a vector of probability measures $\mathbb{Q} \ll \mathbb{P}$, with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^{\Phi^*}$,

$$\pi^\mathbb{Q}(A) := \sup \left\{ \sum_{n=1}^N \mathbb{E}[u_n (X^n + Y^n)] \mid Y \in M^\Phi, \sum_{n=1}^N E_{\mathbb{Q}^n} [Y^n] \leq A \right\}. \quad (19)$$

Notice that in (19) the vector $Y$ is not required to belong to $\mathcal{C}_R$, but only to the vector space $M^\Phi$. In order to show the existence of the optimal solution to the problem $\pi^\mathbb{Q}(A)$, it is necessary to enlarge the domain in (19).

4. $\mathbb{Q}$ is the set of vectors of probability measures defined by

$$\mathbb{Q} := \left\{ \mathbb{Q} \ll \mathbb{P} \left| \frac{d\mathbb{Q}^1}{d\mathbb{P}}, \ldots, \frac{d\mathbb{Q}^N}{d\mathbb{P}} \right. \in L^{\Phi^*}, \sum_{n=1}^N \mathbb{E} \left[ Y^n \frac{d\mathbb{Q}^n}{d\mathbb{P}} \right] \leq 0, \forall Y \in \mathcal{B}_0 \cap M^\Phi \right\}. \quad (20)$$

Identifying Radon-Nikodym derivatives and measures in the natural way, $\mathbb{Q}$ turns out to be the set of normalized (i.e. with componentwise expectations equal to 1), non negative vectors in the polar of $\mathcal{B}_0 \cap M^\Phi$ in the dual system $(M^\Phi, L^{\Phi^*})$. In our $N$-dimensional systemic one-period setting, the set $\mathbb{Q}$ plays the same crucial role as the set of martingale measures in multiperiod stochastic securities markets.

5. We introduce the following convex subset of $\mathbb{Q}$:

$$\mathbb{Q}_\nu := \mathbb{Q} \cap \left\{ \frac{d\mathbb{Q}^1}{d\mathbb{P}}, \ldots, \frac{d\mathbb{Q}^N}{d\mathbb{P}} \right| \frac{d\mathbb{Q}^n}{d\mathbb{P}} \geq 0 \forall n \in \{1, \ldots, N\}, \sum_{n=1}^N \mathbb{E} \left[ v_n \frac{d\mathbb{Q}^n}{d\mathbb{P}} \right] < +\infty \right\}. \quad (20)$$
6. Set
\[ \mathcal{L} := \bigcap_{Q \in \mathcal{Q}} L^1(Q^1) \times \cdots \times L^1(Q^N), \quad \mathcal{Q} := \mathcal{Q}_v. \] (21)

Note that \( M^\Phi \subseteq \mathcal{L} \). We will consider the optimization problems (11), (12) and (13) with the particular choice of \( (\mathcal{L}, \mathcal{Q}) \) in (21) and will show that, with such choice, \( \pi^Q(A) = \Pi^Q(A) \).

Observe that if all utilities are bounded from above, the requirement \( \sum_{n=1}^N E \left[ \frac{dQ^n}{dP^n} \right] \leq +\infty \) is redundant, but it becomes important if we allow utilities to be unbounded.

We also require some additional definitions and notations:

a) \( \overline{B}_0 \) is the polar of the cone \( co(Q_v) \) in the dual pair
\[ \left( \bigcap_{Q \in \mathcal{Q}_v} L^1(Q^1) \times \cdots \times L^1(Q^N), \bigcap_{Q \in \mathcal{Q}_v} L^1(Q^1) \times \cdots \times L^1(Q^N) \right), \]

that is
\[ \overline{B}_0 := \left\{ Y \in \bigcap_{Q \in \mathcal{Q}_v} L^1(Q^1) \times \cdots \times L^1(Q^N) \bigg| \sum_{n=1}^N E_{Q^n}[Y^n] \leq 0, \forall Q \in \mathcal{Q}_v \right\}. \]

It is easy to verify that \( \overline{B}_0 \cap M^\Phi \subseteq \overline{B}_0 \).

b) For any \( A \in \mathbb{R} \) we define \( \overline{B}_A \) as the set
\[ \overline{B}_A := \left\{ Y \in \bigcap_{Q \in \mathcal{Q}_v} L^1(Q^1) \times \cdots \times L^1(Q^N) \bigg| \sum_{n=1}^N E_{Q^n}[Y^n] \leq A, \forall Q \in \mathcal{Q}_v \right\}. \]

We will prove that \( \overline{B}_A \) is the correct enlargement of the domain \( \mathcal{B}_A \cap M^\Phi \) in order to obtain the existence of the optimal solution of the primal problem.

c) \( \{ e_i \}_{i=1,\ldots,N} \) is the canonical base of \( \mathbb{R}^N \).

**Lemma 4.3.** In the dual pair \( (M^\Phi, L^\Phi^*) \), consider the polar \( (\overline{B}_0 \cap M^\Phi)^0 \) of \( \overline{B}_0 \cap M^\Phi \). Then \( (\overline{B}_0 \cap M^\Phi)^0 \cap (L_+^0)^N \) is the cone generated by \( Q \).

**Proof.** From the definition of \( \overline{B}_0 \) and the fact that \( \mathcal{B} \) contains all constant vectors, we may conclude that all vectors in \( \mathbb{R}^N \) of the form \( e_i - e_j \) belong to \( \overline{B}_0 \cap M^\Phi \). Then for all \( Z \in (\overline{B}_0 \cap M^\Phi)^0 \) and for all \( i, j \in \{1, \ldots, N\} \) we must have: \( \mathbb{E} [Z^i] - \mathbb{E} [Z^j] \leq 0 \). As a consequence, \( Z \in (\overline{B}_0 \cap M^\Phi)^0 \) implies \( \mathbb{E} [Z^1] = \cdots = \mathbb{E} [Z^N] \) and so
\[ (\overline{B}_0 \cap M^\Phi)^0 \cap (L_+^0)^N = \mathbb{R}_+ \cdot Q, \] (22)
where \( \mathbb{R}_+ := \{ b \in \mathbb{R}, b \geq 0 \} \).

**Lemma 4.4.** \( Q_v^+ := \{ Q \in Q_v \ s.t. \ Q \sim P \} \neq \emptyset \).
Proof. The condition $\mathcal{B} \subseteq C_{R}$ implies $\mathcal{B}_0 \cap M^\phi \subseteq (C_{R} \cap M^\phi \cap \{\sum_{n=1}^{N} Y^n \leq 0\})$, so that the polars satisfy the opposite inclusion: $(C_{R} \cap M^\phi \cap \{\sum_{n=1}^{N} Y^n \leq 0\})^0 \subseteq (\mathcal{B}_0 \cap M^\phi)^0$. Observe now that any vector $(Z, \ldots, Z)$, for $Z \in L_+^\infty$, belongs to $(C_{R} \cap M^\phi \cap \{\sum_{n=1}^{N} Y^n \leq 0\})^0$. In particular, $(\mathcal{B}_0 \cap M^\phi)^0$ contains vectors in the form $(\frac{\epsilon}{1+\epsilon} Z, \ldots, \frac{\epsilon}{1+\epsilon} Z)$ with $\epsilon > 0$ and $Z \in L_+^\infty$, $E[Z] = 1$. Each component of such a vector has expectation equal to 1, belongs to $L_+^\infty$ and satisfies $\frac{\epsilon}{1+\epsilon} Z \geq \frac{1}{1+\epsilon} Z$. All these conditions imply that there exists a probability vector $Q \in \mathcal{Q}$ such that $\frac{dQ}{dP} > 0$ $P-a.s.$ with $\sum_{n=1}^{N} E\left[v_n\left(\frac{dQ}{dP}\right)\right] < \infty$, hence $Q_v \neq \emptyset$. \hfill \Box

4.2 Scheme of the proof

The proof of Theorem 3.10 is inspired by the classical duality theory in utility maximization, see for example [16] and [26] and by the minimax approach developed in [3]. More precisely, our road map will be the following:

1. First we show, in Remark 4.6, how we may reduce the problem to the case $A = 0$.

2. We consider

$$\pi(A) := \sup \left\{ \sum_{n=1}^{N} E\left[u_n(X^n + Y^n)\right] \mid Y \in M^\phi \cap \mathcal{B}, \sum_{n=1}^{N} Y^n \leq A, P-a.s. \right\}. \quad (23)$$

In Theorem 4.7 we specialize the duality, obtained in Theorem A.3 for a generic convex cone $C$, for the maximization problem $\pi(0)$ over the convex cone $C = B_0 \cap M^\phi$ and prove: (i) the existence of the optimizer $\bar{Y}$ of $\pi(0)$, which belongs to $B_0$; (ii) the existence of the optimizer $\bar{Q}$ to the dual problem of $\pi(0)$. Here we need all the assumptions on the utility functions and on the set $B$ and an auxiliary result stated in Theorem A.4 in Appendix.

3. Proposition 4.9 will show that also the elements in the closure of $B \cap M^\phi$ satisfy the key condition $\sum_{n=1}^{N} E_{Q^n}[Y^n] \leq \sum_{n=1}^{N} Y^n \in \mathbb{R}$ for all $Q \in \mathcal{Q}$.

4. Theorem A.3 is then again applied, to a different set $C = \left\{ Y \in M^\phi \mid \sum_{n=1}^{N} E_{Q^n}[Y^n] \leq 0 \right\}$, to derive Proposition 4.11, which establishes the duality for $\pi(0)$ and $\pi(A)$ in case a fixed probability vector $Q$ is assigned.

5. The minimax duality:

$$\pi(A) = \min_{Q \in \mathcal{Q}} \pi^Q(A) = \pi^{\bar{Q}}(A),$$

is then a simple consequence of the above results (see Corollary 4.12). This duality is the key tool to prove the existence of a SORTE (see Theorem 4.14).

6. Uniqueness and Pareto optimality are then proved in Theorem 4.19.

Remark 4.5. Notice that in the definition of $\pi(A)$ there is no reference to a probability formulation vector $Q$. However, the optimizer of the dual formulation of $\pi(A)$ is a probability vector $\tilde{Q}$ (that will be the equilibrium pricing vector in the SORTE). Even if in the equations (8), (9), (10) we do not a priori require pricing functional of the form $p^\nu(\cdot) = E_{Q^n}[\cdot]$, this particular linear expression naturally appears from the dual formulation.
4.3 Minimax Approach

Remark 4.6. Only in this Remark, we need to change the notation a bit: we make the dependence of our maximization problems on the initial point explicit. To this end we will write

\[
\pi_X(A) := \sup \left\{ \sum_{j=1}^{N} \mathbb{E} [u_j (X^j + Y^j)] \middle| Y \in B_A \cap M^\Phi \right\},
\]

\[
\pi_Q^X(A) := \sup \left\{ \sum_{j=1}^{N} \mathbb{E} [u_j (X^j + Y^j)] \middle| Y \in M^\Phi, \sum_{j=1}^{N} E_{Q^j} [Y^j] \leq A \right\}.
\]

It is possible to reduce the maximization problem expressed by \(\pi_X(A)\) (and similarly by \(\pi_Q^X(A)\)) to the problem related to \(\pi_X(0)\) (respectively, \(\pi_Q^X(0)\)) by using the following simple observation: for any \(a_0 \in \mathbb{R}^N\) with \(\sum_{j=1}^{N} a_{0j} = A\) consider

\[
\pi_X(A) = \sup \left\{ \sum_{j=1}^{N} \mathbb{E} [u_j (X^j + Y^j)] \middle| Y \in B \cap M^\Phi, \sum_{j=1}^{N} Y^j \leq A \right\}
\]

\[
= \sup \left\{ \sum_{j=1}^{N} \mathbb{E} [u_j (X^j + a_{0j} + (Y^j - a_{0j}))] \middle| Y \in B \cap M^\Phi, \sum_{j=1}^{N} (Y^j - a_{0j}) \leq 0 \right\}
\]

\[
= \sup \left\{ \sum_{j=1}^{N} \mathbb{E} [u_j (X^j + a_{0j} + Z^j)] \middle| Z \in B_0 \cap M^\Phi \right\},
\]

where last equality holds as we are assuming that \(\mathbb{R}^N + B = B\). The last line represents the original problem, but with \(A = 0\) and a different initial point. This fact will be used in the conclusion of the proof of Theorem 4.7.

In the following Theorem we follow a minimax procedure inspired by the technique adopted in [6].

**Theorem 4.7.** Under Assumption 4.1 we have

\[
\pi(A) := \sup_{Y \in B_A} \sum_{j=1}^{N} \mathbb{E} [u_j (X^j + Y^j)] = \max_{Y \in B_A} \sum_{j=1}^{N} \mathbb{E} [u_j (X^j + Y^j)]
\]

\[
= \min_{Q \in \mathcal{Q}} \min_{\lambda \in \mathbb{R}^+} \left( \lambda \left( \sum_{j=1}^{N} E_{Q^j} [X^j] + A \right) + \sum_{j=1}^{N} \mathbb{E} \left[ v_j \left( \frac{dQ^j}{dP} \right) \right] \right) \quad \text{(25)}
\]

The minimization problem in (25) admits a unique optimum \((\hat{\lambda}, \hat{Q}) \in \mathbb{R}^+ \times \mathcal{Q}\) with \(\hat{Q} \sim P\). The maximization problem in (24) admits a unique optimum \(\hat{Y} \in B_A\), given by

\[
\hat{Y}^j = -X^j - v'_j \left( \frac{d\hat{Q}^j}{dP} \right), \quad j = 1, \ldots, N,
\]

which belongs to \(B_A\). In addition,

\[
\sum_{j=1}^{N} E_{\hat{Q}^j} [\hat{Y}^j] = A \quad \text{and} \quad \sum_{j=1}^{N} E_{Q^j} [\hat{Y}^j] \leq A \quad \forall Q \in \mathcal{Q}_v.
\]

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Proof. We first prove the result for the case $A = 0$.

**STEP 1**

We first show that

$$
\sup_{B_0 \cap M^\Phi} \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] < \sum_{j=1}^{N} v_j(0) = \sum_{j=1}^{N} u_j(+\infty) \ \forall X \in M^\Phi \ 
$$

(28)

so that we will be able to apply Theorem A.3 with the choice $C := B_0 \cap M^\Phi$. We distinguish two possible cases: $\sum_{j=1}^{N} u_j(+\infty) = +\infty$ or $\sum_{j=1}^{N} u_j(+\infty) < +\infty$.

For $\sum_{j=1}^{N} u_j(+\infty) = +\infty$: observe that for any $Q \in \mathcal{Q}_u$ (which is nonempty by Lemma 4.4) and $\lambda > 0$ we have

$$
\sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] \leq \sum_{j=1}^{N} \mathbb{E} \left[ (X^j + Y^j) \left( \lambda \frac{dQ^j}{dP} \right) \right] + \sum_{j=1}^{N} \mathbb{E} \left[ v_j \left( \lambda \frac{dQ^j}{dP} \right) \right] \leq \sum_{j=1}^{N} \mathbb{E} \left[ X^j \left( \lambda \frac{dQ^j}{dP} \right) \right] + \sum_{j=1}^{N} \mathbb{E} \left[ v_j \left( \lambda \frac{dQ^j}{dP} \right) \right].
$$

We exploited above Fenchel’s Inequality and the definition of $\mathcal{Q}_u$. Observing that the last line does not depend on $Y$ and is finite, and using the well known relation $v_j(0) = u_j(+\infty), j = 1, \ldots, N$, we conclude that

$$
\sup_{B_0 \cap M^\Phi} \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] < +\infty = \sum_{j=1}^{N} v_j(0).
$$

For $\sum_{j=1}^{N} u_j(+\infty) < +\infty$: if the inequality in (28) were not strict, for any maximizing sequence $(Y_m)_m$ we would have, by monotone convergence, that

$$
\sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( +\infty \right) \right] - \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j_m \right) \right] = \mathbb{E} \left[ \sum_{j=1}^{N} \left( u_j \left( +\infty \right) - u_j \left( X^j + Y^j_m \right) \right) \right] \to 0.
$$

Up to taking a subsequence we can assume the convergence is also almost sure. Since all the terms in $\sum_{j=1}^{N} \left( u_j \left( +\infty \right) - u_j \left( X^j + Y^j_m \right) \right)$ are non negative, we also see that $u_j(X^j + Y^j_m) \to m u_j(+\infty)$ almost surely for every $j = 1, \ldots, N$. By strict monotonicity of the utilities, this would imply that, for each $j$, $Y^j_m \to m + \infty$. This clearly contradicts the constraint $Y_m \in B_0$.

**STEP 2**

We will prove equations (24) and (25), with a supremum over $\overline{B}_A$ in place of a maximum, since we will show in later steps (STEP 4) that this supremum is in fact a maximum.

We observe that since $B_0 \cap M^\Phi \subseteq \overline{B}_0$

$$
\sup_{B_0 \cap M^\Phi} \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] \leq \sup_{\overline{B}_0} \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right].
$$

Moreover, by the Fenchel inequality

$$
\sup_{\overline{B}_0} \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] \leq \inf_{\lambda \in \mathbb{R}^+, Q \in \mathcal{Q}} \left( \lambda \sum_{j=1}^{N} E_{Q^j} \left[ X^j \right] + \sum_{j=1}^{N} \mathbb{E} \left[ v_j \left( \lambda \frac{dQ^j}{dP} \right) \right] \right).
$$

Equations (24) and (25) follow from Theorem A.3 replacing there the convex cone $C$ with $B_0 \cap M^\Phi$ and using equation (22), which shows that $(c_1^0)^+ = \mathcal{Q}$.
STEP 3

We prove that if $\hat{\lambda}$ and $\hat{Q}$ are optima in equation (25), then $
abla^j := -X^j - v'_j \left( \frac{\lambda d\hat{Q}^j}{dP} \right)$ defines an element in $\mathcal{B}_0$. Observe that $\hat{\lambda}$ minimizes the function

$$\mathbb{R}_{++} \ni \gamma \mapsto \psi(\gamma) := \sum_{j=1}^{N} \left( \gamma E_{\hat{Q}^j} \left[ X^j \right] + \mathbb{E} \left[ v_j \left( \gamma \frac{d\hat{Q}^j}{dP} \right) \right] \right)$$

which is real valued and convex. Also we have by Monotone Convergence Theorem and Lemma A.2.(1) that the right and left derivatives, which exist by convexity, satisfy

$$\frac{d^+ \psi}{d\gamma} (\gamma) = \sum_{j=1}^{N} \mathbb{E} \left[ X^j \frac{d\hat{Q}^j}{dP} \right] + \sum_{j=1}^{N} \mathbb{E} \left[ v'_j \left( \hat{\lambda} \frac{d\hat{Q}^j}{dP} \right) \frac{d\hat{Q}^j}{dP} \right],$$

hence the function is differentiable. Since $\hat{\lambda}$ is a minimum for $\psi$, this implies $\psi'(\hat{\lambda}) = 0$, which can be rephrased as

$$\sum_{j=1}^{N} \left( \mathbb{E} \left[ X^j \frac{d\hat{Q}^j}{dP} \right] + \mathbb{E} \left[ v'_j \left( \hat{\lambda} \frac{d\hat{Q}^j}{dP} \right) \right] \right) = 0,$$

i.e.,

$$\sum_{j=1}^{N} E_{\hat{Q}^j} \left[ \nabla^j \right] = 0.$$  

(30)

Now consider minimizing

$$\mathbb{Q} \ni \sum_{j=1}^{N} \left( \hat{\lambda} E_{\hat{Q}^j} \left[ X^j \right] + \mathbb{E} \left[ v_j \left( \hat{\lambda} \frac{d\hat{Q}^j}{dP} \right) \right] \right)$$

for fixed $\hat{\lambda}$ and $\mathbb{Q}$ varying in $\mathcal{Q}_v$. Let again $\hat{Q}$, with $\hat{\eta} := \frac{d\hat{Q}}{dP}$, be an optimum and consider another $\mathbb{Q} \in \mathcal{Q}_v$, with $\eta := \frac{d\mathbb{Q}}{dP}$. By Assumption 4.1, the expression $\sum_{j=1}^{N} \mathbb{E} \left[ v_j \left( \lambda \frac{d\mathbb{Q}^j}{dP} \right) \right]$ is finite for all choices of $\lambda$. Observe that $\psi'$ by convexity and differentiability of $v_j$ we have

$$\hat{\lambda} \eta' v'_j \left( \hat{\lambda} \hat{\eta} \right) \leq \hat{\lambda} \eta' v'_j \left( \hat{\lambda} \hat{\eta} \right) + v_j \left( \hat{\lambda} \hat{\eta} \right) - v_j \left( \hat{\lambda} \hat{\eta} \right).$$

Hence by Lemma A.2.(1) and $\hat{Q}, \mathbb{Q} \in \mathcal{Q}_v$ we conclude that

$$\left( \eta' v'_j \left( \hat{\lambda} \hat{\eta} \right) \right)^+ \in L^1(P).$$

(31)

To prove that also the negative part is integrable, we take a convex combination of $\hat{Q}, \mathbb{Q} \in \mathcal{Q}_v$, which still belongs to $\mathcal{Q}_v$. By optimality of $\hat{\eta}$ the function

$$x \mapsto \varphi(x) := \sum_{j=1}^{N} \left( \lambda \mathbb{E} \left[ X^j \left( (1-x)\hat{\eta}^j + x\eta^j \right) \right] + \mathbb{E} \left[ v_j \left( \lambda \left( (1-x)\hat{\eta}^j + x\eta^j \right) \right) \right] \right), 0 \leq x \leq 1,$$

has a minimum at 0, thus the right derivative of $\varphi$ at 0 must be non negative, so that:

$$\sum_{j=1}^{N} \frac{d}{dx} \bigg|_{0} \left( (1-x)\lambda \mathbb{E} \left[ X^j \hat{\eta}^j + x\lambda \mathbb{E} \left[ X^j \eta^j \right] \right] \right) \geq - \sum_{j=1}^{N} \frac{d}{dx} \bigg|_{0} \left( \mathbb{E} \left[ v_j \left( (1-x)\lambda \hat{\eta}^j + x\lambda \eta^j \right) \right] \right).$$

(32)

Define $H_j(x) := v_j \left( (1-x)\lambda \hat{\eta}^j + x\lambda \eta^j \right)$ and observe that as $x \downarrow 0$ by convexity

$$0 \leq \left( -\frac{1}{x} \left( H_j(x) - H_j(0) \right) + H_j(1) - H_j(0) \right) \uparrow \left( -\lambda v'_j \left( \hat{\lambda} \hat{\eta} \right) \eta^j + \hat{\lambda} v'_j \left( \hat{\lambda} \hat{\eta} \right) \hat{\eta} + H_j(1) - H_j(0) \right).$$

(33)
Write now explicitly equation (32) in terms of incremental ratios and add and subtract the real number $E \left[ \sum_{j=1}^{N} (H_j(1) - H_j(0)) \right]$ to get

$$
\lim_{x \to 0} \sum_{j=1}^{N} \left( \frac{1}{x} \left[ \left( 1 - \frac{x}{\tilde{\lambda}E[T_j]} \right) \tilde{\lambda}E[T_j] + x \tilde{\lambda}E[T_j] \right] - \tilde{\lambda}E[T_j] \right) + E[H_j(1) - H_j(0)] \tag{34}
$$

$$
\geq \lim_{x \to 0} \sum_{j=1}^{N} \left( E \left[ \frac{1}{x} (H_j(x) - H_j(0)) + H_j(1) - H_j(0) \right] \right). \tag{35}
$$

The first limit is trivial. Observe that by (33) and Monotone Convergence Theorem we also may compute the second limit and then deduce:

$$
\sum_{j=1}^{N} \left( \tilde{\lambda}E[T_j (\eta^j - \tilde{\eta}^j)] + E[H_j(1) - H_j(0)] \right)
$$

and therefore

$$
+\infty > \sum_{j=1}^{N} \tilde{\lambda}E[T_j (\eta^j - \tilde{\eta}^j)] \geq E \left[ \sum_{j=1}^{N} \left( -\tilde{\lambda}v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j + \tilde{\lambda}v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \tilde{\eta}^j + H_j(1) - H_j(0) \right) \right]
$$

$$
= E \left[ \sum_{j=1}^{N} \left( \tilde{\lambda} \left( v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j \right) - \tilde{\lambda} \left( v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j \right)^+ + \tilde{\lambda}v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right)^{\tilde{\eta}^j} \right) \right].
$$

Since $\sum_{j=1}^{N} v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right)^{\tilde{\eta}^j} \in L^1(P)$ by Lemma A.2 (1), and $\sum_{j=1}^{N} \left( v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j \right)^+ \in L^1(P)$ by equation (31), we deduce that $\sum_{j=1}^{N} \left( v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j \right)^- \in L^1(P)$ so that

$$
0 \leq \left( v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j \right)^- \leq \sum_{j=1}^{N} \left( v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j \right)^- \in L^1(P).
$$

We conclude that $v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j$ defines a vector in $L^1(P) \times \cdots \times L^1(P)$, hence

$$
\tilde{\Psi} \in L^1(Q^1) \times \cdots \times L^1(Q^N) \quad \forall Q \in Q_v. \tag{36}
$$

Moreover equation (32) can be rewritten as:

$$
0 \leq \sum_{j=1}^{N} \tilde{\lambda}E[T_j (\eta^j - \tilde{\eta}^j)] + \sum_{j=1}^{N} \tilde{\lambda}E\left[ v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j \right]. \tag{37}
$$

Now rearrange the terms in (37)

$$
0 \leq -\sum_{j=1}^{N} \tilde{\lambda} \left( E[T_j \eta^j] + E[v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j] \right) + \sum_{j=1}^{N} \tilde{\lambda} \left( E[T_j \eta^j] + E[v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \eta^j] \right)
$$

and use (29):

$$
0 \leq 0 - \sum_{j=1}^{N} \tilde{\lambda} \left( E \left[ \left( -X_j - v_j^j \left( \tilde{\lambda}\tilde{\eta}^j \right) \right) \eta^j \right] \right) = -\tilde{\lambda} \sum_{j=1}^{N} E \left[ \tilde{\lambda} \frac{dQ^j}{dP} \right].
$$
The conditions in (27) are proved in (30) and (38). We conclude with uniqueness. By the strict convexity of the utilities and the convexity of $\mathcal{B}_0$, it is evident that the maximization problem given by \( \sup_{\mathcal{Q}^n} \sum_{j=1}^N E_{Q^n_j} [X^j] \) admits at most one optimum. Now clearly if $\hat{\lambda}, \hat{Q}$ and $\tilde{\lambda}, \tilde{Q}$ are optima for the minimax expression (25), they both give rise to two optima $\hat{Y}$, $\tilde{Y}$ as in previous steps. Uniqueness of the solution for the primal problem implies $\hat{Y} = \tilde{Y}$. Under Assumption
4.1. (a) the functions \( v_1', \ldots, v_N' \) are injective and therefore we conclude that \( \frac{d\tilde{Q}}{dP} = \frac{d\hat{Q}}{dP} \). Taking expectations we get \( \hat{\lambda} = \tilde{\lambda} \) and then \( (\hat{\lambda}, \hat{Q}) = (\tilde{\lambda}, \tilde{Q}) \).

**Conclusion**

The more general case \( A \neq 0 \) can be obtained using Remark 4.6. We just sketch one step of the proof, as the other steps follows similarly. Using \( a_0 \) as in Remark 4.6, in STEP 3 we see that

\[
0 \leq -\tilde{\lambda} \sum_{j=1}^{N} \mathbb{E} \left[ \tilde{Y}^j \frac{dQ^j}{dP} \right] + \hat{\lambda} \sum_{j=1}^{N} a_j^0
\]

which yields that \( \tilde{Y} \in \overline{B_A} \).

**Remark 4.8.** Notice that \( Y \in B \cap M^\Phi \) implies that \( Z \in B_0 \), where \( Z \) is defined by \( Z^j := Y^j - x^j \sum_{k=1}^{N} Y^k \) for any \( x \in \mathbb{R}^N \) such that \( \sum_{j=1}^{N} x^j = 1 \). To see this, recall that we are assuming that \( \mathbb{R}^N + B = B \). As \( \sum_{j=1}^{N} Y^j \in \mathbb{R} \), then \( Z \in B \) and, since also trivially integrability is preserved and \( \sum_{j=1}^{N} Z^j = 0 \), we conclude that \( Z \in B_0 \).

**Proposition 4.9.** For all \( Y \in B \cap M^\Phi \) and \( Q \in Q \)

\[
\sum_{j=1}^{N} E_{Q^j} [Y^j] \leq \sum_{j=1}^{N} Y^j.
\]

Moreover, denoting by \( cl_Q (B \cap M^\Phi) \) the \( L^1(Q^1) \times \cdots \times L^1(Q^N) \)-norm closure of \( B \cap M^\Phi \), inequality (41) holds for all \( Y \in cl_Q (B \cap M^\Phi) \) and \( Q \in Q \). In particular, (41) holds for \( \tilde{Q} \sim P \) and \( \tilde{Y} \in cl_{Q} (B_0 \cap M^\Phi) \) defined in Theorem 4.7.

**Proof.** Take \( Y \in B \cap M^\Phi \) and argue as in Remark 4.8, with the notation introduced there. By the definition of the polar, \( \sum_{j=1}^{N} \mathbb{E} \left[ Z^j \varphi^j \right] \leq 0 \) for all \( \varphi \in (B \cap M^\Phi)^0 \), and in particular for all \( Q \in Q \)

\[
0 \geq \sum_{j=1}^{N} \mathbb{E} \left[ Z^j \frac{dQ^j}{dP} \right] = \sum_{j=1}^{N} \mathbb{E} \left[ Y^j \frac{dQ^j}{dP} \right] - \sum_{j=1}^{N} \mathbb{E} \left[ x^j \left( \sum_{k=1}^{N} Y^k \right) \frac{dQ^j}{dP} \right] = \sum_{j=1}^{N} E_{Q^j} [Y^j] - \sum_{j=1}^{N} Y^j.
\]

As to the second claim, take a sequence \( (k_n) \) in \( B \cap M^\Phi \) converging both \( Q \)-almost surely (hence \( P - a.s. \)) and in norm to \( Y \) and apply (41) to \( k_n \) to get

\[
\sum_{j=1}^{N} E_{Q^j} [Y^j] = \lim_n \sum_{j=1}^{N} E_{Q^j} [k_n^j] \overset{P-a.s.}{\leq} \liminf_n \left( \sum_{j=1}^{N} k_n^j \right) \overset{P-a.s.}{=} \sum_{j=1}^{N} Y^j.
\]

**Remark 4.10.** In particular (41) shows that \( \forall Q \in Q \)

\[
\left\{ Y \in B \cap M^\Phi \mid \sum_{j=1}^{N} Y^j \leq A \right\} \subseteq \left\{ Y \in M^\Phi \mid \sum_{j=1}^{N} E_{Q^j} [Y^j] \leq A \right\}
\]

and therefore \( \pi(A) \leq \pi^Q(A) \).
4.4 Utility Maximization with a fixed probability measure

The following represents a counterpart to Theorem 4.7, once a measure is fixed a priori.

**Proposition 4.11.** Fix $Q \in Q_\circ$. If $\pi^Q(A) < +\infty$, then

$$\pi^Q(A) = \Pi^Q(A) = \sup \left\{ \sum_{j=1}^N \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] \middle| Y \in L^1(Q), \sum_{j=1}^N E_{Q^j} \left[ Y^j \right] \leq A \right\}$$

(43)

$$= \min_{\lambda \in \mathbb{R}^+} \left( \lambda \left( \sum_{j=1}^N E_{Q^j} \left[ X^j \right] + A \right) + \sum_{j=1}^N \mathbb{E} \left[ v_j \left( \lambda \frac{dQ^j}{dP} \right) \right] \right).$$

If additionally any of the two expressions is strictly less than $\sum_{j=1}^N u_j(+\infty)$, then

$$\pi^Q(A) = \min_{\lambda \in \mathbb{R}^+} \left( \lambda \left( \sum_{j=1}^N E_{Q^j} \left[ X^j \right] + A \right) + \sum_{j=1}^N \mathbb{E} \left[ v_j \left( \lambda \frac{dQ^j}{dP} \right) \right] \right).$$

(44)

**Proof.** Again, we prove the case $A = 0$ since Remark 4.6 can be used to obtain the general case $A \neq 0$. From $M^\phi \subseteq L \subseteq L^1(Q)$ we obtain:

$$\pi^Q(0) := \sup \left\{ \sum_{j=1}^N \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] \middle| Y \in M^\phi, \sum_{j=1}^N E_{Q^j} \left[ Y^j \right] \leq 0 \right\} \leq \Pi^Q(0)$$

$$\leq \sup \left\{ \sum_{j=1}^N \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] \middle| Y \in L^1(Q), \sum_{j=1}^N E_{Q^j} \left[ Y^j \right] \leq 0 \right\}$$

$$\leq \min_{\lambda \in \mathbb{R}^+} \left( \lambda \sum_{j=1}^N E_{Q^j} \left[ X^j \right] + \sum_{j=1}^N \mathbb{E} \left[ v_j \left( \lambda \frac{dQ^j}{dP} \right) \right] \right)$$

(45)

by the Fenchel inequality. Define the convex cone

$$C := \left\{ Y \in M^\phi \mid \sum_{j=1}^N E_{Q^j} \left[ Y^j \right] \leq 0 \right\}.$$

The hypotheses on $C$ of Theorem A.3 hold true and inequality (45) shows that $\pi^Q(0) < +\infty$ for all $X \in M^\phi$. The finite dimensional cone $\{ \lambda \left[ \frac{dQ^1}{dP}, \ldots, \frac{dQ^N}{dP} \right], \lambda \geq 0 \} \subseteq L^\phi^*$ is closed, and then by the Bipolar Theorem $C^0 = \{ \lambda \left[ \frac{dQ^1}{dP}, \ldots, \frac{dQ^N}{dP} \right], \lambda \geq 0 \}$. Hence the set $(C^0)^+$ in the statement of Theorem A.3 is exactly $\left[ \frac{dQ^1}{dP}, \ldots, \frac{dQ^N}{dP} \right]$ and Theorem A.3 proves that $\pi^Q(0)$ is equal to the RHS of (45). We can similarly argue to prove (44).

To conclude, we provide the minimax duality between the maximization problems with and without a fixed measure

**Corollary 4.12.** The following holds:

$$\pi(A) = \min_{Q \in Q_\circ} \pi^Q(A) = \pi^Q(A) < +\infty,$$

where $\tilde{Q}$ is the minimax measure from Theorem 4.7.
Proof. It is an immediate consequence of Theorem 4.7 and Proposition 4.11. $\square$

**Lemma 4.13.** For all $Q \in \mathcal{Q}$ we have $\Pi^Q(A) = S^Q(A)$ and, if $\hat{Q}$ is the minimax measure from Theorem 4.7, then
\[
\pi(A) = \hat{\pi}(A) = \Pi^\hat{Q}(A) = S^\hat{Q}(A).
\] (46)

**Proof.** Let $Y \in \mathcal{L}$, $Q \in \mathcal{Q}$, $a^n := EQ^n[Y^n]$ and $Z^n := Y^n - a^n$. As $\mathcal{L} + \mathbb{R}^N = \mathcal{L}$, $Z^n \in \mathcal{L}^n$ and
\[
\Pi^Q(A) = \sup_{Y \in \mathcal{L}} \left\{ \mathbb{E} \left[ \sum_{n=1}^N u_n(X^n + Y^n) \right] \mid \sum_{n=1}^N EQ^n[Y^n] = A \right\}
\]
\[
= \sup_{a \in \mathbb{R}^N, \sum_{n=1}^N a^n = A} \left\{ \mathbb{E} \left[ \sum_{n=1}^N u_n(X^n + Z^n + a^n) \right] \mid EQ^n[Z^n] = 0, \sum_{n=1}^N a^n = A \right\}
\]
\[
= \sup_{a \in \mathbb{R}^N, \sum_{n=1}^N a^n = A} \left\{ \mathbb{E} \left[ \sum_{n=1}^N u_n(X^n + Z^n + a^n) \right] \mid EQ^n[Z^n] = 0 \right\}
\]
\[
= \sup_{a \in \mathbb{R}^N, \sum_{n=1}^N a^n = A} \left\{ \mathbb{E} \left[ u_n(X^n + Y^n) \mid EQ^n[Y^n] = a^n \right] \right\}
\]
\[
= \sup_{a \in \mathbb{R}^N, \sum_{n=1}^N a^n = A} \left\{ \sum_{n=1}^N U_{Q^n}^\hat{Q}(a^n) \right\}
\]
The first equality in (46) follows from Corollary 4.12 and the second one from (43). $\square$

### 4.5 Main results

**Theorem 4.14.** Take $\mathcal{Q} = Q_v$ and set $\mathcal{L} = \bigcap_{Q \in Q_v} L^1(Q)$. Under Assumption 4.1, for any $X \in M^b$ and any $A \in \mathbb{R}$ a SORTE exists, namely $(\hat{Y}, \hat{Q}) \in \mathcal{B}_A \times Q_v$ defined in Theorem 4.7 and
\[
\hat{a}^n := EQ^n[\hat{Y}^n], \quad n = 1, \ldots, N,
\] (47)

satisfy:

1. $\hat{Y}^n$ is an optimum for $U_{Q_n}^\hat{Q}(\hat{a}^n)$, for each $n \in \{1, \ldots, N\}$,
2. $\hat{a}$ is an optimum for $S^\hat{Q}(A)$,
3. $\hat{Y} \in \mathcal{B}$ and $\sum_{n=1}^N \hat{Y}^n = A$ $P$-a.s.

**Proof.**

1) We prove that $U_{Q_n}^\hat{Q}(\hat{a}^n) = \mathbb{E} \left[ u_n \left( X^n + \hat{Y}^n \right) \right] < u_n(+\infty)$, for all $n = 1, \ldots, N$, thus showing that $\hat{Y}^n$ is an optimum for $U_{Q_n}^\hat{Q}(\hat{a}^n)$. As $\hat{Y}^n \in \mathcal{L}^n$ for all $n = 1, \ldots, N$, then by definition of $U_{Q_n}^\hat{Q}(\hat{a}^n)$ we obtain:
\[
\sup \left\{ \mathbb{E} \left[ u_n(X^n + Z) \right] \mid Z \in \mathcal{L}^n, EQ^n[Z] \leq \hat{a}^n \right\} =: U_{Q_n}^\hat{Q}(\hat{a}^n) \geq \mathbb{E} \left[ u_n \left( X^n + \hat{Y}^n \right) \right].
\]
If, for some index, the last inequality were strict we would obtain the contradiction
\[
\pi^Q(A) = \pi(A) \overset{\text{Thm. 4.7}}{=} \sum_{n=1}^N \mathbb{E} \left[ u_n \left( X^n + \hat{Y}^n \right) \right] < \sum_{n=1}^N U_{Q_n}^\hat{Q}(\hat{a}^n) \leq S^\hat{Q}(A) = \pi^\hat{Q}(A),
\] (48)
where we used (46) in the first and last equality.

In particular then $E \left[ u_n \left( X^n + \hat{Y}^n \right) \right] < u_n(\infty)$, for all $n = 1, \ldots, N$. Indeed, if the latter were equal to $u_n(\infty)$, then $u_n$ would attain its maximum over a compact subset of $\mathbb{R}$, which is not the case.

2): From (27) we know that $A = \sum_{n=1}^{N} E \hat{Q}_n^{\ast} [\hat{Y}^n] = \sum_{n=1}^{N} \hat{a}^n$. From (46) we have

$$S\hat{Q}(A) = \pi(A) \overset{\text{Thm. 4.7}}{=} \sum_{n=1}^{N} E \left[ u_n \left( X^n + \hat{Y}^n \right) \right] = \sum_{n=1}^{N} U_n^{\hat{Q}^\ast} (\hat{a}^n) \leq S\hat{Q}(A).$$

3): We already know that $\hat{Y} \in B_A := B \cap \{ Y \in (L^0(P))^N | \sum_{n=1}^{N} Y^n \leq A \}$. From Proposition 4.9 we deduce

$$A = \sum_{n=1}^{N} E \hat{Q}_n^{\ast} [\hat{Y}^n] \leq \sum_{n=1}^{N} \hat{Y}^n \leq A. \quad \square$$

We now turn our attention to uniqueness and Pareto optimality, but we will need an additional property and auxiliary result.

**Definition 4.15 (Def. 4.18 in [4]).** We say that $B \subseteq (L^0(P))^N$ is **closed under truncation** if for each $Y \in B$ there exists $m_Y \in \mathbb{N}$ and $c_Y = (c^1_Y, \ldots, c^n_Y) \in \mathbb{R}^N$ such that $\sum_{n=1}^{N} c^n_Y = \sum_{n=1}^{N} Y^n := c_Y \in \mathbb{R}$ and for all $m \geq m_Y$

$$Y_m := Y_I \{ c^n_{n-1}(|Y^n| < m) \} + c_Y I \{ \cup_{n=1}^{N} (|Y^n| \geq m) \} \in B. \quad (49)$$

**Remark 4.16.** We stress the fact that all the sets introduced in Example 3.15 satisfy closedness under truncation.

**Lemma 4.17.** Let $B$ be closed under truncation. Then for every $A \in \mathbb{R}$

$$B_A \cap L \subseteq \overline{B_A}. \quad \text{Proof.} \text{ Fix any } Q \in Q_v \text{ and argue as in Proposition 4.20 in [4]: let } Y \in B_A \cap L \subseteq L^1(Q) \text{ and consider } Y_m \text{ for } m \in \mathbb{N} \text{ as defined in (49), where w.l.o.g. we assume } m_Y = 1. \text{ Note that } \sum_{n=1}^{N} Y^n_m = c_Y (= \sum_{n=1}^{N} Y^n \leq A) \text{ for all } m \in \mathbb{N}. \text{ By boundedness of } Y_m, \text{ and (49), we have } Y_m \in B \cap M^\phi \text{ for all } m \in \mathbb{N}. \text{ Further, } Y_m \to Y \text{ P-a.s. for } m \to \infty, \text{ and thus, since } |Y_m| \leq \max\{|Y|, |c_Y|\} \in L^1(Q) \text{ for all } m \in \mathbb{N}, \text{ also } Y_m \to Y \text{ in } L^1(Q) \text{ for } m \to \infty \text{ by dominated convergence. Now, if } Q \sim P \text{ we can directly apply Proposition 4.9 to get that } \sum_{n=1}^{N} E_{Q^n}[Y^n] \leq \sum_{n=1}^{N} Y^n \leq A. \text{ If we only have } Q \ll P \text{ we can see that (42) still holds, with the particular choice of } (Y_m)_m \text{ in place of } (k_n)_n, \text{ because the construction of } Y_m \text{ is made } P\text{-almost surely.} \quad \square$$

Define

$$\Pi(A) := \sup \left\{ E \left[ \sum_{n=1}^{N} u_n(X^n + Y^n) \right] \mid Y \in L \cap B, \sum_{n=1}^{N} Y^n \leq A \right\}. \quad (50)$$

**Lemma 4.18.** Let $B$ be closed under truncation. If $\hat{Q}$ is the minimax measure from Theorem 4.7, then

$$\pi(A) = \Pi(A) = \pi^{\hat{Q}}(A) = \Pi^{\hat{Q}}(A) = S^{\hat{Q}}(A). \quad (51)$$
Proof. It is clear that since \( \mathcal{B}_A \cap M^\Phi \subseteq \mathcal{B}_A \cap \mathcal{L} \) we have \( \pi(A) \leq \Pi(A) \) just by definitions (23) and (50). Now observe that by Lemma 4.17 we have \( \mathcal{B}_A \cap \mathcal{L} \subseteq \mathcal{B}_A \), so that \( \Pi(A) \leq \Pi^\Phi(A) \). The chain of equalities then follows by Lemma 4.13.

**Theorem 4.19.** Let \( \mathcal{B} \) be closed under truncation. Under the same assumptions of Theorem 4.14, for any \( X \in M^\Phi \) and \( A \in \mathbb{R} \) the SORTE is unique and is a Pareto optimal allocation for both the sets

\[
\mathcal{Y} = \left\{ Y \in \mathcal{L} \cap \mathcal{B} \left| \sum_{n=1}^{N} Y^n \leq A \right. \mbox{ P-a.s.} \right\} \quad \mbox{and} \quad \mathcal{Y} = \left\{ Y \in \mathcal{L} \left| \sum_{n=1}^{N} E_Q^n [Y^n] \leq A \right. \mbox{ (52)} \right\}.
\]

Proof. Use Proposition 4.11 and Corollary 4.12 to get that for any \( Q \in \mathcal{Q}_e \)

\[
\Pi^Q(A) = \pi^Q(A) \geq \pi(A). \quad (53)
\]

Let \((\hat{Y}, \hat{Q}, \hat{a})\) be a SORTE and \((\check{Y}, \check{Q}, \check{a})\) be the one from Theorem 4.14.

By (1) and (2) in the definition of SORTE, together with Lemma 4.13, we see that \( \check{Y} \) is an optimum for \( \Pi^Q(A) = S^\Phi(A) \). Also, \( \check{Y} \in \mathcal{B}_A \cap \mathcal{B}_A \) by Lemma 4.17. We can conclude by equation (24) that

\[
\pi(A) \geq \sum_{n=1}^{N} \mathbb{E} \left[ u_n \left( X^n + \check{Y}^n \right) \right] = \Pi^Q(A) \overset{\text{eq.}(43)}{=} \pi^\hat{Q}(A) \overset{\text{Cor.4.12}}{\geq} \pi(A),
\]

which tells us that \( \pi(A) = \pi^\hat{Q}(A) = \sum_{n=1}^{N} \mathbb{E} \left[ u_n \left( X^n + \check{Y}^n \right) \right] \).

By Theorem 4.7, we also have \( \pi(A) = \sum_{n=1}^{N} \mathbb{E} \left[ u_n \left( X^n + \check{Y}^n \right) \right] \). Then \( \check{Y}, \check{Q} \in \mathcal{B}_A \) (Lemma 4.17) and \( \Pi(A) = \pi(A) \) (Lemma 4.18) imply that both \( \check{Y}, \check{Q} \) are optima for \( \Pi(A) \). By strict concavity of the utilities \( u_1, \ldots, u_N \), \( \Pi(A) \) has at most one optimum. From this, together with uniqueness of the minimax measure (see Theorem 4.7), we get \( (\check{Y}, \check{Q}) = (\check{Y}, \check{Q}) \). We infer from equation (47) and Remark 3.8 that also \( \check{a} = \hat{a} \).

To prove the Pareto optimality observe that Theorem 4.7 proves that \( \check{Y} \in \mathcal{B}_A \subseteq \mathcal{L} \) is the unique optimum for \( \Pi(A) \) (see Lemma 4.18) and so it is also the unique optimum for \( \Pi^\hat{Q}(A) \). Pareto optimality then follows from Proposition 3.2, noticing that \( \Pi(\mathcal{Y}) \) for the two sets in (52) are \( \Pi(A) \) and \( \Pi^\check{Q}(A) \) respectively.

### 4.6 Dependence of the SORTE on X and on B

We see from the proof of Theorem 4.14 that the triple defining the SORTE (obviously) depends on the choice of \( A \). We now focus on the study of how such triple depends on \( X \). To this end, we first specialize to the case \( B = C_R \).

**Proposition 4.20.** Under the hypotheses of Theorem 4.14 and for \( B = C_R \), the variables \( \frac{d\hat{Q}}{d\bar{Y}} \) and \( X + \check{Y} \) are \( \sigma(X^1 + \cdots + X^N) \) (essentially) measurable.

Proof. By Theorem 4.14 and Theorem 4.19 we have that \((\hat{\lambda}, \hat{Q})\) is an optimum of the RHS of equation (25). Notice that in this specific case \( Y := e_i 1_A - e_j 1_A \in \mathcal{B} \cap \mathcal{M}^\Phi \) for all \( i, j \) and all measurable sets \( A \in \mathcal{F} \). Let \( Q \in \mathcal{Q} \). Then from (41) \( \sum_{n=1}^{N} (E_Q^n [Y^n] - Y^n) \leq 0 \) and so \( Q^i(A) - 1_A - Q^j(A) + 1_A \leq 0 \), i.e., \( Q^i(A) - Q^j(A) \leq 0 \). Similarly taking \( Y := -e_i 1_A + e_j 1_A \in \mathcal{B} \),
we get \( Q^i(A) - Q^i(A) \leq 0 \). Hence all the components of vectors in \( Q \) are equal. Let \( G := \sigma(X^1 + \cdots + X^N) \). Then for any \( \lambda \in \mathbb{R}^+ \) and any \( Q = [Q_1, \ldots, Q] \in Q \) we have:

\[
\lambda \left( \sum_{n=1}^{N} E_{Q^n}[X^n] + A \right) + \sum_{n=1}^{N} \mathbb{E} \left[ v_n \left( \lambda \frac{dQ^n}{dP} \right) \right] = \lambda \left( \mathbb{E} \left[ \left( \sum_{n=1}^{N} X^n \right) \frac{dQ}{dP} \right] + A \right) + \sum_{n=1}^{N} \mathbb{E} \left[ v_n \left( \lambda \frac{dQ}{dP} \right) \right] =
\]

\[
\lambda \left( \mathbb{E} \left[ \left( \sum_{n=1}^{N} X^n \right) \frac{dQ}{dP} \right] + A \right) + \sum_{n=1}^{N} \mathbb{E} \left[ v_n \left( \lambda \frac{dQ}{dP} \right) \right] \geq 0.
\]

where in the last inequality we exploited tower property and Jensen inequality, as \( v_1, \ldots, v_N \) are convex. Notice now that \( \mathbb{E} \left[ \frac{dQ}{dP} \bigg| G \right] \) defines again a probability measure (on the whole \( \mathcal{F} \), the initial sigma algebra) and that this measure still belongs to \( Q \) since all its components are equal. As a consequence, the minimum in equation (25) can be equivalently taken over \( \lambda \in \mathbb{R}^+ \) (as before) and \( Q \in Q \cap (L^0(\Omega, G, P))^N \). The claim for \( \hat{Y} \) follows from (26).

It is interesting to notice that this dependence on the componentwise sum of \( X \) also holds in the case of Bühlmann’s equilibrium (see [12] page 16 and [10]).

**Remark 4.21.** In the case a cluster of agents, see the Example 3.15, the above result can be clearly generalized: the \( i \)-th component of the vector \( \hat{Q} \), for \( i \) belonging to the \( m \)-th group, only depends on the sum of those components of \( X \) whose corresponding indexes belong to the \( m \)-th group itself. It is also worth mentioning that if we took \( B^{(1)} = \mathbb{R}^N \), we would see that each component of \( \hat{Q} \) and of \( \hat{Y} \) solely depends on the corresponding component of \( X \). This is reasonable, since in this case each agent would be only allowed to share and exchange risk with herself/himself and the systemic features of the model we are considering would be lost.

We provide now some additional examples, to the ones in example 3.15, of possible feasible sets \( B \) and study the dependence of the probability measures from \( B \).

**Example 4.22.** Consider a measurable partition \( A_1, \ldots, A_K \) of \( \Omega \) and a collection of partitions \( I^1, \ldots, I^K \) of \( \{1, \ldots, N\} \) as in Example 3.15. Take the associated clusterings \( B^{(I^1)}, \ldots, B^{(I^K)} \) defined as in (17). Then the set

\[
B := \left( \sum_{i=1}^{K} B^{(I^i)} 1_{A_i} \right) \cap \mathcal{C}_\mathbb{R} \tag{54}
\]

satisfies Assumptions 4.1 and is closed under truncation, as it can be checked directly.

The set in (54) can be seen as a scenario-dependent clustering. A particular simple case of (54) is the following. For a measurable set \( A_1 \in \mathcal{F} \) take \( A_2 = \Omega \setminus A_1 \). Then set \( C_\mathbb{R} 1_{A_1} + \mathbb{R}^N 1_{A_2} \) is of the form (54) and consists of all the \( Y \in (L^0)^N \) such that (i) there exists a real number \( \sigma \in \mathbb{R} \) with \( \sum_{n=1}^{N} Y^n = \sigma P \)-a.s. on \( A_1 \), (ii) there exists a vector \( b \in \mathbb{R}^N \) such that \( Y = b \ P \)-a.s. on \( A_2 \) and (iii) \( \sigma = \sum_{n=1}^{N} b^n \) (recall that \( Y \in C_\mathbb{R} \) by (54)).

Let us motivate Example 4.22 with the following practical example. Suppose for each bank \( i \) a regulator establishes an excessive exposure threshold \( D^i \). If the position of bank \( i \) falls below such threshold, we can think that it is too dangerous for the system to let that bank take part to the risk exchange. As a consequence, in the clustering example, on the event \( \{X^i \leq D^i\} \) we can
require the bank to be left alone. Also the symmetric situation can be considered: a bank \( j \) whose position is too good, say exceeding a value \( A' \), will not be willing to share risk with all others, thus entering the game only as isolated individual or as a member of the groups of “safer” banks.

Both these requirements, and many others (say considering random thresholds) can be modelled with the constraints introduced in Example 4.22

It is interesting to notice that, as in Example 3.15, assuming a constraint set of the form given in Example 4.22 forces a particular behavior on the probability vectors in \( \mathcal{Q}_v \).

**Lemma 4.23.** Let \( \mathcal{B} \) be as in example 4.22 and let \( \mathcal{Q} \in \mathcal{Q}_v \). Fix any \( i \in \{1,\ldots,K\} \) and any group \( \Gamma_m \) of the partition \( \Gamma = (\Gamma_m)_m \). Then all the components \( Q^j, j \in \Gamma_m \), agree on \( \mathcal{F}|_{A_i} := \{F \cap A_i, F \in \mathcal{F}\} \).

**Proof.** We think it is more illuminating to prove the statement in a simplified case, rather than providing a fully formal proof (which would require unnecessarily complicated notation). This is "without loss of generality" in the sense that it is clear how to generalize the method. To this end, let us consider the case \( K = 2 \) (i.e. \( A_2 = A'_1 \)) and \( \mathcal{B}^{\{1\}} := \mathcal{C}_R, \mathcal{B}^{\{2\}} := \mathbb{R}^N \). For any \( F \in \mathcal{F} \) and \( i,j \in \{1,\ldots,N\} \) we can take \( Y := (1_F(e_i - e_j))1_{A_1} + 01_{A_2} \) to obtain \( Y \in C_R1_{A_1} + \mathbb{R}^N1_{A_2} \), \( \sum_{j=1}^N Y_j = 0 \). By definition of \( \mathcal{Q}_v \) we get for any \( \mathcal{Q} \in \mathcal{Q}_v \) that \( Q^i(A \cap F) - Q^j(A \cap F) \leq 0 \), and interchanging \( i,j \) yields \( Q^j(A \cap F) - Q^i(A \cap F) \leq 0 \) for any \( i,j = 1,\ldots,N, F \in \mathcal{F} \). \( \square \)

### 4.7 A toy Example

In the following two examples we compare a Bühlmann’s Equilibrium with a SORTE in the simplest case where \( X = 0 := (0,\ldots,0) \) and \( A = 0 \). We consider the exponential case where \( u_n(x) = 1 - e^{-\alpha_n x} \), so that \( u_n(0) = 0 \). In the formula below we use the well known fact:

\[
\sup_{Y \in L^1(Q)} \{E[u_n(Y)] \mid E_Q[Y] \leq x\} = 1 - e^{-\alpha_n x - H(Q,P)},
\]

where \( H(Q,P) = E[\frac{dQ}{dP} \ln(\frac{dQ}{dP})] \) is the relative entropy, for \( Q \ll P \).

**Example 4.24** (Bühlmann’s equilibrium solution). As \( X := 0 \) then \( X_N = \sum_{k=1}^N X^k = 0 \) and therefore the optimal probability measure \( Q_X \) defined in Bühlmann is:

\[
\frac{dQ_X}{dP} := \frac{e^{-\frac{1}{2} X_N}}{E[e^{-\frac{1}{2} X_N}]} = 1,
\]

i.e. \( Q_X = P \). Take \( a = 0 = (0,\ldots,0) \). We compute

\[
U_n^{Q_X}(0) = U_n^P(0) := \sup \{E[u_n(0 + Y)] \mid E_P[Y] \leq 0\} = 1 - e^{-\alpha_n 0 - H(P,P)} = 1 - 1 = 0,
\]

as \( H(P,P) = 0 \), so that

\[
\sum_{n=1}^N U_n^P(0) = 0.
\]

As a consequence, and as \( u_n(0) = 0 \), the optimal solution for each single \( n \) is obviously \( Y_n^Q = 0 \).

**Conclusion:** The Bühlmann’s equilibrium solution associated to \( X := 0 \) (and \( A = 0 \)) is the couple \( (Y_X, Q_X) = (0, P) \). Here the vector \( a \) is taken a priori to be equal to \( (0,\ldots,0) \).
**Example 4.25 (SORTE).** From the formula

\[
\sup_{\mathbf{Q}} \sum_{j=1}^{N} \mathbb{E} \left[ u_{j} (X^{j} + Y^{j}) \right] = \min_{\lambda \in \mathbb{R}^{++}, \mathbf{Q} \in \mathcal{Q}} \left( \lambda \left( \sum_{j=1}^{N} \mathbb{E}_{\mathbf{Q}} [X^{j}] + A \right) + \sum_{j=1}^{N} \mathbb{E} \left[ v_{j} \left( \frac{dQ^{j}}{dP} \right) \right] \right) \tag{56}
\]

with \( \mathbf{X} := \mathbf{0} \) and \( A = 0 \) we deduce that the optimal probability measure \( \hat{\mathbf{Q}} \) coincides again with \( P \). As \( \mathbf{X} = \mathbf{0} \) and \( \hat{\mathbf{Q}} = P \) we may compute explicitly

\[
U_{n}^{Qx} (a^{n}) = U_{n}^{P} (a^{n}) := \sup \left\{ \mathbb{E} [u_{n} (0 + Y)] \mid \mathbb{E}_{P} [Y] \leq a^{n} \right\} = 1 - e^{-\alpha_{n} a^{n} - H(P;P)} = 1 - e^{-\alpha_{n} a^{n}}.
\]

The optimal solution is then deterministic and is given by \( \hat{Y}^{n} = a^{n} \) and we have:

\[
S^{Qx} (A) = S^{P} (0) := \sup \left\{ \sum_{n=1}^{N} U_{n}^{P} (a^{n}) \mid a \in \mathbb{R}^{N} \text{ s.t. } \sum_{n=1}^{N} a^{n} \leq 0 \right\}
\]

\[
= \sup \left\{ 1 - e^{-\alpha_{n} a^{n}} \mid a \in \mathbb{R}^{N} \text{ s.t. } \sum_{n=1}^{N} a^{n} = 0 \right\}. \tag{57}
\]

Solving this simple optimization problem we obtain:

\[
S^{P} (0) = N - \beta e^{-\frac{A_{N}}{\beta}},
\]

where \( A_{N} := \sum_{k=1}^{N} \frac{1}{\alpha_{k}} \log \left( \frac{1}{\alpha_{k}} \right) \) and \( \beta := \sum_{k=1}^{N} \frac{1}{\alpha_{k}} \), and the optimal solution \( \hat{a} \) is:

\[
\hat{a}^{n} = \frac{1}{\alpha_{n}} \left[ \ln(\alpha_{n}) - E_{R} [\ln(\alpha)] \right], \tag{58}
\]

where \( R \) is defined by

\[
R(n) := \frac{1}{\sum_{k=1}^{N} \frac{1}{\alpha_{k}}}, \quad n = 1, \ldots, N, \text{ and } \alpha := (\alpha_{1}, \ldots, \alpha_{N}).
\]

Notice that if the \( \alpha_{n} \) are equal for all \( n \), then \( S^{P} (0) = 0 \), but in general

\[
S^{P} (0) = N - \beta e^{-\frac{A_{N}}{\beta}} \geq 0.
\]

Indeed, by Jensen inequality:

\[
e^{-\frac{A_{N}}{\beta}} = e^{E_{R} [\ln(\alpha)]} \leq E_{R} [e^{\ln(\alpha)}] = E_{R} [\alpha] := \sum_{n=1}^{N} \frac{1}{\alpha_{n}} \alpha_{n} = \frac{N}{\beta}.
\]

From (58) we deduce that the \( \alpha_{n} \) are equal for all \( n \) if and only if \( \hat{a}^{n} = 0 \) for all \( n \), but in general \( \hat{a}^{n} \) may differ from 0. As \( \hat{Y}^{n} = \hat{a}^{n} \), the same holds also for the optimal solution \( \hat{Y}^{n} \). When \( \hat{a}^{n} < 0 \) a violation of Individual Rationality occurs.

**Conclusion:** The SORTE solution associated to \( \mathbf{X} := \mathbf{0} \) (and \( A = 0 \)) is the triplet \( (\hat{Y}, P, \hat{a}) \) where \( \hat{Y} = \hat{a} \) is assigned in equation (58).

The above comparison shows that a SORTE is not a Bühlmann equilibrium, even when \( \mathbf{X} := \mathbf{0} \) and \( A = 0 \). When the \( \alpha_{n} \) are all equal, then the Bühlmann and the SORTE solution coincide, as all agents are assumed to have the same risk aversion.
Remark 4.26. In this example, notice that we may control the risk sharing components $Y^n$ of agent $n$ in the SORTE by:

$$|Y^n| \leq \frac{1}{\alpha_{\text{min}}} [\ln(\alpha_{\text{max}}) - \ln(\alpha_{\text{min}})].$$

Suppose that $\alpha_{\text{min}} < \alpha_{\text{max}}$ and consider the expression for $\tilde{Y}^n = \hat{a}^n$ in (58). If $\alpha_j = \alpha_{\text{min}}$ then the corresponding $\tilde{Y}^j < 0$ is in absolute value relatively large (divide by $\alpha_{\text{min}}$), while if $\alpha_k = \alpha_{\text{max}}$ the corresponding $\tilde{Y}^k > 0$ is in absolute value relatively small (divide by $\alpha_{\text{max}}$).

A Appendix

A.1 Orlicz Spaces and Utility Functions

We consider the utility maximization problem defined on Orlicz spaces, see [29] for further details on Orlicz spaces. This presents several advantages. From a mathematical point of view, it is a more general setting than $L^\infty$, but at the same time it simplifies the analysis, since the topology is order continuous and there are no singular elements in the dual space. Furthermore, it has been shown in [7] that the Orlicz setting is the natural one to embed utility maximization problems, as the natural integrability condition $E[u(X)] > -\infty$ is implied by $E[\phi(X)] < +\infty$.

Let $u : \mathbb{R} \to \mathbb{R}$ be a concave and increasing function satisfying $\lim_{x \to -\infty} \frac{u(x)}{x} = +\infty$. Consider $\phi(x) := -u(-|x|) + u(0)$. Then $\phi : \mathbb{R} \to [0, +\infty)$ is a strict Young function, i.e., it is finite valued, even and convex on $\mathbb{R}$ with $\phi(0) = 0$ and $\lim_{x \to +\infty} \frac{\phi(x)}{x} = +\infty$. The Orlicz space $L^\phi$ and Orlicz Heart $M^\phi$ are respectively defined by

$$L^\phi := \{ X \in L^0(\mathbb{R}) \mid E[\phi(\alpha X)] < +\infty \text{ for some } \alpha > 0 \},$$

$$M^\phi := \{ X \in L^0(\mathbb{R}) \mid E[\phi(\alpha X)] < +\infty \text{ for all } \alpha > 0 \},$$

and they are Banach spaces when endowed with the Luxemburg norm. The topological dual of $M^\phi$ is the Orlicz space $L^{\phi^*}$, where the convex conjugate $\phi^*$ of $\phi$, defined by

$$\phi^*(y) := \sup_{x \in \mathbb{R}} \{ xy - \phi(x) \}, \quad y \in \mathbb{R},$$

is also a strict Young function. Note that

$$E[u(X)] > -\infty \text{ if } E[\phi(X)] < +\infty.$$  \hspace{1cm} (61)

Remark A.1. It is well known that $L^\infty(\mathbb{P}; \mathbb{R}) \subseteq M^\phi \subseteq L^\phi \subseteq L^1(\mathbb{P}; \mathbb{R})$. In addition, from the Fenchel inequality $xy \leq \phi(x) + \phi^*(y)$ we obtain

$$(\alpha |X|) \left( \lambda dQ \right) \leq \phi(\alpha |X|) + \phi^*(\lambda dQ)$$

for some probability measure $Q \ll \mathbb{P}$, and we immediately deduce that $dQ/d\mathbb{P} \in L^{\phi^*}$ implies $L^\phi \subseteq L^1(Q; \mathbb{R})$.

Given the utility functions $u_1, \cdots, u_N : \mathbb{R} \to \mathbb{R}$, satisfying the above conditions, with associated Young functions $\phi_1, \cdots, \phi_N$, we define

$$M^\Phi := M^{\phi_1} \times \cdots \times M^{\phi_N}, \quad L^\Phi := L^{\phi_1} \times \cdots \times L^{\phi_N}.$$  \hspace{1cm} (62)
A.2 Auxiliary results

Lemma A.2. Let \( v : [0, +\infty) \rightarrow \mathbb{R} \cup \{+\infty\} \) be a convex function, and suppose that its restriction to \((0, +\infty)\) is real valued and differentiable. Let \( Q \ll P \) be a given probability measure with \( v \left( \frac{dQ}{dP} \right) \in L^1(P) \) for all \( \lambda > 0 \). Then

1. \( v' \) is defined on \((0, +\infty)\) and real valued there and extendable to \([0, +\infty)\) by taking \( \lim_{x \to 0} v'(x) \in \mathbb{R} \cup \{-\infty\} \). Also, \( \frac{dQ}{dP} v' \left( \frac{dQ}{dP} \right) \in L^1(P) \) for all \( \lambda > 0 \).

2. If \( g \) is such that \( g + \frac{1}{2} \in L^\infty_+(P) \), then \( v \left( \frac{dQ}{dP} \right) \in L^1(P) \).

3. If \( v'(0+) = -\infty \), \( v'(+\infty) = +\infty \) and \( v \) is strictly convex \( F(\gamma) := E \left[ \frac{dQ}{dP} v' \left( \frac{dQ}{dP} \right) \right] \) is a well defined bijection between \((0, +\infty)\) and \( \mathbb{R} \).

Proof. Lemma 2 of [6].

The following minimax theorem holds:

Theorem A.3. Let \( u_1, \ldots, u_n : \mathbb{R} \rightarrow \mathbb{R} \) be strictly increasing and concave functions. Let \( C \subseteq M^\Phi \) be a convex cone such that for every \( i, j = 1, \ldots, N, e_i - e_j \in C \). Denote by \( C^0 \) the polar of the cone \( C \) in the dual pair \((M^\Phi, L^\Phi^*)\)

\[
C^0 := \left\{ Z \in L^{\Phi^*} \ s.t. \ \sum_{j=1}^N E \left[ Y^j Z^j \right] \leq 0 \ \forall \ Y \in C \right\}.
\]

Set

\[
C^0_1 := \left\{ Z \in C^0 \ s.t. \ E \left[ Z^1 \right] = \ldots = E \left[ Z^N \right] = 1 \right\}, \quad (C^0_1)^+ := \left\{ Z \in C^0_1 \ s.t. \ Z^j \geq 0 \ \forall \ j \right\}
\]

and suppose that

\[
\sup_{Y \in C} \left( \sum_{j=1}^N E \left[ u_j \left( X^j + Y^j \right) \right] \right) < +\infty \ \forall X \in M^\Phi.
\]

Then

\[
\sup_{Y \in C} \left( \sum_{j=1}^N E \left[ u_j \left( X^j + Y^j \right) \right] \right) = \min_{\lambda \in \mathbb{R}^+, Q \in (C^0_1)^+} \left( \lambda \sum_{j=1}^N E \left[ X^j \frac{dQ^j}{dP} \right] + \sum_{j=1}^N E \left[ v_j \left( \lambda \frac{dQ^j}{dP} \right) \right] \right).
\]

If any of the two expressions above is strictly smaller than \( \sum_{j=1}^N u_j(+\infty) \), then

\[
\sup_{Y \in C} \left( \sum_{j=1}^N E \left[ u_j \left( X^j + Y^j \right) \right] \right) = \min_{\lambda \in \mathbb{R}^+, Q \in (C^0_1)^+} \left( \lambda \sum_{j=1}^N E \left[ X^j \frac{dQ^j}{dP} \right] + \sum_{j=1}^N E \left[ v_j \left( \lambda \frac{dQ^j}{dP} \right) \right] \right).
\]

Proof.

Observe first that \( X \mapsto \rho(X) := -\sup_{Y \in C} \left( \sum_{j=1}^N E \left[ u_j \left( X^j + Y^j \right) \right] \right) \) is a non increasing, finite valued, convex functional on the Fréchet lattice \( M^\Phi \). Only convexity is non-evident: to show it,
Consider \( \mathbf{X}, \mathbf{Z} \in M^\Phi \) and \( \mathbf{Y}, \mathbf{W} \in \mathcal{C} \). For any \( 0 \leq \lambda \leq 1 \), we have by concavity
\[
\lambda \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] + (1 - \lambda) \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( Z^j + W^j \right) \right]
\leq \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( \lambda X^j + Y^j + (1 - \lambda)(Z^j + W^j) \right) \right]
\]
\[
= \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( \lambda X^j + (1 - \lambda)Z^j + (\lambda Y^j + (1 - \lambda)W^j) \right) \right] \leq -\rho(\lambda \mathbf{X} + (1 - \lambda)\mathbf{Z})
\]
as \( \lambda \mathbf{Y} + (1 - \lambda)\mathbf{W} \in \mathcal{C} \). Thus taking suprema over \( \mathbf{Y}, \mathbf{W} \in \mathcal{C} \) we get
\[
\lambda(-\rho(\mathbf{X})) + (1 - \lambda)(-\rho(\mathbf{Z})) \leq -\rho(\lambda \mathbf{X} + (1 - \lambda)\mathbf{Z}).
\]
Now the Extended Namioka-Klee Theorem (see [8] Theorem A.3) can be applied and we obtain
\[
\rho(\mathbf{X}) = \max_{0 \leq \mathbf{Z} \in L^\Phi} \left( \sum_{j=1}^{N} E_p \left[ X^j(-Z^j) \right] - \alpha(\mathbf{Z}) \right),
\]
where
\[
\alpha(\mathbf{Z}) := \sup_{\mathbf{X} \in M^\Phi} \left( \sum_{j=1}^{N} E_p \left[ X^j(-Z^j) \right] - \rho(\mathbf{X}) \right) = \sup_{\mathbf{X} \in M^\Phi} \left( \sum_{j=1}^{N} E_p \left[ X^j(-Z^j) \right] + \sup_{\mathbf{Y} \in \mathcal{C}} \left( \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] \right) \right)
\]
\[
= \sup_{\mathbf{Y} \in \mathcal{C}} \left( \sum_{j=1}^{N} E_p \left[ Y^j(Z^j) \right] + \sup_{\mathbf{W} \in M^\Phi} \left( \sum_{j=1}^{N} E_p \left[ W^j(-Z^j) \right] + \left( \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( W^j \right) \right] \right) \right) \right).
\]
Observe now that \( -U(\mathbf{z}) := \sum_{j=1}^{N} -u_j(z^j) \) for \( \mathbf{z} \in \mathbb{R}^N \) defines a continuous, convex, proper function whose Fenchel transform is
\[
(-U)^*(\mathbf{w}) := \sup_{\mathbf{z} \in \mathbb{R}^N} \langle \mathbf{z}, \mathbf{w} \rangle - (-U(\mathbf{z})) = \sup_{\mathbf{z} \in \mathbb{R}^N} \langle \mathbf{z}, \mathbf{w} \rangle + U(\mathbf{z}) = \sup_{\mathbf{z} \in \mathbb{R}^N} \langle U(\mathbf{z}) - \langle \mathbf{z}, -\mathbf{w} \rangle \rangle = \sum_{j=1}^{N} v_j(-w^j).
\]
Now we apply Corollary on page 534 of [30] with \( L = M^\Phi, L^* = L^{\Phi^*}, F(\mathbf{x}) = -U(\mathbf{x}) \) to see that
\[
\sup_{\mathbf{W} \in M^\Phi} \left( \sum_{j=1}^{N} E_p \left[ W^j(-Z^j) \right] + \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( W^j \right) \right] \right) = E_p \left[ \sum_{j=1}^{N} v_j(Z^j) \right]
\]
and replacing this in (63) we get:
\[
\alpha(\mathbf{Z}) = \sup_{\mathbf{Y} \in \mathcal{C}} \left( \sum_{j=1}^{N} E_p \left[ Y^j Z^j \right] + E_p \left[ \sum_{j=1}^{N} v_j(Z^j) \right] \right).
\]
Now observe that there are two possibilities:

- either \( \mathbf{Z} \in \mathcal{C}^0 \), and in this case \( \alpha(\mathbf{Z}) = E_p \left[ \sum_{j=1}^{N} v_j(Z^j) \right] \) since \( \mathbf{0} \in \mathcal{C} \)
• or $\alpha(Z) = +\infty$, since $v_1, \ldots, v_N$ are bounded from below.

Hence

$$
- \sup_{Y \in C} \left( \sum_{j=1}^{N} \mathbb{E} \left[ u_j \left( X^j + Y^j \right) \right] \right) = \max_{0 \leq Z \in L_v^*} \left( \sum_{j=1}^{N} E_{P} \left[ X^j (-Z^j) \right] - \alpha(Z) \right)
$$

$$
= \max_{0 \leq Z \in C^0} \left( - \left( \sum_{j=1}^{N} E_{P} \left[ X^j Z^j \right] + E_{P} \left[ \sum_{j=1}^{N} v_j(Z^j) \right] \right) \right)
$$

$$
= - \min_{0 \leq Z \in C^0} \left( \sum_{j=1}^{N} E_{P} \left[ X^j Z^j \right] + E_{P} \left[ \sum_{j=1}^{N} v_j(Z^j) \right] \right). \tag{64}
$$

Moreover, since for every $i, j = 1, \ldots, N$ $e_i - e_j \in C$ we can argue as in Lemma 4.3 to deduce that $C^0 \cap (L^0_v)^N = \mathbb{R}_+ \cdot (C_v^0)^+$. Replacing this in the expression (64) we get

$$
\sup_{Y \in C} \left( \sum_{j=1}^{N} E \left[ u_j \left( X^j + Y^j \right) \right] \right) = \min_{\lambda \in \mathbb{R}_+, Q \in (C_v^0)^+} \left( \lambda \sum_{j=1}^{N} E \left[ X^j \frac{dQ^j}{dP} \right] + \sum_{j=1}^{N} E \left[ v_j \left( \lambda \frac{dQ^j}{dP} \right) \right] \right).
$$

To prove the last claim, observe that if the optimum $\lambda$ in the right hand side was 0, we would have

$$
\sup_{Y \in C} \left( \sum_{j=1}^{N} E \left[ u_j \left( X^j + Y^j \right) \right] \right) = \sum_{j=1}^{N} E \left[ u_j(0) \right] = \sum_{j=1}^{N} u_j(+\infty),
$$

which contradicts our hypotheses. \hfill \square

**Theorem A.4.** Let $u_1, \ldots, u_N$ satisfy Assumption 4.1. Let $K \subseteq M$ be a convex cone such that for all $i, j \in \{1, \ldots, N\}$ $e_i - e_j \in K$ and suppose that $Q_v^c \neq \emptyset$, where

$$
Q_v^c := \left\{ Q \sim P \mid \frac{dQ^j}{dP} \in L_v^*, \mathbb{E} \left[ v_j \left( \frac{dQ^j}{dP} \right) \right] < +\infty, \sum_{j=1}^{N} E_{Q^j} \left[ k^j \right] \leq 0 \ \forall k \in K \right\} \subseteq L_v^*.
$$

Then denoting by $cl(Q\ldots)$ the closure in $L^1(Q^1) \times \cdots \times L^1(Q^N)$ with respect to the norm $\|X\|_Q := \sum_{j=1}^{N} \|X^j\|_{L^1(Q^j)}$ we have

$$
\bigcap_{Q \in Q_v^c} cl(Q \left( K - L^1_v(Q) \right)) = \left\{ W \in \bigcap_{Q \in Q_v^c} L^1(Q) \mid \sum_{j=1}^{N} E_{Q^j} \left[ W^j \right] \leq 0 \ \forall Q \in Q_v^c \right\}.
$$

**Proof.** We modify the procedure in [6] Theorem 4. The inclusion $(LHS \subseteq RHS)$ can be checked directly. As to the opposite one $(RHS \subseteq LHS)$, suppose we had a $k \in RHS$ and a $Q \in Q_v^c$ with $k \notin cl(Q \left( K - L^1_v(Q) \right))$, that is $k \notin LHS$. We stress that by construction

$$
\sum_{j=1}^{N} E_{Q^j} \left[ k^j \right] \leq 0 \ \forall Q \in Q_v^c. \tag{65}
$$

In the dual system

$$
(L^1(Q), L^\infty(Q))
$$

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the set $cl_Q \left( K - L^1_+ (Q) \right)$ is convex and $\sigma \left( L^1(Q), L^\infty(Q) \right)$-closed by compatibility of the latter topology with the norm topology. Thus we can use Hahn-Banach Separation Theorem to get a class $\hat{\xi} \in L^\infty(Q)$ with

$$ 0 = \sup_{\mathbf{w} \in (K - L^1_+ (Q))} \left( \sum_{j=1}^N \mathbb{E} \left[ \hat{\xi}^j W^j \frac{dQ^j}{dP} \right] \right) < \sum_{j=1}^N \mathbb{E} \left[ \hat{\xi}^j k^j \frac{dQ^j}{dP} \right]. \quad (66) $$

We now work componentwise. First observe that $\hat{\xi}^j \geq 0$ P-a.s. for every $j = 1, \ldots, N$. Hence $\hat{\xi}^j \frac{dQ^j}{dP} \geq 0$ P-a.s. for every $j = 1, \ldots, N$. Moreover, since for all $i, j \in \{1, \ldots, N\} \in \mathbf{e}_i - \mathbf{e}_j \in K$, we have

$$ \mathbb{E} \left[ \hat{\xi}^j \frac{dQ^j}{dP} \right] = \cdots = \mathbb{E} \left[ \hat{\xi}^N \frac{dQ^N}{dP} \right]. \quad (67) $$

It follows that for every $j = 1, \ldots, N$

$$ P \left( \hat{\xi}^j \frac{dQ^j}{dP} > 0 \right) > 0 $$

since if this were not the case all the terms in equation (67) would be null, which would yield $\hat{\xi}^1 \frac{dQ^1}{dP} = \cdots = \hat{\xi}^N \frac{dQ^N}{dP} = 0$, a contradiction with (66).

Hence the vector

$$ \frac{dQ^j}{dP} := \frac{1}{\mathbb{E} \left[ \hat{\xi}^j \frac{dQ^j}{dP} \right]} \hat{\xi}^j \frac{dQ^j}{dP} $$

is well defined and identifies a vector of probability measures $[Q^1, \ldots, Q^N]$. We trivially have that

$$ Q^1 \ll P, \frac{dQ^1}{dP} \in L^\infty, $$

and by equation (66), together with (67)

$$ \sup_{\mathbf{w} \in K} \left( \sum_{j=1}^N \mathbb{E} \left[ W^j \frac{dQ^j}{dP} \right] \right) \leq 0 < \sum_{j=1}^N \mathbb{E} \left[ k^j \frac{dQ^j}{dP} \right]. \quad (68) $$

We observe that if we could prove $Q_1 \in Q^c_\nu$, we would get a contradiction with (65). However this needs not to be true, since we cannot guarantee $Q^1_1, \ldots, Q^N_1 \sim P$.

As $Q \in Q^c_\nu$, we have $Q \sim P$, and for $Q_1$ above we have $Q_1 \ll Q, \frac{dQ^k}{dQ^\nu} \in L^\infty(Q^k) = L^\infty(P)$. Take $\lambda \in (0, 1]$ and define $Q_\lambda$ via

$$ \frac{dQ^k_\lambda}{dP} := \lambda \frac{dQ^k}{dP} + (1 - \lambda) \frac{dQ^k_\nu}{dP}. $$

We now prove that $Q_\lambda \in Q^c_\nu$. It is easy to check that

$$ 0 < \lambda \leq \frac{dQ^k_\lambda}{dQ^k} \leq (1 - \lambda) \frac{dQ^k}{dQ^k} + \lambda, $$

so that Lemma A.2.2 with $g = g^k := \frac{dQ^k}{dQ^\nu}$, together with $\mathbb{E} \left[ v_k \left( \frac{dQ^k}{dP} \right) \right] < +\infty \forall k = 1, \ldots, N$ ($Q \in Q^c_\nu$ by construction), yields

$$ \mathbb{E} \left[ v_k \left( \frac{dQ^k_\lambda}{dP} \right) \right] = \mathbb{E} \left[ v_k \left( \frac{dQ^k}{dQ^\nu} \frac{dQ^\nu}{dP} \right) \right] = \mathbb{E} \left[ v_k \left( g^k \frac{dQ^k}{dP} \right) \right] < +\infty, \forall k = \{1, \ldots, N\}, \lambda \in (0, 1]. $$
Moreover $Q \in Q_v^c$ and $\lambda > 0$ imply $Q^k \sim P$ for all $k = 1, \ldots, N$. This, together with equation (68), yields
\[
\sum_{j=1}^N \mathbb{E} \left[ W^j \frac{dQ^j}{dP} \right] \leq 0 \quad \forall \mathbf{W} \in K, \forall \lambda \in (0, 1).
\]
We can conclude that $Q_\lambda \in Q_v^c, \forall \lambda \in (0, 1]$. At the same time
\[
\sum_{j=1}^N \mathbb{E} \left[ k^j \frac{dQ^j}{dP} \right] = \lambda \sum_{j=1}^N \mathbb{E} \left[ k^j \frac{dQ^j}{dP} \right] + (1 - \lambda) \sum_{j=1}^N \mathbb{E} \left[ k^j \frac{dQ^j}{dP} \right] \xrightarrow{\lambda \rightarrow 0} \sum_{j=1}^N \mathbb{E} \left[ k^j \frac{dQ^j}{dP} \right] \quad \text{Eq. (68)},
\]
which gives a contradiction with Equation (65). We conclude that $\text{RHS} \subseteq \text{LHS}$. $\square$

A.3 Proof of Proposition 3.4

We recall that we adopt the following notation: $\mathcal{Q} = \{Q\}$ for some $Q \in Q_v, Q \sim P$ and $\mathcal{L} = L^1(Q^1) \times \cdots \times L^1(Q^N)$. We claim that for $Q \in \mathcal{Q}$ and $A \in \mathbb{R}$
\[
S^Q(A) = \max_{a \in \mathbb{R}^N : \sum_{n=1}^N a^n = A} \sum_{n=1}^N U^Q_n(a^n). \quad (69)
\]
First note that $\sum_{n=1}^N U^Q_n(a^n) < +\infty$ for $X \in M^a, a \in \mathbb{R}^N, Q \in \mathcal{Q}$ by Fenchel Inequality. The condition $X^n \in M^{a_n}$ implies that $U^Q_n(a^n) > -\infty$. Due to the monotonicity and concavity of $u_n$, the functions $U^Q_1(\cdot), \ldots, U^Q_N(\cdot)$ are monotone nondecreasing, concave and continuous on $\mathbb{R}$. By Lemma 4.13
\[
\Pi^Q(A) = S^Q(A) = \sup_{\sum_{n=1}^N a^n = A} \sum_{n=1}^N U^Q_n(a^n). \quad (70)
\]
As shown in Lemma 4.7 [4], for arbitrary constants $A, B \in \mathbb{R}$ the set
\[
K := \left\{ a \in \mathbb{R}^N \mid \sum_{n=1}^N a^n \leq A, \sum_{n=1}^N U^Q_n(a^n) \geq B \right\}
\]
is a compact subset of $\mathbb{R}^N$. The supremum in (70) can be taken equivalently on a compact set $K$ of the above form, for some $B$. As the functions $U^Q_1(\cdot), \ldots, U^Q_N(\cdot)$ are continuous the supremum is therefore a maximum. This proves the claim and Item 2) in Definition 3.3. Item 1) follows by classical results, see for example [9] Proposition 3.6: for every $Q \in Q_v, Q \sim P$ and $a \in \mathbb{R}^N$
\[
\sup \{ \mathbb{E}[u_n(X^n + Y)] \mid Y \in L^1(Q^n), E_{Q^n}[Y] \leq a^n \} \quad \forall n = 1, \ldots, N,
\]
admits a unique optimum.

A.4 Proofs of two claims in Section 1

1. We prove that:
\[
\sup_{Y^n} \mathbb{E} \left[ u_n(x^n + X^n + \tilde{Y}^n - E_{Q_x}[\tilde{Y}^n]) \right] = \sup_{Y^n} \{ \mathbb{E}[u_n(X^n + Y^n)] \mid E_{Q_x}[Y^n] \leq x^n \}.
\]
By a simple change of variable
\[
\sup_{\tilde{Y}^n} \mathbb{E} \left[ u_n(x^n + X^n + \tilde{Y}^n - E_{Q_X}[\tilde{Y}^n]) \right] \\
\leq \sup_{Y^n} \{ \mathbb{E}[u_n(X^n + Y^n)] \mid E_{Q_X}[Y^n] = x^n \} \\
= \sup_{Y^n} \{ \mathbb{E}[u_n(x^n + X^n + Y^n - E_{Q_X}[Y^n])] \mid E_{Q_X}[Y^n] = x^n \} \\
\leq \sup_{Y^n} \mathbb{E}[u_n(x^n + X^n + Y^n - E_{Q_X}[Y^n])].
\]

The monotonicity of \(u_n\) also implies
\[
\sup_{Y^n} \{ \mathbb{E}[u_n(X^n + Y^n)] \mid E_{Q_X}[Y^n] = x^n \} = \sup_{Y^n} \{ \mathbb{E}[u_n(X^n + Y^n)] \mid E_{Q_X}[Y^n] \leq x^n \}. 
\]

2. We prove that from (5) and (6) a solution \((\tilde{Y}^n, p^n_X, a^n_X)\) fulfills
\[
\sum_{n=1}^N p^n_X(\tilde{Y}^n_X) = 0.
\]
Assume by contradiction that \(\sum_{n=1}^N p^n_X(\tilde{Y}^n_X) = a > 0\). Then \(\hat{Y}^n_X\) defined by \(\hat{Y}^1_X := \tilde{Y}^1_X - p^1_X(\tilde{Y}^1_X) + a, \hat{Y}^n_X := \tilde{Y}^n_X - p^n_X(\tilde{Y}^n_X)\) for \(n = 2, ..., N\) fulfills \(\sum_{n=1}^N \hat{Y}^n_X = 0\). Hence, by monotonicity of the functions \(u_n\) and by cash additivity of \(p^n_X, n = 1, ..., N\), problem (5) would be strictly greater for \(\hat{Y}^n_X\) than for \(\tilde{Y}^n_X\), contradicting the assumed optimality of \(\tilde{Y}^n_X\). Similarly if \(\sum_{n=1}^N p^n_X(\tilde{Y}^n_X) = a < 0\).

References


