# Universal Arbitrage Aggregator in Discrete Time Markets under Uncertainty *


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**Abstract**

In a model independent discrete time financial market, we discuss the richness of the family of martingale measures in relation to different notions of Arbitrage, generated by a class $\mathcal{S}$ of *significant* sets, which we call Arbitrage *de la classe* $\mathcal{S}$. The choice of $\mathcal{S}$ reflects into the intrinsic properties of the class of polar sets of martingale measures. In particular: for $\mathcal{S} = \{\Omega\}$, absence of Model Independent Arbitrage is equivalent to the existence of a martingale measure; for $\mathcal{S}$ being the open sets, absence of Open Arbitrage is equivalent to the existence of full support martingale measures. These results are obtained by adopting a technical filtration enlargement and by constructing a universal aggregator of all arbitrage opportunities. We further introduce the notion of market feasibility and provide its characterization via arbitrage conditions. We conclude providing a dual representation of Open Arbitrage in terms of weakly open sets of probability measures, which highlights the robust nature of this concept.



**Keywords**: Model Uncertainty, First Fundamental Theorem of Asset Pricing, Feasible Market, Open Arbitrage, Full Support Martingale Measure.
**MSC (2010):** primary 60G42, 91B24, 91G99, 60H99; secondary 46A20, 46E27.


## 1 Introduction: No Arbitrage under Uncertainty

The introduction of Knightian Uncertainty in mathematical models for Finance has recently renewed the attention on foundational issues such as option pricing rules, super-hedging, and arbitrage conditions.

We can distinguish two extreme cases:

1. We are completely sure about the reference probability measure $P$. In this case, the classical notion of No Arbitrage or NFLVR can be successfully applied (as in [DMW90, DS94, DS98]).


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2. We face complete uncertainty about any probabilistic model and therefore we must describe our model independently by any probability. In this case we might adopt a model independent (weak) notion of No Arbitrage

In the second case, a pioneering contribution was given in the paper by Hobson [Ho98] where the problem of pricing exotic options is tackled under model mis-specification. In his approach the key assumption is the existence of a martingale measure for the market, consistent with the prices of some observed vanilla options (see also [BHR01, CO11, DOR14] for further developments). In [DH07], Davis and Hobson relate the previous problem to the absence of Model Independent Arbitrages, by the mean of semi-static strategies. A step forward towards a model-free version of the First Fundamental Theorem of Asset Pricing in discrete time was formerly achieved by Riedel [Ri11] in a one period market and by Acciao at al. [AB13] in a more general setup.

Between cases 1. and 2., there is the possibility to accept that the model could be described in a probabilistic setting, but we cannot assume the knowledge of a specific reference probability measure but at most of a set of priors, which leads to the new theory of Quasi-sure Stochastic Analysis as in [BK12, C12, DHP11, DM06, Pe10, STZ11, STZ11a].

The idea is that the classical probability theory can be reformulated as far as the single reference probability $P$ is replaced by a class of (possibly non-dominated) probability measures $\mathcal{P}'$. This is the case, for example, of uncertain volatility (e.g. [STZ11a]) where, in a general continuous time market model, the volatility is only known to lie in a certain interval $[\sigma_m, \sigma_M]$.
In the theory of arbitrage for non-dominated sets of priors, important results were provided by Bouchard and Nutz [BN13] in discrete time. A suitable notion of arbitrage opportunity with respect to a class $\mathcal{P}'$, named $NA(\mathcal{P}')$, was introduced and it was shown that the no arbitrage condition is equivalent to the existence of a family $\mathcal{Q}'$ of martingale measure having the same polar sets of $\mathcal{P}'$. In continuous time markets, a similar topic has been recently investigated also by Biagini et al. [BBKN14].

Bouchard and Nutz [BN13] answer the following question: which is a good notion of arbitrage opportunity for all **admissible** probabilistic models $P \in \mathcal{P}'$ (i.e. one single $H$ that works as an arbitrage for all admissible models) ? To pose this question one has to know **a priori** which are the admissible models, i.e. we have to exhibit a subset of probabilities $\mathcal{P}'$.

*In this paper our aim is to investigate arbitrage conditions and robustness properties of markets that are described independently of any reference probability or set of priors.*

We consider a financial market described by a discrete time adapted stochastic process $S := (S_t)_{t\in I}$, $I = \{0, \dots, T\}$, defined on $(\Omega, \mathcal{F}, \mathbb{F})$, $\mathbb{F} := (\mathcal{F}_t)_{t\in I}$, with $T < \infty$ and taking values in $\mathbb{R}^d$ (see Section 2). Note we are not imposing any restriction on $S$ so that it may describe generic financial securities (for examples, stocks and/or options). Differently from previous approaches in literature, in our setting the measurable space $(\Omega, \mathcal{F})$ and the price process $S$ defined on it are given, and we investigate the properties of martingale measures for $S$ induced by no arbitrage conditions. The class $\mathcal{H}$ of admissible trading strategies is formed by all $\mathbb{F}$-predictable $d$-dimensional stochastic processes and we denote with $\mathcal{M}$ the set of all probability measures under which $S$ is an $\mathbb{F}$-martingale and with $\mathcal{P}$ the set of all probability measures on $(\Omega, \mathcal{F})$. We introduce therefore a flexible definition of Arbitrage which allows us to characterize the richness of the set $\mathcal{M}$ in a unified framework.

**Arbitrage *de la classe* $\mathcal{S}$.** We look for a single strategy $H$ in $\mathcal{H}$ which represents an Arbitrage opportunity in some appropriate sense. Let:

$$\mathcal{V}_H^+ = \{\omega \in \Omega \mid \ V_T(H)(\omega) > 0\},$$

where $V_T(H) = \sum_{t=1}^T H_t \cdot (S_t - S_{t-1})$ is the final value of the strategy $H$. It is natural to introduce several notion of Arbitrage accordingly to the properties of the set $\mathcal{V}_H^+$.

**Definition 1** *Let $\mathcal{S}$ be a class of measurable subsets of $\Omega$ such that $\varnothing \notin \mathcal{S}$. A trading strategy $H \in \mathcal{H}$ is an Arbitrage de la classe $\mathcal{S}$ if*

- *$V_0(H) = 0$, $V_T(H)(\omega) \geq 0\ \forall \omega \in \Omega$ and $\mathcal{V}_H^+$ contains a set in $\mathcal{S}$.*

The class $\mathcal{S}$ has the role to translate mathematically the meaning of a "true gain". When a probability $P$ is given (the "reference probability") then we agree on representing a true gain as $P(V_T(H) > 0) > 0$ and therefore the classical no arbitrage condition can be expressed as: no losses $P(V_T(H) < 0) = 0$ implies no true gain $P(V_T(H) > 0) = 0$. In a similar fashion, when a subset $\mathcal{P}'$ of probability measures is given, one may replace the $P$-a.s. conditions above with $\mathcal{P}$-q.s conditions, as in [BN13]. However, if we can not or do not want to rely on a priory assigned set of probability measures, we may well use another concept: there is a true gain if the set $\mathcal{V}_H^+$ contains a set considered *significant*. This is exactly the role attributed to the class $\mathcal{S}$ which is the core of Section 3. Families of sets, not determined by some probability measures, have been already used in the context of the first and second fundamental theorem of asset pricing respectively by Battig Jarrow [BJ99] and Cassese [C08] (see Section 4.1 for a more specific comparison).

In order to investigate the properties of the martingale measures induced by No Arbitrage conditions of this kind we first study (see Section 4) the structural properties of the market adopting a geometrical approach in the spirit of [Pl97] but with $\Omega$ being a general Polish space, instead of a finite sample space. In particular, we characterize the class $\mathcal{N}$ of the $\mathcal{M}$-polar sets i.e. those $B \subset \Omega$ such that there is no martingale measure that can assign a positive measure to $B$. In the model independent framework the set $\mathcal{N}$ is induced by the market since the set of martingale measure has not to withstand to any additional condition (such as being equivalent to a certain $P$). Once these polar sets are identified we explicitly build in Section 4.6 a process $H^\bullet$ which depends only on the price process $S$ and satisfies:

- $V_T(H^\bullet)(\omega) \geq 0\ \forall \omega \in \Omega$
- $N \subseteq \mathcal{V}_{H^\bullet}^+$ for every $N \in \mathcal{N}$.

This strategy is a measurable selection of a set valued process $\mathbb{H}$, that we baptize **Universal Arbitrage Aggregator** since for any $P$, which is not absolutely continuous with respect to $\mathcal{M}$, an arbitrage opportunity $H^P$ (in the classical sense) can be found among the values of $\mathbb{H}$. All the inefficiencies of the market are captured by the process $H^\bullet$ but, in general, it fails to be $\mathbb{F}$-predictable. To recover predictability we need to enlarge the natural filtration of the process $S$ by considering a suitable technical filtration $\widetilde{\mathbb{F}} := \{\widetilde{\mathcal{F}}_t\}_{t \in I}$ which does not affect the set of martingale measures, i.e. any martingale measure $Q \in \mathcal{M}$ can be uniquely extended to a martingale measure $\widetilde{Q}$ on the enlarged filtration.
This allows us to prove, in Section 4.6, the main result of the paper:

**Theorem 2** *Let $(\Omega, \widetilde{\mathcal{F}}_T, \widetilde{\mathbb{F}})$ be the enlarged filtered space as in Section 4.5 and let $\widetilde{\mathcal{H}}$ be the set of d-dimensional discrete time $\widetilde{\mathbb{F}}$-predictable stochastic process. Then*

$$\textit{No Arbitrage de la classe } \mathcal{S} \textit{ in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M} \neq \varnothing \textit{ and } \mathcal{N} \textit{ does not contain sets of } \mathcal{S}$$

In other words, properties of the family $\mathcal{S}$ have a dual counterpart in terms of polar sets of the pricing functional.

In Section 4.6 we further provide our version of the Fundamental Theorem of Asset Pricing: the equivalence between absence of Arbitrage de la classe $\mathcal{S}$ in $\widetilde{\mathcal{H}}$ and the existence of martingale measures $Q \in \mathcal{M}$ with the property that $Q(C) > 0$ for all $C \in \mathcal{S}$.

**Model Independent Arbitrage.** When $\mathcal{S} := \{\Omega\}$ then the Arbitrage de la classe $\mathcal{S}$ corresponds to the notion of a Model Independent Arbitrage. As $\Omega$ never belongs to the class of polar sets $\mathcal{N}$, from Theorem 2 we directly obtain the following result.

**Theorem 3**

$$\textit{No Model Independent Arbitrage in } \widetilde{\mathcal{H}} \iff \mathcal{M} \neq \varnothing.$$

An analogous result has been obtained in [AB13] when considering a single risky asset $S$ as the canonical process on the path space $\Omega = \mathbb{R}_+^T$, a possibly uncountable collection of options $(\varphi_\alpha)_{\alpha \in A}$ whose prices are known at time 0, and when trading is possible through semi-static strategies (see also [Ho11] for a detailed discussion). Assuming the existence of an option $\varphi_0$ with a specific payoff, equivalence in Theorem 3 is achieved in the original measurable space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{H})$. In our setup, although we are free to choose a $(d+k)$-dimensional process $S$ for modeling a finite number of options $(k)$ on possibly different underlying $(d)$, the class $\widetilde{\mathcal{H}}$ of admissible strategies are dynamic in every $S^i$ for $i = 1, \dots d+k$. In order to incorporate the case of semi-static strategies we would need to consider restrictions on $\widetilde{\mathcal{H}}$ and for this reason the two results are not directly comparable.

**Arbitrage with respect to open sets.** In the topological context, in order to obtain full support martingale measures, the suitable choice for $\mathcal{S}$ is the class of open sets. This selection determines the notion of Arbitrage with respect to open sets, which we shorten as "Open Arbitrage":

- **Open Arbitrage** is a trading strategy $H \in \mathcal{H}$ such that $V_0(H) = 0$, $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$ and $\mathcal{V}_H^+$ contains an open set.

This concept admits the following dual reformulation (see Section 6, Proposition 64).

$$\begin{array}{l}\text{An Open Arbitrage consists in a trading strategy } H \in \mathcal{H} \text{ and a non empty} \\ \text{weakly open set } \mathcal{U} \subseteq \mathcal{P} \text{ such that for all } P \in \mathcal{U},\ V_T(H) \geq 0\ P\text{-a.s. and } P(\mathcal{V}_H^+) > 0.\end{array} \quad (1)$$

The robust feature of an open arbitrage is therefore evident from this dual formulation, as a certain strategy $H$ satisfies (1) if it represents an arbitrage in the classical sense for a whole open set of probabilities. In addition, if $H$ is such strategy and we disregard any finite subset of probabilities then $H$ remains an Open Arbitrage. Moreover every weakly open subset of $\mathcal{U}$ contains a full support probability $P$ (see Lemma 57) under which $H$ is a $P$-Arbitrage in the classical sense. Full support martingale measures can be efficiently used whenever we face model mis-specification, since they have a well spread support that captures the features of the sample space of events without neglecting significantly large parts. In Dolinski and Soner [DS14] the equivalence of a local version of NA and the existence of full support martingale measures has been proved (see Section 2.5, [DS14]) in a continuous time market determined by one risky asset with proportional transaction costs.

**Feasibility and approximating measures.** In Section 5 we answer the question: which are the markets that are feasible in the sense that the properties of the market are nice for "**most**" probabilistic models? Clearly this problem depends on the choice of the feasibility criterion, but to this aim we do not need to exhibit a priori a subset of probabilities. On the opposite, given a market (described without reference probability), the induced set of No Arbitrage models (probabilities) for that market will determine if the market itself is feasible or not. What is needed here is a good notion of "**most**" probabilistic models.
More precisely given the price process $S$ defined on $(\Omega, \mathcal{F})$, we introduce the set $\mathcal{P}_0$ of probability measures that exhibit No Arbitrage in the classical sense:

$$\mathcal{P}_0 = \{P \in \mathcal{P} \mid \text{No Arbitrage with respect to } P\}. \tag{2}$$

When

$$\overline{\mathcal{P}_0}^{\tau} = \mathcal{P}$$

with respect to some topology $\tau$ the market is feasible in the sense that any "bad" reference probability can be approximated by No Arbitrage probability models. We show in Proposition 58 that this property is equivalent to the existence of a full support martingale measure if we choose $\tau$ as the weak* topology.

One other contribution of the paper, proved in Section 5, is the following characterization of feasible markets and absence of Open Arbitrage in terms of existence of full support martingale measures. We denote with $\mathcal{P}^+ \subset \mathcal{P}$ the set of full support probability measures.

**Theorem 4** *The following are equivalent:*

1. *The market is feasible, i.e* $\overline{\mathcal{P}_0}^{\sigma(\mathcal{P}, C_b)} = \mathcal{P}$*;*
2. *There exists* $P \in \mathcal{P}_+$ *s.t. No Arbitrage w.r.to* $P$ *(in the classical sense) holds true;*
3. $\mathcal{M} \cap \mathcal{P}_+ \neq \varnothing$*;*
4. *No Open Arbitrage holds with respect to admissible strategies* $\widetilde{\mathcal{H}}$*.*

Riedel [Ri11] already pointed out the relevance of the concept of full support martingale measures in a probability-free set up. Indeed in a one period market model and under the assumption that the price process is continuous with respect to the state variable, he showed that the absence of a one point arbitrage (non-negative payoff, with strict positivity in at least one point) is equivalent to the existence of a full support martingale measure. As shown in Section 6.1, this equivalence is no longer true in a multiperiod model (or in a single period model with non trivial initial sigma algebra), even for price processes continuous in $\omega$. In this paper we consider a multi-assets multi-period model without $\omega$-continuity assumptions on the price processes and we develop the concept of open arbitrage, as well as its dual reformulation, that allows for the equivalence stated in the above theorem.

Finally, we present a number of simple examples that point out: the differences between single period and multi-period models (examples 13, 66, 67); the geometric approach to absence of arbitrage and existence of martingale measures (Section 4.1); the need in the multi-period setting of the disintegration of the atoms (example 26); the need of the one period anticipation of some polar sets (example 32). The consequences of our version of the FTAP for the robust formulation of the superhedging duality will be analyzed in a forthcoming paper.

# 2 Financial Markets

We will assume that $(\Omega, d)$ is a Polish space and $\mathcal{F} = \mathcal{B}(\Omega)$ is the Borel sigma algebra induced by the metric $d$. The requirement that $\Omega$ is Polish is used in Section 4.3 to guarantee the existence of a proper regular conditional probability, see Theorem 28. We fix a finite time horizon $T \geq 1$, a finite set of time indices $I := \{0, \dots, T\}$ and we set: $I_1 := \{1, \dots, T\}$. Let $\mathbb{F} := \{\mathcal{F}_t\}_{t \in I}$ be a filtration with $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_T \subseteq \mathcal{F}$. We denote with $\mathcal{L}(\Omega, \mathcal{F}_t; \mathbb{R}^d)$ the set of $\mathcal{F}_t$-measurable random variables $X : \Omega \to \mathbb{R}^d$ and with $\mathcal{L}(\Omega, \mathbb{F}; \mathbb{R}^d)$ the set of adapted processes $X = (X_t)_{t \in I}$ with $X_t \in \mathcal{L}(\Omega, \mathcal{F}_t; \mathbb{R}^d)$.

The market consists of one non-risky asset $S_t^0 = 1$ for all $t \in I$, constantly equal to 1, and $d \geq 1$ risky assets $S^j = (S_t^j)_{t \in I}$ , $j = 1, \dots, d$, that are real-valued adapted stochastic processes. Let $S = [S^1, \dots, S^d] \in \mathcal{L}(\Omega, \mathbb{F}; \mathbb{R}^d)$ be the $d$-dimensional vector of the (discounted) price processes. In this paper we focus on arbitrage conditions, and therefore without loss of generality we will restrict our attention to self-financing trading strategies of zero initial cost. Therefore, we may assume that a trading strategy $H = (H_t)_{t \in I_1}$ is a $\mathbb{R}^d$-valued predictable stochastic process: $H = [H^1, \dots, H^d]$, with $H_t \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d)$, and we denote with $\mathcal{H}$ the class of all trading strategies. The (discounted) value process $V(H) = (V_t(H))_{t \in I}$ is defined by:

$$V_0(H) := 0, \quad V_t(H) := \sum_{i=1}^{t} H_i \cdot (S_i - S_{i-1}), \quad t \geq 1.$$

A (discrete time) financial market is therefore assigned, without any reference probability measure, by the quadruple $[(\Omega, d); (\mathcal{B}(\Omega), \mathbb{F}); S; \mathcal{H}]$ satisfying the previous conditions.

**Notation 5** *For $\mathcal{F}$-measurable random variables $X$ and $Y$, we write $X > Y$ (resp. $X \geq Y$, $X = Y$) if $X(\omega) > Y(\omega)$ for all $\omega \in \Omega$ (resp. $X(\omega) \geq Y(\omega)$, $X(\omega) = Y(\omega)$ for all $\omega \in \Omega$).*

## 2.1 Probability and martingale measures

Let $\mathcal{P} := \mathcal{P}(\Omega)$ be the set of all probabilities on $(\Omega, \mathcal{F})$ and $C_b := C_b(\Omega)$ the space of continuous and bounded functions on $\Omega$. Except when explicitly stated, we endow $\mathcal{P}$ with the weak* topology $\sigma(\mathcal{P}, C_b)$, so that $(\mathcal{P}, \sigma(\mathcal{P}, C_b))$ is a Polish space (see [AB06] Chapter 15 for further details). The convergence of $P_n$ to $P$ in the topology $\sigma(\mathcal{P}, C_b)$ will be denoted by $P_n \overset{w}{\to} P$ and the $\sigma(\mathcal{P}, C_b)$-closure of a set $\mathcal{Q} \subseteq \mathcal{P}$ will be denoted with $\overline{\mathcal{Q}}$.

We define the *support* of an element $P \in \mathcal{P}$ as

$$supp(P) = \bigcap \{C \in \mathcal{C} \mid P(C) = 1\}$$

where $\mathcal{C}$ are the closed sets in $(\Omega, d)$. Under our assumptions the support is given by

$$supp(P) = \{\omega \in \Omega \mid P(B_\varepsilon(\omega)) > 0 \text{ for all } \varepsilon > 0\},$$

where $B_\varepsilon(\omega)$ is the open ball with radius $\varepsilon$ centered in $\omega$.

**Definition 6** *We say that $P \in \mathcal{P}$ has full support if $supp(P) = \Omega$ and we denote with*

$$\mathcal{P}_+ := \{P \in \mathcal{P} \mid supp(P) = \Omega\}$$

*the set of all probability measures having full support.*

Observe that $P \in \mathcal{P}_+$ if and only if $P(A) > 0$ for every open set $A$. Full support measures are therefore important, from a topological point of view, since they assign positive probability to all open sets.

**Definition 7** *The set of* $\mathbb{F}$*-martingale measures is*

$$\mathcal{M}(\mathbb{F}) = \{Q \in \mathcal{P} \mid S \text{ is a } (Q, \mathbb{F})\text{-martingale}\}. \tag{3}$$

*and we set:* $\mathcal{M} := \mathcal{M}(\mathbb{F})$, *when the filtration is not ambiguous, and*

$$\mathcal{M}_+ = \mathcal{M} \cap \mathcal{P}_+.$$

**Definition 8** *Let* $P \in \mathcal{P}$ *and* $\mathcal{G} \subseteq \mathcal{F}$ *be a sub-sigma algebra of* $\mathcal{F}$. *The generalized conditional expectation of a non negative* $X \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ *is defined by:*

$$E_P[X \mid \mathcal{G}] := \lim_{n \to +\infty} E_P[X \wedge n \mid \mathcal{G}],$$

*and for* $X \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{R})$ *we set* $E_P[X \mid \mathcal{G}] := E_P[X^+ \mid \mathcal{G}] - E_P[X^- \mid \mathcal{G}]$, *where we adopt the convention* $\infty - \infty = -\infty$. *All basic properties of the conditional expectation still hold true (see for example [FKV09]). In particular if* $Q \in \mathcal{M}$ *and* $H \in \mathcal{H}$ *then* $E_Q[H_t \cdot (S_t - S_{t-1}) \mid \mathcal{F}_{t-1}] = H_t \cdot E_Q[(S_t - S_{t-1}) \mid \mathcal{F}_{t-1}] = 0$ *Q-a.s., so that* $E_Q[V_T(H)] = 0$ *Q-a.s.*

# 3 Arbitrage de la classe $\mathcal{S}$

Let $H \in \mathcal{H}$ and recall that $\mathcal{V}_H^+ := \{\omega \in \Omega \mid V_T(H)(\omega) > 0\}$ and that $V_0(H) = 0$.

**Definition 9** *Let* $P \in \mathcal{P}$. *A P-**Classical Arbitrage** is a trading strategy* $H \in \mathcal{H}$ *such that:*

- $V_T(H) \geq 0$ $P-$*a.s., and* $P(\mathcal{V}_H^+) > 0$

*We denote with* $NA(P)$ *the absence of P-Classical Arbitrage.*

Recall the definition of Arbitrage de la classe $\mathcal{S}$ stated in the Introduction.

**Definition 10** *Some examples of Arbitrage de la classe* $\mathcal{S}$:

1. *H is a 1p-Arbitrage when* $\mathcal{S} = \{C \in \mathcal{F} \mid C \neq \varnothing\}$. *This is the weakest notion of arbitrage since* $\mathcal{V}_H^+$ *might reduce to a single point. The 1p-Arbitrage corresponds to the definition given in [Ri11]. This can be easily generalized to the following notion of n point Arbitrage: H is an np-Arbitrage when*

$$\mathcal{S} = \{C \in \mathcal{F} \mid C \text{ has at least } n \text{ elements}\},$$

   *and might be significant for* $\Omega$ *(at most) countable.*

2. *H is an Open Arbitrage when* $\mathcal{S} = \{C \in \mathcal{B}(\Omega) \mid C \text{ open non-empty}\}$.

3. *H is a* $\mathcal{P}'$*-q.s. Arbitrage when* $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0 \text{ for some } P \in \mathcal{P}'\}$, *for a fixed family* $\mathcal{P}' \subseteq \mathcal{P}$. *Notice that* $\mathcal{S} = (\mathcal{N}(\mathcal{P}'))^c$, *the complements of the polar sets of* $\mathcal{P}'$. *Then there are No* $\mathcal{P}'$*-q.s. Arbitrage if:*

$$H \in \mathcal{H} \text{ such that } V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega \Rightarrow V_T(H) = 0 \ \mathcal{P}'\text{-q.s.}$$

   *This definition is similar to the No Arbitrage condition in [BN13], the only difference being that here we require* $V_T(H)(\omega) \geq 0$ $\forall \omega \in \Omega$, *while in the cited reference it is only required* $V_T(H) \geq 0$ $\mathcal{P}'$*-q.s.. Hence No* $\mathcal{P}'$*-q.s. Arbitrage is a condition weaker than No Arbitrage in [BN13].*

4. *H is a P-a.s. Arbitrage when* $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0\}$ *for a fixed* $P \in \mathcal{P}$*. As in the previous example the No P-a.s. Arbitrage is a weaker condition than the No P-Classical Arbitrage condition, the only difference being that here we require* $V_T(H)(\omega) \geq 0\ \forall \omega \in \Omega$*, while in the classical definition it is only required* $V_T(H) \geq 0$ *P-a.s.*

5. *H is a Model Independent Arbitrage when* $\mathcal{S} = \{\Omega\}$*, in the spirit of [AB13, DH07, CO11].*

6. *H is an* $\varepsilon$*-Arbitrage when* $\mathcal{S} = \{C_\varepsilon(\omega) \mid \omega \in \Omega\}$*, where* $\varepsilon > 0$ *is fixed and* $C_\varepsilon(\omega)$ *is the closed ball in* $(\Omega, d)$ *of radius* $\varepsilon$ *and centered in* $\omega$*.*

Obviously, for any class $\mathcal{S}$,

$$\text{No } 1p\text{-Arbitrage} \Rightarrow \text{No Arbitrage de la classe } \mathcal{S} \Rightarrow \text{No Model Ind. Arbitrage} \tag{4}$$

and these notions depend only on the properties of the financial market and are not necessarily related to any probabilistic models.

**Remark 11** *The No Arbitrage concepts defined above, as well as the possible generalization of No Free Lunch de la classe* $\mathcal{S}$*, can be considered also in more general, continuous time, financial market models. We choose to present our theory in the discrete time framework, as the subsequent results in the next sections will rely crucially on the discrete time setting.*

**Example 12** *The flexibility of our approach relies on the arbitrary choice of the class* $\mathcal{S}$*. Consider* $\Omega = C^0([0,T];\mathbb{R})$ *which is a Polish space once endowed with the supremum norm* $\|\cdot\|_\infty$*. We may consider two classes*

$$\mathcal{S}^\infty = \{ \text{ open balls in } \|\cdot\|_\infty\} \quad \text{and} \quad \mathcal{S}^1 = \{ \text{ open balls in } \|\cdot\|_1\}$$

*where* $\|\omega\|_1 = \int_0^T |\omega(t)|dt$*. Notice that since the integral operator* $\int_0^T |\cdot|dt : C^0([0,T];\mathbb{R}) \to \mathbb{R}$ *is* $\|\cdot\|_\infty$*-continuous every open ball in* $\|\cdot\|_1$ *is also open in* $\|\cdot\|_\infty$*. Hence every Arbitrage de la classe* $\mathcal{S}^1$ *is also an Arbitrage de la classe* $\mathcal{S}^\infty$ *but not the converse.*
*For instance consider a market described by an underlying process* $S^1$ *and a digital option* $S^2$*, where trading is allowed only in a set of finite times* $\{0, 1, ..., T-1\}$*. Define* $S^1_0(\omega) = s_0$ *for every* $\omega \in \Omega$ *and* $S^1_t(\omega) = \omega(t)$ *for the underlying and* $S^2_t(\omega) = \mathbf{1}_B(\omega)\mathbf{1}_T(t)$ *for the option where* $B := \{\omega \mid S^1_t(\omega) \in (s_0 - \varepsilon, s_0 + \varepsilon)\ \forall t \in [0,T]\}$*. A long position in the option at time* $T-1$ *is an arbitrage de la classe* $\mathcal{S}^\infty$ *even though there does not exist any arbitrage de la classe* $\mathcal{S}^1$*.*

## 3.1 Defragmentation

When the reference probability $P \in \mathcal{P}$ is fixed, the market admits a $P$-Classical Arbitrage if and only if there exists a time $t \in \{1, \ldots, T\}$ and a random vector $\eta \in L^0(\Omega, \mathcal{F}_{t-1}, P; \mathbb{R}^d)$ such that $\eta \cdot (S_t - S_{t-1}) \geq 0$ $P$-a.s. and $P(\eta \cdot (S_t - S_{t-1}) > 0) > 0$ (see [DMW90] or [FS04], Proposition 5.11). In our context the existence of an Arbitrage de la classe $\mathcal{S}$, over a certain time interval $[0,T]$, does not necessarily imply the existence of a single time step where the arbitrage is realized on a set in $\mathcal{S}$. It might happen, instead, that the agent needs to implement a strategy over multiple time steps to achieve an arbitrage de la classe $\mathcal{S}$. The following example shows exactly a simple case in which this phenomenon occurs. Recall that $\mathcal{L}(\Omega, \mathcal{F}; \mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued $\mathcal{F}$-measurable random variables on $\Omega$.

**Example 13** *Consider a 2 periods market model composed by two risky assets* $S^1, S^2$ *on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *which are described by the following trajectories*

$$
S^1 : \begin{array}{ccccc}
 & & 3 \to & 3 & \omega \in A_1 \\
 & \nearrow & & 5 & \omega \in A_2 \\
 & & & \nearrow & \\
2 \to & & 2 & & \\
 & & & \searrow & \\
 & \searrow & & 1 & \omega \in A_3 \\
 & & 1 \to & 1 & \omega \in A_4
\end{array}
\qquad
S^2 : \begin{array}{ccccc}
 & & 7 \to & 7 & \omega \in A_1 \\
 & \nearrow & & 3 & \omega \in A_2 \\
 & & & \nearrow & \\
2 \to & & 2 & & \\
 & & & \searrow & \\
 & \searrow & & 1 & \omega \in A_3 \\
 & & 1 \to & 1 & \omega \in A_4
\end{array}
$$

*Consider* $H_1 = (-1, +1)$ *and* $H_2 = (\mathbf{1}_{A_2 \cup A_3}, -\mathbf{1}_{A_2 \cup A_3})$. *Then* $H_1 \cdot (S_1 - S_0) = 4\mathbf{1}_{A_1}$ *and* $H_2 \cdot (S_2 - S_1) = 2\mathbf{1}_{A_2}$. *Choosing* $A_1 = \mathbb{Q} \cap (0,1), A_2 = (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1)$ *and* $A_3 = [1, +\infty), A_4 = (-\infty, 0]$ *we observe that an Open Arbitrage can be obtained only by a two step strategy, while in each step we have* only $1p$*-Arbitrages.*
*In general the multi step strategy realizes the Arbitrage de la classe* $\mathcal{S}$ *at time* $T$ *even though it does not yield necessarily a positive gain at each time: i.e. there might exist a* $t < T$ *such that* $\{V_t(H) < 0\} \neq \varnothing$. *This is the case of Example 32.*

In the remaining of this section $\Delta S_t = [S_t^1 - S_{t-1}^1, \ldots, S_t^d - S_{t-1}^d]$.

**Lemma 14** *The strategy* $H \in \mathcal{H}$ *is a* $1p$*-Arbitrage if and only if there exists a time* $t \in I_1$*, an* $\alpha \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d)$ *and a non empty* $A \in \mathcal{F}_t$ *such that*

$$
\begin{array}{ll}
\alpha(\omega) \cdot \Delta S_t(\omega) \geq 0 & \forall\, \omega \in \Omega \\
\alpha(\omega) \cdot \Delta S_t(\omega) > 0 & \text{on } A.
\end{array} \tag{5}
$$

**Proof.** $(\Rightarrow)$ Let $H \in \mathcal{H}$ be a $1p$-Arbitrage. Set

$$\bar{t} = \min\{t \in \{1, \ldots, T\} \mid V_t(H) \geq 0 \text{ with } V_t(H)(\omega) > 0 \text{ for some } \omega \in \Omega\}.$$

If $\bar{t} = 1$, $\alpha = H_1$ satisfies the requirements. If $\bar{t} > 1$, by definition, $\{V_{\bar{t}-1}(H) < 0\} \neq \varnothing$ or $\{V_{\bar{t}-1}(H) = 0\} = \Omega$ . In the first case, for $\alpha = H_{\bar{t}} \mathbf{1}_{\{V_{\bar{t}-1}(H) < 0\}}$ we have $\alpha \cdot \Delta S_{\bar{t}} \geq 0$ with strict inequality on $\{V_{\bar{t}-1}(H) < 0\}$. In the latter case $\alpha = H_{\bar{t}}$ satisfies the requirements.
$(\Leftarrow)$ Take $\alpha \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d)$ as by assumption and define $H \in \mathcal{H}$ by $H_s = 0$ for every $s \neq t$ and $H_t = \alpha$. Hence $V_T(H)(\omega) = V_t(H)(\omega)$ for every $\omega \in \Omega$ so that $V_T(H) \geq 0$ and $\{\omega \in \Omega \mid V_T(H)(\omega) > 0\} = \{\omega \in \Omega \mid \alpha \cdot \Delta S_t(\omega) > 0\}$ which ends the proof. ∎

**Remark 15** *Notice that only the implication* $(\Leftarrow)$ *of the previous Lemma holds true for Open Arbitrage. This means that there exists an Open Arbitrage if we can find a time* $t \in I_1$*, an* $\alpha \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d)$ *and a set* $A \in \mathcal{F}_t$ *containing an open set such that (5) holds true. Similarly for Arbitrage de la classe* $\mathcal{S}$. *On the other hand the converse is false in general as shown by Example 13.*

The following Lemma provides a full characterization of Arbitrages de la classe $\mathcal{S}$ by the mean of a multi-step decomposition of the strategy.

**Lemma 16 (Defragmentation)** *The strategy* $H \in \mathcal{H}$ *is an Arbitrage de la classe* $\mathcal{S}$ *if and only if there exists:*

- *a finite family* $\{U_t\}_{t \in I}$ *with* $U_t \in \mathcal{F}_t$, $U_t \cap U_s = \varnothing$ *for every* $t \neq s$ *and* $\bigcup_{t \in I} U_t$ *contains a set in* $\mathcal{S}$*;*

- *a strategy* $\widehat{H} \in \mathcal{H}$ *such that* $V_T(\widehat{H}) \geq 0$ *on* $\Omega$, *and* $\widehat{H}_t \cdot \Delta S_t > 0$ *on* $U_t$ *for any* $U_t \neq \varnothing$.

**Proof.** ($\Rightarrow$) Let $H \in \mathcal{H}$ be an Arbitrage de la classe $\mathcal{S}$. Define $B_t = \{V_t(H) > 0\}$ and

$$\begin{aligned}
U_1 = B_1 &\quad\Rightarrow\quad H_1 \cdot \Delta S_1(\omega) > 0 \quad \forall\, \omega \in U_1 \\
U_2 = B_1^c \cap B_2 &\quad\Rightarrow\quad H_2 \cdot \Delta S_2(\omega) > 0 \quad \forall\, \omega \in U_2 \\
U_{T-1} = B_1^c \cap \ldots \cap B_{T-2}^c \cap B_{T-1} &\quad\Rightarrow\quad H_{T-1} \cdot \Delta S_{T-1}(\omega) > 0 \quad \forall\, \omega \in U_{T-1} \\
U_T = B_1^c \cap \ldots \cap B_{T-2}^c \cap B_{T-1}^c \cap \mathcal{V}_H^+ &\quad\Rightarrow\quad H_T \cdot \Delta S_T(\omega) > 0 \quad \forall\, \omega \in U_T
\end{aligned}$$

From the definition of $\{U_1, U_2, \ldots, U_T\}$ we have that $\mathcal{V}_H^+ \subseteq \bigcup_{i=1}^T U_i$. Set $\widehat{H}_1 = H_1$ and consider the strategy for every $2 \leq t \leq T$ given by

$$\widehat{H}_t(\omega) = H_t(\omega)\mathbf{1}_{D_{t-1}}(\omega) \quad \text{where } D_{t-1} = \left(\bigcup_{s=1}^{t-1} U_s\right)^c.$$

By construction $\widehat{H} \in \mathcal{H}$ and $\widehat{H}_t \cdot \Delta S_t(\omega) > 0$ for every $\omega \in U_t$.
($\Leftarrow$) The converse implication is trivial. ■

# 4 Arbitrage de la classe $\mathcal{S}$ and Martingale Measures

Before addressing this topic in its full generality we provide some insights into the problem and we introduce some examples that will help to develop the intuition on the approach that we adopt. The required technical tools will then be stated in Sections 4.2 and 4.3.

Consider the family of polar sets of $\mathcal{M}$

$$\mathcal{N} := \{A \subseteq A' \in \mathcal{F} \;\mid\; Q(A') = 0 \;\forall\; Q \in \mathcal{M}\}.$$

In Nutz and Bouchard [BN13] the notion of $NA(\mathcal{P}')$ for any fixed family $\mathcal{P}' \subseteq \mathcal{P}$ is defined by:

$$V_T(H) \geq 0 \;\mathcal{P}' - q.s. \Rightarrow V_T(H) = 0 \;\mathcal{P}' - q.s.$$

where $H$ is a predictable process which is measurable with respect to the universal completion of $\mathbb{F}$. One of the main results in [BN13] asserts that, under $NA(\mathcal{P}')$, there exists a class $\mathcal{Q}'$ of martingale measures which shares the same polar sets of $\mathcal{P}'$. If we take $\mathcal{P}' = \mathcal{P}$ then $NA(\mathcal{P})$ is equivalent to No (universally measurable) $1p$-Arbitrage, since $\mathcal{P}$ contains all Dirac measures. In addition, the class of polar sets of $\mathcal{P}$ is empty. In Section 4.4 we will show that this same result is true also in our setting as a consequence of Proposition 34. The existence of a class of martingale measures with no polar sets implies that $\forall \omega \in \Omega$ there exists $Q \in \mathcal{M}$ such that $Q(\{\omega\}) > 0$ and since $\Omega$ is a separable space we can find a dense set $D := \{\omega_n\}_{n=1}^\infty$, with associated $Q^n \in \mathcal{M}$, such that $\sum_{n=1}^\infty \frac{1}{2^n} Q^n$ is a full support martingale measure (see Lemma 76).

**Proposition 17** *We have the following implications*

1. *No* $1p$*-Arbitrage* $\Longrightarrow \mathcal{M}_+ \neq \emptyset$.

2. $\mathcal{M}_+ \neq \emptyset \Longrightarrow$ *No Open Arbitrage.*

**Proof.** The proof of 1. is postponed to Section 4.4.
We prove 2. by observing that for any open set $O$ and $Q \in \mathcal{M}_+$ we have $Q(O) > 0$. Since for any $H \in \mathcal{H}$ such that $V_T(H) \geq 0$ we have $Q(\mathcal{V}_H^+) = 0$, then $\mathcal{V}_H^+$ does not contain any open set. ■

**Example 18** *Note however that the existence of a full support martingale measure is compatible with $1p$-Arbitrage so that the converse implication of 1. in Proposition 17 does not hold. Let $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Consider the market with one risky asset: $S_0 = 2$ and*

$$S_1 = \begin{cases} 3 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega \in \mathbb{Q}^+ \end{cases} \tag{6}$$

*Then obviously there exists a $1p$-Arbitrage even though there exist full support martingale measures (those probabilities assigning positive mass only to each rational).*

As soon as we weaken No $1p$-Arbitrage, by adopting any other no arbitrage conditions in Definition 10, there is no guarantee of the existence of martingale measures, as shown in Section 4.1. In order to obtain the equivalence between $\mathcal{M} \neq \varnothing$ and No Model Independent Arbitrage (the weakest among the No Arbitrage conditions de la classe $\mathcal{S}$) we will enlarge the filtration, as explained in Section 4.5.

## 4.1 Examples

This section provides a variety of counterexamples to many possible conjectures on the formulation of the FTAP in the model-free framework. A financially meaningful example is the one of two call options with the same spot price $p_1 = p_2$ but with strike prices $K_1 > K_2$, formulated in [DH07], which already highlights that the equivalence between absence of model independent arbitrage and existence of martingale measures is not possible.

We consider a one period market (i.e. $T = 1$) with $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and with $d = 2$ risky assets $S = [S^1, S^2]$, in addition to the riskless asset $S^0 = 1$. Admissible trading strategies are represented by vectors $H = (\alpha, \beta) \in \mathbb{R}^2$ so that

$$V_T(H) = \alpha \Delta S^1 + \beta \Delta S^2,$$

where $\Delta S^i = S^i_1 - S^i_0$ for $i = 1, 2$. Let $S_0 = [S^1_0, S^2_0] = [2, 2]$,

$$S^1_1 = \begin{cases} 3 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega \in \mathbb{Q}^+ \end{cases} \quad ; \quad S^2_1 = \begin{cases} 1 + \exp(\omega) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega = 0 \\ 1 + \exp(-\omega) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases} \tag{7}$$

and $\mathcal{F} = \mathcal{F}^S$. We notice the following simple facts.

1. There are no martingale measures:
$$\mathcal{M} = \varnothing.$$
Indeed, if we denote by $\mathcal{M}_i$ the set of martingale measures for the $i^{th}$ asset we have $\mathcal{M}_1 = \{Q \in \mathcal{P} \mid Q(\mathbb{R}^+ \setminus \mathbb{Q}) = 0\}$ and $\forall Q \in \mathcal{M}_2,\ Q(\mathbb{R}^+ \setminus \mathbb{Q}) > 0$.

2. The final value of the strategy $H = (\alpha, \beta) \in \mathbb{R}^2$ is
$$V_T(H) = \begin{cases} \alpha + \beta(\exp(\omega) - 1) & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ -\beta & \omega = 0 \\ \beta(\exp(-\omega) - 1) & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases}.$$
Only the strategies $H \in \mathbb{R}^2$ having $\beta = 0$ and $\alpha \geq 0$ satisfy $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. For $\beta = 0$ and $\alpha > 0$, $\mathcal{V}^+_H = \mathbb{R}^+ \setminus \mathbb{Q}$ and therefore there are **No Open Arbitrage** and **No Model Independent Arbitrage** (but $\mathcal{M} = \varnothing$). This fact persists even if we impose

boundedness restrictions on the process $S$ or on the admissible strategies $H$, as the following modification of the example shows: let $S_0 = [2,2]$ and take $S_1^1 = [2+\exp(-\omega)]\mathbf{1}_{\mathbb{R}^+\setminus\mathbb{Q}} + 2\mathbf{1}_{\mathbb{Q}^+}$ and $S_1^2 = [1+\exp(\omega)\wedge 4]\mathbf{1}_{\mathbb{R}^+\setminus\mathbb{Q}} + \mathbf{1}_{\{0\}} + [1+\exp(-\omega)]\mathbf{1}_{\mathbb{Q}^+\setminus\{0\}}$.

3. Set $\mathcal{H}^+ := \{H \in \mathcal{H} \mid \; V_T(H) \geq 0 \text{ and } V_0(H) = 0\}$ so that we have $\bigcup_{H\in\mathcal{H}^+} \mathcal{V}_H^+ = \mathbb{R}^+\setminus\mathbb{Q} \subsetneq \Omega$. This shows that the condition $\mathcal{M} = \varnothing$ is not equivalent to $\bigcup_{H\in\mathcal{H}^+} \mathcal{V}_H^+ = \Omega$ i.e. it is not true that the set of martingale measures is empty iff for every $\omega$ there exists a strategy $H$ that gives positive wealth on $\omega$ and $V_0(H) = 0$. In order to recover the equivalence between these two concepts (as in Proposition 43) we need to enlarge the filtration in the way explained in Section 4.5.

4. By fixing any probability $P$ there exists a $P$-Classical Arbitrage, since the FTAP holds true and $\mathcal{M} = \varnothing$. Indeed:

   (a) If $P(\mathbb{R}^+ \setminus \mathbb{Q}) = 0$, then $\beta = -1$ ($\alpha = 0$) yield a $P$-Classical arbitrage, since $\mathcal{V}_H^+ = \mathbb{Q}^+$ and $P(\mathcal{V}_H^+) = 1$

   (b) If $P(\mathbb{R}^+\setminus\mathbb{Q}) > 0$ then $\beta = 0$ and $\alpha = 1$ yield a $P$-Classical arbitrage, since $\mathcal{V}_H^+ = \mathbb{R}^+\setminus\mathbb{Q}$ and $P(\mathcal{V}_H^+) > 0$.

5. Instead, by adopting the definition of a $P$-a.s. Arbitrage ($V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$ and $P(\mathcal{V}_H^+) > 0$), there are two possibilities:

   (a) If $P(\mathbb{R}^+ \setminus \mathbb{Q}) = 0$, **No $P$-a.s. Arbitrage** holds, since only the strategies $H \in \mathbb{R}^2$ having $\beta = 0$ and $\alpha \geq 0$ satisfies $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$ and $\mathcal{V}_H^+ = \mathbb{R}^+ \setminus \mathbb{Q}$.

   (b) If $P(\mathbb{R}^+\setminus\mathbb{Q}) > 0$, then $\beta = 0$ and $\alpha = 1$ **yield a $P$-a.s. arbitrage**, since $\mathcal{V}_H^+ = \mathbb{R}^+\setminus\mathbb{Q}$ and $P(\mathcal{V}_H^+) > 0$.

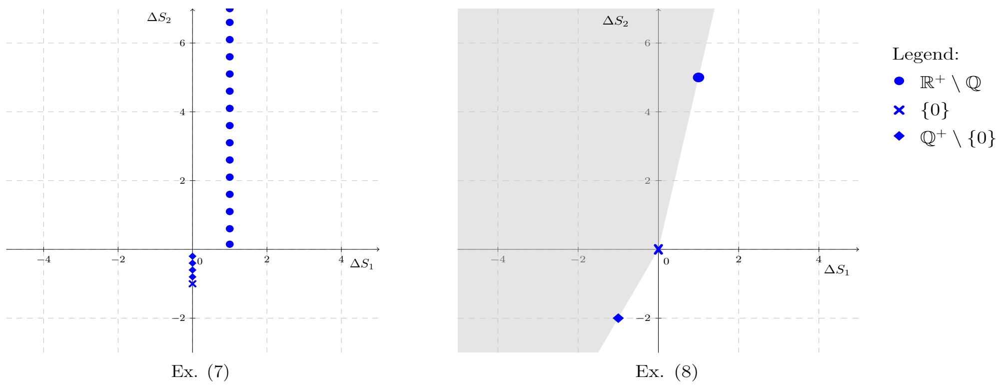
6. Geometric approach: If we plot the vector $[\Delta S^1, \Delta S^2]$ on the real plane (see Figure 1) we see that there exists a unique separating hyperplane given by the vertical axis. As a consequence 1p-Arbitrage can arise only by investment in the first asset ($\beta = 0$). For a separating hyperplane we mean an hyperplane in $\mathbb{R}^d$ passing by the origin and such that one of the associated half-space contains (not necessarily strictly contains) all the image points of the random vector $[\Delta S^1, \Delta S^2]$. Let us now consider this other example on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+)$. Let $S_0 = [2,2]$, and

$$S_1^1 = \begin{cases} 3 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega = 0 \\ 1 & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases} \qquad S_1^2 = \begin{cases} 7 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega = 0 \\ 0 & \omega \in \mathbb{Q}^+ \setminus \{0\} \end{cases} \tag{8}$$

   In both examples (7) and (8) there exist separating hyperplanes i.e. a $1p$-Arbitrage can be obtained (see Figure 1). In example (7) $\mathcal{M}$ is empty and we find a unique separating hyperplane: this hyperplane cannot give a strict separation of the set $[\Delta S^1(\omega), \Delta S^2(\omega)]_{\omega\in\mathbb{Q}^+}$ even though $\mathbb{Q}^+$ does not support any martingale measure. In example (8) $\mathcal{M} = \{\delta_{\omega=0}\}$, only the event $\{\omega = 0\}$ supports a martingale measure and there exists an infinite number of hyperplanes which strictly separates the image of both polar sets $\mathbb{R}^+ \setminus \mathbb{Q}$ and $\mathbb{Q}^+ \setminus \{0\}$, namely, those separating the convex grey region in Figure 1.

In conclusion the previous examples show that in a model-free environment the existence of a martingale measure can not be implied by arbitrage conditions - *at least of the type considered so far*. This is an important difference between the model-free and quasi-sure analysis approach (see for example [BN13]):

Figure 1: In examples (7) and (8), 0 does not belong to the relative interior of the convex set generated by the points $\{[\Delta S^1(\omega), \Delta S^2(\omega)]\}_{\omega\in\Omega}$ and hence there exists an hyperplane which separates the points.




- Model free approach: we deduce the 'richness' of the set $\mathcal{M}$ of martingale measures starting directly from the underlying market structure $(\Omega, \mathcal{F}, S)$ and we analyze the class of polar sets with respect to $\mathcal{M}$.
- Quasi sure approach: the class of priors $\mathcal{P}' \subseteq \mathcal{P}$ and its polar sets are given and one formulates a No-Arbitrage type condition to guarantee the existence of a class of martingale measures which has the same polar sets as the set of priors.

**An alternative definition of Arbitrage.** The notion of No $P$-Classical Arbitrage ($P(V_T(H) < 0) = 0 \Rightarrow P(V_T(H) > 0) = 0$) can be rephrased as: $V_0(H) = 0$ and

$$\{V_T(H) < 0\} \text{ is negligible } \Rightarrow \{V_T(H) > 0\} \text{ is negligible} \tag{9}$$

or in our setting

$$\mathcal{V}_H^- \text{ does not contain sets in } \mathcal{S} \Rightarrow \mathcal{V}_H^+ \text{ does not contain sets in } \mathcal{S}. \tag{10}$$

where $\mathcal{V}_H^- := \{\omega \in \Omega \mid V_T(H)(\omega) < 0\}$. In the definition (10) we are giving up the requirement $V_T(H) \geq 0$, and so the differences with respect to the existence of arbitrage opportunities showed in Item 5 of the example in this section disappear. However, this alternative definition of arbitrage does not work well, as shown by the following example. Consider $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+)$, a one period market with one risky asset: $S_0 = 2$,

$$S_1 = \begin{cases} 3 & \omega \in [1, \infty) \\ 2 & \omega = [0,1) \setminus \mathbb{Q} \\ 1 & \omega \in [0,1) \cap \mathbb{Q} \end{cases} \tag{11}$$

Consider the strategy of buying the risky asset: $H = 1$. Then $\mathcal{V}_H^- = [0,1) \cap \mathbb{Q}$ does not contain an open set, $\mathcal{V}_H^+ = [1, \infty)$ contains open sets. Therefore, there is an Open Arbitrage (in the modified definition obtained from (10)) but there are full support martingale measures, for example $Q([0,1) \cap \mathbb{Q}) = Q([1,\infty)) = \frac{1}{2}$. Notice also that by enlarging the filtration the Open Arbitrage would persist.

A concept of no arbitrage similar to (9) was introduced by Cassese [C08], by adopting an ideal $\mathcal{N}$ of "negligible" sets - not necessarily derived from probability measures. In a continuous time setting, he proves that the absence of such an arbitrage is equivalent to the existence of a finitely additive "martingale measure". Our results are not comparable with those by [C08] since the markets are clearly different, we do not require any structure on the family $\mathcal{S}$ and [C08] works with finitely additive measures. In addition, the example (11) just discussed shows the limitation in our setting of the definition (9) for finding martingale probability measures with the appropriate properties.

## 4.2 Technical Lemmata

Recall that $S = (S_t)_{t\in I}$ is an $\mathbb{R}^d$-valued stochastic process defined on a Polish space $\Omega$ endowed with its Borel $\sigma$-algebra $\mathcal{F} = \mathcal{B}(\Omega)$ and $I_1 := \{1, ..., T\}$.

*Through the rest of the paper we will make use of the natural filtration $\mathcal{F}^S = \{\mathcal{F}_t^S\}_{t\in I}$ of the process $S$ and for ease of notation we will not specify $S$, but simply write $\mathcal{F}_t$ for $\mathcal{F}_t^S$.*

For the sake of simplicity we indicate by $\mathbf{Z} := Mat(d \times (T+1); \mathbb{R})$ the space of $d \times (T+1)$ matrices with real entries representing the space of all the possible trajectories of the price process. Namely for every $\omega \in \Omega$ we have $(S_0(\omega), S_1(\omega), ..., S_T(\omega)) = (z_0, z_1, ..., z_T) =: z \in \mathbf{Z}$. Fix $s \le t$: for any $z \in \mathbf{Z}$ we indicate , the components from $s$ to $t$ by $z_{s:t} = (z_s, ..., z_t)$ and $z_{t:t} = z_t$. Similarly $S_{s:t} = (S_s, S_{s+1}, ..., S_t)$ represents the process from time $s$ to $t$.
We denote with $ri(K)$ the relative interior of a set $K \subseteq \mathbb{R}^d$. In this section we will make extensive use of the geometric properties of the image in $\mathbb{R}^d$ of the increments of the price process $\Delta S_t := S_t - S_{t-1}$ relative to a set $\Gamma \subseteq \Omega$. The typical sets that we will consider are the level sets $\Gamma = \Sigma_{t-1}^z$, where:

$$\Sigma_{t-1}^z := \{\omega \in \Omega \mid S_{0:t-1}(\omega) = z_{0:t-1}\} \in \mathcal{F}_{t-1}, \;\; z \in \mathbf{Z}, \, t \in I_1 \tag{12}$$

and $\Gamma = A_{t-1}^z$, the intersection of the level set $\Sigma_{t-1}^z$ with a set $A \in \mathcal{F}_{t-1}$:

$$A_{t-1}^z := \{\omega \in A \mid S_{0:t-1}(\omega) = z_{0:t-1}\} \in \mathcal{F}_{t-1}. \tag{13}$$

For any $\Gamma \subseteq \Omega$ define the convex cone:

$$(\Delta S_t(\Gamma))^{cc} := co\left(conv(\Delta S_t(\Gamma))\right) \cup \{0\} \subseteq \mathbb{R}^d. \tag{14}$$

If $0 \in ri(\Delta S_t(\Gamma))^{cc}$ we cannot apply the hyperplane separating theorem to the convex sets $\{0\}$ and $ri(\Delta S_t(\Gamma))^{cc}$, namely, there is no $H \in \mathbb{R}^d$ that satisfies $H \cdot \Delta S_t(\omega) \ge 0$ for all $\omega \in \Gamma$ with strict inequality for some of them. As intuitively evident, and shown in Corollary 21 below, $0 \in ri(\Delta S_t(\Gamma))^{cc}$ if and only if No $1p$-Arbitrage are possible on the set $\Gamma$, since a trading strategy on $\Gamma$ with a non-zero payoff always yields both positive and negative outcomes.

In this situation, for $\Gamma = \Sigma_{t-1}^z$, the level set is not suitable for the construction of a $1p$-Arbitrage opportunity and sets with this property are naturally important for the construction of a martingale measure. We wish then to identify, for $\Gamma = \Sigma_{t-1}^z$ satisfying $0 \notin ri(\Delta S_t(\Gamma))^{cc}$, those subset of $\Sigma_{t-1}^z$ that retain this property. This result is contained in the following key Lemma 20.

Observe first that for a convex cone $K \subseteq \mathbb{R}^d$ such that $0 \notin ri(K)$ we can consider the family $V = \{v \in \mathbb{R}^d \mid \|v\| = 1 \text{ and } v \cdot y \ge 0 \; \forall y \in K\}$ so that

$$\overline{K} = \bigcap_{v\in V} \{y \in \mathbb{R}^d \mid v \cdot y \ge 0\} = \bigcap_{n\in\mathbb{N}} \{y \in \mathbb{R}^d \mid v_n \cdot y \ge 0\},$$

where $\{v_n\} = (\mathbb{Q}^d \cap V) \setminus \{0\}$.

**Definition 19** *Adopting the above notations, we will call $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} v_n \in V$ the standard separator.*

**Lemma 20** *Fix* $t \in I_1$ *and* $\Gamma \neq \varnothing$. *If* $0 \notin ri(\Delta S_t(\Gamma))^{cc}$ *then there exist* $\beta \in \{1,\dots,d\}$, $H^1,\dots,H^\beta$, $B^1,\dots,B^\beta, B^*$ *with* $H^i \in \mathbb{R}^d$, $B^i \subseteq \Gamma$ *and* $B^* := \Gamma \setminus \big(\cup_{j=1}^{\beta} B^j\big)$ *such that:*

1. $B^i \neq \varnothing$ *for all* $i = 1,\dots\beta$, *and* $\{\omega \in \Gamma \mid \Delta S_t(\omega) = 0\} \subseteq B^*$ *which may be empty;*
2. $B^i \cap B^j = \varnothing$ *if* $i \neq j$;
3. $\forall i \leq \beta,\ H^i \cdot \Delta S_t(\omega) > 0$ *for all* $\omega \in B^i$ *and* $H^i \cdot \Delta S_t(\omega) \geq 0$ *for all* $\omega \in \cup_{j=i}^{\beta} B^j \cup B^*$.
4. $\forall H \in \mathbb{R}^d$ *s.t.* $H \cdot \Delta S_t \geq 0$ *on* $B^*$ *we have* $H \cdot \Delta S_t = 0$ *on* $B^*$.

*Moreover, for* $z \in \mathbf{Z}$, $A \in \mathcal{F}_{t-1}$ *and* $\Gamma = A_{t-1}^z$ *(or* $\Gamma = \Sigma_{t-1}^z$*) we have* $B^i, B^* \in \mathcal{F}_t$ *and*

$$H(\omega) := \sum_{i=1}^{\beta} H^i \mathbf{1}_{B^i}(\omega) \tag{15}$$

*is an* $\mathcal{F}_t$*-measurable random variable that is uniquely determined when we adopt for each* $H^i$ *the standard separator.*
*Clearly in these cases,* $\beta$, $H^i$, $H$, $B^i$ *and* $B^*$ *will depend on* $t$ *and* $z$ *and whenever necessary they will be denoted by* $\beta_{t,z}$, $H_{t,z}^i$, $H_{t,z}$, $B_{t,z}^i$ *and* $B_{t,z}^*$.

**Proof.** Set $A^0 := \Gamma$ and $K^0 = (\Delta S_t(\Gamma))^{cc} \subseteq \mathbb{R}^d$ and let $\Delta_0 := \{\omega \in A^0 \mid \Delta S_t(\omega) = 0\}$, which may be empty.

**Step 1:** The set $K^0 \subseteq \mathbb{R}^d$ is non-empty and convex and so $ri\big(K^0\big) \neq \varnothing$. From the assumption $0 \notin ri\big(K^0\big)$ there exists a standard separator $H^1 \in \mathbb{R}^d$ such that (i) $H^1 \cdot \Delta S_t(\omega) \geq 0$ for all $\omega \in A^0$; (ii) $B^1 := \{\omega \in A^0 \mid H^1 \cdot \Delta S_t(\omega) > 0\}$ is non-empty. Set $A^1 := (A^0 \setminus B^1) = \{\omega \in A^0 \mid H^1 \cdot \Delta S_t(\omega) = 0\}$ and let $K^1 := (\Delta S_t(A^1))^{cc}$, which is a non-empty convex set with $dim(K^1) \leq d-1$.
If $0 \in ri\big(K^1\big)$ (this includes the case $K^1 = \{0\}$) the procedure is complete: one cannot separate $\{0\}$ from the relative interior of $K^1$. The conclusion is that $\beta = 1$, $B^* = A^1 = A^0 \setminus B^1$ which might be empty, and $\Delta_0 \subseteq B^*$. Notice that if $K^1 = \{0\}$ then $B^* = \Delta_0$ which might be empty. Otherwise:

**Step 2:** If $0 \notin ri\big(K^1\big)$ we find the standard separator $H^2 \in \mathbb{R}^d$ such that $H^2 \cdot \Delta S_t(\omega) \geq 0$, for all $\omega \in A^1$, and the set $B^2 := \{\omega \in A^1 \mid H^2 \cdot \Delta S_t(\omega) > 0\}$ is non-empty. Denote $A^2 := (A^1 \setminus B^2)$ and let $K^2 = (\Delta S_t(A^2))^{cc}$ with $dim(K^2) \leq d-2$.
If $0 \in ri\big(K^2\big)$ (this includes $K^2 = \{0\}$) the procedure is complete and we have the conclusions with $\beta = 2$ and $B^* = A^1 \setminus B^2 = A^0 \setminus (B^1 \cup B^2)$, and $\Delta_0 \subseteq B^*$. Notice that if $K^1 = \{0\}$ then $B^* = \Delta_0$. Otherwise:

**...**

**Step d-1** If $0 \notin ri\big(K^{d-2}\big)$ ...Set $B^{d-1} \neq \varnothing$, $A^{d-1} = (A^{d-2} \setminus B^{d-1})$, $K^{d-1} = (\Delta S_t(A^{d-1}))^{cc}$ with $dim(K^{d-1}) \leq 1$. If $0 \in ri\big(K^{d-1}\big)$ the procedure is complete. Otherwise:

**Step d:** We necessarily have $0 \notin ri\big(K^{d-1}\big)$, so that $dim(K^{d-1}) = 1$, and the convex cone $K^{d-1}$ necessarily coincides with a half-line with origin in 0. Then we find a standard separator $H^d \in \mathbb{R}^d$ such that: $B^d := \{\omega \in A^{d-1} \mid H^d \cdot \Delta S_t(\omega) > 0\} \neq \varnothing$ and the set

$$B^* := \{\omega \in A^{d-1} \mid \Delta S_t(\omega) = 0\} = \{\omega \in A^0 \mid \Delta S_t(\omega) = 0\} = \Delta_0$$

satisfies: $B^* = A^{d-1} \setminus B^d$. Set $A^d := A^{d-1} \setminus B^d = B^* = \Delta_0$ and $K^d := (\Delta S_t(A^d))^{cc}$. Then $K^d = \{0\}$.

Since $dim(\Delta S_t(\Gamma))^{cc} \leq d$ we have at most $d$ steps. In case $\beta = d$ we have $\Gamma = A^0 = \bigcup_{i=1}^{d} B^i \cup \Delta_0$. To prove the last assertion we note that for any fixed $t$ and $z$, $B^i$ are $\mathcal{F}_t$-measurable since $B^i = A_{t-1}^z \cap (f \circ S_t)^{-1}((0,\infty))$ where $f : \mathbb{R}^d \mapsto \mathbb{R}$ is the continuous function $f(x) = H^i \cdot (x - z_{t-1})$ with $H^i \in \mathbb{R}^d$ fixed. ■

**Corollary 21** *Let $t \in I_1$, $z \in \mathbf{Z}$, $A \in \mathcal{F}_{t-1}$, $\Gamma = A_{t-1}^z$. Then $0 \in ri(\Delta S_t(\Gamma))^{cc}$ if and only if there are No 1p-Arbitrage on $\Gamma$, i.e.:*

$$\textit{for all } H \in \mathbb{R}^d \textit{ s.t. } H(S_t - z_{t-1}) \geq 0 \textit{ on } \Gamma \textit{ we have } H(S_t - z_{t-1}) = 0 \textit{ on } \Gamma. \tag{16}$$

**Proof.** Let $0 \notin ri(\Delta S_t(\Gamma))^{cc}$. Then from Lemma 20-3) with $i = 1$ we obtain a 1$p$-Arbitrage $H^1$ on $\Gamma = \cup_{j=1}^{\beta} B^j \cup B^*$, since $B^1 \neq \varnothing$. Viceversa, if $0 \in ri(\Delta S_t(\Gamma))^{cc}$ we obtain (16) from the argument following equation (14). ■

**Definition 22** *For $A \in \mathcal{F}_{t-1}$ and $\Gamma = A_{t-1}^z$ we naturally extend the definition of $\beta_{t,z}$ in Lemma 20 to the case of $0 \in ri(\Delta S_t(\Gamma))^{cc}$ using*

$$\beta_{t,z} = 0 \dot{\Leftrightarrow} 0 \in ri(\Delta S_t(\Gamma))^{cc}$$

*with $B_{t,z}^0 = \varnothing$ and $B_{t,z}^* = A_{t-1}^z \in \mathcal{F}_{t-1}$. In this case, we also extend the definition of the random variable in (15) as $H_{t,z}(\omega) \equiv 0$.*

**Corollary 23** *Let $t \in I_1$, $z \in \mathbf{Z}$, $A \in \mathcal{F}_{t-1}$ and $\Gamma = A_{t-1}^z$ with $0 \notin ri(\Delta S_t(\Gamma))^{cc}$. For any $P \in \mathcal{P}$ s.t. $P(\Gamma) > 0$ let $j := \inf\{1 \leq i \leq \beta \mid P(B_{t,z}^i) > 0\}$. If $j < \infty$ the trading strategy $H(s,\omega) := H^j \mathbf{1}_\Gamma(\omega) \mathbf{1}_{\{t\}}(s)$ is a P-Classical Arbitrage.*

**Proof.** From Lemma 20-3) we obtain that: $H^j \Delta S_t(\omega) > 0$ on $B_{t,z}^j$ with $P(B_{t,z}^j) > 0$; $H^j \Delta S_t(\omega) \geq 0$ on $\bigcup_{i=j}^{\beta_{t,z}} B_{t,z}^i \cup B_{t,z}^*$ and $P(B_{t,z}^k) = 0$ for $1 \leq k < j$. ■

**Remark 24** *Let $D \subseteq \mathbb{R}^d$ and $C := (D)^{cc} \subseteq \mathbb{R}^d$ be the convex cone generated by $D$. If $0 \in ri(C)$ then for any $x \in D$ there exist a finite number of elements $x_j \in D$ such that 0 is a convex combination of $\{x, x_1, ..., x_m\}$ with a strictly positive coefficient of $x$. Indeed, fix $x \in D$ and recall that for any convex set $C \subseteq \mathbb{R}^d$ we have*

$$ri(C) := \{z \in C \mid \forall x \in C \ \exists \varepsilon > 0 \text{ s.t. } z - \varepsilon(x - z) \in C\}.$$

*As $0 \in ri(C)$ and $x \in D \subseteq C$ we obtain $-\varepsilon x \in C$, for some $\varepsilon > 0$, and therefore: $\frac{\varepsilon}{1+\varepsilon}x + \frac{1}{1+\varepsilon}(-\varepsilon x) = 0$. Since $-\varepsilon x \in C$ then it is a linear combination with non negative coefficients of elements of $D$ and we obtain: $\frac{\varepsilon}{1+\varepsilon}x + \frac{1}{1+\varepsilon}\sum_{j=1}^m \alpha_j x_j = 0$, which can be rewritten - possibly normalizing the coefficients - as: $\lambda x + \sum_{j=1}^m \lambda_j x_j = 0$, with $x_j \in D$, $\lambda + \sum_{j=1}^m \lambda_j = 1$, $\lambda > 0$ and $\lambda_j \geq 0$. When the set $D \subseteq \mathbb{R}^d$ is the set of the image points of the increment of the price process $[\Delta S_t(\omega)]_{\omega \in \Gamma}$, for a fixed time $t$, this observation shows that, however we choose $\omega \in \Gamma$ we can construct a conditional martingale measure, relatively to the period $[t-1, t]$, which assigns a strictly positive weight to $\omega$ and has finite support. The measure is determined by the coefficients $\{\lambda, \lambda_1, ..., \lambda_m\}$ in the equation: $0 = \lambda \Delta S_t(\omega) + \sum_{j=1}^m \lambda_j \Delta S_t(\omega_j)$. This heuristic argument is made precise in the following Corollary and it will be used also in the proof of Proposition 34.*

**Corollary 25** *Let $z$, $t$, $\Gamma = A_{t-1}^z$ and $B_{t,z}^*$ as in Lemma 20.*

$$\textit{For all } U \subseteq B_{t,z}^*,\ U \in \mathcal{F} \quad \textit{there exists } Q \in \mathcal{M}(B_{t,z}^*) \textit{ s.t. } Q(U) > 0$$

*where $\mathcal{M}(B) = \{Q \in \mathcal{P} \mid Q(B) = 1$ and $E_Q[S_t \mid \mathcal{F}_{t-1}] = S_{t-1}$ Q-a.s.$\}$, for $B \in \mathcal{F}$.*

**Proof.** From Lemma 20-4) there are no $1p$-Arbitrage restricted to $\Gamma = B^*_{t,z}$. Applying Corollary 21 this implies that $0 \in ri(\Delta S_t(B^*_{t,z}))^{cc}$. Take any $\omega \in U \subseteq B^*_{t,z}$ . Applying Remark 24 to the set $D := \Delta S_t(B^*_{t,z})$ and to $x := \Delta S_t(\omega) \in D$, we deduce the existence of $\{\omega_1, \dots, \omega_m\} \subseteq B^*_{t,z}$ and non negative coefficients $\{\lambda_t(\omega_1), ..., \lambda_t(\omega_m)\}$ and $\lambda_t(\omega) > 0$ such that: $\lambda_t(\omega) + \sum_{j=1}^m \lambda_t(\omega_j) = 1$ and

$$0 = \lambda_t(\omega)\Delta S_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j)\Delta S_t(\omega_j).$$

Since $\{\omega_1, \dots, \omega_m\} \subseteq B^*_{t,z}$ and $\omega \in B^*_{t,z}$ we have $S_{t-1}(\omega_j) = z_{t-1}$ and $S_{t-1}(\omega) = z_{t-1}$. Therefore:

$$0 = \lambda_t(\omega)(S_t(\omega) - z_{t-1}) + \sum_{j=1}^{m} \lambda_t(\omega_j)(S_t(\omega_j) - z_{t-1}), \tag{17}$$

so that $Q(\{\omega\}) = \lambda_t(\omega)$ and $Q(\{\omega_j\}) = \lambda_t(\omega_j)$, for all $j$, give the desired probability. ■

**Example 26** *Let $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and consider a single period market with $d = 3$ risky asset $S_t = [S^1_t, S^2_t, S^3_t]$ with $t = 0, 1$ and $S_0 = [2, 2, 2]$. Let*

$$S^1_1(\omega) = \begin{cases} 1 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega \in \mathbb{Q} \cap [1/2, +\infty) \\ 3 & \omega \in \mathbb{Q} \cap [0, 1/2) \end{cases} \quad S^2_1(\omega) = \begin{cases} 2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1+\omega^2 & \omega \in \mathbb{Q} \cap [1/2, +\infty) \\ 1+\omega^2 & \omega \in \mathbb{Q} \cap [0, 1/2) \end{cases}$$

$$S^3_1(\omega) = \begin{cases} 2+\omega^2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 2 & \omega \in \mathbb{Q} \cap [1/2, +\infty) \\ 2 & \omega \in \mathbb{Q} \cap [0, 1/2) \end{cases}$$

Fix $t = 1$ and $z \in \mathbf{Z}$ with $z_0 = S_0$. It is easy to check that in this case $\beta_{t,z} = 2$ with $B^1_{t,z} = \mathbb{R}^+ \setminus \mathbb{Q}$, $B^2_{t,z} = \mathbb{Q} \cap [0, 1/2)$, $B^*_{t,z} = \mathbb{Q} \cap [1/2, +\infty)$. The corresponding strategies $H = [h_1, h_2, h_3]$ (standard in the sense of Lemma 20) are given by $H^1_{t,z} = [0, 0, 1]$ and $H^2_{t,z} = [1, 0, 0]$. Note that $H^1_{t,z}$ is a 1p-arbitrage with $\mathcal{V}^+_{H^1_{t,z}} = B^1_{t,z}$. We have therefore that $B^1_{t,z}$ is a null set with respect to any martingale measure. The strategy $H^2_{t,z}$ satisfies $V_T(H^2_{t,z}) \geq 0$ on $(B^1_{t,z})^c$ with $\mathcal{V}^+_{H^2_{t,z}} = B^2_{t,z}$ hence, $B^2_{t,z}$ is also an $\mathcal{M}$-polar set. This example shows the need of a multiple separation argument, as it is not possible to find a *single* separating hyperplane $H \in \mathbb{R}^d$ such that the image points of $B^1_{t,z} \cup B^2_{t,z}$ (which is $\mathcal{M}$-polar), through the random vector $\Delta S$, are strictly contained in one of the associated half-spaces. We have indeed that $B^2_{t,z}$ is a subset of $\{\omega \in \Omega \mid H^1_{t,z}(S_1 - S_0) = 0\}$ where $H^1_{t,z}$ is the only 1p-arbitrage in this market.

The corollaries 23 and 25 show the difference between the sets $B^i$ and $B^*$. Restricted to the time interval $[t-1, t]$, a probability measure whose mass is concentrated on $B^*$ admits an equivalent martingale measure while for those probabilities that assign positive mass to at least one $B^i$ an arbitrage opportunity can be constructed. We can summarize the possible situations as follows.

**Corollary 27** *For $\Gamma = A^z_{t-1}$, with $A \in \mathcal{F}_{t-1}$, and $\mathcal{M}(B)$ defined in Corollary 25 we have:*

1. *$B^*_{t,z} = A^z_{t-1} \Leftrightarrow$ No $1p$-Arbitrage on $A^z_{t-1} \dot{\Leftrightarrow} 0 \in ri(\Delta S_t(A^z_{t-1}))^{cc}$.*
2. *$B^*_{t,z} = \varnothing \Leftrightarrow 0 \notin conv\big(\Delta S_t(A^z_{t-1})\big)$*
3. *$\beta_{t,z} = 1$ and $B^*_{t,z} \neq \varnothing \implies \exists H \in \mathbb{R}^d \setminus \{0\}$ s.t. $B^*_{t,z} = \{\omega \in A^z_{t-1} \mid H(S_t(\omega) - z_{t-1}) = 0\}$ is "martingalizable" i.e. $\forall U \subset B^*_{t,z}$, $U \in \mathcal{F}$ there exists $Q \in \mathcal{M}(B^*_{t,z})$ s.t. $Q(U) > 0$.*

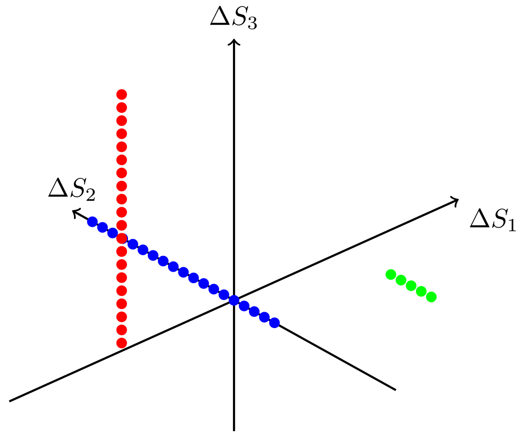



Figure 2: Decomposition of $\Omega$ in Example 26: $B^1_{t,z}$ in red $B^2_{t,z}$ in green and $B^*_{t,z}$ in blue

**Proof.** Equivalence 1. immediately follows from Corollary 21 and Definition 22.
To show 2. we use the sets $K^i$ for $i = 1, \dots, \beta_{t,z}$ and the other notations from the proof of Lemma 20. Suppose first that $0 \notin conv\big(\Delta S_t(\Gamma)\big)$ which implies $0 \notin ri(\Delta S_t(\Gamma))^{cc}$ and $\Delta_0 = \varnothing$. From the assumption we have $0 \notin conv\big(\Delta S_t(C)\big)$ for any subset $C \subseteq \Gamma$ so, in particular, $0 \notin ri(K^i)$ unless $K^i = \{0\}$. This implies $B^*_{t,z} = \Delta_0 = \varnothing$.
Suppose now $0 \in conv\big(\Delta S_t(\Gamma)\big)$. If $0 \in ri(\Delta S_t(\Gamma))^{cc}$, by Definition 22 we have $B^*_{t,z} = \Gamma \neq \varnothing$. Suppose then $0 \notin ri(\Delta S_t(\Gamma))^{cc}$. As $0 \in conv\big(\Delta S_t(\Gamma)\big)$ there exists $n \geq 1$ such that: $0 = \sum_{j=1}^n \lambda_j (S_t(\omega_j) - z_{t-1})$, with $\sum_{j=1}^n \lambda_j = 1$, $\lambda_j > 0$ and $\omega_j \in \Gamma$ for all $j$. If 0 is extremal then $n = 1$, $S_t(\omega_1) - z_{t-1} = 0$ and $\{\omega_1\} \in \Delta_0 \subseteq B^*_{t,z}$. If $n \geq 2$ we have $0 \in conv\,(\Delta S_t\{\omega_1, \dots, \omega_n\})$ so that for any $H \in \mathbb{R}^d$ that satisfies $H \cdot \Delta S_t(\omega_i) \geq 0$ for any $i = 1, \dots, n$ we have $H \cdot \Delta S_t(\omega_i) = 0$. Hence $\{\omega_1, \dots, \omega_n\} \subseteq B^*_{t,z}$ by definition of $B^*_{t,z}$.
We conclude by showing 3. From Lemma 20 items 3 and 4, if we select $H = H^1$ then $\{\omega \in \Gamma \mid H^1(S_t(\omega) - z_{t-1}) = 0\} = \Gamma \setminus B^1_{t,z} = B^*_{t,z} \neq \varnothing$ and on $B^*_{t,z}$ we may apply Corollary 25. ■

## 4.3 On $\mathcal{M}$-polar sets

We consider for any $t \in I$ the sigma-algebra $\mathfrak{F}_t := \bigcap_{Q \in \mathcal{M}} \mathcal{F}^Q_t$, where $\mathcal{F}^Q_t$ is the $Q$-completion of $\mathcal{F}_t$. $\mathfrak{F}_t$ is the universal completion of $\mathcal{F}_t$ with respect to $\mathcal{M} = \mathcal{M}(\mathbb{F})$. Notice that the introduction of this enlarged filtration needs the knowledge *a priori* of the whole class $\mathcal{M}$ of martingale measures. Recall that any measure $Q \in \mathcal{M}$ can be uniquely extended to a measure $\overline{Q}$ on the enlarged sigma algebra $\mathfrak{F}_T$ so that we can write with slight abuse of notation $\mathcal{M}(\mathbb{F}) = \mathcal{M}(\mathfrak{F})$ where $\mathfrak{F} := \{\mathfrak{F}_t\}_{t \in I}$.

We wish to show now that under any martingale measure the sets $B^i_{t,z}$ (and their arbitrary unions) introduced in Lemma 20 must be null-sets. To this purpose we need to recall some properties of a proper regular conditional probability (see Theorems 1.1.6, 1.1.7 and 1.1.8 in Stroock-Varadhan [SV06]).

**Theorem 28** *Let $(\Omega, \mathcal{F}, Q)$ be a probability space, where $\Omega$ is a Polish space, $\mathcal{F}$ is the Borel $\sigma$-algebra, $Q \in \mathcal{P}$. Let $\mathcal{A} \subseteq \mathcal{F}$ be a countably generated sub sigma algebra of $\mathcal{F}$. Then there exists a proper regular conditional probability, i.e. a function $Q_{\mathcal{A}}(\cdot, \cdot) : (\Omega, \mathcal{F}) \mapsto [0, 1]$ such that:*

a) for all $\omega \in \Omega$, $Q_{\mathcal{A}}(\omega, \cdot)$ is a probability measure on $\mathcal{F}$;

b) for each $B \in \mathcal{F}$, the function $Q_{\mathcal{A}}(\cdot, B)$ is a version of $Q(B \mid \mathcal{A})(\cdot)$;

c) $\exists N \in \mathcal{A}$ with $Q(N) = 0$ such that $Q_{\mathcal{A}}(\omega, B) = 1_B(\omega)$ for $\omega \in \Omega \setminus N$ and $B \in \mathcal{A}$

d) In addition, if $X \in L^1(\Omega, \mathcal{F}, Q)$ then $E_Q[X \mid \mathcal{A}](\omega) = \int_\Omega X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega})\ Q - a.s.$

Recall that $\mathcal{F}_t = \mathcal{F}_t^S$, $t \in I$, is countably generated.

**Lemma 29** *Fix $t \in I_1 = \{1, ..., T\}$, $A \in \mathcal{F}_{t-1}$, $Q \in \mathcal{M}$ and for $z \in \mathbf{Z}$ consider $A_{t-1}^z := \{\omega \in A \mid S_{0:t-1}(\omega) = z_{0:t-1}\}$. Then*

$$\bigcup_{z \in \mathbf{Z}} \{\omega \in A_{t-1}^z \ s.t.\ Q_{\mathcal{F}_{t-1}}(\omega, \cup_{i=1}^{\beta_{t,z}} B_{t,z}^i) > 0\}$$

*is a subset of an $\mathcal{F}_{t-1}$-measurable $Q$-null set.*

**Proof.** If $Q(A) = 0$ there is nothing to show. Suppose now $Q(A) > 0$. In this proof we set for the sake of simplicity $X := S_t$, $Y := E_Q[X \mid \mathcal{F}_{t-1}] = S_{t-1}$ $Q$-a.s. $\beta := \beta_{t,z}$ and $\mathcal{A} := \mathcal{F}_{t-1} = \mathcal{F}_{t-1}^S$. Set

$$D_t^z := \left\{\omega \in A_{t-1}^z \text{ such that } Q_{\mathcal{A}}(\omega, \cup_{i=1}^{\beta} B_{t,z}^i) > 0\right\}.$$

If $z \in \mathbf{Z}$ is such that $0 \in ri(\Delta S_t(A_{t-1}^z))^{cc}$ then $\cup_{i=1}^{\beta} B_{t,z}^i = \varnothing$ and $D_t^z = \varnothing$. So we can consider only those $z \in \mathbf{Z}$ such that $0 \notin ri(\Delta S_t(A_{t-1}^z))^{cc}$. Fix such $z$.
Since $\mathcal{A} = \mathcal{F}_{t-1}^S$ is countably generated, $Q$ admits a proper regular conditional probability $Q_{\mathcal{A}}$. From Theorem 28 d) we obtain:

$$Y(\omega) = \int_\Omega X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) \quad Q - a.s.$$

As $A_{t-1}^z \in \mathcal{A}$, by Theorem 28 c) there exists a set $N \in \mathcal{A}$ with $Q(N) = 0$ so that $Q_{\mathcal{A}}(\omega, A_{t-1}^z) = 1$ on $A_{t-1}^z \setminus N$ and therefore we have

$$\int_\Omega X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) = \int_{A_{t-1}^z} X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) \quad \forall \omega \in A_{t-1}^z \setminus N. \tag{18}$$

Since $0 \notin ri(\Delta S_t(\Gamma))^{cc}$ we may apply Lemma 20: for any $i = 1, \ldots, \beta$, there exists $H^i \in \mathbb{R}^d$ such that $H^i \cdot (X(\tilde{\omega}) - z_{t-1}) \geq 0$ for all $\tilde{\omega} \in \cup_{l=i}^{\beta} B_{t,z}^l \cup B_{t,z}^*$ and $H^i \cdot (X(\tilde{\omega}) - z_{t-1}) > 0$ for every $\tilde{\omega} \in B_{t,z}^i$.

Now we fix $\omega \in D_t^z \setminus N \subseteq A_{t-1}^z \setminus N$. Then the index $j := \min\{1 \leq i \leq \beta \mid Q_{\mathcal{A}}(\omega, B_{t,z}^i) > 0\}$ is well defined and: i) $H^j \cdot (X(\tilde{\omega}) - z_{t-1}) > 0$ on $B_{t,z}^j$, (ii) $Q_{\mathcal{A}}(\omega, B_{t,z}^j) > 0$ iii) $H^j \cdot (X(\tilde{\omega}) - z_{t-1}) \geq 0$ on $\cup_{l=j}^{\beta} B_{t,z}^l \cup B_{t,z}^*$; iv) $Q_{\mathcal{A}}(\omega, B_{t,z}^i) = 0$ for $i < j$. From i) and ii) we obtain

$$Q_{\mathcal{A}}(\omega, A_{t-1}^z \cap \{H^j \cdot (X - z_{t-1}) > 0\}) \geq Q_{\mathcal{A}}(\omega, B_{t,z}^j) > 0.$$

From iii) and iv) we obtain:

$$\begin{aligned} Q_{\mathcal{A}}(\omega, \{H^j \cdot (X - z_{t-1}) \geq 0\}) &\geq Q_{\mathcal{A}}(\omega, \cup_{l=j}^{\beta} B_{t,z}^l \cup B_{t,z}^*) \\ &\geq Q_{\mathcal{A}}(\omega, A_{t-1}^z) - Q_{\mathcal{A}}(\omega, \cup_{i<j} B_{t,z}^i) = 1. \end{aligned}$$

Hence

$$H^j \cdot \left(\int_{A_{t-1}^z} X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) - z_{t-1}\right) = \int_{A_{t-1}^z} H^j \cdot (X(\tilde{\omega}) - z_{t-1}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) > 0$$

and therefore, from equation (18) and from $z_{t-1} = Y(\omega)$, we have:

$$H^j \cdot \left(\int_\Omega X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega}) - Y(\omega)\right) > 0.$$

As this holds for any $\omega \in D_t^z \setminus N$ we obtain:

$$D_t^z \setminus N \subseteq \{\omega \in \Omega \mid Y(\omega) \neq \int_\Omega X(\tilde{\omega}) Q_{\mathcal{A}}(\omega, d\tilde{\omega})\} =: N^* \in \mathcal{F}_{t-1}$$

with $Q(N^*) = 0$. Hence, $D_t^z \subseteq N \cup N^* := N_0$ with $Q(N_0) = 0$ and $N_0$ not dependent on $z$. As this holds for every $z \in \mathbf{Z}$ we conclude that $\bigcup_{z \in \mathbf{Z}} D_t^z \subseteq N_0$. ∎

**Corollary 30** *Fix $t \in I_1$ and $Q \in \mathcal{M}$. If*

$$\mathfrak{B}_t := \bigcup_{z \in \mathbf{Z}} \left\{ \bigcup_{i=1}^{\beta_{t,z}} B_{t,z}^i \right\}$$

*for $B_{t,z}^i$ given in Lemma 20 with $\Gamma = \Sigma_{t-1}^z$ or $\Gamma = A_{t-1}^z$ (defined in equations* (12) *and* (13)*), then $\mathfrak{B}_t$ is a subset of an $\mathcal{F}_t$-measurable $Q$ null set.*

**Proof.** First we consider the case $\Gamma = \Sigma_{t-1}^z$ and $B_{t,z}^i$ given in Lemma 20 with $\Gamma = \Sigma_{t-1}^z$. As in the previous proof, we denote the sigma-algebra $\mathcal{F}_{t-1}$ with $\mathcal{A} := \mathcal{F}_{t-1}$. Notice that if $z \in \mathbf{Z}$ is such that $0 \in ri(\Delta S_t(\Gamma))^{cc}$ then $\cup_{i=1}^{\beta_{t,z}} B_{t,z}^i = \varnothing$, hence we may assume that $0 \notin ri(\Delta S_t(\Gamma))^{cc}$ . From the proof of Lemma 29

$$\bigcup_{z \in \mathbf{Z}} D_t^z \subseteq N_0 = N \cup N^*$$

with $Q(N_0) = 0$. Notice that if $\omega \in \Omega \backslash N_0$ then, for all $z \in \mathbf{Z}$, either $\omega \notin \Sigma_{t-1}^z$ or $Q_{\mathcal{A}}(\omega, \cup_{i=1}^{\beta_{t,z}} B_{t,z}^i) = 0$. Hence $\omega \in \Sigma_{t-1}^z \backslash N_0$ implies $Q_{\mathcal{A}}(\omega, \cup_{i=1}^{\beta_{t,z}} B_{t,z}^i) = 0$. By Theorem 28 c) we have $Q_{\mathcal{A}}(\omega, (\Sigma_{t-1}^z)^c) = 0$ for all $\omega \in \Sigma_{t-1}^z \setminus N_0$.

Fix now $\omega \in \Sigma_{t-1}^z \setminus N_0$ and consider the completion $\mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ of $\mathcal{F}_t$ and the unique extension on $\mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ of $Q_{\mathcal{A}}(\omega, \cdot)$, which we denote with $\widehat{Q}_{\mathcal{A}}(\omega, \cdot) : \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)} \to [0,1]$.

From $\mathfrak{B}_t \cap (\Sigma_{t-1}^z)^c \subseteq (\Sigma_{t-1}^z)^c$ and $Q_{\mathcal{A}}(\omega, (\Sigma_{t-1}^z)^c) = 0$ we deduce: $\mathfrak{B}_t \cap (\Sigma_{t-1}^z)^c \in \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ and $\widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t \cap (\Sigma_{t-1}^z)^c) = 0$. From $\mathfrak{B}_t \cap \Sigma_{t-1}^z = \cup_{i=1}^{\beta_{t,z}} B_{t,z}^i$ and $Q_{\mathcal{A}}(\omega, \cup_{i=1}^{\beta_{t,z}} B_{t,z}^i) = 0$ we deduce: $\mathfrak{B}_t \cap \Sigma_{t-1}^z \in \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ and $\widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t \cap \Sigma_{t-1}^z) = 0$. Then $\mathfrak{B}_t = (\mathfrak{B}_t \cap \Sigma_{t-1}^z) \cup (\mathfrak{B}_t \cap (\Sigma_{t-1}^z)^c) \in \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$ and $\widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t) = 0$. Since $\omega \in \Sigma_{t-1}^z \setminus N_0$ was arbitrary, we showed that $\widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t) = 0$ for all $\omega \in \Sigma_{t-1}^z \setminus N_0$ and all $z \in \mathbf{Z}$. Since $\bigcup_{z \in \mathbf{Z}} (\Sigma_{t-1}^z \setminus N_0) = \Omega \setminus N_0$ we have:

$$\mathfrak{B}_t \in \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)} \text{ and } \widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t) = 0 \text{ for all } \omega \in \Omega \setminus N_0 \text{ with } Q(N_0) = 0. \tag{19}$$

Now consider the sigma algebra

$$\widehat{\mathcal{F}}_t = \bigcap_{\omega \in \Omega \backslash N_0} \mathcal{F}_t^{Q_{\mathcal{A}}(\omega,\cdot)}$$

and observe that $\mathfrak{B}_t \in \widehat{\mathcal{F}}_t$. Notice that if a subset $B \subseteq \Omega$ satisfies: $B \subseteq C$ for some $C \in \mathcal{F}_t$ with $Q_{\mathcal{A}}(\omega, C) = 0$ for all $\omega \in \Omega \setminus N_0$, then

$$Q(C) = \int_\Omega Q_{\mathcal{A}}(\omega, C) Q(d\omega) = \int_{\Omega \backslash N_0} Q_{\mathcal{A}}(\omega, C) Q(d\omega) = 0,$$

so that $B \in \mathcal{F}_t^Q$. This shows that $\mathcal{F}_t \subseteq \widehat{\mathcal{F}}_t \subseteq \mathcal{F}_t^Q$. Hence $\mathfrak{B}_t \in \mathcal{F}_t^Q$. Let $\widehat{Q} : \widehat{\mathcal{F}}_t \to [0,1]$ be defined by $\widehat{Q}(\cdot) := \int_\Omega \widehat{Q}_{\mathcal{A}}(\omega, \cdot) Q(d\omega)$. Then $\widehat{Q}$ is a probability which satisfies $\widehat{Q}(B) = Q(B)$ for every $B \in \mathcal{F}_t$ and therefore is an extension on $\widehat{\mathcal{F}}_t$ of $Q$. Since $\overline{Q} : \mathcal{F}_t^Q \to [0,1]$ is the unique extension on $\mathcal{F}_t^Q$ of $Q$ and $\mathcal{F}_t \subseteq \widehat{\mathcal{F}}_t \subseteq \mathcal{F}_t^Q$ then $\widehat{Q}$ is the restriction of $\overline{Q}$ on $\widehat{\mathcal{F}}_t$ and

$$\overline{Q}(\mathfrak{B}_t) = \widehat{Q}(\mathfrak{B}_t) = \int_\Omega \widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t) Q(d\omega) = \int_{\Omega \backslash N_0} \widehat{Q}_{\mathcal{A}}(\omega, \mathfrak{B}_t) Q(d\omega) = 0.$$

Suppose now $A \in \mathcal{F}_{t-1}$, $\Gamma = A^z_{t-1}$ and set $\mathfrak{C}_t := \bigcup_{z\in\mathbf{Z}} \left\{ \cup_{i=1}^{\beta_{t,z}} B^i_{t,z} \right\}$ where $B^i_{t,z}$ is given in Lemma 20 with $\Gamma = A^z_{t-1}$. Fix any $\omega \in A$. Then $\Sigma_t^{S_{0:T}(\omega)} \subseteq A$ since $A \in \mathcal{F}_{t-1}$. As a consequence $\mathfrak{C}_t \subseteq \mathfrak{B}_t$.
■

**Corollary 31** *Fix $t \in I_1 = \{1, ..., T\}$ and for any $A \in \mathcal{F}_{t-1}$ consider $A^z_{t-1} = \{\omega \in A \mid S_{0:t-1}(\omega) = z_{0:t-1}\} \neq \varnothing$. Then for any $Q \in \mathcal{M}$ the set $\bigcup\{A^z_{t-1} \mid 0 \notin conv\big(\Delta S_t(A^z_{t-1})\big)\}$ is a subset of an $\mathcal{F}_{t-1}$-measurable $Q$-null set and as a consequence is an $\mathcal{M}$-polar set.*

**Proof.** From Corollary 27 2), the condition $0 \notin conv\big(\Delta S_t(A^z_{t-1})\big)$ implies that $\cup_{i=1}^{\beta_{t,z}} B^i_{t,z} = A^z_{t-1}$, thus: $Q_{\mathcal{A}}(\omega, A^z_{t-1}) = 1$ on $A^z_{t-1} \setminus N$, $D^z_t = \{\omega \in A^z_{t-1}$ s.t. $Q_{\mathcal{F}_{t-1}}(\omega, A^z_{t-1}) > 0\} \supseteq A^z_{t-1} \setminus N$ and

$$\Big(\bigcup\{A^z_{t-1} \mid 0 \notin conv\big(\Delta S_t(A^z_{t-1})\big)\} \setminus N\Big) \subseteq \bigcup_{z\in\mathbf{Z}} D^z_t \subseteq N_0 \in \mathcal{F}_{t-1}.$$

■

### 4.3.1 Backward effect in the multiperiod case

The following example shows that additional care is required in the multi-period setting:

**Example 32** *Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and consider a single risky asset $S_t$ with $t = 0, 1, 2$.*

$$S_0 = 7 \qquad S_1(\omega) = \begin{cases} 8 & \omega \in \{\omega_1, \omega_2\} \\ 3 & \omega \in \{\omega_3, \omega_4\} \end{cases} \qquad S_2(\omega) = \begin{cases} 9 & \omega = \omega_1 \\ 6 & \omega = \omega_2 \\ 5 & \omega = \omega_3 \\ 4 & \omega = \omega_4 \end{cases}$$

*Fix $z \in \mathbf{Z}$ with the first two components $(z_0, z_1)$ equal to $(7, 3)$.*
***First period:*** *$\Sigma^z_0 = \Omega$ and $0 \in ri(conv(\Delta S_1(\Sigma^z_0))) = (-4, 1)$ and there exists $Q_1$ such that $Q_1(\omega_i) > 0$ for $i = 1, 2, 3, 4$ and $S_0 = E_{Q_1}[S_1]$. If we restrict the problem to the first period only, there exists a full support martingale measure for $(S_0, S_1)$ and there are no $\mathcal{M}$-polar sets.*
***Second period:*** *$\Sigma^z_1 = \{\omega_3, \omega_4\}$, $0 \notin conv(\Delta S_2(\Sigma^z_1)) = [1, 2]$ and hence $\Sigma^z_1$ is not supported by any martingale measure for $S$, i.e. if $Q \in \mathcal{M}$ then $Q(\{\omega_3, \omega_4\}) = 0$.*
***Backward:*** *As $\{\omega_3, \omega_4\}$ is a $Q$ null set for any martingale measure $Q \in \mathcal{M}$, then $Q(\{\omega_1, \omega_2\}) = 1$. This reflects into the first period as $0 \notin conv(\Delta S_1(\{\omega_1, \omega_2\})) = \{1\}$ and we deduce that also $\{\omega_1, \omega_2\}$ is not supported by any martingale measure, implying $\mathcal{M} = \varnothing$.*

*This example thus shows that new $\mathcal{M}$-polar sets (as $\{\omega_3, \omega_4\}$) can arise at later times creating a backward effect on the existence martingale measures. In order to detect these situations at time $t$, we shall need to anticipate certain polar sets at posterior times.*

More formally we need to consider the following iterative procedure. Let

$$\begin{aligned} \Omega_T &:= \quad \Omega \\ \Omega_{t-1} &:= \quad \Omega_t \setminus \bigcup_{z\in\mathbf{Z}} \{\Sigma^z_{t-1} \mid 0 \notin conv\big(\Delta S_t(\widetilde{\Sigma}^z_{t-1})\big)\}, \quad t \in I_1, \end{aligned}$$

where

$$\widetilde{\Sigma}^z_{t-1} := \{\omega \in \Omega_t \mid S_{0:t-1} = z_{0:t-1}\}, \quad t \in I_1.$$

We show that the set $B^i_{t,z}$ obtained from Lemma 20 with $\Gamma = \widetilde{\Sigma}^z_t$ belong to the family of polar set of $\mathcal{M}(\mathbb{F})$:

$$\mathcal{N} := \{A \subseteq A' \in \mathcal{F} \ \mid \ Q(A') = 0 \ \forall \ Q \in \mathcal{M}(\mathbb{F})\}$$

More precisely,

**Lemma 33** *For all $t \in I_1$ and $z \in \mathbf{Z}$ consider the sets $B^i_{t,z}$ from Lemma 20 with $\Gamma = \widetilde{\Sigma}^z_{t-1}$. Let*

$$\widetilde{\mathfrak{B}}_t := \bigcup_{z \in \mathbf{Z}} \left\{ \cup_{i=1}^{\beta_{t,z}} B^i_{t,z} \right\} \quad \mathfrak{D}_{t-1} := \bigcup_{z \in \mathbf{Z}} \left\{ \Sigma^z_{t-1} \mid 0 \notin conv(\Delta S_t(\widetilde{\Sigma}^z_{t-1})) \right\}$$

*For any $Q \in \mathcal{M}$, $\widetilde{\mathfrak{B}}_t$ is a subset of a $\mathcal{F}_t$-measurable $Q$-null set and $\mathfrak{D}_{t-1}$ is a subset of an $\mathcal{F}_{t-1}$-measurable $Q$-null set.*

**Proof.** We prove this by backward induction. For $t = T$ the assertion is true from Corollary 30 and Corollary 31. Suppose now the claim holds true for any $k+1 \leq t \leq T$. From the inductive hypothesis there exists $N^Q_k \in \mathcal{F}_k$ such that $\mathfrak{D}_k \subseteq N^Q_k$ with $Q(N^Q_k) = 0$. Introduce the auxiliary $\mathcal{F}_k$-measurable random variable

$$X^Q_k := S_{k-1} \mathbf{1}_{N^Q_k} + S_k \mathbf{1}_{(N^Q_k)^c} \tag{20}$$

and notice that $E_Q[X^Q_k \mid \mathcal{F}_{k-1}] = S_{k-1}$ $Q$-a.s. Moreover from $\Delta X^Q_k := X^Q_k - S_{k-1} = 0$ on $N^Q_k$ and $\Omega \setminus N^Q_k \subseteq \Omega \setminus \mathfrak{D}_k$, we can deduce that

$$0 \notin ri(\Delta S_k(\widetilde{\Sigma}^z_{k-1}))^{cc} \Longrightarrow 0 \notin ri(\Delta X^Q_k(\Sigma^z_{k-1}))^{cc} \tag{21}$$

which implies $\widetilde{\mathfrak{B}}_k \subseteq \mathfrak{B}_k(X^Q_k) \cup N^Q_k$ where we denote $\mathfrak{B}_k(X^Q_k)$ the set obtained from Corollary 30 with $\Gamma = \Sigma^z_{k-1}$ and $X^Q_k$ which replaces $S_k$. According to Corollary 30 we find $M^Q_k \in \mathcal{F}_k$ with $Q(M^Q_k) = 0$ such that $\widetilde{\mathfrak{B}}_k \subseteq \mathfrak{B}_k(X^Q_k) \cup N^Q_k \subseteq M^Q_k \cup N^Q_k$. Since $Q$ is arbitrary we have the thesis. We now show the second assertion.
For every $Q \in \mathcal{M}$ and $\underline{\varepsilon} = (\varepsilon, ..., \varepsilon) \in \mathbb{R}^d$ with $\varepsilon > 0$ we can define

$$S^Q_k = (S_{k-1} + \underline{\varepsilon}) \mathbf{1}_{N^Q_k \cup M^Q_k} + S_k \mathbf{1}_{(N^Q_k \cup M^Q_k)^c} \tag{22}$$

and $E_Q[S^Q_k \mid \mathcal{F}_{k-1}] = S_{k-1}$. With $\Delta S^Q_k := S^Q_k - S_{k-1}$ we claim

$$\mathfrak{D}_{k-1} \subseteq \bigcup_{z \in \mathbf{Z}} \{ \Sigma^z_{k-1} \mid 0 \notin conv(\Delta S^Q_k(\Sigma^z_{k-1})) \}. \tag{23}$$

Indeed let $z \in \mathbf{Z}$ such that $\Sigma^z_{k-1} \subseteq \mathfrak{D}_{k-1}$ and observe that

$$0 \notin conv(\Delta S_k(\widetilde{\Sigma}^z_{k-1})) \Leftrightarrow 0 \notin conv(\Delta S_k(\Sigma^z_{k-1} \setminus \mathfrak{D}_k)). \tag{24}$$

Since $\Sigma^z_{k-1} \setminus N^Q_k \subseteq \Sigma^z_{k-1} \setminus \mathfrak{D}_k \subseteq \widetilde{\mathfrak{B}}_k \subseteq N^Q_k \cup M^Q_k$, then

$$\begin{aligned} \Sigma^z_{k-1} &= (\Sigma^z_{k-1} \cap N^Q_k) \cup (\Sigma^z_{k-1} \setminus N^Q_k) \subseteq N^Q_k \cup M^Q_k \\ &\subseteq \bigcup_{z \in \mathbf{Z}} \{ \Sigma^z_{k-1} \mid 0 \notin conv(\Delta S^Q_k(\Sigma^z_{k-1})) \} \end{aligned}$$

for any $\Sigma^z_{k-1} \subseteq \mathfrak{D}_{k-1}$. Hence the claim since $\bigcup_z \{ \Sigma^z_{k-1} \mid 0 \notin conv(\Delta S^Q_k(\Sigma^z_{k-1})) \}$ is a subset of an $\mathcal{F}_{k-1}$-measurable $Q$-null set. ■

## 4.4 On the maximal $\mathcal{M}$-polar set and the support of martingale measures

The sets introduced in Sections 4.2 and 4.3.1 provide a geometric decomposition of $\Omega$ in two parts, $\Omega = \Omega_* \cup (\Omega_*)^c$ specified in Proposition 34 below. The set $\Omega_*$ contains those events $\omega$ supported by martingale measures, namely, for any of those events it is possible to construct a martingale measure (even with finite support) that assign positive probability to $\omega$. Observe that such a decompostion is induced by $S$ and it is determined prior to arbitrage considerations.

**Proposition 34** *Let* $\{\Omega_t\}_{t\in I}$ *as defined in Section 4.5 and, for any* $z \in \mathbf{Z}$, *let* $\beta_{t,z}$ *and* $B^*_{t,z}$ *be the index* $\beta$ *and the set* $B^*$ *from Lemma 20 with* $\Gamma = \widetilde{\Sigma}^z_{t-1}$. *Define*

$$\Omega_* := \bigcap_{t=1}^{T} \left( \bigcup_{z\in\mathbf{Z}} B^*_{t,z} \right).$$

*We have the following*

$$\mathcal{M} \neq \varnothing \Longleftrightarrow \Omega_* \neq \varnothing \Longleftrightarrow \mathcal{M} \cap \mathcal{P}_f \neq \varnothing,$$

*where*

$$\mathcal{P}_f := \{P \in \mathcal{P} \mid supp(P) \text{ is finite}\}$$

*is the set of probability measures whose support is a finite number of* $\omega \in \Omega$.

*If* $\mathcal{M} \neq \varnothing$ *then for any* $\omega_* \in \Omega_*$ *there exists* $Q \in \mathcal{M}$ *such that* $Q(\{\omega_*\}) > 0$, *so that* $(\Omega_*)^c$ *is the maximal* $\mathcal{M}$*-polar set, i.e.* $(\Omega_*)^c$ *is an* $\mathcal{M}$*-polar set and*

$$\forall N \in \mathcal{N} \text{ we have } N \subseteq (\Omega_*)^c. \tag{25}$$

**Proof.** Observe first that:

$$(\Omega_*)^c = \bigcup_{t=1}^{T} \widetilde{\mathfrak{B}}_t.$$

From Lemma 33, $\widetilde{\mathfrak{B}}_t$ is an $\mathcal{M}$-polar set for any $t \in I_1$, which implies $(\Omega_*)^c$ is an $\mathcal{M}$-polar set.
Suppose now that $\Omega_* = \varnothing$ so that $\Omega = \bigcup_{t=1}^{T} \widetilde{\mathfrak{B}}_t$ is a polar set. We can conclude that $\mathcal{M} = \varnothing$.
Suppose now that $\Omega_* \neq \varnothing$. We show that for every $\omega_* \in \Omega_*$ there exists a $Q \in \mathcal{M}$ such that $Q(\{\omega_*\}) > 0$. Observe now that for any $t \in I_1$ and for any $\omega \in \Omega_*$, $0 \in ri(\Delta S_t(B^*_{t,z}))^{cc}$ with $z = S_{0:T}(\omega)$. As we did in Corollary 25, we apply Remark 24 and conclude that there exists a finite number of elements of $B^*_{t,z}$, named $C_t(\omega) := \{\omega, \omega_1, \ldots, \omega_m\} \subseteq B^*_{t,z}$, such that

$$S_{t-1}(\omega) = \lambda_t(\omega) S_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j) S_t(\omega_j) \tag{26}$$

where $\lambda_t(\omega) > 0$ and $\lambda_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j) = 1$.
Fix now $\omega_* \in \Omega_*$. We iteratively build a set $\Omega^T_f$ which is suitable for being the finite support of a discrete martingale measure (and contains $\omega_*$).
Start with $\Omega^1_f = C_1(\omega_*)$ which satisfies (26) for $t = 1$. For any $t > 1$, given $\Omega^{t-1}_f$, define $\Omega^t_f := \left\{ C_t(\omega) \mid \omega \in \Omega^{t-1}_f \right\}$. Once $\Omega^T_f$ is settled, it is easy to construct a martingale measure via (26):

$$Q(\{\omega\}) = \prod_{t=1}^{T} \lambda_t(\omega) \quad \forall \omega \in \Omega^T_f$$

Since, by construction, $\lambda_t(\omega_*) > 0$ for any $t \in I_1$, we have $Q(\{\omega_*\}) > 0$ and $Q \in \mathcal{M} \cap \mathcal{P}_f$.

To show (25) just observe from the previous line that $\Omega_*$ is not $\mathcal{M}$-polar, while $(\Omega_*)^c = \bigcup_{t=1}^{T} \widetilde{\mathfrak{B}}_t$ is $\mathcal{M}$-polar thanks to Lemma 33. ■

**Proof of Proposition 17.** The absence of $1p$-Arbitrages readly implies that $\Omega_* = \Omega$ (see Corollary 27). Take a dense subset $\{\omega_n\}_{n\in\mathbb{N}}$ of $\Omega$: from Proposition 34 for any $\omega_n$ there exists a martingale measure $Q^n \in \mathcal{M}$ such that $Q^n(\{\omega_n\}) > 0$. From Lemma 76 in the Appendix $Q := \sum_{i=1}^{\infty} \frac{1}{2^i} Q^i \in \mathcal{M}$, moreover $Q(\{\omega_n\}) > 0$ $\forall n \in \mathbb{N}$. Since $\{\omega_n\}_{n\in\mathbb{N}}$ is dense, $Q$ is a full support martingale measure. ■

## 4.5 Enlarged Filtration and Universal Arbitrage Aggregator

In Sections 4.2 and 4.3 we solve the problem of characterizing the $\mathcal{M}$-polar sets of a certain market model on a fixed time interval $[t-1,t]$ for $t \in I_1 = \{1,...,T\}$. In particular, if we look at the level sets $\Sigma_{t-1}^z$ of the price process $(S_t)_{t\in I}$, we can identify the component of these sets that must be polar (Corollary 30) which coincides with the whole level set when $0 \notin conv(\Delta S_t(\Sigma_{t-1}^z))$ (Corollary 31). Further care is required in the multiperiod case due to the backward effects (see Section 4.3.1), but nevertheless a full characterization of $\mathcal{M}$-polar sets is obtained in Section 4.4 .

In this section we build a predictable strategy that embrace all the inefficiencies of the market. Unfortunately, even on a single time-step, the polar set given by Corollary 30 belongs, in general, to $\mathfrak{F}_t$ (the universal $\mathcal{M}$-completion), hence the trading strategies suggested by equation (15) in Lemma 20 fail to be predictable. This reflects into the necessity of enlargement of the original filtration by anticipating some one step-head information. Under this filtration enlargement, which depends only on the underlying structure of the market, the set of martingale measures will not change (see Lemma 41).

**Definition 35** *We call Universal Arbitrage Aggregator the strategy*

$$H_t^\bullet(\omega)\mathbf{1}_{\Sigma_{t-1}^z} := \begin{cases} H_{t,z}(\omega) & on \ \bigcup_{i=1}^{\beta_{t,z}} B_{t,z}^i \\ 0 & on \ \Sigma_{t-1}^z \setminus \bigcup_{i=1}^{\beta_{t,z}} B_{t,z}^i \end{cases}, \quad t \in I_1 = \{1,...,T\}, \tag{27}$$

*where* $z \in \mathbf{Z}$ *satisfies* $z_{0:t-1} = S_{0:t-1}(\omega)$, $H_{t,z}$, $B_{t,z}^i, B_{t,z}^*$ *comes from* (15) *and Lemma 20 with* $\Gamma = \widetilde{\Sigma}_{t-1}^z$.
*This strategy is predictable with respect to the enlarged filtration* $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}_t\}_{t\in I}$ *given by*

$$\widetilde{\mathcal{F}}_t \ : \ = \mathcal{F}_t \vee \sigma(H_1^\bullet,...,H_{t+1}^\bullet), \ t \in \{0,...,T-1\} \tag{28}$$

$$\widetilde{\mathcal{F}}_T \ : \ = \mathcal{F}_T \vee \sigma(H_1^\bullet,...,H_T^\bullet). \tag{29}$$

**Remark 36** *The strategy* $H^\bullet$ *in equation* (27) *satisfies* $V_T(H^\bullet) \geq 0$ *and*

$$\mathcal{V}_{H^\bullet}^+ = \bigcup_{t=1}^T \widetilde{\mathfrak{B}}_t. \tag{30}$$

*Indeed, from Lemma 20* $H_{t,z} \cdot \Delta S_t > 0$ *on* $\bigcup_{i=1}^{\beta_{t,z}} B_{t,z}^i$, *so that* $\bigcup_{t=1}^T \widetilde{\mathfrak{B}}_t \subseteq \mathcal{V}_{H^\bullet}^+$. *On the other hand,* $\mathcal{V}_{H^\bullet}^+ \subseteq \{H_t^\bullet \neq 0 \text{ for some } t\} \subseteq \bigcup_{t=1}^T \widetilde{\mathfrak{B}}_t$.
*For* $t < T$ *we therefore conclude that* $\widetilde{\mathcal{F}}_t \subseteq \mathcal{F}_t \vee \bigcup_{s=1}^{t+1} \mathcal{N}_s \subseteq \mathfrak{F}_t$, *where*

$$\mathcal{N}_t := \left\{ A = \bigcup_{z\in\mathbf{V}} \bigcup_{i\in J(z)} B_{t,z}^i \mid \text{for some} \quad \begin{matrix} \mathbf{V} \subseteq \mathbf{Z} \\ J(z) \subseteq \{1,...,\beta_{t,z}\} \end{matrix} \right\} \cup \mathfrak{D}_t,$$

*while for* $t = T$, $\widetilde{\mathcal{F}}_T \subseteq \mathcal{F}_T \vee \bigcup_{s=1}^T \mathcal{N}_s \subseteq \mathfrak{F}_T$.

*For any* $Q \in \mathcal{M}$ *and* $t \in I$, *any element of* $\mathcal{N}_t$ *is a subset of a* $\mathcal{F}_t$*-measurable* $Q$*-null set .*

From now on we will assume that the class of admissible trading strategies $\widetilde{\mathcal{H}}$ is given by all $\widetilde{\mathbb{F}}$ predictable processes. We can rewrite the definition of Arbitrage de la classe $\mathcal{S}$ using strategies adapted to $\widetilde{\mathbb{F}}$. Namely, an Arbitrage de la classe $\mathcal{S}$ with respect to $\widetilde{\mathcal{H}}$ is an $\widetilde{\mathbb{F}}$-predictable processes $H = [H^1, \ldots, H^d]$ such that $V_T(H) \geq 0$ and $\{V_T(H) > 0\}$ contains a set in $\mathcal{S}$.

**Remark 37** *No Arbitrage de la classe* $\mathcal{S}$ *with respect to* $\widetilde{\mathcal{H}}$ *implies No Arbitrage de la classe* $\mathcal{S}$ *with respect to* $\mathcal{H}$.

**Remark 38 (Financial interpretation of the filtration enlargement)** *Fix $t \in I_1$, $z \in \mathbf{Z}$, the event $\Sigma_{t-1}^z = \{S_{0:t-1} = z_{0:t-1}\}$ and suppose the market presents the opportunity given by $0 \notin ri(\Delta S_t(\Sigma_{t-1}^z))^{cc}$. Consider two probabilities $P_k \in \mathcal{P}$, $k = 1, 2$, for which $P_k(\Sigma_{t-1}^z) > 0$. Following Lemma 20, if $j_k := \inf\{i = 1, \ldots, \beta \mid P_k(B_{t,z}^i) > 0\} < \infty$, then the rational choice for the strategy is $H^{j_k}$, as shown in Corollary 23. Thus it is possible that $j_k < \infty$ holds for both probabilities, so that the two agents represented by $P_1$ and $P_2$ agree that $\Sigma_{t-1}^z$ is a non-efficient level set of the market, although it is possible that $j_1 \neq j_2$ so that they might not agree on the trading strategy $H^{j_k}$ that establish the $P_k$-Classical Arbitrage on $\Sigma_{t-1}^z$. In such case, these two arbitrages are realized on different subsets of $\Sigma_{t-1}^z$ and generate different payoffs. Nevertheless note that any of these agents is able to find an arbitrage opportunity among the finite number of trading strategies $\{H_{t,z}^i\}_{i=1}^{\beta_{t,z}}$ given by Lemma 20 (recall $\beta_{t,z} \leq d$). The filtration enlargement allows to embrace them all. This can be referred to the analogous discussion in [DH07]: "A weak arbitrage opportunity is a situation where we know there must be an arbitrage but we cannot tell, without further information, what strategy will realize it".*

We expand on this argument more formally. Recall that Lemma 20 provides a partition of any level set $\widetilde{\Sigma}_{t-1}^z$ with the following property: for any $\omega \in (\Omega_*)^c$ there exists a unique set $B_{t,z}^i$, identified by $i = i(\omega)$, such that $\omega \in B_{t,z}^i$ with $z = S_{0:T}(\omega)$. Define therefore, for any $t \in I_1$ the multifunction

$$\mathbb{H}_t(\omega) := \left\{ H \in \mathbb{R}^d \text{ s.t. } H \cdot \Delta S_t(\widehat{\omega}) \geq 0 \text{ for any } \widehat{\omega} \in \cup_{j=i(\omega)}^{\beta_{t,z}} B_{t,z}^j \cup B_{t,z}^* \right\} \quad \text{if } \omega \in (\Omega_*)^c \tag{31}$$

and $\mathbb{H}_t(\omega) = \{0\}$ otherwise.
Observe that for any $t \in I_1$, if $\omega_1, \omega_2$ satisfy $S_{0:t-1}(\omega_1) = S_{0:t-1}(\omega_2)$ and $i(\omega_1) = i(\omega_2)$ they belong to the same $B_{t,z}^i$ and $\mathbb{H}_t(\omega_1) = \mathbb{H}_t(\omega_2)$. In other words $\mathbb{H}_t$ is constant on any $B_{t,z}^i$ and therefore for any open set $V \subseteq \mathbb{R}^d$ we have

$$\{\omega \in \Omega \mid \mathbb{H}_t(\omega) \cap V \neq \varnothing\} = \bigcup_{z \in \mathbf{Z}} \bigcup_{i=1}^{\beta_{t,z}} \{B_{t,z}^i \mid \mathbb{H}_t(B_{t,z}^i) \cap V \neq \varnothing\}$$

from which $\mathbb{H}_t$ is measurable with respect to $\mathcal{F}_{t-1} \vee \bigcup_{s=1}^t \mathcal{N}_s$. Note that since $H_t^\bullet(\omega) \in \mathbb{H}_t(\omega)$ for any $\omega \in \Omega$, we have that $H_t^\bullet$ is a selection of $\mathbb{H}_t$ with the same measurability. We now show how the process $\mathbb{H} := (\mathbb{H}_t)_{t \in I_1}$ provides $P$-Classical Arbitrage as soon as we choose a probabilistic model $P \in \mathcal{P}$ which is not absolutely continuous with respect to $\nu := \sup_{Q \in \mathcal{M}} Q$. The case of $P \ll \nu$ is discussed in Remark 40.

**Theorem 39** *Let $\mathbb{H}$ be defined in (31). If $P \in \mathcal{P}$ is not absolutely continuous with respect to $\nu$ then there exists an $\mathbb{F}^P$-predictable trading strategy $H^P$ which is a $P$-Classical Arbitrage and*

$$H^P(\omega) \in \mathbb{H}(\omega) \qquad P\text{-}a.s.$$

*where $\mathcal{F}_t^P$ denote the $P$-completion of $\mathcal{F}_t$ and $\mathbb{F}^P := \{\mathcal{F}_t^P\}_{t \in I}$.*

**Proof.** See Appendix 7.1. ■

From Lemma 43 if $P \in \mathcal{P}$ fulfills the hypothesis of Theorem 39 there exists an $\mathcal{F}$-measurable set $F \subseteq (\Omega_*)^c$ with $P(F) > 0$. Note that from Remark 69 such a $P$ always exists if $(\Omega_*)^c \neq \varnothing$. Theorem 39 asserts therefore that for any probabilistic models which supports $(\Omega_*)^c$ an $\mathbb{F}^P$-*predictable* arbitrage opportunity can be found among the values of the set-valued process $\mathbb{H}$. This property suggested us to baptize $\mathbb{H}$ as the Universal Arbitrage Aggregator and thus $H^\bullet$ as a (standard) selection of the Universal Arbitrage Aggregator. Note that we could have considered a different selection of $\mathbb{H}$ satisfying the essential requirement (30). Since this choice does not affect any of our results we simply take $H^\bullet$.

**Remark 40** *Recall from (2) that any $P \in (\mathcal{P}_0)^c$ admits a $P$-Classical Arbitrage opportunity. We can distinguish between two different classes in $(\mathcal{P}_0)^c$. The first one is: $\mathcal{P}_{\mathcal{M}} := \{P \in (\mathcal{P}_0)^c \mid P << \nu\}$ or, in other words, an element $P \in (\mathcal{P}_0)^c$ belong to $\mathcal{P}_{\mathcal{M}}$ iff any subset of $(\Omega_*)^c$ is $P$-null. Then for each probability $P$ in this class, there exists a probability $P'$ with larger support that annihilates any $P$-Classical Arbitrage opportunity. Recall Example 26 where $\Omega_* = \mathbb{Q} \cap [1/2, +\infty)$. By choosing $P = \delta_{\{\frac{1}{2}\}} \in \mathcal{P}_{\mathcal{M}}$ we clearly have $P$-Classical Arbitrages. Nevertheless by simply taking $P' = \lambda\delta_{\{\frac{1}{2}\}} + (1-\lambda)\delta_{\{2\}}$ for some $0 < \lambda < 1$ this market is arbitrage free. From a model-independent point of view these situations must not considered as market inefficiencies since they vanish as soon as more trajectories are considered. This feature is captured by the Universal Arbitrage Aggregator by means of the property: $H^\bullet = 0$ on $\Omega_*$.*

*On the other hand when $P \in (\mathcal{P}_0)^c \setminus \mathcal{P}_{\mathcal{M}}$ then $P$ assigns a positive measure to some $\mathcal{M}$-polar $\mathcal{F}$-measurable set $F \in \mathcal{N}$. Therefore, any other $P' \in \mathcal{P}$ with larger support will satisfy $P'(F) > 0$ and the probabilistic model $(\Omega, \mathcal{F}, \mathbb{F}, S, P')$ will also exhibit $P'$-Classical Arbitrages. In the case of Example 26 $(\Omega_*)^c = B^1 \cup B^2$ where $B^1 = \mathbb{R}^+ \setminus \mathbb{Q}$ and $B^2 = \mathbb{Q} \cap [0, 1/2)$. If $P((\Omega_*)^c) > 0$ the market exhibits a $P$-Classical Arbitrage, but this is still valid for any probabilistic model given by $P'$ with $P << P'$. In particular if $P'(B^1) > 0$ then $H^1 := [0, 0, 1]$ is a $P'$-Classical Arbitrage, while if $P'(B^1) = 0$ and $P'(B^2) > 0$ then $H^2 := [1, 0, 0]$ is the desired strategy. In this example, $H_1^\bullet = H^1 1_{\{\mathbb{R}^+ \setminus \mathbb{Q}\}} + H^2 1_{\{\mathbb{Q} \cap [0,1/2)\}}$.*

**Lemma 41** *$\mathcal{M}(\mathbb{F}) \leftrightarrows \mathcal{M}(\widetilde{\mathbb{F}})$ with the following meaning*

- *the restriction of any $\widetilde{Q} \in \mathcal{M}(\widetilde{\mathbb{F}})$ to $\mathcal{F}_T$ belongs to $\mathcal{M}(\mathbb{F})$;*
- *any $Q \in \mathcal{M}(\mathbb{F})$ can be uniquely extended to an element of $\mathcal{M}(\widetilde{\mathbb{F}})$*

**Proof.** Let $\widetilde{Q} \in \mathcal{M}(\widetilde{\mathbb{F}})$ and $Q \in \mathcal{P}(\Omega)$ be the restriction to $\mathcal{F}_T$. For any $t \in I_1$ and $A \in \mathcal{F}_{t-1}$ we have $E_Q[(S_t - S_{t-1})\mathbf{1}_A] = E_{\widetilde{Q}}[(S_t - S_{t-1})\mathbf{1}_A] = 0$. Let now $Q \in \mathcal{M}(\mathbb{F})$. There exists a unique extension to $\widetilde{\mathcal{F}}_t$ of $Q$ that we call $\widetilde{Q}$. For any $\widetilde{A} \in \widetilde{\mathcal{F}}_{t-1}$ with $t \in I_1$ there exists $A \in \mathcal{F}_{t-1}$ such that $\widetilde{Q}(\widetilde{A}) = \widetilde{Q}(A) = Q(A)$. Hence $E_{\widetilde{Q}}[(S_t - S_{t-1})\mathbf{1}_{\widetilde{A}}] = E_Q[(S_t - S_{t-1})\mathbf{1}_A] = 0$. We conclude that $E_{\widetilde{Q}}[S_t \mid \widetilde{\mathcal{F}}_{t-1}] = S_{t-1}$, hence $\widetilde{Q} \in \mathcal{M}(\widetilde{\mathbb{F}})$. ■

**Remark 42** *The filtration enlargement $\widetilde{\mathbb{F}}$ has been introduced to guarantee the aggregation of 1p-Arbitrages on the sets $B^i_{t,z}$ obtained from Lemma 20 with $\Gamma = \widetilde{\Sigma}^z_{t-1}$. If indeed we follow [C12] we can consider any collection of probability measures $\Theta_t := \{P^i_{t,z}\}$ on $(\Omega, \mathcal{F})$ such that $P^i_{t,z}(B^i_{t,z}) = 1$. Observe first that*

$$\mathcal{F}_t^{\Theta_t} \supseteq \sigma\left(\bigcup\{B^i_{t,z} \mid z \in \mathbf{V}, i \in J(z)\}\right)$$

*with $\mathbf{V}$ and $J(z)$ arbitrary. For any $P^i_{t,z}$ we have indeed that $\mathcal{F}_t^{P^i_{t,z}}$ contains any subset of $(B^i_{t,z})^c$. Therefore if $A = \bigcup\{B^i_{t,z} \mid z \in \mathbf{V},\ i \in J(z)\}$ we have*

- *if $z \notin \mathbf{V}$ or $i \notin J(z)$ then $A \in \mathcal{F}_t^{P^i_{t,z}}$ trivially because $A \subset (B^i_{t,z})^c$*
- *if $z \in \mathbf{V}$ and $i \in J(z)$ then $A \in \mathcal{F}_t^{P^i_{t,z}}$ because $A = B^i_{t,z} \cup \bar{A}$ with $\bar{A} \subseteq (B^i_{t,z})^c$*

*It is easy to check that $\Theta_t$ has the Hahn property on $\mathcal{F}_t$ as defined in Definition 3.2, [C12], with $\Phi_t := \Theta_t \mid_{\mathcal{F}_t}$. We can therefore apply Theorem 3.16 in [C12] to find an $\mathcal{F}_t^{\Theta_t}$- measurable function $H_t$ such that $H_t = H^i_{t,z}$ $P^i_{t,z}$-a.s. which means that $H_t(\omega) = H^i_{t,z}$ for every $\omega \in B^i_{t,z}$.*

### 4.6 Main Results

Our aim now is to show how the absence of arbitrage de la classe $\mathcal{S}$ provides a pricing functional *via* the existence of a martingale measure with nice properties.
Clearly the "No 1$p$-Arbitrage" condition is the strongest that one can assume in this model independent framework and we have shown in Proposition 17 that it automatically implies the existence of a full support martingale measure. On the other hand we are interested in characterizing those markets which can exhibit 1$p$-Arbitrages but nevertheless admits a rational system of pricing rules.

The set $\Omega_*$ introduced in Section 4.4 has a clear financial interpretation as it represents the set of events for which No 1p-Arbitrage can be found. This is the content of the following Proposition. Let $(\Omega, \widetilde{\mathcal{F}}_T, \widetilde{\mathbb{F}})$, $\widetilde{\mathcal{H}}$ as in Section 4.5 and define

$$\widetilde{\mathcal{H}}^+ := \left\{ H \in \widetilde{\mathcal{H}} \mid \; V_T(H)(\omega) \geq 0 \, \forall \, \omega \in \Omega \text{ and } V_0(H) = 0 \right\}.$$

**Proposition 43** *(1)* $\mathcal{V}^+_{H^\bullet} = \bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H = (\Omega_*)^c$

*(2)* $\mathcal{M} \neq \varnothing$ *if and only if* $\bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H$ *is strictly contained in* $\Omega$.

**Proof.** (2) follows from (1) and Proposition 34. Indeed: $\mathcal{M} \neq \varnothing$ iff $\Omega_* \neq \varnothing$ iff $(\Omega_*)^c \subsetneqq \Omega$ iff $\bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H \subsetneq \Omega$. Now we prove (1). Given (30), we only need to show the inclusion $\bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H \subseteq (\Omega_*)^c$. Let $\overline{\omega} \in \bigcup_{H \in \widetilde{\mathcal{H}}^+} \mathcal{V}^+_H$, then there exists $\overline{H} \in \widetilde{\mathcal{H}}^+$ and $t \in I_1$ such that $\overline{H}_t(\omega) \cdot \Delta S_t(\omega) \geq 0$ $\forall \omega \in \Omega$ and $\overline{H}_t(\overline{\omega}) \cdot \Delta S_t(\overline{\omega}) > 0$. Let $z = S_{0:T}(\overline{\omega})$. From Lemma 20 there exists $i \in \{1, \ldots, \beta_{t,z}\}$ such that $\overline{\omega} \in B^i_{t,z}$ hence we conclude that $\overline{\omega} \in \widetilde{\mathfrak{B}}_t$ and therefore $\overline{\omega} \in (\Omega_*)^c$. ■

**Proof of Theorem 2.** We prove that

$$\exists \text{ an Arbitrage de la classe } \mathcal{S} \text{ in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M} = \varnothing \text{ or } \mathcal{N} \text{ contains sets of } \mathcal{S}.$$

Notice that if $H \in \widetilde{\mathcal{H}}$ satisfies $V_T(H)(\omega) \geq 0$ $\forall \omega \in \Omega$ then, if $\mathcal{M} \neq \varnothing$, $\mathcal{V}^+_H \in \mathcal{N}$, otherwise $0 < \mathbb{E}_Q[V_T(H)] = V_0(H) = 0$ for $Q \in \mathcal{M}$. If there exists an $\widetilde{\mathcal{H}}$-Arbitrage de la classe $\mathcal{S}$ then $\mathcal{V}^+_H$ contains a set in $\mathcal{S}$ and therefore $\mathcal{N}$ contains a set in $\mathcal{S}$. If instead $\mathcal{M} = \varnothing$ we already have the thesis. For the opposite implication, we exploit the Universal Arbitrage $H^\bullet \in \widetilde{\mathcal{H}}$ as defined in equation (27) satisfying $V_T(H^\bullet)(\omega) \geq 0$ $\forall \omega \in \Omega$ and $\mathcal{V}^+_{H^\bullet} = \bigcup_{t=1}^T \widetilde{\mathfrak{B}}_t = (\Omega_*)^c$. If $\mathcal{M} = \varnothing$ then, by Proposition 34, $(\Omega_*)^c = \Omega$ and $H^\bullet$ is an $\widetilde{\mathcal{H}}$-Model Independent Arbitrage and hence (from (4)) $H^\bullet$ is also an Arbitrage de la classe $\mathcal{S}$. If $\mathcal{M} \neq \varnothing$ and $\mathcal{N}$ contains a set $C$ in $\mathcal{S}$ then $C \subseteq (\Omega_*)^c = \mathcal{V}^+_{H^\bullet}$ from (25) and Proposition 43, item 1. Therefore $H^\bullet$ is an $\widetilde{\mathcal{H}}$-Arbitrage de la classe $\mathcal{S}$. ■

**Definition 44** *Define the following convex subset of* $\mathcal{P}$*:*

$$\mathcal{R}_{\mathcal{S}} := \{ Q \in \mathcal{P} \mid Q(C) > 0 \textit{ for all } C \in \mathcal{S} \}. \tag{32}$$

The martingale measures having the property of the class $\mathcal{R}_{\mathcal{S}}$ will be associated to the Arbitrage de la classe $\mathcal{S}$.

**Example 45** *We consider the examples introduced in Definition 10. Suppose there are no Model Independent Arbitrage in* $\widetilde{\mathcal{H}}$*. From Theorem 2 we obtain:*

1. *1$p$-Arbitrage:* $\mathcal{S} = \{ C \in \mathcal{F} \mid C \neq \varnothing \}$.
   - *No 1$p$-Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N} = \varnothing$;
   - $\mathcal{R}_{\mathcal{S}} = \mathcal{P}_+$, *if* $\Omega$ *finite or countable; otherwise* $\mathcal{R}_{\mathcal{S}} = \varnothing$.

- *In the case of np-Arbitrage we have:*

$$\mathcal{R}_{\mathcal{S}} = \{Q \in \mathcal{P} \mid Q(A) > 0 \text{ for all } A \subseteq \Omega \text{ having at least } n \text{ elements}\}$$

*No np-Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *does not contain elements having more than* $n-1$ *elements.*

2. *Open Arbitrage:* $\mathcal{S} = \{C \in \mathcal{B}(\Omega) \mid C \text{ open non-empty}\}$.
   - *No Open Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *does not contain non-empty open sets;*
   - $\mathcal{R}_{\mathcal{S}} = \mathcal{P}_+$.
3. $\mathcal{P}'$*-q.s. Arbitrage:* $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0 \text{ for some } P \in \mathcal{P}'\}$, $\mathcal{P}' \subseteq \mathcal{P}$.
   - *No* $\mathcal{P}'$*-q.s. Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *may contain only* $\mathcal{P}'$*-polar sets;*
   - $\mathcal{R}_{\mathcal{S}} = \{Q \in \mathcal{P} \mid P' \ll Q \text{ for all } P' \in \mathcal{P}'\}$.
4. $P$*-a.s. Arbitrage:* $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0\}$, $P \in \mathcal{P}$.
   - *No* $P$*-a.s. Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *may contain only* $P$*-null sets;*
   - $\mathcal{R}_{\mathcal{S}} = \{Q \in \mathcal{P} \mid P \ll Q\}$.
5. *Model Independent Arbitrage:* $\mathcal{S} = \{\Omega\}$.
   - $\mathcal{R}_{\mathcal{S}} = \mathcal{P}$.
6. $\varepsilon$*-Arbitrage:* $\mathcal{S} = \{C_\varepsilon(\omega) \mid \omega \in \Omega\}$, *where* $\varepsilon > 0$ *is fixed and* $C_\varepsilon(\omega)$ *is the closed ball in* $(\Omega, d)$ *of radius* $\varepsilon$ *and centered in* $\omega$.
   - *No* $\varepsilon$*-Arbitrage in* $\widetilde{\mathcal{H}}$ *iff* $\mathcal{N}$ *does not contain closed balls of radius* $\varepsilon$*;*
   - $\mathcal{R}_{\mathcal{S}} = \{Q \in \mathcal{P} \mid Q(C_\varepsilon(\omega)) > 0 \text{ for all } \omega \in \Omega\}$.

**Corollary 46** *Suppose that the class* $\mathcal{S}$ *has the property:*

$$\exists \{C_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S} \text{ s.t. } \forall C \in \mathcal{S}\ \exists \overline{n} \text{ s.t. } C_{\overline{n}} \subseteq C. \tag{33}$$

*Then:*

$$\text{No Arb. de la classe } \mathcal{S} \text{ in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M} \cap \mathcal{R}_{\mathcal{S}} \neq \varnothing. \tag{34}$$

**Proof.** Suppose $Q \in \mathcal{M} \cap \mathcal{R}_{\mathcal{S}} \neq \varnothing$. Then any polar set $N \in \mathcal{N}$ does not contain sets in $\mathcal{S}$ (otherwise, if $C \in \mathcal{S}$ and $C \subseteq N$ then $Q(C) > 0$ and $Q(N) = 0$, a contradiction). Then, from Theorem 2, No Arbitrage de la classe $\mathcal{S}$ holds true. Conversely, suppose that No Arbitrage de la classe $\mathcal{S}$ holds true so that $\mathcal{M} \neq \varnothing$ and let $\{C_n\}_{n\in C} \subseteq \mathcal{S}$ be the collection of sets in the assumption. From Theorem 2, we obtain that $N \in \mathcal{N}$ does not contain any set in $\mathcal{S}$, and so each set $C_n$ is not a polar set, hence for each $n$ there exists $Q_n \in \mathcal{M}$ such that $Q_n(C_n) > 0$. Set $Q := \sum_{n=1}^{\infty} \frac{1}{2^n} Q_n \in \mathcal{M}$ (see Lemma 76). Take any $C \in \mathcal{S}$ and let $C_{\overline{n}} \subseteq C$. Then

$$Q(C) \geq \frac{1}{2^{\overline{n}}} Q_{\overline{n}}(C) \geq \frac{1}{2^{\overline{n}}} Q_{\overline{n}}(C_{\overline{n}}) > 0$$

and $Q \in \mathcal{M} \cap \mathcal{R}_{\mathcal{S}}$. ■

**Corollary 47** *Let* $\mathcal{S}$ *be the class of non empty open sets. Then the condition (33) is satisfied and therefore*

$$\text{No Open Arbitrage in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M}_+ \neq \varnothing. \tag{35}$$

**Proof.** Consider a dense countable subset $\{\omega_n\}_{n\in\mathbb{N}}$ of $\Omega$, as $\Omega$ is Polish. Consider the open balls:

$$B^m(\omega_n) := \left\{\omega \in \Omega \mid d(\omega, \omega_n) < \frac{1}{m}\right\},\ m \in \mathbb{N},$$

The density of $\{\omega_n\}_{n\in\mathbb{N}}$ implies that $\Omega = \bigcup_{n\in\mathbb{N}} B^m(\omega_n)$ for any $m \in \mathbb{N}$. Take any open set $C \subseteq \Omega$. Then there exists some $\overline{n}$ such that $\omega_{\overline{n}} \in C$. Take $\overline{m} \in \mathbb{N}$ sufficiently big so that $B^{\overline{m}}(\omega_{\overline{n}}) \subseteq C$. ■

**Corollary 48** *Suppose that $\Omega$ is finite or countable. Then the condition (33) is fulfilled and therefore:*

$$\text{No Arb. de la classe } \mathcal{S} \text{ in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M} \cap \mathcal{R}_{\mathcal{S}} \neq \varnothing. \tag{36}$$

*In particular:*

$$\text{No 1p-Arbitrage in } \widetilde{\mathcal{H}} \iff \mathcal{M}_+ \neq \emptyset. \tag{37}$$

$$\text{No } P\text{-a.s. Arbitrage in } \widetilde{\mathcal{H}} \iff \exists\, Q \in \mathcal{M} \text{ s.t. } P \ll Q. \tag{38}$$

$$\text{No } \mathcal{P}'\text{-q.s. Arbitrage in } \widetilde{\mathcal{H}} \iff \exists\, Q \in \mathcal{M} \text{ s.t. } P' \ll Q\ \forall P' \in \mathcal{P}'. \tag{39}$$

**Proof.** Define $\mathcal{S}_0 := \{\{\omega\} \mid \omega \in \Omega$ such that there exists $C \in \mathcal{S}$ with $\omega \in C\}$. Then $\mathcal{S}_0$ is at most a countable set and satisfies condition (33). ■

**Remark 49** *While (37) holds also for 1p-Arbitrage in $\mathcal{H}$ (see Proposition 65)), (38) and (39) can not be improved. Indeed, by replacing in the example (7) $\mathbb{R}^+$ with $\mathbb{Q}^+$ and $\mathbb{Q}^+$ with $\mathbb{N}$, $\Omega$ is countable, we still have $\mathcal{M} = \varnothing$ but there are No $P$-a.s. Arbitrage in $\mathcal{H}$ if $P(\mathbb{Q}^+ \diagdown \mathbb{N}) = 0$ (see Section 4.1, item 5 (a)).*

**Remark 50** *There are other families of sets satisfying condition (33). For example, in a topological setting, nowhere dense subset of $\Omega$ (those having closure with empty interior) are often considered "negligible" sets. Then the class of sets which are the complement of nowhere dense sets, satisfies condition (33).*

**Remark 51** *Condition (33) is not necessary to obtain the desired equivalence (34). Consider for example the class $\mathcal{S}$ defining $\varepsilon$-Arbitrage in Example 45 item 6. In such a case condition (33) fails, as soon as $\Omega$ is uncountable. However, we now prove that (34) holds true, when $\Omega = \mathbb{R}$. We already know by the previous proof that $\mathcal{M} \cap \mathcal{R}_{\mathcal{S}} \neq \varnothing$ implies No Arbitrage de la classe $\mathcal{S}$ in $\widetilde{\mathcal{H}}$. For the converse, from No Arbitrage de la classe $\mathcal{S}$ in $\widetilde{\mathcal{H}}$ we know that each element in $\mathcal{S} := \{[r-\varepsilon, r+\varepsilon] \mid r \in \mathbb{R}\}$ is not a polar set. Consider the countable class*

$$G := \{[q-\varepsilon, q+\varepsilon] \mid q \in \mathbb{Q}\} \subseteq \mathcal{S}.$$

*Each set $G_n \in G$ is not a polar set, hence for each $n$ there exists $Q_n \in \mathcal{M}$ such that $Q_n(G_n) > 0$. Set $\overline{Q} := \sum_{n=1}^{\infty} \frac{1}{2^n} Q_n \in \mathcal{M}$ (see Lemma 76). The set*

$$D := \{r \in \mathbb{R} \mid \overline{Q}([r-\varepsilon, r+\varepsilon]) = 0\}$$

*is at most countable. Indeed, any two distinct intervals $J := [r-\varepsilon, r+\varepsilon]$ and $J' := [r'-\varepsilon, r'+\varepsilon]$, with $r, r' \in D$, must be disjoint, otherwise for a rational $q$ between $r$ and $r'$ we would have: $[q-\varepsilon, q+\varepsilon] \subseteq J \cup J'$ and thus $\overline{Q}([q-\varepsilon, q+\varepsilon]) = 0$, which is impossible by construction of $\overline{Q}$. For each $r_n \in D$ the set $[r_n-\varepsilon, r_n+\varepsilon] \in \mathcal{S}$ is not a polar set, hence for each $n$ there exists $\widehat{Q}_n \in \mathcal{M}$ such that $\widehat{Q}_n([r_n-\varepsilon, r_n+\varepsilon]) > 0$. Set $\widehat{Q} := \sum_{n=1}^{\infty} \frac{1}{2^n} \widehat{Q}_n \in \mathcal{M}$. Thus $Q := \frac{1}{2}\overline{Q} + \frac{1}{2}\widehat{Q} \in \mathcal{M} \cap \mathcal{R}_{\mathcal{S}}$ is the desired measure.*

## 5 Feasible Markets

We extend the classical notion of arbitrage with respect to a single probability measure $P \in \mathcal{P}$ to a class of probabilities $\mathcal{R} \subseteq \mathcal{P}$ as follows:

**Definition 52** *The market admits $\mathcal{R}$-**Arbitrage** if*

- *for all $P \in \mathcal{R}$ there exists a $P$-Classical Arbitrage.*

*We denote with No $\mathcal{R}$-Arbitrage the property: for some $P \in \mathcal{R}$, $NA(P)$ holds true.*

**Remark 53 (Financial interpretation of $\mathcal{R}$-Arbitrage.)** *If a model admits an $\mathcal{R}$-Arbitrage then the agent will not be able to find a fair pricing rule, whatever model $P \in \mathcal{R}$ he will choose. However, the presence of an $\mathcal{R}$-Arbitrage only implies that for each $P$ there exists a trading strategy $H^P$ which is a $P$-Classical Arbitrage and this is a different concept respect to the existence of one single trading strategy $H$ that realizes an arbitrage for all $P \in \mathcal{R}$. In the particular case of $\mathcal{R} = \mathcal{P}$ this notion was firstly introduced in [DH07] as "Weak Arbitrage opportunity" and further studied in [CO11, DOR14] and the reference therein. The No $\mathcal{R}$-Arbitrage property above should not be confused with the condition $NA(\mathcal{R})$ introduced by Bouchard and Nutz [BN13] and recalled in Section 4 as well as in Definition 10, item 3.*

We set:

$$\mathcal{P}_e(P) = \{P' \in \mathcal{P} \mid P' \sim P\}, \qquad \mathcal{M}_e(P) = \{Q \in \mathcal{M} \mid Q \sim P\}$$

In discrete time financial markets the Dalang-Morton-Willinger Theorem applies, so that $NA(P)$ iff $\mathcal{M}_e(P) \neq \varnothing$.

**Proposition 54** *Suppose that $\mathcal{R} \subseteq \mathcal{P}$ has the property: $P \in \mathcal{R}$ implies $\mathcal{P}_e(P) \subseteq \mathcal{R}$. Then*

$$\text{No } \mathcal{R}\text{-Arbitrage iff } \mathcal{M} \cap \mathcal{R} \neq \varnothing.$$

*In particular*

$$\begin{aligned} \text{No } \mathcal{R}_{\mathcal{S}}\text{-Arbitrage iff } \mathcal{M} \cap \mathcal{R}_{\mathcal{S}} &\neq \varnothing, \\ \text{No } \mathcal{P}_{+}\text{-Arbitrage iff } \mathcal{M}_{+} &\neq \varnothing, \\ \text{No } \mathcal{P}\text{-Arbitrage iff } \mathcal{M} &\neq \varnothing. \end{aligned}$$

*where $\mathcal{R}_{\mathcal{S}}$ is defined in* (32) *and all arbitrage conditions here are with respect to $\mathcal{H}$.*

**Proof.** Suppose $Q \in \mathcal{M} \cap \mathcal{R} \neq \varnothing$. Since $Q \in \mathcal{R}$ and $NA(Q)$ holds true we have No $\mathcal{R}$-Arbitrage. Viceversa, suppose No $\mathcal{R}$-Arbitrage holds true. Then there exists $P \in \mathcal{R}$ for which $NA(P)$ holds true and therefore there exists $Q \in \mathcal{M}_e(P)$. The assumption $\mathcal{P}_e(P) \subseteq \mathcal{R}$ implies that $Q \in \mathcal{M}_e(P) := \mathcal{M} \cap \mathcal{P}_e(P) \subseteq \mathcal{M} \cap \mathcal{R}$. The particular cases follows from the fact that $\mathcal{R}_{\mathcal{S}}$ has the property: $P \in \mathcal{R}_{\mathcal{S}}$ implies $\mathcal{P}_e(P) \subseteq \mathcal{R}_{\mathcal{S}}$. ■

**Remark 55** *As a result of the previous proposition, whenever (34), (35), (36) hold true each (equivalent) condition in (34), (35), (36) is also equivalent to: No $\mathcal{R}_{\mathcal{S}}$-Arbitrage in $\mathcal{H}$ (with $\mathcal{S} := \{$open sets$\}$ for (35)).*

Given the measurable space $(\Omega, \mathcal{F})$ and the price process $S$ defined on it, in this section we investigate the properties of the set of arbitrage free (for $S$) probabilities on $(\Omega, \mathcal{F})$. A minimal reasonable requirement on the financial market is the existence of at least one probability $P \in \mathcal{P}$

that does not allow any $P$-Classical Arbitrage. Recall from the Introduction the definition of the set

$$\mathcal{P}_0 = \{P \in \mathcal{P} \mid \mathcal{M}_e(P) \neq \varnothing\}.$$

By Proposition 54 and the definition of $\mathcal{P}_0$ it is clear that:

$$\text{No } \mathcal{P}\text{-Arbitrage} \Leftrightarrow \mathcal{M} \neq \varnothing \Leftrightarrow \mathcal{P}_0 \neq \varnothing,$$

and each one of these conditions is equivalent to No Model Independent Arbitrage with respect to $\widetilde{\mathcal{H}}$ (Theorem 3). When $\mathcal{P}_0 \neq \varnothing$, it is possible that only very few models (i.e. a "small" set of probability measures - the extreme case being $|\mathcal{P}_0| = 1$) are arbitrage free. On the other hand, the financial market could be very "well posed", so that for "most" models no arbitrage is assured - the extreme case being $\mathcal{P}_0 = \mathcal{P}$.

To distinguish these two possible occurrences we analyze the conditions under which the set $\mathcal{P}_0$ is dense in $\mathcal{P}$: in this case even if there could be some particular models allowing arbitrage opportunities, the financial market is well posed for most models.

**Definition 56** *The market is feasible if* $\overline{\mathcal{P}_0} = \mathcal{P}$

Recall that we are here considering the $\sigma(\mathcal{P}, C_b)$- closure.
In Proposition 58 we characterize feasibility with the existence of a full support martingale measure, a condition independent of any a priori fixed probability.

**Lemma 57** *For all* $P \in \mathcal{P}_+$

$$\overline{\mathcal{P}_e(P)} = \mathcal{P} \text{ and } \mathcal{P}_+ \text{ is } \sigma(\mathcal{P}, C_b)\text{-dense in } \mathcal{P}.$$

**Proof.** It is well know that under the assumption that $(\Omega, d)$ is separable, $\mathcal{P}_+ \neq \varnothing$. Let us first show that $\forall a \in \Omega$ we have that $\delta_a \in \overline{\mathcal{P}_e(P)}$ where $P \in \mathcal{P}_+$ and $\delta_a$ is the point mass probability measure in $a$. Let

$$A_n := \left\{ \omega \in \Omega \ : \quad d(a, \omega) < \frac{1}{n} \right\}.$$

This set is open in the topology induced by $d$ and, since $P$ has full support, $0 < P(A_n) < 1$. Define the conditional probability measure $P_n := P(\cdot \mid A_n)$. For all $0 < \lambda < 1$, $P_\lambda := \lambda P(\cdot \mid A_n^c) + (1-\lambda) P(\cdot \mid A_n)$ is a full support measure equivalent to $P$ and $P_\lambda$ converges weakly to $P(\cdot \mid A_n)$ as $\lambda \downarrow 0$. Hence: $P_n \in \overline{\mathcal{P}_e(P)}$. In order to show that $P_n \overset{w}{\to} \delta_a$ we prove that $\forall\ G$ open $\liminf P_n(G) \geq \delta_a(G)$. If $a \in G$ then $\delta_a(G) = 1$ and $P(G \cap A_n) = P(A_n)$ eventually so we have that $\liminf P_n(G) = 1 = \delta_a(G)$. Otherwise if $a \notin G$ then $\delta_a(G) = 0$ and the inequality is obvious. Since $\forall a \in \Omega$ we have that $\delta_a \in \overline{\mathcal{P}_e(P)}$ then $co(\{\delta_a \ : \ a \in \Omega\}) \subseteq \overline{\mathcal{P}_e(P)}$ and from the density of the probability measures with finite support in $\mathcal{P}$ (respect to the weak topology) it follows that $\overline{\mathcal{P}_e(P)} = \mathcal{P}$. The last assertion is obvious since $\mathcal{P}_e(P) \subseteq \mathcal{P}_+$ for each $P \in \mathcal{P}_+$. ■

**Proposition 58** *The following assertions are equivalent:*

1. $\mathcal{M}_+ \neq \emptyset$;
2. *No* $\mathcal{P}_+$*-Arbitrage;*
3. $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$;
4. $\overline{\mathcal{P}_0 \cap \mathcal{P}_+} = \mathcal{P}$;
5. $\overline{\mathcal{P}_0} = \mathcal{P}$.

**Proof.** Since $\mathcal{M}_+ \neq \emptyset \Leftrightarrow$ No $\mathcal{P}_+$-Arbitrage by Proposition 54 and No $\mathcal{P}_+$-Arbitrage $\Leftrightarrow$ $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$ by definition, $1), 2), 3)$ are clearly equivalent.
Let us show that 3) $\Rightarrow$ 4): Assume that $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$ and observe that if $P \in \mathcal{P}_0 \cap \mathcal{P}_+$ then $\mathcal{P}_e(P) \subseteq \mathcal{P}_0 \cap \mathcal{P}_+$, which implies that $\overline{\mathcal{P}_e(P)} \subseteq \overline{\mathcal{P}_0 \cap \mathcal{P}_+} \subseteq \mathcal{P}$. From Lemma 57 we conclude that 4) holds.
Observe now that the implication 4) $\Rightarrow$ 5) holds trivially, so we just need to show that 5) $\Rightarrow$ 3). Let $P \in \mathcal{P}_+$. If $P$ satisfies NA$(P)$ there is nothing to show, otherwise by 5) there exist a sequence of probabilities $P_n \in \mathcal{P}_0$ such that $P_n \overset{w}{\to} P$ and the condition NA$(P_n)$ holds $\forall n \in \mathbb{N}$. Define $P^* := \sum_{n=1}^{+\infty} \frac{1}{2^n} P_n$ and note that for this probability the condition NA$(P^*)$ holds true, so we just need to show that $P^*$ has full support. Assume by contradiction that $supp(P^*) \subset \Omega$. Then there exist an open set $O$ such that $P^*(O) = 0$ and $P(O) > 0$ since $P$ has full support. From $P_n(O) = 0$ for all $n$, and $P_n \overset{w}{\to} P$ we obtain $0 = \liminf P_n(O) \geq P(O) > 0$, a contradiction. ■

**Remark 59** *From the previous proof we observe that if the market is feasible then* $\overline{\bigcup_{P \in \mathcal{P}_0} supp(P)} = \Omega$ *and no "significantly large parts" of* $\Omega$ *are neglected by no arbitrage models* $P \in \mathcal{P}_0$*.*

**Proof of Theorem 4.** Proposition 58 guarantees: 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. and Corollary 47 assures: 3. $\Leftrightarrow$ 4. ■

**The case of a countable space $\Omega$.** When $\Omega = \{\omega_n \mid n \in \mathbb{N}\}$ is countable it is possible to provide another characterization of feasibility using the norm topology instead of the weak topology on $\mathcal{P}$. More precisely, we consider the topology induced by the total variation norm. A sequence of probabilities $P_n$ *converges in variation* to $P$ if $\|P_n - P\| \to 0$, where the variation norm of a signed measure $R$ is defined by:

$$\|R\| = \sup_{(A_i,\ldots,A_n) \in \mathcal{F}} \sum_{i=1}^{n} |R(A_i)|, \tag{40}$$

and $(A_i, \ldots, A_n)$ is a finite partition of $\Omega$.

**Lemma 60** *Let* $\Omega$ *be a countable space. Then* $\forall P \in \mathcal{P}_+$

$$\overline{\mathcal{P}_e(P)}^{\|\cdot\|} = \overline{\mathcal{P}_+}^{\|\cdot\|} = \mathcal{P}.$$

**Proof.** Since $\Omega$ is countable we have that

$$\mathcal{P} = \{P := \{p_n\}_1^\infty \in \ell^1 \ \mid \ p_n \geq 0 \ \forall n \in \mathbb{N}, \ \|P\|_1 = 1\},$$

$$\mathcal{P}_+ = \{P \in \mathcal{P} \ \mid \ p_n > 0 \ \forall n \in \mathbb{N}\},$$

with $\|\cdot\|_1$ the $\ell^1$ norm. Observe that in the countable case $\mathcal{P}_e(P) = \mathcal{P}_+$ for every $P \in \mathcal{P}_+$. So we only need to show that for any $P \in \mathcal{P}$ and any $\varepsilon > 0$ there exists $P' \in \mathcal{P}_+$ s.t. $\|P - P'\|_1 \leq \varepsilon$.

Let $P \in \mathcal{P} \setminus \mathcal{P}_+$. Then $P = \{p_n\}_1^\infty \in \ell^1$ and there exists at least one index $n$ for which $p_n = 0$. Let $\alpha > 0$ be the constant satisfying

$$\sum_{n \in \mathbb{N} \text{ s.t. } p_n = 0} \frac{\alpha}{2^n} = 1.$$

There also exists one index $n$, say $n_1$, for which $1 \geq p_{n_1} > 0$. Let $p := p_{n_1} > 0$.

For any positive $\varepsilon < p$, define $P' = \{p'_n\}$ by: $p'_{n_1} = p - \frac{\varepsilon}{2}$, $p'_n = p_n$ for all $n \neq n_1$ s.t. $p_n > 0$, $p'_n = \frac{\alpha}{2^n} \frac{\varepsilon}{2}$ for all $n$ s.t. $p_n = 0$. Then $p'_n > 0$ for all $n$ and $\sum_{n=1}^\infty p'_n = \sum_{n \text{ s.t.} p_n > 0} p_n = 1$ , so that $P' \in \mathcal{P}_+$ and $\|P - P'\|_1 = \varepsilon$. ■

**Remark 61** *In the general case, when* $\Omega$ *is uncountable, while it is still true that* $\overline{\mathcal{P}_+}^{\|\cdot\|} = \mathcal{P}$, *it is no longer true that* $\overline{\mathcal{P}_e(P)}^{\|\cdot\|} = \mathcal{P}$ *for any* $P \in \mathcal{P}_+$.
*Take* $\Omega = [0,1]$ *and* $\mathcal{P}_e(\lambda)$ *the set of probability measures equivalent to Lebesgue. It is easy to see that* $\delta_0 \notin \overline{\mathcal{P}_e(\lambda)}^{\|\cdot\|}$ *since* $\|P - \delta_0\| \geq P((0,1]) = 1$ .

**Proposition 62** *If* $\Omega$ *is countable, the following conditions are equivalent:*

1. $\mathcal{M}_+ \neq \emptyset$;
2. *No* $\mathcal{P}_+$*-Arbitrage;*
3. $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$;
4. $\overline{\mathcal{P}_0}^{\|\cdot\|} = \mathcal{P}$,

*where* $\|\cdot\|$ *is the total variation norm on* $\mathcal{P}$

**Proof.** Using Lemma 60 the proof is straightforward using the same techniques as in Proposition 58. ∎

# 6 On Open Arbitrage

In the introduction we already illustrated the interpretation and robust features of the dual formulation of Open Arbitrage. In order to prove the equivalence between Open Arbitrage and (1) consider the following definition and recall that $\mathcal{V}_H^+ := \{\omega \in \Omega \mid V_T(H)(\omega) > 0\}$.

**Definition 63** *Let* $\tau$ *be a topology on* $\mathcal{P}$ *and* $\mathfrak{H}$ *be a class of trading strategies. Set*

$$W(\tau, \mathfrak{H}) = \left\{ H \in \mathfrak{H} \mid \begin{array}{c} \text{there exists a non empty } \tau-\text{open set } \mathcal{U} \subseteq \mathcal{P} \text{ such that} \\ \forall P \in \mathcal{U} \quad V_T(H) \geq 0 \ P\text{-a.s.} \quad \text{and} \quad P(\mathcal{V}_H^+) > 0 \end{array} \right\}$$

Clearly, $W(\tau, \mathfrak{H})$ consists of the trading strategies satisfying condition (1) with respect to the appropriate topology and the measurability requirement. The first item in the next proposition is the announced equivalence. The second item shows that the analogue equivalence is true also with respect to the class $\widetilde{\mathcal{H}}$. Therefore, in Theorem 4 we could add to the four equivalent conditions also the dual formulation of Open Arbitrage with respect to $\widetilde{\mathcal{H}}$.

**Proposition 64** *(1) Let* $\sigma := \sigma(\mathcal{P}, C_b)$ *and* $\|\cdot\|$ *the variation norm defined in (40). Then:*

$$\begin{array}{rcl} H & \in & W(\|\cdot\|, \mathcal{H}) \Longleftrightarrow \quad H \in \mathcal{H} \text{ is a 1p-Arbitrage} \\ & \Uparrow & \\ H & \in & W(\sigma, \mathcal{H}) \quad \Longleftrightarrow \quad H \in \mathcal{H} \text{ is an Open Arbitrage} \end{array}$$

*In addition, if* $H \in W(\sigma, \mathcal{H})$ *then* $V_T(H)(\omega) \geq 0$ *for all* $\omega \in \Omega$.

*(2) Let* $\mathcal{F} = \mathcal{B}(\Omega)$ *be the Borel sigma algebra and let* $\widetilde{\mathcal{F}}$ *be a sigma algebra such that* $\mathcal{F} \subseteq \widetilde{\mathcal{F}}$. *Define the set*

$$\widetilde{\mathcal{P}} := \{\widetilde{P} : \widetilde{\mathcal{F}} \to [0,1] \mid \widetilde{P} \text{ is a probability}\},$$

*and endow* $\widetilde{\mathcal{P}}$ *with the topology* $\widetilde{\sigma} := \sigma(\widetilde{\mathcal{P}}, C_b)$. *The class of admissible trading strategies* $\widetilde{\mathcal{H}}$ *is given by all* $\widetilde{\mathbb{F}}$*- predictable processes. Then*

$$H \in W(\widetilde{\sigma}, \widetilde{\mathcal{H}}) \iff H \in \widetilde{\mathcal{H}} \text{ is an Open Arbitrage in } \widetilde{\mathcal{H}}$$

*In addition, if* $H \in W(\widetilde{\sigma}, \widetilde{\mathcal{H}})$ *then* $V_T(H)(\omega) \geq 0$ *for all* $\omega \in \Omega$.

**Proof.** We prove (1) and we postpone the proof of (2) to the Appendix.

(a) $H$ is a $1p$-Arbitrage $\Rightarrow H \in W(\|\cdot\|, \mathcal{H})$. Let $H \in \mathcal{H}$ be a $1p$-Arbitrage. Then $V_T(H)(\omega) \geq 0$ $\forall \omega \in \Omega$ and there exists a probability $P$ such that $P(\mathcal{V}_H^+) > \varepsilon > 0$. From the implication $\|P - Q\| < \varepsilon \Rightarrow |P(C) - Q(C)| < \varepsilon$ for every $C \in \mathcal{F}$, we obtain: $\bar{P}(\mathcal{V}_H^+) > 0$ $\forall \bar{P} \in B_\varepsilon(P)$, where $B_\varepsilon(P)$ is the ball of radius $\varepsilon$ centered in $P$. Hence $H \in W(\|\cdot\|, \mathcal{H})$.

(b) $H \in W(\|\cdot\|, \mathcal{H}) \Rightarrow H$ is a $1p$-Arbitrage. If $H \in W(\|\cdot\|, \mathcal{H})$ then $V_T(H) \geq 0$ $P$-a.s. for all $P$ in the open set $\mathcal{U}$. We need only to show that $B := \{\omega \in \Omega \mid V_T(H)(\omega) < 0\}$ is empty. By contradiction, let $\omega \in B$, take any $P \in \mathcal{U}$ and define the probability $P_\lambda := \lambda\delta_\omega + (1-\lambda)P$. Since $V_T(H) \geq 0$ $P$-a.s. we must have $P(\omega) = 0$, otherwise $P(B) > 0$. However, $P_\lambda(B) \geq P_\lambda(\omega) = \lambda > 0$ for all positive $\lambda$ and $P_\lambda$ will eventually belongs to $\mathcal{U}$, as $\lambda \downarrow 0$, which contradicts $V_T(H) \geq 0$ $P$-a.s. for any $P \in \mathcal{U}$.

(c) $H \in W(\sigma, \mathcal{H}) \Rightarrow H \in W(\|\cdot\|, \mathcal{H})$. This claim is trivial because every weakly open set is also open in the norm topology.

(d) If $H \in W(\sigma, \mathcal{H})$ then $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. This follows from (c) and (b).

(e) $H \in W(\sigma, \mathcal{H}) \Rightarrow H$ is an Open Arbitrage. Suppose $H \in W(\sigma, \mathcal{H})$, so that $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. We claim that $(\mathcal{V}_H^+)^c = \{\omega \in \Omega \mid V_T(H) = 0\}$ is not dense in $\Omega$. This will imply the thesis as $int(\mathcal{V}_H^+)$ will then be a non empty open set on which $V_T(H) > 0$. Suppose by contradiction that $\overline{(\mathcal{V}_H^+)^c} = \Omega$. We know by Lemma 77 in the Appendix that the set $\mathcal{Q}$ of embedded probabilities $co(\{\delta_\omega\} \mid \omega \in (\mathcal{V}_H^+)^c)$ is weakly dense in $\mathcal{P}$ and hence it intersects, in particular, the weakly open set $\mathcal{U}$ in the definition of $W(\sigma, \mathcal{H})$. However, for every $P \in \mathcal{Q}$ we have $V_T(H) = 0$ $P$-a.s. and so $H$ is not in $W(\sigma, \mathcal{H})$.

(f) $H$ is an Open Arbitrage $\Rightarrow H \in W(\sigma, \mathcal{H})$. Note first that if $F$ is a closed subset of $\Omega$, then $\mathcal{P}(F) := \{P \in \mathcal{P} \mid supp(P) \subset F\}$ is a $\sigma(\mathcal{P}, C_b)$-closed face of $\mathcal{P}$ from Th. 15.19 in [AB06]. If $H$ is an Open Arbitrage then $\mathcal{V}_H^+$ contains an open set and in particular $G := \overline{(\mathcal{V}_H^+)^c}$ is a closed set strictly contained in $\Omega$. Observe then that $\mathcal{U} := (\mathcal{P}(G))^c$ is a non empty open set of probabilities that fulfills the properties in the definition of $W(\sigma, \mathcal{H})$. ■

The following proposition is an improvement of (37), as the $1p$-Arbitrage is defined with respect to $\mathcal{H}$.

**Proposition 65** *For $\Omega$ countable: No $1p$-Arbitrage in $\mathcal{H} \Longleftrightarrow \mathcal{M}_+ \neq \emptyset$.*

**Proof.** From Propositions 17 and 64 we only need to prove $\mathcal{M}_+ \neq \emptyset \Longrightarrow W(\|\cdot\|, \mathcal{H}) = \varnothing$. From Proposition 62 item 4) we have $\mathcal{M}_+ \neq \emptyset \Longrightarrow \overline{\mathcal{P}_0}^{\|\cdot\|} = \mathcal{P}$ and so for every (norm) open set $\mathcal{U} \subseteq \mathcal{P}$ there exists $P \in \mathcal{P}_0 \cap \mathcal{U}$ for which $NA(P)$ holds, which implies $W(\|\cdot\|, \mathcal{H}) = \varnothing$. ■

## 6.1 On the continuity of S with respect to $\omega$

Consider first a one period market $I = \{0, 1\}$ with $S_0 = s_0 \in \mathbb{R}^d$ and $S_1$ a random outcome *continuous* in $\omega$. Then every $1p$-Arbitrage generates an Open Arbitrage (this was shown by [Ri11] and is intuitively clear). From Proposition 17, No $1p$-Arbitrage implies $\mathcal{M}_+ \neq \varnothing$ and therefore No Open Arbitrage. We then conclude that, in this particular case, the three conditions are all equivalent and Theorem 4 holds without the enlargement of the natural filtration so that we recover in particular the result stated in [Ri11].

Differently from the one period case, in the multi-period setting it is no longer true that No Open Arbitrage and No $1p$-Arbitrage (with respect to admissible strategies $\mathcal{H}$) are equivalent, as shown by the following examples. Moreover, even with $S$ continuous in $\omega$, No Open Arbitrage is not equivalent to $\mathcal{M}_+ \neq \varnothing$ as long as we do not enlarge the filtration as in Section 4.5.

**Example 66** *Consider* $\Omega = [0,1] \times [0,1]$, $\mathcal{F} = \mathcal{B}_{[0,1]} \otimes \mathcal{B}_{[0,1]}$ *and the canonical process given by* $S_1(\omega) = \omega_1$ *and* $S_2(\omega) = \omega_2$. *Clearly for any* $\omega = (\omega_1, \omega_2)$ *such that* $\omega_1 \in (0,1)$ *we have that* $0 \in ri(\Delta S_2(\Sigma_1^{\omega}))^{cc}$. *On the other hands for* $\overline{\omega} = (1, \omega_2)$ *or* $\widehat{\omega} = (0, \omega_2)$ *we have* 1p*-Arbitrages since* $S_2(\overline{\omega}) \le S_1(\overline{\omega})$ *with* $<$ *for any* $\omega_2 \neq 1$ *and* $S_2(\widehat{\omega}) \ge S_1(\widehat{\omega})$ *with* $>$ *for any* $\omega_2 \neq 0$. *Denote by* $\Sigma^1 = \{S_1 = 1\}$ *and* $\Sigma^0 = \{S_1 = 0\}$ *then* $\alpha(\omega) = -\mathbf{1}_{\Sigma^1} + \mathbf{1}_{\Sigma^0}$ *is a* 1p*-Arbitrage which does not admit any open arbitrage since neither* $\Sigma^1$ *nor* $\Sigma^0$ *are open sets, and any strategy which is not zero on* $(\Sigma^1 \cup \Sigma^0)^c$ *gives both positive and negative payoffs.*

**Example 67** *We show an example of a market with* $S$ *continuous in* $\omega$, *with no Open Arbitrage in* $\mathcal{H}$ *and* $\mathcal{M}_+ = \varnothing$. *Let us first introduce the following continuous functions on* $\Omega = [0, +\infty)$

$$\varphi^m_{a,b}(\omega) := \begin{cases} m(\omega - a) & \omega \in [a, \frac{a+b}{2}] \\ -m(\omega - b) & \omega \in [\frac{a+b}{2}, b] \\ 0 & otherwise \end{cases} \quad \phi^m_{a,b}(\omega) := \begin{cases} m(\omega - a) & \omega \in [a, a+1] \\ m & \omega \in [a+1, b-1] \\ -m(\omega - b) & \omega \in [b-1, b] \\ 0 & otherwise \end{cases}$$

*with* $a, b, m \in \mathbb{R}$. *Define the continuous (in* $\omega$*) stochastic process* $(S_t)_{t=0,1,2,3}$

$$\begin{aligned} S_0(\omega) &:= \frac{1}{2} \\ S_1(\omega) &:= \phi^1_{[0,3]}(\omega) + \phi^1_{[3,6]}(\omega) + \sum_{k=3}^{\infty} \varphi^1_{[2k,2k+2]}(\omega) \\ S_2(\omega) &:= \phi^{\frac{1}{2}}_{[0,3]}(\omega) + \phi^{\frac{1}{2}}_{[3,6]}(\omega) + \sum_{k=3}^{\infty} \varphi^2_{[2k,2k+2]}(\omega) \\ S_3(\omega) &:= \varphi^2_{[0,3]}(\omega) + \varphi^{\frac{1}{4}}_{[3,6]}(\omega) + \varphi^4_{[6,8]}(\omega) + \sum_{k=4}^{\infty} \varphi^{4k}_{[2k+1-\frac{1}{k}, 2k+1+\frac{1}{k}]}(\omega) \end{aligned}$$

*It is easy to check that given* $z \in \mathbf{Z}$ *such that* $z_{0:2} = [\frac{1}{2}, 1, 2]$, *we have* $\Sigma_2^z = \{2k+1\}_{k \ge 3}$ *and* $H := \mathbf{1}_{\Sigma_2^z}$ *is the only* 1p*-Arbitrage opportunity in the market. One can also check that* $\mathcal{V}_H^+ = \Sigma_2^z$, *as a consequence,* $H$ *is not an Open Arbitrage and*

$$Q(\{2k+1\}_{k \ge 3}) = 0 \text{ for any } Q \in \mathcal{M} \tag{41}$$

*Consider now* $\hat{z} \in \mathbf{Z}$ *with* $\hat{z}_{0:2} = [\frac{1}{2}, 1, \frac{1}{2}]$ *and the corresponding level set* $\Sigma_2^{\hat{z}}$. *It is easy to check that*

$$\Sigma_2^{\hat{z}} = [1,2] \cup [4,5] \quad \text{and} \quad \Delta S_2 < 0 \text{ on } \Sigma_2^{\hat{z}} \tag{42}$$

*Observe now that* $z_{0:1} = \hat{z}_{0:1}$ *and that* $\Sigma_1^z = [1,2] \cup [4,5] \cup \{2k+1\}_{k \ge 3}$. *We therefore have*

$$S_2(\omega) = \begin{cases} 2 & \omega \in \{2k+1\}_{k \ge 3} \\ \frac{1}{2} & \omega \in [1,2] \cup [4,5] \end{cases} \quad \text{for } \omega \in \Sigma_1^z$$

*From* $S_1(\omega) = 1$ *on* $\Sigma_1^z$, (41) *and* (42) *we have that any martingale measure must satisfy* $Q([1,2] \cup [4,5]) = 0$. *In other words there exist polar sets with non-empty interior which implies* $\mathcal{M}_+ = \varnothing$.

# 7 Appendix

## 7.1 proof of Theorem 39

**Lemma 68 (Lebesgue decomposition of** $P$**)** *Let* $\nu := \sup_{Q \in \mathcal{M}} Q$. *For any* $P \in \mathcal{P}$ *there exists a set* $F \in \mathcal{F}$ *such that* $F \subseteq (\Omega_*)^c$, *and the measures* $P_c(\cdot) := P(\cdot \setminus F)$ *and* $P_s(\cdot) := P(\cdot \cap F)$ *satisfy*

$$P_c \ll \nu, \; P_s \perp \nu \qquad \text{and} \qquad P = P_c + P_s \tag{43}$$

**Proof.** We wish to apply Theorem 4.1 in [LYL07] to $\mu = P \in \mathcal{P}$ and $\nu = \sup_{Q\in\mathcal{M}} Q$. It is easy to check that: 1) $\mu$ and $\nu$ are monotone $[0,1]$-valued set functions on $\mathcal{F}$ satisfying $\mu(\varnothing) = 0$ and $\nu(\varnothing) = 0$; 2) $P$ is *exhaustive*, i.e. if $\{A_n\}_{n\in\mathbb{N}}$ is a disjoint sequence then $P(A_n) \to 0$ (indeed, $1 \geq P(\cup_n A_n) = \sum_n P(A_n) \geq 0 \Rightarrow P(A_n) \to 0$; 3) $\nu$ *is weakly null additive*: if $A, B \in \mathcal{F}$ with $\nu(A) = \nu(B) = 0$ then $\nu(A\cup B) = 0$ (indeed, if $\nu(A) = \nu(B) = 0$ then for any $Q \in \mathcal{M}$, $Q(A) = Q(B) = 0$ which implies $Q(A\cup B) = 0$ and $\nu(A\cup B) = 0$); 4) $\nu$ is *continuous from below.* Indeed if $A_n \nearrow A$ then $Q(A_n) \uparrow Q(A)$, $Q(A) = \sup_n Q(A_n)$ and

$$\lim_{n\to\infty} \nu(A_n) = \sup_n \nu(A_n) = \sup_n \sup_{Q\in\mathcal{M}} Q(A_n) = \sup_{Q\in\mathcal{M}} \sup_n Q(A_n) = \nu(A).$$

Hence $\mu$ and $\nu$ satisfy all the assumptions of Theorem 4.1 in [LYL07] and hence we obtain the existence of $F \in \mathcal{F}$ such that $\nu(F) = 0$ and the decomposition in (43) holds true. From Proposition 34, $\forall A \in \mathcal{F}$ such that $A \subseteq \Omega_*$ we have $\nu(A) > 0$. Therefore, $F \subseteq (\Omega_*)^c$ and this concludes the proof. ∎

**Remark 69** *Observe that if $(\Omega_*)^c \neq \varnothing$ the set of probability measures with non trivial singular part $P_s$ is non-empty. Simply take, for instance, any convex combination of $\{\delta_\omega \mid \omega \in (\Omega_*)^c\}$.*

**Preliminary considerations.** We want to consider now the probabilistic model $(\Omega, \{\mathcal{F}_t^P\}_{t\in I}, S, P)$ and we need therefore to pass from $\omega$-wise considerations to $P$-a.s considerations. For this reason we first need to construct an auxiliary process $S_t^P$ with the property $S_t^P = S_t$ $P$-a.s for any $t \in I$ in the same spirit of Lemma 33.
Let $P_{\Delta S_T}(\cdot,\cdot) : \Omega \times \mathcal{B}(\mathbb{R}^d) \mapsto [0,1]$ be the conditional distribution of $\Delta S_T$ and denote $\Upsilon_{\Delta S_T}$ its random support. Define as in Rokhlin [Ro08] the set $A_{\Delta S_T} := \{0 \notin \text{ri}(\text{conv}\Upsilon_{\Delta S_T})\}$. It may happen that $P(A_{\Delta S_T}) = 0$. In this case $\mathfrak{B}_T$ and $\mathfrak{D}_{T-1}$ as in Lemma 33 are subset of $P$-null sets (respectively in $\mathcal{F}_T$ and $\mathcal{F}_{T-1}$). Construct iteratively $X_t^P$ and $S_t^P$ as in (20) and (22). Denote $\Delta X_t^P := X_t^P - S_{t-1}$ and let

$$\tau := \min\left\{t \in I_1 \mid P\left(A_{\Delta X_t^P}\right) > 0\right\}. \tag{44}$$

Observe that $\tau$ is well defined since, from Lemma 33, if $P(A_{\Delta X_t^P}) = 0$ for any $t \geq 1$ we have that $\bigcup_{t\in I_1} \widetilde{\mathfrak{B}}_t = (\Omega_*)^c$ is a subset of a $P$-null set (cfr (21)). This is a contradiction since $P$ is not absolutely continuous with respect to $\nu$, henceforth the set $F$ from Lemma 43 satisfies $F \subseteq (\Omega_*)^c$ and $P(F) > 0$. From now on denote $S_t = S_t \mathbf{1}_{t<\tau} + X_t^P \mathbf{1}_{t\geq\tau}$ which is a $P$-a.s. version of $S$.

**Remark 70** *For any $t \in I_1$ denote $P_{t-1}(\cdot,\cdot) : (\Omega, \mathcal{F}) \mapsto [0,1]$ the conditional probability of $P$ on $\mathcal{F}_{t-1}$. Recall from Theorem 28 c] that there exists $N_1 \in \mathcal{F}_{t-1}$ with $P(N_1) = 0$ such that for any $\omega \in \Omega \setminus N_1$ we have $P_{t-1}\left(\omega, \Sigma_{t-1}^{z(\omega)}\right) = 1$ where $z(\omega) = S_{0:T}(\omega)$.*

**Construction of a $P$-arbitrage from $\mathbb{H}$.** Fix time $\tau$ and denote $A_\tau := A_{\Delta S_\tau}$. For any $\omega \in \Omega$ the level set $\Sigma_{\tau-1}^z$ can be decomposed as $\Sigma_{\tau-1}^z = \cup_{i=1}^{\beta_{\tau,z}} B_{\tau,z}^i \cup B_{\tau,z}^*$. Define for any $z \in \mathbf{Z}$

$$j_z := \inf\left\{j \in \{1,\dots,\beta_{\tau,z}\} \mid P(\omega, B_{\tau,z}^j) > 0 \;\forall \omega \in \Sigma_{\tau-1}^z\right\}$$

and recall that $P(\cdot, B_{\tau,z}^j)$ is constant on $\Sigma_{\tau-1}^z$ (Theorem 28 b]). Define $N_2 := \bigcup_{z\in\mathbf{Z}_f} \cup_{i=1}^{j_z-1} B_{\tau,z}^i$ where $\mathbf{Z}_f := \{z \in \mathbf{Z} \mid j_z < \infty\}$. $N_2$ is a $\bar{P}$-null set since for any $\omega \in N_1^c$ we have $\bar{P}(\omega, N_2) = \bar{P}(\omega, \cup_{i=1}^{j_z-1} B_{\tau,z}^i) = 0$ hence $\bar{P}(N_2) = \bar{P}(N_1 \cap N_2) + \bar{P}(N_1^c \cap N_2) = 0$ (see also Lemma 73 below). Recall that $\bar{P}(\cdot)$ and $\bar{P}(\omega,\cdot)$ denote the completion of $P(\cdot)$ and $P(\omega,\cdot)$ respectively.

Denote $N := N_1 \cup N_2$. We are now able to define the following multifunction $\Psi : \Omega \mapsto 2^{\mathbb{R}^d}$ with values in the power set of $\mathbb{R}^d$.

$$\Psi(\omega) := \begin{cases} \Delta S_\tau \left( \Sigma_{\tau-1}^{z(\omega)} \cap N^c \right) & \omega \in N^c \\ \varnothing & \text{otherwise} \end{cases} \tag{45}$$

In Lemma 71 we show that $\Psi$ is $\mathcal{F}_{\tau-1}^P$-measurable. We apply now an argument similar to [Ro08]. Denote $\mathbb{S}_1^d$ the unitary closed ball in $\mathbb{R}^d$, $\text{lin}(\chi)$ the linear space generated by $\chi$ and $\chi^\circ$ the polar cone of $\chi$. By preservation of measurability (see Proposition 75) the (closed-valued) multifunction

$$\omega \mapsto G_0(\omega) := \text{lin}(\Psi(\omega)) \cap (-\text{cone } \Psi(\omega))^\circ \cap \mathbb{S}_1^d$$

is also $\mathcal{F}_{\tau-1}^P$-measurable and $G_0(\omega) \neq \varnothing$ iff $\omega \in A_\tau \cap N^c$, hence $A_\tau = \{0 \notin \text{ri}(\text{conv}\Upsilon_{\Delta S_\tau^P})\}$ is $\mathcal{F}_{\tau-1}^P$-measurable. Note that we already have that $G_0(\omega) \subseteq \mathbb{H}_\tau(\omega)$ for $P$-a.e. $\omega \in \Omega$. Indeed fix $\omega \notin N$ and consider the level set $\Sigma_{\tau-1}^{z(\omega)}$ and its decomposition as in Lemma 20. By construction of $G_0$ we have that any $g \in G_0(\omega) \neq \varnothing$ satisfies $g \cdot \Delta S_\tau(\omega) \geq 0$ for any $\omega \in \cup_{i=j_z}^{\beta_{\tau,z}} B_{\tau,z}^i \cup B_{\tau,z}^*$ and thus $g \in \mathbb{H}(\omega)$.

Nevertheless, the random set $G_0(\omega)$ contains those $g \in \mathbb{S}_1^d$ such that $g \cdot \Delta S_\tau(\omega) = 0$. Thus, we will not extract a measurable selection from $G_0$ but we will rather consider for any $n \in \mathbb{N}$ the following closed-valued multifunction

$$\omega \mapsto G_n(\omega) := \text{lin}(\Psi(\omega)) \cap \left\{ v \in \mathbb{R}^d \mid \langle v, s \rangle \geq \frac{1}{n} \quad \forall s \in \Psi(\omega) \setminus \{0\} \right\} \cap \mathbb{S}_1^d, \quad n \geq 1$$

and seek for a measurable selection of $G := \cup_{n=0}^\infty G_n$. From Lemma 74 all the random sets $G_n$ are $\mathcal{F}_{\tau-1}^P$-measurable and therefore the same is true for $G$. Now, for any $n \geq 0$, let $\tilde{H}_n$ a measurable selection of $G_n$ on $\{G_n \neq \varnothing\}$ which always exists for a (measurable) closed-valued multifunction with $\tilde{H}_n(\omega) = 0$ if $G_n(\omega) = \varnothing$. Define therefore

$$H_k := \sum_{n=0}^k \tilde{H}_n \qquad \text{and} \quad B_k := \mathcal{V}_{H_k}^+ \tag{46}$$

By construction $B_k$ is an increasing sequence of sets converging to $\cup_z B_{\tau,z}^{j_z}$ which is therefore measurable and it satisfies

$$P(\cup_z B_{\tau,z}^{j_z}) = \int_\Omega P(\omega, \cup_z B_{\tau,z}^{j_z}) dP(\omega) = \int_{\Omega \setminus N} P(\omega, B_{\tau,z}^{j_z}) dP(\omega) \geq \int_{A_\tau \setminus N} P(\omega, B_{\tau,z}^{j_z}) dP(\omega) > 0$$

which follows from the definition of conditional probability, $P(A_\tau) > 0$ and $P(\omega, B_{\tau,z}^{j_z}) > 0$ for every $\omega \in A_\tau \setminus N$. We can therefore conclude that there exists $m \geq 0$ such that $P(B_m) > 0$ and since obviously $H_m \Delta S_\tau \geq 0$ we have that $H_m$ is a $P$-arbitrage. The normalized random variable $H_\tau^P := H_m(\omega)/\|H_m(\omega)\|$ is a measurable selector of the multifunction $G_0$ since it satisfies $H_\tau^P(\omega) \in \cup_{n=1}^m G_n(\omega) \subseteq G(\omega) \subseteq \mathbb{H}_\tau(\omega)$ $P$-a.s. and thus the desired strategy is given by $H_s^P = H_\tau^P \mathbf{1}_\tau(s)$.

**Lemma 71** *The multifunction $\Psi$ defined in (45) is $\mathcal{F}_{\tau-1}^P$-measurable.*

**Proof.** Recall that by definition the multifunction $\Psi$ is measurable iff for any open set $V \subseteq \mathbb{R}^d$ we have $\{\omega \mid \Psi(\omega) \cap V \neq \varnothing\}$ is a measurable set. Observe that

$$\Psi^{-1}(V) := \{\omega \mid \Psi(\omega) \cap V \neq \varnothing\} = S_{\tau-1}^{-1} \left[ S_{\tau-1} \left( \Delta S_\tau^{-1}(V) \cap N^c \right) \right] \cap N^c$$

Let us show that the complement of this set is $\mathcal{F}^P_{\tau-1}$-measurable from which the thesis will follow. Observe that for any function $f$ and for any set $A$ we have $(f^{-1}(A))^c = f^{-1}(A^c)$ so that

$$\begin{aligned}(\Psi^{-1}(V))^c &= S^{-1}_{\tau-1}\left[S_{\tau-1}\left(\Delta S^{-1}_{\bar{t}}(V)\cap N^c\right)^c\right]\cup N\\ &= S^{-1}_{\tau-1}\left[S_{\tau-1}\left((\Delta S^{-1}_{\tau}(V))^c\cup N\right)\right]\cup N\\ &= S^{-1}_{\tau-1}\left[S_{\tau-1}\left(\Delta S^{-1}_{\tau}(V^c)\cup N\right)\right]\cup N\end{aligned}$$

Note now that $A_1 := \Delta S^{-1}_{\tau}(V^c)\cup N$ is an analytic set since it is union of a Borel set and a $\bar{P}$-null set. The set $B_1 := S_{\tau-1}(A_1)$ is an analytic subset of $\mathbb{R}^d$ since $S$ is a Borel function and image of an analytic set through a Borel measurable function is analytic. Finally $A_2 := S^{-1}_{\tau-1}(B_1)$ is an analytic subset of $\Omega$ since pre-image of an analytic set through a Borel measurable function is analytic. Since $P$-completion of $\mathcal{F}$ contains any analytic set, $A_2 \cup N$ is also analytic and belongs to $\mathcal{F}^P_{\tau-1}$.

**Remark 72** *For sure $A_2\cup N$ is analytic and belongs to $\mathcal{F}^P$. The heuristic for $A_2\cup N$ belonging to $\mathcal{F}^P_{\tau-1}$ should be that this set is union of atoms of $\mathcal{F}^P_{\tau-1}$. More formally, since $B_1$ is analytic in $\mathbb{R}^d$ for any measure $\mu$ there exists $F,G$ such that $B_1 = F\cup G$ with $F$ a Borel set and $G$ a subset of $\mu$-null measure (because analytic sets are in the completion of $\mathcal{B}$ respect to any measure $\mu$). If we take $\mu$ as the distribution of $S_{\tau-1}$ under $P$ we have $A_2 = S^{-1}_{\tau-1}(F)\cup S^{-1}_{\tau-1}(G)$. Since $S^{-1}_{\tau-1}(F)\in\mathcal{F}_{\tau-1}$ and $S^{-1}_{\tau-1}(G)$ is a subset of a $\mathcal{F}_{\tau-1}$-measurable $P$-null set, we have $A_2\in\mathcal{F}^P_{\tau-1}$ and hence also $A_2\cup N$.*

■

**Lemma 73** *Let $(\Omega,\mathcal{F},P)$ a probability space and $\mathcal{G}$ a subsigma-algebra of $\mathcal{F}$. Let $P_{\mathcal{G}}(\omega,\cdot)$ the conditional probability of $P$ on $\mathcal{G}$. Then*

$$\bar{P}(A) = \int_{\Omega\setminus N(A)} \bar{P}_{\mathcal{G}}(\omega,A)dP(\omega) \qquad A\in\mathcal{F}^P \tag{47}$$

*where $\bar{P}_{\mathcal{G}}(\omega,\cdot)$ is the completion of $P_{\mathcal{G}}(\omega,\cdot)$ and $N(A)\in\mathcal{G}$ is a $P$-null set which depends on $A$.*

**Proof.** It is easy to see that every set in $\mathcal{F}^P$ is union of a set $F\in\mathcal{F}$ and a subset of a $P$-null set. For any $F\in\mathcal{F}$, $\bar{P}(F) = P(F)$ and $P_{\mathcal{G}}(\omega,F) = \bar{P}_{\mathcal{G}}(\omega,F)$ so equality (47) is obvious from the definition of conditional probability (with $N(F)=\varnothing$). Let $A$ be a subset of a P-null set $A_1$. $0 = P(A_1) = \int_\Omega P_{\mathcal{G}}(\omega,A_1)dP(\omega)$ which means that $P_{\mathcal{G}}(\omega,A_1) = 0$ $P$-a.s. Thus, we also have $\bar{P}_{\mathcal{G}}(\omega,A) = 0$ $P$-a.s. from which the equality (47) follows with $N(A) = \{\omega\in\Omega : P_{\mathcal{G}}(\omega,A_1) > 0\}\in\mathcal{G}$. ■

**Measurable selection results.**

**Lemma 74** *Let $(\Omega,\mathcal{A})$ a measurable space and $\Psi:\Omega\mapsto 2^{\mathbb{R}^d}$ an $\mathcal{A}$-measurable multifunction. Let $\varepsilon>0$ then*

$$\Psi^\varepsilon : \omega\mapsto\left\{v\in\mathbb{R}^d \mid \langle v,s\rangle\geq\varepsilon \quad \forall s\in\Psi(\omega)\setminus\{0\}\right\}$$

*is an $\mathcal{A}$-measurable multifunction.*

**Proof.** Observe first that for $v\in\mathbb{R}^d$

$$\begin{aligned}\langle v,s\rangle\geq\varepsilon \quad \forall s\in\Psi(\omega)\setminus\{0\} \;&\Leftrightarrow\; \langle v,s\rangle\geq\varepsilon \quad \forall s\in\overline{\Psi}(\omega)\setminus\{0\}\\ &\Leftrightarrow\; \langle v,s\rangle\geq\varepsilon \quad \forall s\in D(\omega)\setminus\{0\}\end{aligned} \tag{48}$$

where $D(\omega)$ is a dense subset of $\Psi(\omega)$. This is obvious by continuity of the scalar product. With no loss of generality we can then consider $\Psi$ closed valued and we denote by $\psi_n$ its Castaing representation (see Theorem 14.5 in [RW98] for details). For any $n \in \mathbb{N}$ consider the following closed-valued multifunction:

$$\Lambda_n(\omega) = \begin{cases} \left\{v \in \mathbb{R}^d \mid \langle v, \psi_n(\omega)\rangle \geq \varepsilon\right\} & \text{if } \omega \in \text{dom } \Psi,\ \psi_n(\omega) \neq 0 \\ \mathbb{R}^d & \text{if } \omega \in \text{dom } \Psi,\ \psi_n(\omega) = 0 \\ \varnothing & \text{otherwise} \end{cases}$$

We claim that $\Lambda_n$ is measurable for any $n \in \mathbb{N}$ from which the map $\omega \mapsto \bigcap_{n\in\mathbb{N}} \Lambda_n(\omega)$ is also measurable (cfr Proposition 75). From (48) we thus conclude that $\Psi^\varepsilon$ is measurable.
We are only left to show the claim. To this end observe that $\Lambda_n(\omega)$ has non-empty interior on $\{\Lambda_n \neq \varnothing\}$. Therefore for any open set $V \subseteq \mathbb{R}^d$ we have $\{\omega \in \Omega \mid \Lambda_n(\omega) \cap V \neq \varnothing\} = \{\omega \in \Omega \mid \text{int}(\Lambda_n(\omega)) \cap V \neq \varnothing\}$. Note now that

$$\{\omega \in \Omega \mid \text{int}(\Lambda_n(\omega)) \cap V \neq \varnothing\} = \psi_n^{-1}\left(\Pi_y\left(\Pi_x^{-1}(V) \cap \langle\cdot,\cdot\rangle^{-1}(\varepsilon,\infty)\right)\right) \cup \psi_n^{-1}(0)$$

which is measurable (when $\psi_n$ is measurable) from the continuity of $\langle\cdot,\cdot\rangle$ and from the open mapping property of the projections $\Pi_x, \Pi_y : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$. ■

**Proposition 75** *[Proposition 14.2-11-12 [RW98]] Consider a class of $\mathcal{A}$-measurable set-valued functions. The following operations preserve $\mathcal{A}$-measurability: countable unions, countable intersections (if the functions are closed-valued), finite linear combination, convex/linear/affine hull, generated cone, polar set, closure.*

## 7.2 Complementary results

Recall that we are assuming that $\Omega$ is a Polish space.

**Lemma 76** *Let $Q_i \in \mathcal{M}$ for any $i \in \mathbb{N}$. Then*

$$Q := \sum_{i\in\mathbb{N}} \frac{1}{2^i} Q_i \in \mathcal{M}$$

**Proof.** We first observe that $Q \in \mathcal{P}$ hence we just need to show that is a martingale measure. Consider the measures $Q_k := \sum_{i=1}^k \frac{1}{2^i} Q_i$, which are not probabilities, and note that for each $k$ we have: $\int_\Omega 1_B \Delta S_t dQ_k = 0$ if $B \in \mathcal{F}_{t-1}$. We observe that $\|Q_k - Q\| \to 0$ for $k \to \infty$, where $\|\cdot\|$ is the total variation norm. We have indeed that

$$\sup_{A\in\mathcal{F}} |Q_k(A) - Q(A)| = \sup_{A\in\mathcal{F}} \sum_{i=k+1}^{\infty} \frac{1}{2^i} Q_i(A) = \sum_{i=k+1}^{\infty} \frac{1}{2^i} \to 0 \text{ as } k \to \infty.$$

In particular we have $Q_k(A) \uparrow Q(A)$ for any $A \in \mathcal{F}$. This implies that for a non negative random variable $X$

$$\lim_{k\to\infty} \int_\Omega X dQ_k = \lim_{k\to\infty} \left(\sup_{f\in\mathfrak{S}} f_j(\omega) Q_k(A_j)\right) = \sup_k \sup_{f\in\mathfrak{S}} f_j(\omega) Q_k(A_j) =$$

$$\sup_{f\in\mathfrak{S}} \sup_k f_j(\omega) Q_k(A_j) = \sup_{f\in\mathfrak{S}} f_j(\omega) Q(A_j) = \int_\Omega X dQ$$

where $\mathfrak{S}$ are the simple function less or equal than $X$. For any $B \in \mathcal{F}_{t-1}$ we then have:

$$E_Q\left[1_B \Delta S_t\right] = \int_\Omega (1_B \Delta S_t)^+ dQ - \int_\Omega (1_B \Delta S_t)^- dQ$$

$$= \lim_{k\to\infty} \int_\Omega (1_B \Delta S_t)^+ dQ_k - \lim_{k\to\infty} \int_\Omega (1_B \Delta S_t)^- dQ_k = \lim_{k\to\infty} \int_\Omega 1_B \Delta S_t dQ_k = 0.$$

■

**Lemma 77** *For any dense set $D \subseteq \Omega$, the set of probabilities $co(\{\delta_\omega\}_{\omega \in D})$ is $\sigma(\mathcal{P}, C_b)$ dense in $\mathcal{P}$.*

**Proof.** Take $\omega^* \notin D$ and let $\omega_n \to \omega^*$. Note that for every open set $G$ we have $\liminf \delta_{\omega_n}(G) \geq \delta_{\omega^*}(G)$ and this is equivalent to the weak convergence $\delta_{\omega_n} \overset{w}{\to} \delta_{\omega^*}$. Observe that for every set $X$ we have

$$\overline{co(X)} = \overline{co}(X) := \bigcap \{C \mid C \text{ convex closed containing } X\} = \overline{co}(\overline{X}).$$

Hence, by taking $X = \{\delta_\omega\}_{\omega \in D}$ and by $\sigma(\mathcal{P}, C_b)$ density of the set of measures with finite support in $\mathcal{P}$, we obtain the thesis. ∎

**Lemma 78** *Let $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel sigma algebra and let $\widetilde{\mathcal{F}}$ be a sigma algebra such that $\mathcal{F} \subseteq \widetilde{\mathcal{F}}$. The set $\widetilde{\mathcal{P}} := \{\widetilde{P} : \widetilde{\mathcal{F}} \to [0,1] \mid \widetilde{P} \text{ is a probability}\}$ is endowed with the topology $\sigma(\widetilde{\mathcal{P}}, C_b)$. Then*

1. *If $A \subseteq \Omega$ is dense in $\Omega$, then $co(\{\delta_\omega\}_{\omega \in A})$ is $\sigma(\widetilde{\mathcal{P}}, C_b)$ dense in $\widetilde{\mathcal{P}}$. Notice that any element $Q \in co(\{\delta_\omega\}_{\omega \in A})$ can be extended to $\widetilde{\mathcal{F}}$.*

2. *If $D \subseteq \Omega$ is closed then*
$$\widetilde{\mathcal{P}}(D) := \{\widetilde{P} \in \widetilde{\mathcal{P}} \mid supp(\widetilde{P}) \subseteq D\}$$
*is $\sigma(\widetilde{\mathcal{P}}, C_b)$ closed, where the support is well-defined by*
$$supp(\widetilde{P}) := \bigcap \{C \in \mathcal{C} \mid \widetilde{P}(C) = 1\}$$
*and $\mathcal{C}$ are the closed sets in $(\Omega, d)$.*

**Proof.** By construction for any $\widetilde{P} \in \widetilde{\mathcal{P}}$ we have $\int f d\widetilde{P} = \int f dP$ for any $f \in C_b$ where $P \in \mathcal{P}$ is the restriction of $\widetilde{P}$ to $\mathcal{F}$.
To show the first claim we choose any $\widetilde{P} \in \widetilde{\mathcal{P}}$. Consider $P \in \mathcal{P}$ the restriction of $\widetilde{P}$ to $\mathcal{F}$. Then from Lemma 77 there exists a sequence $Q_n \in co(\{\delta_\omega\}_{\omega \in A})$ such that $\int f dQ_n \to \int f dP$ for every $f \in C_b$. As a consequence $\int f dQ_n \to \int f d\widetilde{P}$, for every $f \in C_b$.
To show the second claim consider any net $\{\widetilde{P}_\alpha\}_\alpha \subset \widetilde{\mathcal{P}}(D)$ such that $\widetilde{P}_\alpha \overset{w}{\longrightarrow} \widetilde{P}$. We want to show that $\widetilde{P} \in \widetilde{\mathcal{P}}(D)$. Consider $P_\alpha, P$ the restriction to $\mathcal{F}$ of $\widetilde{P}_\alpha, \widetilde{P}$ respectively. Then $P_\alpha \overset{w}{\longrightarrow} P$. Notice that by definition $supp(P_\alpha) = supp(\widetilde{P}_\alpha) \subseteq D$ and $supp(P) = supp(\widetilde{P})$. Moreover the set $\mathcal{P}(D) = \{P \in \mathcal{P} \mid supp(P) \subseteq D\}$ is $\sigma(\mathcal{P}, C_b)$ closed (Theorem 15.19 in [AB06]) so that $D \supseteq supp(P) = supp(\widetilde{P})$. ∎

**Proof of Proposition 64, item (2).** Recall that an Open Arbitrage in $\widetilde{\mathcal{H}}$ is a $\{\widetilde{\mathcal{F}}\}$-predictable processes $H = [H^1, \ldots, H^d]$ such that $V_T(H) \geq 0$ and $\mathcal{V}_H^+ = \{V_T(H) > 0\}$ contains an open set.

First we show that $H \in W(\widetilde{\sigma}, \widetilde{\mathcal{H}})$ implies $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. We need only to show that $B := \{\omega \in \Omega \mid V_T(H)(\omega) < 0\}$ is empty. By contradiction, let $\omega \in B$, take any $P \in \mathcal{U}$ and define the probability $P_\lambda := \lambda \delta_\omega + (1 - \lambda) P$. Since $V_T(H) \geq 0$ $P$-a.s. we must have $P(\omega) = 0$, otherwise $P(B) > 0$. However, $P_\lambda(B) \geq P_\lambda(\omega) = \lambda > 0$ for all positive $\lambda$ and $P_\lambda$ will belongs to $\mathcal{U}$, as $\lambda \downarrow 0$, which contradicts $V_T(H) \geq 0$ $P$-a.s. for any $P \in \mathcal{U}$.

To prove the equivalence, assume first that $H \in W(\widetilde{\sigma}, \widetilde{\mathcal{H}})$. We claim that $(\mathcal{V}_H^+)^c = \{\omega \in \Omega \mid V_T(H) = 0\}$ is not dense in $\Omega$. This will imply the thesis as the open set $int(\mathcal{V}_H^+)$ will then be a not empty on which $V_T(H) > 0$. Suppose by contradiction that $\overline{(\mathcal{V}_H^+)^c} = \Omega$. We know by Lemma 78 that the corresponding set $\mathcal{Q}$ of embedded probabilities $co(\{\delta_\omega\}_{\omega \in (\mathcal{V}_H^+)^c})$ is weakly dense in $\widetilde{\mathcal{P}}$ and hence it intersects, in particular, the weakly open set $\mathcal{U}$. However, for every $P \in \mathcal{Q}$ we have $V_T(H) = 0$ $P$-a.s. and so this contradicts the assumption.

Suppose now that $H \in \widetilde{H}$ is an Open Arbitrage. Note that from Lemma 78 if $F$ is a closed subset of $\Omega$, then $\widetilde{\mathcal{P}}(F) := \{P \in \widetilde{\mathcal{P}} \mid supp(P) \subset F\}$ is $\sigma(\widetilde{\mathcal{P}}, C_b)$-closed. Since $H$ is an Open

Arbitrage then $\mathcal{V}_H^+$ contains an open set and in particular $G := \overline{(\mathcal{V}_H^+)^c}$ is a closed set strictly contained in $\Omega$. Observe then that $\left(\widetilde{\mathcal{P}}(G)\right)^c$ is a non empty $\sigma(\widetilde{\mathcal{P}}, C_b)$-open set of probabilities such that for all $P \in \mathcal{U}$ we have $V_T(H) \geq 0$, $P$-a.s. and $P(\mathcal{V}_H^+) > 0$. ■

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