III. Symmetries of Dynamical Systems

The application of symmetry methods for differential equations for dynamical systems (as defined in Chap. II) presents some special aspects [146,305], due to the structure of the differential equations under study. In a sense, to be clear in the following, when having to deal with the specially simple class of equations consisting of first order autonomous ODEs, one is faced with a specially difficult problem when trying to determine its symmetries. Indeed one has, in general, an infinite dimensional symmetry algebra. Moreover, as we will see below, the structure of this is related to the constants of motion for the system.

Thus, one studies the symmetry of the system to learn more about its behaviour, periodic and otherwise noteworthy solutions, conserved quantities, etc., but on the other hand information about these solutions, conserved quantities, etc. is needed to unravel the structure of the symmetry algebra!

The two problems are strongly related, and indeed we will see that determining the full symmetry algebra of the dynamical system is not any easier than determining its most general solution. Luckily, on the one hand we can make good use of symmetry methods based on a partial knowledge of the symmetry algebra, and on the other hand in the study of dynamical systems we are often satisfied – or at least we are used to be satisfied, as we cannot usually go beyond this stage – with qualitative and/or semiquantitative, and anyway partial, descriptions of its behaviour.

Another important fact is that we can proceed in a perturbative way, not only in the study of the dynamical system (Chap. II), but also in the determination of its symmetries. Moreover, an approximate determination of a symmetry – or even the determination of an approximate symmetry\(^1\) (see below) – gives useful information on the perturbative expansion of the dynamical systems. Actually, as we will argue below, once we are studying the dynamical system by a perturbative approach, approximate systems are at least as useful as exact ones [94,159].

The specific structure of dynamical systems will lead to the above mentioned relation between symmetries and constants of motion, and the infinite dimensional structure of the symmetry algebra (actually, as we will see below, it would be more appropriate to speak of the symmetry module), but it does also naturally lead to consider a more restricted class of symmetries, i.e. those which leave

\(^1\)This includes not only the case of a vector field that is obtained as an approximation of an exact symmetry, but also that of a vector field such that its commutator with the dynamical vector field is in some sense “small”, but which is not “near” an exact symmetry.
globally invariant the class of dynamical systems ODEs. From a physical point of view, these are the symmetries which "recognize" the special role of time, i.e. of the independent variable. Considering these will also focus our attention on the trajectories of the dynamical system (rather than on full solutions, i.e. integral curves in space-time), and this will in turn lead to relations between such symmetries and the topology of trajectories for solutions to the dynamical system.

In the final section, we will consider a topic that lies somewhat out of the main line of development of these notes, but is, however, (in our opinion) very relevant, i.e. approximate symmetries. For these, we will only introduce the basic concepts and study their algebraic structure, which is equivalent to that of exact symmetries. Applications of these will not be considered, but are discussed in [94].

In the Appendix, we discuss some points concerning the module structure of $\mathcal{G}_X$. This will be further discussed in the next chapter, but here we want to warn the reader about some relatively common misunderstandings, and clarify some points.

1. Symmetries of Dynamical Systems

We will write, as in Chap. II, a dynamical system (DS) in the form

$$\dot{x} = f(x) \ ; \ x \in M \ ; \ f : M \to TM$$

where $M \subseteq \mathbb{R}^n$ is a smooth manifold (regularly embedded in $\mathbb{R}^n$), $x$ are cartesian coordinates in $\mathbb{R}^n$, and $f$ a smooth function. We also associate to (1) the vector field

$$X_f = f^i(x) \frac{\partial}{\partial x^i} .$$

According to our general procedure, we look for symmetry vector fields$^2$ in the form

$$X = \tau(x,t) \partial_t + \varphi^i(x,t) \partial_i ,$$

where we have written $\partial_i \equiv (\partial/\partial x^i)$. The first prolongation of this is

$$X^{(1)} = X + \dot{\varphi}^{(i)} \frac{\partial}{\partial \dot{x}^i} ,$$

and according to the general prolongation formula the coefficients $\dot{\varphi}^{(i)}$ are given by

$$\dot{\varphi}^{(i)} = \left[ \partial_i \varphi^j + \dot{x}^j \partial_j \varphi^i - \dot{x}^i (\partial_t \tau + \dot{x}^j \partial_j \tau) \right] .$$

Thus, by applying $X^{(1)}$ on the ODE (1) describing the DS, and by requiring the vanishing of this on the solution to (1), we get a rather complicate system of linear PDEs for $\tau$ and the $\varphi$'s, which can be written explicitly as

$^2$ Note we are restricting, as nearly always in this volume, our attention to continuous symmetry; for dicrete symmetries see e.g. [202].
\[
\frac{\partial \varphi^i}{\partial t} + f^j(x) \frac{\partial \varphi^i}{\partial x^j} - \frac{\partial f^i(x)}{\partial x^j} \varphi^j = f^i(x) \left( \frac{\partial \tau}{\partial t} + f^j(x) \frac{\partial \tau}{\partial x^j} \right). \quad (6)
\]

This is in general, clearly, an underdetermined system: indeed, we have \( n \) equations for the \((n + 1)\) functions \( \varphi^1, ..., \varphi^n; \tau \).

2. Lie-Point Time-Independent Symmetries

When we apply a symmetry transformation like (3) – or more precisely, its first prolongation (4) – on a DS (1), we do not, in general, again obtain a DS.

This is a general situation, i.e. we have a similar problem also for general ODEs, and for PDEs, when we allow dependent and independent variables to mix through the symmetry transformation. In order to guarantee that an evolution equation (in this case an ODE) is transformed into an equation of the same class\(^3\), we have to require that the time transformation does not depend on spatial independent variables (absent in this case), nor on the dependent variables. In the present notation, this means that we have to ask that \( \tau \) be a function of \( t \) alone, i.e. that the effect of \( X \) on the time variable amounts to a reparametrization of time.

It should be emphasized that this requirement is sufficient to guarantee that a (system of) first order ODE is transformed into a system of the same kind, but not that an autonomous system remains autonomous. By direct inspection, we see that even with this hypothesis (i.e. assuming \( \tau_j = 0 \)) the system is changed, up to first order terms, into

\[
\dot{x}^i = f^i(x) + \varepsilon \left[ (\varphi^j \partial_j f^i - f^j \partial_j \varphi^i) + (\tau_k f^i - \partial_k \varphi^i) \right], \quad (7)
\]

which in general is not an autonomous equation.

The condition for \( X \) to be a symmetry, i.e. to leave the DS unchanged, can be read directly from (7) to be, in this case,

\[
\partial_t \varphi^i - \tau_k f^i = (\varphi^j \partial_j f^i) - (f^j \partial_j) \varphi^i. \quad (8)
\]

The right hand side of this has a special relevance in all the theory of symmetries for dynamical systems. Thus we have a special notation for it, i.e. we define (Chaps. I and II) the Lie-Poisson bracket \( \{.,.\} : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M} \) (where \( \mathcal{M} \) represents the algebra of, now possibly time-dependent, vector fields on \( \mathcal{M} \)) by

\[
\{f, g\} = (f \cdot \nabla) g - (g \cdot \nabla) f. \quad (9)
\]

Note that (as already noted in Chap. I) this is just the commutator of vector fields read on their components. Indeed, if we have vector fields \( X_f = f^i \partial_i \) and

---

\(^3\) It should be stressed that we require this to be true for the full group of transformations generated by the symmetry vector field \( X \), i.e. for all values of the group parameter \( \lambda \) in the transformation \( e^{\lambda X} \).
X_g = g^i \partial_i$, then their commutator $[X_f, X_g] = X_h = h^i \partial_i$ has components given by $h = \{f, g\}$.

In this notation, (8) reads simply

$$\varphi_t - \tau f = \{\varphi, f\}.$$  \hspace{1cm} (10)

If we disregard the possibility of time reparametrizations, i.e. we require that our symmetry transformations do not affect the temporal coordinate, we have $\tau \equiv 0$, and (10) reduces to

$$\varphi_t = \{\varphi, f\}.$$  \hspace{1cm} (11)

If we allow a nonzero $\tau$ (with $\tau_j = 0$ for all $j$), then we can define $\psi(x, t) = \varphi(x, t) - \tau(t)f(x)$, and (10) is then rewritten as

$$\psi_t = \{\psi, f\}.$$  \hspace{1cm} (12)

Thus, once we impose the condition $\tau = \tau(t)$, we can limit ourselves to considering symmetries with $\tau = 0$. From now on we will indeed proceed in this way.

The reader familiar with the language of symmetry methods for differential equations will recognize at once that we are in this way essentially considering the evolutionary representative $[262]$ of the vector field $X$.

However, a vector field of this form, i.e. of the form $X = \varphi^i(x, t) \partial_i$, is not in general transforming dynamical systems into dynamical systems, although it can of course leave unchanged a given class of dynamical systems. To be sure that the class of dynamical systems is left globally invariant, we should also ask that $\varphi_t = 0$ (or $\psi_t = 0$ if we are dealing with the evolutionary representative). In this case the symmetry condition takes an especially simple form, i.e. it reduces to

$$\{f(x), \varphi(x)\} = 0.$$  \hspace{1cm} (13)

Such symmetries, which do not depend on, neither affect, the time, are also called Lie-point time-independent (or LPTI for short), and we are primarily interested in them.

Note incidentally that $\varphi \equiv f$ is an obvious solution of (13), this indeed corresponds to the “trivial” symmetry $Y \equiv X_f = f^i \partial_i$ which is just the generator of the dynamical flow: $x(t) \rightarrow x(t + t')$.

It should also be mentioned that, from a purely geometrical point of view, in which a dynamical system is just a vector field on $M$, it is entirely natural to consider only symmetry vector fields $Y$ which are themselves vector fields on $M$, i.e. LPTI symmetries. That this is geometrically natural is reflected in the connection with the topological features of trajectories, discussed in Sect. 4 below.

**Exercise 1.** Given a system $\dot{x} = f(x)$, this is put in Lax form – if possible – by giving a matrix $B$ and a matrix function $L(x)$ such that, taking into account the

\footnote{More precisely, the evolutionary representative would be given by $X_e = (\varphi^i - \tau \dot{x}^i) \partial_i$. Thus, we are also imposing $\dot{x} = f(x)$.}
time evolution of \( x(t) \), these satisfy \( \dot{L} = [B, L] \). Consider the system \( \dot{L} = X(L) \), with the (linear) vector field \( X \) determined by \( X(L) = [B, L] \), and look for symmetries of \( X \) in the form \( Y = \sum_{i,j} \phi^{ij} (\partial/\partial L_{ij}) \). Characterize these, and the time evolution of the corresponding \( \phi \).

3. Constants of Motion
   and the Module Structure of the Symmetry Algebra

We first reintroduce some formal notation, already seen in previous chapters. We denote by \( \mathcal{M} \) the Lie algebra of vector fields on \( M \) (equipped with the Lie commutator), and by \( \mathcal{V} \) the algebra of functions \( f : M \rightarrow TM \); this is a Lie algebra if equipped with the Lie-Poisson bracket \( \{.,.\} \), and there is an obvious isomorphism between \( \mathcal{M} \) and \( \mathcal{V} \). We also denote by \( \mathcal{F} \) the algebra (actually, the ring) of scalar functions\(^5\) \( f : M \rightarrow \mathbb{R} \). In all these sets we consider not only analytic or smooth functions (those we are interested in analyzing) but also formal power series.

We fix a given dynamical system in \( M, \dot{x} = f(x) \), with \( f \in \mathcal{V} \); equivalently, we fix a given vector field \( X = f^i \partial_i \in \mathcal{M} \). The Lie algebra of vector fields \( Y \in \mathcal{M} \) such that \( [X, Y] = 0 \) is denoted by \( \mathcal{G}_X \). This is also characterized as the algebra of vector fields \( Y = g^i(x) \partial_i \) such that \( \{f, g\} = 0 \), and correspondingly the Lie algebra of functions \( g \in \mathcal{V} \) which satisfy \( \{f, g\} = 0 \) is denoted by \( \mathcal{G}_f \).

The constants of motion for \( X \) are the functions \( s(x) \in \mathcal{F} \) which are conserved along the flow of \( X \), i.e. such that \( X(s) = 0 \) or equivalently

\[
\sum_{i=1}^{n} f^i(x) \frac{\partial s(x)}{\partial x^i} = 0 \, .
\]

One sees immediately these form an algebra (actually a ring), as indeed \( X(s_1 + s_2) = X(s_1) + X(s_2) \). This will be denoted by \( \mathcal{I} \subseteq \mathcal{F} \) (or also by \( \mathcal{I}_f \) when necessary to avoid confusion).

Now, let us consider \( \alpha(x) \in \mathcal{I} \) and \( g(x) \in \mathcal{G}_f \), and the function \( \alpha(x)g(x) \in \mathcal{V} \). This is easily seen to be again in \( \mathcal{G}_f \), as

\[
\{f, \alpha g\} = \alpha \{f, g\} + X(\alpha) g \, .
\]

In this way, we conclude that the symmetry algebra \( \mathcal{G}_f \) (equivalently, \( \mathcal{G}_X \)) has, beyond the structure of Lie algebra, also the structure of a module over \( \mathcal{I} \). More precisely, we have [330]:

**Proposition** (Walcher). The set \( \mathcal{G}_f \) is a finitely generated module over \( \mathcal{I}_f \).

It should be emphasized that, whenever \( X \) admits a nontrivial \( \mathcal{I} \), \( \mathcal{G}_X \) will be infinite-dimensional as a Lie algebra, just as a consequence of the possibility of

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\(^5\) Clearly, \( \mathcal{V} \) stands for "vector" and we continue to use \( \mathcal{F} \) instead than \( \mathcal{S} \) for the scalar function to avoid any confusion with sets of "symmetries".
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multiplying any \( Y \in \mathcal{G}_X \) (e.g. \( X \) itself) by a function in \( \mathcal{I}_1 \), i.e. of the module structure. Obviously, we can – and in general will – have a \( \mathcal{G} \) which is infinite-dimensional as an algebra, but finite dimensional as a module.\(^6\)

As an example of this, consider the simple system

\[
\begin{align*}
\dot{x} &= \left[1 + e^{-(x^2+y^2)}\right] y \\
\dot{y} &= -\left[1 + e^{-(x^2+y^2)}\right] x .
\end{align*}
\]

Clearly, \( Y = x \partial_y - y \partial_x \in \mathcal{G}_X \), and \( \rho = (x^2 + y^2) \in \mathcal{I}_1 \), and thus \( \mathcal{G}_X \) is infinite-dimensional as a Lie algebra, as it contains\(^7\) the algebra \( \mathcal{G}_0 \) of linearly (but not functionally) independent vector fields \( Y_k = \rho^k Y \). This is, however, one-dimensional as a module over \( \mathcal{I}_1 \).

**Exercise 2.** Consider the system

\[
\begin{align*}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 .
\end{align*}
\]

Show that its Lie-point symmetry algebra \( \mathcal{G}_X \), made of vector fields of the form \( Y = \alpha(x) \partial_1 + \beta(x) \partial_2 \) (\( \alpha, \beta \) polynomials), is infinite-dimensional, and more precisely consists of all the vector fields of the form

\[
Y = [a(r^2)x_1 - b(r^2)x_2] \partial_1 + [b(r^2)x_1 + a(r^2)x_2] \partial_2
\]

with \( a, b \) arbitrary polynomials. Determine now \( \mathcal{I}_1 \), and check that indeed \( \mathcal{G}_X \) is a module over \( \mathcal{I}_1 \). Check that any vector field \( Y \in \mathcal{G}_X \) can be written in the form

\[
Y = \xi(x) \sum_{k=0}^{1} c_k A^k x
\]

where \( (DX)(0) = Ax \) and \( \xi \in \mathcal{I}_A \) is a constant of motion for the linearized equation \( \dot{x} = Ax \).

4. Symmetry and Topology of Trajectories

A solution to the DS \( \dot{x} = f(x) \) is a curve \( \gamma \in \mathbb{R} \times M \), i.e. a curve

\[
\gamma = (t, x(t))
\]

in the extended phase space. However, as the ODEs representing a DS (according to our definition in Chap. II) are autonomous, it would also be entirely natural to consider just the phase space and the trajectory \( \vartheta \) corresponding to \( \gamma \), i.e. the

---

\(^6\) The situation is similar to the one encountered when studying the invariant (scalar) and equivariant (vector) polynomials for a group representation.

\(^7\) Actually, one could see that \( \mathcal{G} = \mathcal{G}_0 \).
set of points in \( M \) touched by \( \gamma \). More precisely, let \( \mu \) denote the projection from \( \mathbb{R} \times M \) to \( M \), then

\[
\vartheta = \{ x \in M : \mu(\gamma(t)) = x, \ t \in \mathbb{R} \}
\]  

or, with an obvious shorthand notation,

\[
\vartheta = \mu(\gamma) .
\]  

We will not discuss the (well known) general relation between the two points of view, and the general advantages of using the standard or the extended phase space [11], and we will confine our discussion to the relation with symmetry methods.

When studying DS, a special role is played by solutions which are in some sense simple (and hopefully easier to determine than a general one) and act as organizing center. In particular, one often studies solutions of increasing order of complexity. By this we mean first of all stationary solutions, i.e. fixed points, or zeroes of the vector field \( X \), then periodic solutions, and then \( n \)-periodic solutions. Clearly, solutions \( x(t) \) of these different kinds are topologically equivalent as curves \( \gamma \in \mathbb{R} \times M \), but are topologically different when we just look at their integral curves in \( M \), i.e. at their trajectories \( \vartheta \). Indeed, when we speak of “increasing complexity” of solutions as above, we are actually referring to the topology, i.e. the number of fundamental cycles in the surface representing the closure of their integral curve (trajectory) in \( M \). For the above mentioned classes of special solutions, we have respectively a point, a circle \( S^1 \), an \( n \)-torus \( T^n = S^1 \times \ldots \times S^1 \), and correspondingly, zero, one, and \( n \) cycles.

If we consider a general vector field of the form

\[
Y = \tau(x, t)\partial_t + \sum_{i=1}^{n} \varphi^i(x, t)\partial_i ,
\]  

or even a time-dependent evolutionary vector field,

\[
Y = \sum_{i=1}^{n} \varphi^i(x, t)\partial_i ,
\]

all the solution curves \( \gamma \) are equivalent under these, i.e. we can a priori transform any curve into any other curve by means of a transformation generated by such a \( Y \).

On the other hand, when we consider \( Y \in \mathcal{M} \), time plays no role in the group of transformations generated by \( X \), and we cannot transform a curve corresponding to a trajectory \( \vartheta \) of a given topology into a curve corresponding

---

8 Of course, other kinds of “notable” solutions exist, e.g. heterocline and homocline solutions, the ones in the stable or unstable manifolds of limit points and cycles, etc. In the present context, we will just focus on (multi)periodic ones.
to a trajectory of a different topology\footnote{It should be clear that when speaking about the “topology of a trajectory” \( \vartheta \) we consider the closure \( T \subseteq M \) of \( \vartheta \) in \( M \), and we mean the topology of \( T \) as a submanifold of \( M \).}, as is seen from the group requirements, i.e. from the requirement that \( e^{\lambda Y} \) must be an invertible transformation, and from the time independence of \( Y \).

We can thus conclude that, with the notation introduced above, our discussion amounts to saying that \( \text{LPTI symmetries preserve the topology of trajectories} \ [144] \).

By the same kind of argument, we also obtain that the period of periodic solutions is preserved under LPTI transformations, and similarly for the \( n \) periods for \( n \)-periodic solutions.

In order to have a (trivial) example of this kind of situation, consider a linear system in \( \mathbb{R}^{2m} \)

\[
\dot{x} = Ax
\]  

with the matrix \( A \) written in block form, in terms of the standard two-dimensional rotation matrix

\[
J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

as

\[
A = \begin{pmatrix} \omega_1 J & 0 & \ldots & 0 \\ 0 & \omega_2 J & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \omega_m J \end{pmatrix}
\]

(These correspond to inhomogeneous rotations and scalings, respectively, in the various two-dimensional invariant subspaces.)

For generic, i.e. independent, \( \omega \)'s, the LPTI symmetries of this are given, with \( \rho_k = (x_{2k}^2 - 1 + x_{2k}^2) \) (these are the generators of \( I \)) and \( \alpha_i, \beta_i \) arbitrary smooth functions, by

\[
Y_\alpha = \sum_{i=1}^{m} \alpha_i(\rho_1, \rho_m) (x_{2i-1} \partial x_{2i} - x_{2i} \partial x_{2i-1})
\]

\[
Z_\beta = \sum_{i=1}^{m} \beta_i(\rho_1, \rho_m) (x_{2i-1} \partial x_{2i-1} + x_{2i} \partial x_{2i})
\]

Exercise 3. Consider the case in which some of the frequencies are multiple, i.e. there exist \( i, j \) distinct such that \( \omega_i = \omega_j \) (this requires introducing a permutation group).
5. Time-Dependent Symmetries

In the previous section, we have seen that LPTI symmetries preserve the topology of trajectories, and we have argued that for more general transformations we are not guaranteed that this is the case.

The purpose of this section is simply to show, by means of explicit examples, that indeed by relaxing our requirement on the symmetry vector fields, e.g. considering time-dependent symmetries — or, as we will also see, symmetries that are only locally analytic — we can connect solutions whose trajectories in phase space are topologically different.

Let us consider as an example the dynamical system

\[
\begin{align*}
\dot{x} &= (1 - r^2)x - y \\
\dot{y} &= (1 - r^2)y + x
\end{align*}
\]  

(27)

where as usual \( r^2 = x^2 + y^2 \). One can check easily that the vector field

\[
\sigma = r^2 e^{-2t} (x\partial_x + y\partial_y)
\]  

(28)

is a symmetry of (27).

If we act with this on the periodic solution \( \xi(t) \) of (27) satisfying \( r(t) = 1 \) (which has a circle as trajectory), applying \( e^{2\sigma} \) on \( \xi(t) \) yields a solution whose trajectory is a spiral having \( \xi(t) \) as asymptotic limit.

Indeed, we have

\[
\begin{align*}
\frac{\partial x}{\partial \varepsilon} &= x r^2 e^{-2t} \\
\frac{\partial y}{\partial \varepsilon} &= y r^2 e^{-2t}
\end{align*}
\]  

(29)

which must be supplemented with the boundary condition \( x^2 + y^2 = 1 \) for \( \varepsilon = 0 \). In this way we get

\[
\frac{\partial r^2}{\partial \varepsilon} = r^4 e^{-2t},
\]  

(30)

and we have explicitly, with obvious notation,

\[
r^2(t) = \left[ 1 - \varepsilon e^{-2t} \right]^{-1}.
\]  

(31)

This is a spiral, internal or external depending on the sign of \( \varepsilon \), approaching the limit cycle \( r^2(t) = 1 \).

As mentioned above, the same effect, i.e. connecting solutions whose trajectories have different topologies, can also be reached by considering time-independent “symmetries”, provided that we are ready to suitably enlarge the class of dynamical systems which we accept as such. In particular, we could consider vector fields which are locally, but not globally, analytic, as the following example shows.

Consider again the system (27), and denote by \( \varphi \) the corresponding vector field generating the time-flow. Apart from the rotation \( \rho = x\partial_y - y\partial_x \), another symmetry is given by
which is a combination of \( \varphi \) and \( \rho \). If we multiply (32) by the (locally analytic) constant of motion

\[
\kappa = \frac{r^2}{1 - r^2} \exp \left[ -2 \arctg(y/x) \right],
\]

we get a (only locally analytic) vector field

\[
\tilde{\sigma} = r^2 \exp \left[ -2 \arctg(y/x) \right] (x \partial_x + y \partial_y)
\]

which satisfies \([\tilde{\sigma}, \varphi] = 0\). Now, on the solutions to (27) this \( \tilde{\sigma} \) coincides with the \( \sigma \) considered above, see (28). Thus (34) transforms limit-cycle solutions to spiralling solutions (or vice versa), i.e. it connects solutions which have trajectories with different topologies.

It should be remarked that these “local” symmetries\(^\text{10}\) are quite common. Indeed, from the fundamental theorem on rectification of flow \([11]\) we know that around any non-singular point for a flow in \( \mathbb{R}^n \) we have, beyond the flow itself, \((n-1)\) local symmetries and \((n-1)\) local constants of motion.

It can be useful to note that, if one finds \( n \) functionally independent constants of motion \( s_i = s_i(x, t) \) of the DS (14) (here one must also consider constants of motion possibly dependent on time), then one can introduce the following quantities \( p_{jk}(x, t) \) by means of the linear system (in fact, \( \det(\partial s_j / \partial x_i) \neq 0 \) in some open domain)

\[
\sum_j p_{jk} \frac{\partial s_j}{\partial x_i} = \delta_{ik} \quad (i, j, k = 1, \ldots, n).
\]

Then one can verify that the vector fields

\[
Y_j = \sum_k p_{jk} \frac{\partial}{\partial x_k}
\]

provide \( n \) independent symmetries of the DS \( \dot{x} = f \) \([266]\).

Let us emphasize that in the following we will only be concerned with LPTI symmetries that are expressed as power series (either formal or possibly convergent in some neighborhood of \( x_0 = 0 \)).

**Exercise 4.** Consider the system

\[
\begin{align*}
\dot{x} &= -r^2 y \\
\dot{y} &= r^2 x
\end{align*}
\]

All of its solutions are periodic, and the period depends on \( r \). Show that

\[
\sigma = (x \partial_x + y \partial_y) + 2 \arctg(y/x) (y \partial_x - x \partial_y)
\]

connects solutions with different periods.

---

\(^{10}\)They should not be confused with a local group of symmetries: here we are considering vector fields that are well defined only locally.
6. Orbital Symmetries

In the previous sections, we focused on the trajectory \( \theta \) rather than on the full solution \( \gamma \) (Sect. 4). However, we considered the effect of proper Lie-point symmetry on these. By “proper” symmetries, we mean vector fields which transform the solution \( \gamma \) into another (possibly, the same) solution \( \gamma' \).

In recent works, Walcher [332,333] considered orbital symmetries, mainly focusing on the case where these leave the (orbit of the) solution invariant\(^{11} \), but do not necessarily leave the solution itself invariant. In this section, we just want to point out the existence of this promising theory, following [332]. For a more complete discussion, see [333].

Given two dynamical systems on \( M \) identified by \( X_f \) and \( X_g \), i.e.
\[
\dot{x} = f(x) \quad \text{and} \quad \dot{x} = g(x),
\]
we say that \( \Phi : M \to M \) is a solution-preserving map from \( f \) to \( g \) if \( \Phi \otimes I \) maps solutions \((x(t), t)\) of \( \dot{x} = X_f(x) \) to solutions \((y(t), t)\) of \( \dot{y} = X_g(x) \). If \( g(x) = \mu(x)f(x) \) with \( \mu(x) \) a scalar function \( \mu : M \to R \), and \( \Phi \) is invertible, we say that \( \Phi \) is an orbital symmetry of \( f \) (a proper orbital symmetry if \( \mu(x) \neq 1 \)). The group \( G \) of maps \( \Phi : M \to M \) is an orbital symmetry group for \( f \) if all the \( \Phi \in G \) are orbital symmetries of \( f \).

The above definition of orbital symmetries is justified by the fact that the vector fields \( X_f \) and \( X_{\mu f} \) have different solutions but the same trajectories (orbits), and conversely two dynamical systems which have locally the same solution trajectories differ only by a scalar factor \( \mu(x) \) which is, moreover, the quotient of two analytic functions [332].

Orbital symmetries are more abundant than proper symmetries (indeed, any proper symmetry is also an orbital symmetry, but the converse is not true). Correspondingly, the reduction procedure by orbital symmetries is more complicated than the one by proper symmetries [332,333]. As far as we know, reduction by orbital symmetries has been considered only for finite linear groups of orbital symmetries [332].

In this case, orbital symmetry groups can be characterized as follows [332,333]: A finite linear group \( G \) is an orbital symmetry group of \( \dot{x} = f(x) \) if and only if there is a homomorphism \( \chi : G \to C^* \) such that
\[
\Phi \cdot f \cdot \Phi^{-1} = \chi(\Phi)f
\]
for all the \( \Phi \in G \). When \( \chi(\Phi) = 1 \), we have a proper symmetry group.

We refer to [333] for further details and applications.

Exercise 5. Is every vector field that leaves invariant the trajectories of solutions to a differential equation \( \dot{x} = f(x) \) an orbital symmetry? What if we consider a time-dependent vector fields?

\(^{11}\)Instead of transforming the orbit — i.e. in our terminology the trajectory — of one solution into that of a different solution.
7. Approximate Symmetries

As we have been seeing, the knowledge of exact symmetries of an ordinary differential equation allows us to reduce it, and sometimes to completely solve it. Thus, it helps in determining exact solutions.

However, in many cases one would be satisfied with approximate rather than exact solutions. It turns out, as we will briefly discuss in this section, that in this case approximate symmetries (to be introduced below) are as good as exact ones. Moreover, it happens that these can be determined perturbatively, which is of not little help.

We will, as usual, consider autonomous dynamical systems written as

\[ \dot{x} = f(x) \]  

(37)

with \( x \in M \subseteq \mathbb{R}^n \), \( f : M \rightarrow TM \), and we will assume that \( f(0) = 0 \). We also write \( X = f'(x) \partial_x \) for the associated vector field.

We recall that under a coordinate transformation generated by the vector field \( Y = s'(x) \partial_x \), i.e.

\[ x \rightarrow \tilde{x} = x + \varepsilon s(x) + o(\varepsilon) \, , \]

(38)

this is transformed to a new system

\[ \tilde{f}(x) := f(x) + \varepsilon \{ f, s \} + o(\varepsilon) \, , \]

(39)

where we have introduced the Lie-Poisson bracket

\[ \{ \varphi, \psi \} := (\varphi \cdot \nabla)\psi - (\psi \cdot \nabla)\varphi \, . \]

(40)

The LPTI symmetries of (37) are therefore given by vector fields \( Y \) with \( s \) solution of

\[ \{ f, s \} = 0 \, . \]

(41)

This is just the commutation condition

\[ [X, Y] = 0 \]

(42)

expressed in terms of the functions \( f, s \).

In order to determine explicitly \( Y \), we should solve (41). We now want to solve this perturbatively. Thus, we expand \( f \) and \( s \) around \( x_0 = 0 \) (note that necessarily \( s(0) = 0 \)); i.e. we write

\[ f(x) = \sum_{m=0}^{\infty} f_m(x) \]

(43)

\[ s(x) = \sum_{m=0}^{\infty} s_m(x) \, , \]

(44)

where \( f_m, s_m \) are homogeneous polynomials of degree \( m + 1 \). Writing (41) in terms of these gives a series of equations.
\[ \sum_{j=0}^{k} \{f_j, s_{k-j}\} = 0 \quad k = 0, 1, 2, \ldots \quad (45) \]

Note that if we have solved the equations in this sequence for \( k < k_0 - 1 \), i.e. have determined \( s_0, \ldots, s_{k_0-1} \), then the \( k_0 \) equation reads

\[ \{f_0, s_{k_0}\} = - \sum_{j=1}^{k_0} \{f_j, s_{k_0-j}\} := F_{k_0}(x) \quad (46) \]

where the r.h.s. is a known function of \( f_0, \ldots, f_{k_0-1} \) and \( s_0, \ldots, s_{k_0-1} \).

**Remark 1.** Notice that \( \{f_0, \cdot\} \) is the homological operator associated with (the linear part of) \( f \) of normal form theory (Chaps. II and IV). With \( f_0(x) = Ax \), we also write\(^\text{12}\)

\[ \{f_0, \cdot\} := L_A(\cdot) \quad (47) \]

so that (46) is an equation of the form

\[ L_A(s_k) = F_k(x) \quad (48) \]

that can be solved if and only if \( F_k(x) \in \text{Ran}(L_A) \). Moreover, \( s_k \) is determined up to a function in \( \text{Ker}(L_A) \).

When (41), i.e. (45), is satisfied not for all \( k \), but only for \( k \leq K \), we say that \( Y \) is an approximate symmetry of order \( K \) for \( X \).

We can also write, for given \( f \) (i.e. \( X \)) and \( s \) (i.e. \( Y \)),

\[ [X, Y] = Z = r^i \partial_i = \left( \sum_k r_k \right) \partial_i \quad (49) \]

which is equivalently written as

\[ \{f, s\} := r ; \quad r(x) = \sum_{m=0}^{\infty} r_m(x) . \quad (50) \]

With this notation, we have the following:

**Definition 1.** If in (49),(50) we have \( r_m = 0 \ \forall m \leq k \), then \( Y = s^i \partial_i \) is an approximate symmetry of order \( k \) for (1).

The set of approximate symmetries of order \( k \) of the vector field \( X \) will be denoted by \( G_X^{(k)} \); obviously, \( G_X^{(k+1)} \subseteq G_X^{(k)} \).

**Lemma 1.** The set \( G_X^{(k)} \) is a a Lie algebra.

**Proof.** Let \( Y_1, Y_2 \) be in \( G_X^{(k)} \). We consider

\(^\text{12}\)We repeat here some statements already presented – and studied – in previous chapters for the sake of completeness of the present chapter.
and want to show that \( \sigma \) is also an approximate symmetry. The commutator

\[
[Y_1, Y_2] \equiv Z = s^i(x) \partial_i \quad ; \quad s(x) = \sum_{m=0}^{\infty} s_m(x) \tag{51}
\]

is also written, using the Jacobi identity, as

\[
[Y_1, [X, Y_2]] - [Y_2, [X, Y_1]] \equiv [Y_1, R_2] - [Y_2, R_1] . \tag{53}
\]

Now, both \( R_1 \) and \( R_2 \) have only terms of order greater than \( k \), so that \( w_m(x) = 0 \) for all the \( m \leq k \); that is, \( Z \in \mathcal{G}^{(k)}_X \).

Actually, as was already the case for the symmetry algebra \( \mathcal{G}_X \), the approximate symmetry algebra of order \( k \) also has the structure of a module.

Let us consider a scalar function \( \zeta(x) \in R \), which is expanded as

\[
\zeta(x) = \sum_{m=0}^{\infty} z_m(x) . \tag{54}
\]

By acting on this with \( X \) we get

\[
X(\zeta) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (f_k \cdot \nabla) z_{m-k}(x) \equiv \sum_{m=0}^{\infty} a_m(x) . \tag{55}
\]

**Definition 2.** If in (55) we have \( a_m(x) = 0 \) for all the \( m \leq k \), we say that \( \zeta \) is an approximate constant of motion of order \( k \) for \( X \).

We denote by \( I^{(k)}_X \) the set of approximate constants of motion of order \( k \) for \( X \). Clearly, \( I^{(k)}_X \) is an (abelian) algebra under the standard product of functions; moreover we have \( I^{(k+1)}_X \subseteq I^{(k)}_X \).

**Lemma 2.** The set \( \mathcal{G}^{(k)}_X \) has, beyond the structure of Lie algebra, the structure of a module over the algebra \( I^{(k)}_X \).

**Proof.** This follows immediately from (55).

**Remark 2.** The determination of approximate symmetries and approximate constants of motion is intimately related to the construction of normal forms. This relation is discussed in [94] for general systems. For the special case of Hamiltonian systems, see [159] or Chapter IV.

**Remark 3.** In many cases\(^{13}\) the study of nonlinear systems goes through consideration of a suitable truncation of the nonlinear equation, followed by a study of the relation between the dynamics of the full system and that of the truncated one. In this respect, approximate symmetries (of suitable order) are as relevant as the exact ones, as they play the same role in the study of the truncated system [94].

\(^{13}\)In particular, but not only, when using the normal forms approach.
Appendix. On the Module Structure

In Sect. 3 we have shown the module structure of the symmetry algebra $\mathcal{G}_X$ for a vector field $X = f^i(x)\partial_i$; a more detailed and complete discussion of this structure will be given in Chap. IV.

The present short discussion is aimed at preventing a few common misunderstandings concerning the structure of $\mathcal{G}_X$, and related questions. Indeed, one could be led to think that the form given in Exercise 2 is the general one for symmetry vector fields, i.e. that the above statement holds for general $X$ and $A$ [with $(Df)(0) = A$ and $A$ diagonalizable]. It should be stressed that, for general systems, this is not true.

The correct relation between linearized systems, their symmetries and constants of motions, and symmetries and constants of motion for the full system, will be discussed in Chap. IV (and is also given in [121]). Here we just want to convince the reader, by explicit example, that this cannot be as simple as the one satisfied for the simple rotation vector field considered above.

Consider the system $\dot{x} = f(x)$ ($x \in \mathbb{R}^2$) with $J$ is the same as in (23)

\[
f(x) = [\alpha(r^2)I + \beta(r^2)J]x.
\] (A.1)

It is easy to see that for general $\alpha, \beta$, there are symmetries of this which cannot be written in the form $Y = G(x)\partial_i$ with

\[
G(x) = \xi(x) \sum_{k=0}^{1} c_k A^k x
\] (A.2)

where $(Df)(0) = A$ and $\xi \in \mathcal{I}_A$ is a constant of motion for the linearized equation $\dot{x} = Ax$.

Indeed, it suffices to consider the case where $\alpha(0) = 1$ and $\beta(0) = 0$, and to notice that $X \in \mathcal{G}_X$ by definition ($X = X_f$ is the vector field corresponding to (A.1)): now $A = I$, and of course $A^n = I$, but $r$ is not a constant of motion of $\dot{x} = Ax$, and $J \neq A^n$ for all $n$, so that there is no way to write $f(x)$ in the form (A.2). A similar reasoning would apply also to the case where $\beta(0) \neq 0$ (so that we have distinct eigenvalues, see below).

It should be stressed, again as a warning against possible errors, that the unfolding of the normal form of vector fields $X$ in $\mathbb{R}^N$ with given linear part $A$ is not – for general systems, i.e. general $A$ – of the form

\[
\tilde{F}(x) = Ax + \sum_{m,k} c_{m,k} \xi_m(x) A^k x,
\] (A.3)

where $\ell = 0, 1, ..., N - 1$ and the $\xi_m$ are a basis for $\mathcal{I}_A$, the ring of constants of motion under $\dot{x} = Ax$.

To see that (A.3) is not correct, it will be enough to consider the case

\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\] (A.4)
Note that in this case we can write the normal form in a way similar to (A.3), provided that we consider not only the $A^k$, but all the matrices which commute with $A$ (again, this will be discussed in full in Chap. IV, see also [330]). Note that in [204], formula (A.3) is given, but the author specifies that $A$ is assumed to be diagonalizable, and to have only eigenvalues with multiplicity one, in which case the $A^k$ provide a basis for the linear space of matrices commuting with $A$.

It seems to us that, at least in some cases, the origin of the error mentioned above can be traced to a rather simple misunderstanding: if $X = f^i(x)\partial_i$, and $Y = S^i\partial_i$ commute with $X$, then

$$\frac{dS^i}{dt} = \frac{\partial f^i}{\partial x^j} S^j .$$

(A.5)

However, if in the same setting we consider $(Df)(0) = A$ then, in general, $Y$ (i.e. its components $S^i$) does not evolve according to the linearized equations:

$$\frac{dS^i}{dt} \neq A S^j .$$

(A.6)

We also stress that, in general, not only are the symmetries not solutions of the linearized equation $\dot{x} = Ax$, but also they are not even symmetries for it.

Note that the situation is, in several regards, different if $X$ is in normal form, as discussed in Chap. IV.