EINSTEIN'S NONCONVENTIONAL CONCEPTION OF THE PHOTON AND THE MODERN THEORY OF DYNAMICAL SYSTEMS

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Abstract. Everyone knows how Einstein introduced in the year 1905 the concept of the photon, by giving some conctreteness to the discretization of energy previously introduced by Planck at a formal level. Here we point out how, till the end of his life, Einstein considered such a conception just a "provisional way out", to be substituted by a conception involving continuous variations of energy. We explain how such a conception is understood by taking into account Einstein's contribution to the first Solvay conference. Finally we show how such a conception can be at least partially implemented in classical mechanics, through results from the modern theory of dynamical systems.

1 Introduction

The critical position of Einstein towards the standard interpretation of quantum mechanics is very well known, and is vividly witnessed by the famous paper he wrote with Podolsky and Rosen, which stimulated in more recent times so many discussions and controversies. There is however another specific point where Einstein manifested his uneasyness with respect to the standard interpretation of quantum mechanics; we refer to the starting point itself of quantum mechanics, namely the dilemma continuity–discontinuity (i.e. the problem of the very existence of energy levels), which came about in connection with Planck's law.

Apparently this fact remained unnoticed, or at least we were unable to find any reference to it in the literature (see [1]); and this might be a sufficient reason for discussing it in the present paper. Another element of interest is the fact that the nonconventional Einstein's conception of the photon we are referring to turns out to be strictly related to some of the most recent advances in the theory of dynamical systems. This was for the first time pointed out in [2], where it was shown how a relevant fluctuation formula of Einstein is a statistical counterpart of a purely dynamical formula that we like to call the Benettin–Jeans formula. So let us pass to illustrate what we mean by Einstein's nonconventional conception of the photon, and how we came to interpret it in terms of concepts from the theory of dynamical systems.

2 Einstein's nonconventional conception of the photon

First of all, a nonconventional Einstein's conception of the photon indeed exists, at least potentially, in our opinion. A hint for this can be found in a famous page of Einstein's scientific autobiography, which was written a few years before his death. Indeed, he first recalls how, by inventing the photon, he had given some concreteness to the discretization of energy, previously introduced by Planck at a purely formal level. In his very words (see [3]): "*This way of considering the problem showed in a definitive and direct way that it is necessary to attribute a certain immediate concreteness to Planck's quanta and that, under the energetic aspect, radiation possesses a sort of molecular structure*". But after a few lines he adds: "*This interpretation, that almost all contemporary physicists consider as essentially definitive, to me appears instead as a simple provisional way out*". These words are actually so sharp that no doubts should be left; and this is indeed the first fact we are referring to. But the problem of understanding what Einstein actually had in mind, as a positive concrete proposal, when referring to a provisional way out, is a quite a different one. We try now to disclose this point.

In our opinion the clue is given by what Einstein wrote in two papers, in 1909 and 1911 (see [4] and [5]); the second of such papers constitutes in fact his contribution to the first Solvay conference, and is the one to which we will mostly make reference. It such a paper Einstein points out the relevance of fluctuations, and shows that formally Planck's formula is equivalent to assuming that there exists a certain functional relation between energy fluctuations and mean energy of a system of identical oscillators, namely

$$\sigma_E^2 = \epsilon U + U^2/N \ , \tag{1}$$

where U and σ_E^2 denote the mean energy and the fluctuation (precisely, the mean square deviation, or variance) respectively of the energy E of a system of N harmonic oscillators of the same frequency ω, while ϵ is the quantum of energy expressed by $\epsilon = \hbar\omega$ in terms of the reduced Planck's constant $\hbar$. The sense in which such a fluctuation formula is equivalent to Planck's formula will be illustrated in the next section; in the subsequent one we will instead show how an analytical formula of Einstein's type arises in classical mechanics.

For the moment however we just recall how Einstein interpreted his fluctuation formula in connection with the dilemma continuity–discontinuity. Indeed he makes reference to the corresponding formula for the relative fluctuations, namely

$$\frac{\sigma_E^2}{U^2} = \frac{\epsilon}{U} + \frac{1}{N} \ .$$

Then, by considering the limit of a large number N of oscillators or small energies, in which the formula takes the simpler form

$$\frac{\sigma_E^2}{U^2} \simeq \frac{\epsilon}{U} \ ,$$

he remarks (see [5]): "*If U becomes of the order of $\hbar\omega$ (namely of ϵ), the relative fluctuation becomes of the order of unity; in other terms, the fluctuation of energy is of the order of magnitude of energy itself, i.e. the total energy is alternatively present or absent, and consequently behaves as if it were not indefinitely divisible. It is not necessary to make the hypothesis that distinct energy elements of a definite magnitude exist*".

This is actually what we mean by Einstein's nonconventional conception of the photon: one can conceive of the harmonic oscillator in a classical sense, as possessing at each time a well defined energy, ranging in the familiar domain $E \geq 0$, and if one finds a mechanics that produces Einstein's fluctuation formula for the energy, then the "level" ϵ turns out to be just that particular value of energy having the property that, correspondingly, "*the fluctuation of energy is of the order of magnitude of the energy itself*", i.e. that "*energy is alternatively present or absent*". In another passage of [5] Einstein even reinforces such an argument by saying that, in virtue of such a formula, "*the statistical properties of the phenomenon are the same as if energy were transferred through integer numbers of quanta $\hbar\omega$*". With this we presume we have given sufficient support to our claim that a nonconventional Einstein's conception of the photon indeed exists: it describes the energy of the oscillator in classical continuous terms, and the apparent quantum discontinuity just corresponds to a concise description of classical processes obeying a certain fluctuation law for energy, namely (1), giving a suitable functional relation between variance and mean. In our opinion, this is exactly what Einstein had in mind when he wrote the passage from his scientific autobiography quoted in the introduction. So apparently we are concerned here with a conception that he nurtured from at least the year 1909 till his death.

3 Einstein's interpretation of Planck's formula in terms of fluctuations

We now illustrate how Einstein came to conceive of his fluctuation formula (1). What he did was to provide a physical substantiation for the original deduction Planck had given of his law on October 19, 1900 (see [6]). To this end, let us recall preliminarly that it was only in later communications (starting from that of december 1900) that Planck gave his familiar deduction involving the standard statistical arguments with a discretization of energy, while in his first communication he was instead proceeding at a phenomenological level, without invoking any discretization at all.

Planck was concerned with the problem of finding a formula for the mean energy U of a system of N oscillators of the same frequency ω in equilibrium with a heat reservoir at absolute temperature T, or inverse temperature $\beta = 1/kT$, where k is Boltzmann's constant, and made the following remark (we are using here a contamination of the notations of Planck and of Einstein). He knew that Wien's law

$$U = C \exp(-\beta\epsilon) \ ,$$

with ϵ proportional to ω and a suitable constant C, fits well the experimental data for large frequencies, while the most recent experimental data available to him, which were referring to lower frequencies, turned out to rather fit the equipartition law

$$U = N/\beta = NkT \ .$$

On the other hand, Wien's law is obviously obtained as a solution of the differential equation

$$\frac{\mathrm{d}U}{\mathrm{d}\beta} = -\epsilon U \ ,$$

while the equipartition law obvioulsy satisfies the differential equation

$$\frac{\mathrm{d}U}{\mathrm{d}\beta} = -\frac{U^2}{N} \ ,$$

with a suitable choice of the integration constant. Just by virtue of imagination, through an interpolation he was then led to conceive of the differential equation

$$\frac{\mathrm{d}U}{\mathrm{d}\beta} = -(\epsilon U + \frac{U^2}{N}) \ , \tag{2}$$

which by integration, and a suitable choice of the integration constant, indeed gives Planck's formula, namely

$$U(\omega, T) = N \left(\frac{\epsilon}{e^{\beta\epsilon} - 1} \right) \ . \tag{3}$$

Planck's constant $\hbar$ was then introduced by fit with the experimental data through the relation $\epsilon = \hbar\omega$, because it was already known, by a general argument of Wien, that ϵ had to be taken proportional to frequency.

So much for what concerns Planck's first communication. The contribution of Einstein was the following one. Already in the year 1903 (see [7]) he had remarked that in the canonical ensemble the fluctuations of energy, described by the corresponding variance σ_E^2, is expressed in terms of the mean energy U through a relation having a kind of universal thermodynamic character, namely $\frac{\mathrm{d}U}{\mathrm{d}\beta} = -\sigma_E^2$. Thus Einstein was led to split Planck's differential equation (2) into two relations, namely

$$\frac{\mathrm{d}U}{\mathrm{d}\beta} = -\sigma_E^2 \tag{4}$$

and

$$\sigma_E^2 = \epsilon U + U^2/N \ ; \tag{5}$$

the former was conceived to be just a kind of general thermodynamic relation, while the latter should rather have a dynamical character, and might in principle be deducible from a microscopic dynamics. In his very words (see [5]): these two relations "*exhaust the thermodynamic content of Planck's*" formula; and: "*a mechanics compatible with the energy fluctuation* $\sigma_E^2 = \epsilon U + U^2/N$ *must then necessasily lead to Planck's*" formula. It was pointed out in [2] that the second of the above equations has indeed a mechanical character, coinciding essentially with what we call the Benettin–Jeans formula with a suitable ϵ (this is a delicate point in our result). So we might say that the "mechanics" conceived by Einstein as leading to Planck's formula perhaps is nothing but the dear old classical mechanics of Newton.

In this connection, however, we address preliminarly an important question of a general character, namely how can one obtain in classical mechanics something quantitatively comparable with quantum mechanics, as the latter involves a quantity, i.e. Planck's constant $\hbar$, which is completely extraneous to classical mechanics. A first answer, to which we limit oureselves in the present paper, is that Planck's constant can be introduced in classical mechanics simply through the molecular parameters. Indeed, consider for example a system of equal particles of the same mass m interacting through a typical interatomic potential, such as the familiar one of Lennard-Jones. Now, this potential contains two parameters, say V_0 and σ, with the dimensions of an energy and a length respectively, and from them and the mass m one constructs an action, namely $\sigma\sqrt{mV_0}$, which a priori can take any value. But if one takes for the parameters m, V_0, σ entering the model just the ones corresponding to actual atoms, as reported in the standard textbooks, one finds that the relation $\sigma\sqrt{mV_0} \simeq 2Z\hbar$ holds, where Z is the atomic number of the considered atoms. This is one way in which Planck's constant can be made to enter classical physics at the level of pure mechanics (see [8]). A more fundamental way would require considering the role of the electromagnetic field, but this interesting point will not be discussed here (see for example [9]).

4 A dynamical implementation of Einstein's fluctuation formula

The road that led us to provide a partial implementation of Einstein's fluctuation formula (1) in classical terms is a long one. It started from a serious attention given, since the early years 70's, to the paradoxical result obtained in the year 1954 by Fermi Pasta and Ulam (FPU; see [10]). Such authors had shown, by numerical computations of the equations of motion of a one–dimensional model of a crystal, that at low energies classical dynamics seems to give results in contradiction with the law of equipartition, predicted by the Maxwell–Boltzmann distribution of classical equilibrium statistical mechanics. The first scientists that took up the problem, namely Izrailev and Chirikov (see [11]), put forward the very natural conjecture that the FPU paradox should disappear in the limit in which the number N of oscillators tends to infinity. Instead, in the paper [12] it was suggested that the lack of equipartition could persist in the limit of infinitely many oscillators, and in the paper [8] it was even found (still by numerical integration of the FPU model) that in classical mechanics Planck–like distributions seemed to occur. In fact, in the latter paper it was also realized, for the first time in a foundational context, that Planck's constant does in fact show up in classical mechanics through the molecular parameters in the way mentioned above.

There was then an intricate road passing through an appreciation of the many new possibilities offered by the modern results in the theory of dynamical systems (especially the stability results provided by KAM theorem and Nekhoroshev's theorem, see [13]). But it was finally realized (see [14]) that the simplest model describing the essence of the problem is that of a system of equal diatomic molecules on a line, where there are "internal" degrees of freedom (the oscillations of each of the molecules about its center of mass) and "external" ones (the center of mass of each molecule). The two subsystems (internal and external degrees of freedom, respectively) were found to go very rapidly to separate equilibria, and the problem remained of how would they go to a mutual equilibrium. It was then found numerically that the relaxation time to mutual equilibrium between the two subsystems increases as a stretched exponential with the frequency, and an analytical proof was provided in [15]. Eventually, the problem was then reduced to its very core, namely: the exchange of energy between a single spring (of frequency ω) and a colliding particle, the system moving on a line and the interaction being given by a potential between the particle and one extreme of the spring. This, by the way, is essentially equivalent to a model first discussed by Kelvin and Poincaré (see [16]) just in connection with the dynamical foundations of classical statistical mechanics.

So let us consider the problem of the exchange of energy δe of a spring on a single collision of one of its extremes with a point particle. An elementary calculation of a few lines made on a simplified version of the model gives the result (see [2])

$$\delta e = \eta^2 + 2\eta\sqrt{e_0}\cos\varphi_0 \ , \tag{6}$$

where e_0 is the initial energy of the spring, φ_0 its initial phase, while η is a quantity which tends to zero as a stretched exponential when the frequency ω of the spring increases and the velocity v of the particles decreases. We like to call formula (6) the Benettin–Jeans formula (see [17]). Now, one expects that the formula (6) should be correct if the frequency of the spring is sufficiently large and its energy sufficiently small, and this was proven in [18] by a quite delicate mathematical analysis, similar to the ones used in order to prove the exponentially small splitting of the separatrices in Melnikov's theorem.

Let us look now at the Benettin–Jeans formula (6), thinking of the impinging particle as mimicking a heat reservoir at a given temperature T, and of the spring as mimicking a crystal at a much lower temperature. Due to the exponential smallness of η, the formula implies that the echange of energy which should lead to equipartition with the reservoir is exponentially small with the characteristic internal frequency ω, so that the number of collisions required to go to equilibrium (i.e. the time required for it) is highly nonuniform in the frequency, being exponentially large with ω. Examples can be given in which there exists a frequency $\bar{\omega}$ that relaxes in 1 second, while the frequency $\bar{\omega}/2$ relaxes in 10^{-8} seconds and the frequency $2\bar{\omega}$ in 10^5 years. This fact is very important, because it explains the most relevant feature of Planck's formula, namely the circumstance that the high frequencies have a very small energy with respect to that of the reservoir: in dynamical terms, this is due to the fact that the high frequencies require an exponentially long time in order to go to equilibrium if they start from a negligible initial energy.

But the fact remains that the coefficients entering the exponentially small quantity η of the Benettin–Jeans formula (6) turn out to depend on the molecular parameters characterizing the particular interatomic potential considered. So one is lackng a formula possessing a sort of thermodynamic character. This fact was, for the whole group of people involved in the research described here, a great conceptual difficulty.

The clue was found by taking into consideration the second term appearing in the Benettin–Jeans formula (6), which produces a fluctuation of energy (depending on the phase φ_0) much larger than the drift term η^2. The relation between the Benettin–Jeans formula (6) and Einstein's fluctuation formula (1) was found in the following way (see [2]). Consider a sequence of k collisions and take the average over the phases (which as usual are assumed to be uniformly distributed). Denoting by u_k and σ_k^2 the mean energy and variance of energy respectively after k collisions, a completely elementary calculation gives the formulæ

$$u_k = e_0 + k\eta^2 \ , \quad \sigma_k^2 = 2e_0 k\eta^2 + (k\eta^2)^2 \ ,$$

which depend on time (i.e. on the number k of collisions). But one immediately sees that the "time" k can be eliminated, so that a functiomal relation exists between variance and mean, namely

$$\sigma_k^2 = 2e_0(u_k - e_0) + (u_k - e_0)^2 \ .$$

A similar relation also holds if one considers a system of N identical independent oscillators of frequency ω. Indeed, the quantities of interest are now the total energy $E_k = \sum_{i=1}^N e_k^{(i)}$ (where $e_k^{(i)}$ denotes the energy of the i–th oscillator after k collisions) and the corresponding exchanged energy $\tilde{E}_k = E_k - E_0$, where E_0 is the initial energy. By the central limit theorem, $\tilde{E}_k$ is normally distributed with a mean $\tilde{U}_k$ and a variance which are obtained by adding up the corresponding quantities for each oscillator, So, denoting by $\tilde{U}$ and $\sigma_{\tilde{E}}^2$ expectation and variance of the exchanged energy at any "time" k, one gets between $\sigma_{\tilde{E}}^2$ and $\tilde{U}$ a functional relation which is independent of "time" k, namely

$$\sigma_{\tilde{E}}^2 = 2a_0\omega\,\tilde{U} + \tilde{U}^2/N \ , \tag{7}$$

where a_0 denotes the initial action per oscillator, $a_0 := E_0/(\omega N)$. Notice that the quantity η, which contains the molecular parameters characterizing the particular system considered, has now completely disappeared, and formula (7) has some kind of thermodynamic universality.

In order to have something comparable to Einstein's fluctuation formula (1), we have however to get rid of the parameter still appearing in formula (7), namely the the quantity $2a_0$, twice the initial action per oscillator, which takes the place of Planck's constant $\hbar$. This is a delicate point that should deserve a deep investigation At the moment we are unable to say anything rigorous, and only present here some heuristic considerations. The point is that, as was recalled above, the Benettin–Jeans dynamical formula was established for small initial energies of the oscillators. So formula (7) should hold only for low enough energy or action per oscillator, say for $a_0 < a_*$, with a certain critical or threshold action a_*. This naturally leads to think of a situation with the N oscillators having random initial actions all smaller than a_*, so that, on averaging over the initial actions, uniformly distributed over the interval $(0, a_*)$, one would get a formula as (7) with $2a_*/2$, namely a_*. in place of Planck's constant. The actual mean energy would correspondingly be given by Planck's formula with the addition of an analog of the zero–point energy, namely $a_*\omega/2$, playing here the role of the initial energy. Notice that in such a fluctuation formula the molecular parameters enter only through the critical action a_* and so Planck's constant $\hbar$ finally appears in the way described above (see [8] and [19] [20]).

5 Conclusions

So we hope we were able to show that a nonconventional Einstein's conception of the photon, involving continuous variations of energy, indeed exists, and how it might be implemented in classical mechanics. We are well aware of the fact that we are still faced with many deep problems, but we like nevertheless to sketch here, in a few words, a perspective that seems now to be opened, in which Planck's law appears just as a first order approximation.

Namely, the law of equipartition can be considered just as a zeroth order approximation, in which the high nonuniformity of the relaxation times with respect to frequency is altogether neglected. Planck's law instead appears as a first order approximation, describing a kind of metaequilibrium state, similar to those occuring in glasses (an analogy first pointed out in [21]). Quantum mechanics would, in this sense, just be a first order approximation within classical mechanics. If our point of view is correct, deviations from Planck's law should show up, especially in the region of low frequencies, where equipartition would actually be present, with an "equipartition front" advancing with time, at an extremely slow pace. Such an effect was indeed predicted already by Jeans (a quotation can be found in [22]). For a review of the experimental data on laboratory black body up to some years ago, see the second part of the work [23], and also [24].

Finally, one also has a critical historical problem. Indeed, the point of view of metastability described above was advocated by Jeans at the beginning of the last century, but such an author then made a retractation (vividly documented in [25] and [26]), after Poincaré had proven (see [27]) that Planck's law seems to imply quantization, namely the existence of energy levels. So the problem is whether Poincaré's argument is really compulsory, but up to now we were unable to settle the question.

Aknowledgment

The work of one of us (A.C.) was made possible by a grant from Fondazione Cariplo per la Ricerca Scientifica.

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