THE THEORY OF DYNAMICAL SYSTEMS AND THE RELATIONS BETWEEN CLASSICAL AND QUANTUM MECHANICS

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July 18, 2000

Abstract

We give a review of some works where it is shown that certain quantum–like features are exhibited by classical systems. Two kinds of problems are considered. The first one concerns the specific heat of crystals (the so called Fermi–Pasta–Ulam problem), where a glassy behavior is observed, and the energy distribution is found to be of Planck–like type. The second kind of problems concerns the self–interaction of a charged particle with the electromagnetic field, where an analog of the tunnel effect is proven to exist, and moreover some nonlocal effects are exhibited, leading to a natural hidden variable theory which violates Bell's inequalities.

1 Introduction

The relations between classical and quantum mechanics are usually studied in the context of the so called semiclassical limit. Indeed it is well known that classical mechanics is recovered in the limit in which Planck's constant h (or its rationalized version $\hbar = h/2\pi$) becomes somehow negligible, in a sense analogous to that in which newtonian mechanics is recovered from relativistic mechanics in the limit in which the speed of light c becomes infinite. In the present paper a review is given of some researches in which the relations between classical mechanics and quantum mechanics are investigated in a somehow reverse way, namely with the aim of showing that classical mechanics (or rather classical physics, inasmuch as we consider also the role of the electromagnetic field) already contains in itself some relevant quantum–like features. Such a line of research was initiated about thirty years ago (see [1] and [2]) in connection with the equipartition problem of classical statistical mechanics, stimulated by the last work of Fermi (1954)

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on the so called Fermi–Pasta–Ulam (FPU) problem (see [3]), and found a huge support from the fact that at the same time the theory of dynamical systems (with the famous KAM theorem) was becoming one of the most popular fields of research in the scientific community. Later, the problem of the self–interaction of a charged particle with the electromagnetic field also started to be investigated (see [4] and [5]) in the spirit of the theory of dynamical systems. There too some quantum–like features came to light, related to the existence of the so–called *runaway solutions* of the Abraham–Lorentz–Dirac (ALD) equation, which is the relevant equation in the point particle limit (see [6] and [7]); we are referring to a possibility of describing pair creation and annihilation (see [8]), the existence of a classical analog of the tunnel effect (see [9] and [10]), and a violation of Bell's inequalities (see [11]).

These are the two main themes (the FPU problem and the ALD equation) that will be discussed below. Preliminarily, it might however be useful to recall how it can be understood that any quantum–like feature at all can show up in classical mechanics, where apparently there is no place for the new quantity, the quantum of action $\hbar$, which characterizes quantum mechanics. The fact is that something quantitatively comparable to $\hbar$ actually occurs in classical mechanics, and indeed at two different levels. The first one is purely mechanical. Consider a system of particles describing an actual molecular system, and thus involving some realistic potential, typically the Lennard–Jones potential $V(r) = 4V_0[(\sigma/r)^{12} - (\sigma/r)^6]$, r being the distance between two molecules. Then one has three characteristic parameters, namely the energy V_0, the length σ and the mass m of the molecules, from which the characteristic action $A = \sqrt{mV_0}\,\sigma$ is obtained. Now, if one chooses arbitrary values for such parameters, the characteristic action A too takes any value; but if one inserts realistic values, as given in the standard textbooks (see [12]), one immediately realizes that A is of the order of magnitude of $\hbar$; actually, with an incredible precision one has $A = 2Z\hbar$, where Z is the atomic number. Thus if it turns out that in some classical model of molecules an action comes into the arena, then most surely such an action will be of the order of magnitude of Planck's constant. This is indeed what was actually realized in [1], where a quantum–like phenomenon in a classical system was first observed. Actually it was found that in the FPU problem at very low energies the energy spectrum exhibited a Planck–like distribution rather than equipartition, and by fit it was found, with a great surprise, that the corresponding action was very nearly equal to Planck's constant. It took some time to find the explanation, namely that definite molecular parameters had been chosen (actually those of Argon), for which the relation $\sqrt{mV_0}\,\sigma = 2Z\hbar$ holds.

The second level at which an action related to Planck's constant $\hbar$ enters in classical physics concerns the interaction of a charged particle with the electromagnetic field. In fact everyone knows that there exists the fine structure constant, namely the dimensionless number α defined by $\alpha = e^2/\hbar c$, where e is the electron charge and c the speed of light, and that one has $\alpha \simeq 1/137$. But such a relation can also be read in the form

$$\hbar \simeq 137 \, \frac{e^2}{c} \ ,$$

namely: the "classical" quantity e^2/c is an action, and is thus proportional to $\hbar$, the proportionality factor being as above. So if in the classical theory of the interaction of charged particles with the electromagnetic field an action comes into the arena, then most surely it will be of the order of magnitude of e^2/c, and possibly not very different from $\hbar$. An example is the following, which is deduced from some simple exercises proposed in the classical textbook of Landau and Lifshitz (see [13]). Consider the scattering of a charge (with initial velocity v) by a repulsive center of forces, and estimate the orbit in the zero–th order, i.e. purely mechanical, approximation. A formula for the emitted energy ΔE is easily found, which turns out to depend on the parameters entering the force. Moreover the emitted spectrum is found to exhibit an exponential decay, with a characteristic cutoff frequency $\bar{\omega}$, which also depends in a complicated way on the parameters defining the model. However one immediately checks that such parameters disappear in the expression for the action $\Delta E/\bar{\omega}$, because one finds

$$\frac{\Delta E}{\bar{\omega}} = a \frac{e^2}{c} \frac{v^2}{c^2}$$

where a is a dimensionless number of the order of unity.

So perhaps it makes sense to try to understand how much does classical physics extend into the domain of quantum physics. Our personal approach is described in the present review, which we would like to dedicate to Martin Gutzwiller and Paolo Raineri.

2 Planck–like distributions in the FPU problem

So we start with the quantum–like features occurring in problems of FPU type (for previous reviews see [14] and [15]; see also [16], section 3.4).

a) The FPU paradox. Fermi Pasta and Ulam were performing numerical computations on a one–dimensional model of a crystal, namely a system of a certain number of particles on a line, with nearest neighbor nonlinear interaction and certain boundary conditions. The corresponding linearized system is equivalent to a system of uncoupled harmonic oscillators (the system's normal modes) and the authors had in mind to check that the actual nonlinear coupling would lead, after a transient, to energy equipartition, namely the same mean energy (in time average) for all modes; this is indeed predicted by the equilibrium distribution of classical statistical mechanics in phase space, namely the Maxwell–Boltzmann or Gibbs distribution. The FPU paradox was that, energy having been given initially only to the low frequency modes, equipartition was not found within the actual observation time: there was a kind of freezing, and energy did not appear to flow to the high frequency modes. FPU also reported a figure (Fig. 9) from which one could guess that the spectrum (energy versus frequency) had an exponential–like decay towards the high frequencies. Notice that this is exactly the qualitative characteristic feature of Planck's law which, on October 19, 1900, gave rise to quantum mechanics.

The late E. Segré, in a conversation with one of us, once mentioned that Fermi did not like very much the standard interpretation of quantum mechanics, but neither did he like ideological discussions. As Ulam reports in his preface to the work of FPU, reprinted in Volume 2 of Fermi's collected papers (n. 266): "*The results of the calculations ... were interesting and quite surprising to Fermi. He expressed to us the opinion that they really constitute a little discovery in providing intimations that the prevalent beliefs in the universality of mixing and thermalization in nonlinear systems may not be always justified*". If one thinks of the attention Fermi had given to such a problem in one of his early works (see [17] and also [18]), such a return to it just before his death appears as not surprising.

b) Different attitudes. The first scientists that took up Fermi's challenge were Izrailev and Chirikov who, in a deep and remarkable paper (see [19]), prompted the solution to the FPU paradox: there should exist some kind of energy threshold, such that for low energies one has "ordered motions", somehow as with KAM tori, while chaotic motions, leading to equipartition, should prevail above threshold. Moreover such a threshold should be a function of the initially excited frequency ω, so that in the frequency energy plane (ω, E) one has a kind of "border of stochasticity" described by some function $E^c = E^c(\omega)$. The paradoxical result of FPU was explained by the fact that initial data below threshold had been taken. Moreover, Izrailev and Chirikov also provided some kind of mathematical theory allowing them to estimate the critical curve $E^c(\omega)$; the main feature being that it tends to the curve $E^c(\omega) = 0$ when the number of particles in the system tends to infinity. This would be the complete solution of FPU's paradox according to Izrailev and Chirikov: for macroscopic systems there is no threshold at all, and one always meets with equipartition, as predicted by classical statistical mechanics. This is the reason why classical mechanics fails, as it should.

This, at least, is the way we understand Izrailev and Chirikov's attitude. In the present paper we will try to explain our point of view, which is a completely different one: we interpret the FPU paradox as pointing out a feature actually existing in classical mechanics, which survives also for macroscopic systems, and intimates that quantum–like features are present at a classical level. Briefly, our point of view consists in stressing (quoting again from Ulam) the role of the "*rate of thermalization*", following an attitude that actually goes back to a famous paper of Boltzmann of the year 1895 (see [20] and [21]): the lack of equipartition is due to the fact that the rate of thermalization is highly nonuniform with respect to frequency, increasing very fast with it, so that the high frequencies, particularly at low temperatures, do not have time to thermalize within the available observation time. This is actually the point of view which is ordinarily taken when studying glasses or supercooled liquids (see [22]), as was first pointed out, in connection with the FPU problem, in the paper [23]. In other words, in the FPU problem there should occur a freezing in the same sense of glasses or supercooled liquids, but this should have a foundational relevance in the sense of Boltzmann, and not just a technological one.

A very general theorem was proven (see [24] and [25]) for the exchange of energy between two subsystems of oscillators of low and high frequencies respectively. It was found that the relaxation time to equipartition increases as a stretched exponential with frequency, the result being independent of the number N of degrees of freedom. Examples of physical systems can be given (see [26]) where there exists a frequency $\bar{\omega}$ (of the order of 10^{14} sec^{-1}) which relaxes in one second, while the frequency $\bar{\omega}/2$ relaxes in 10^{-8} seconds and the frequency $2\bar{\omega}$ in 10^5 years. This is indeed the familiar barrier–like effect of the exponential.

So this is in short the reason why among people working on the FPU problem, particularly in Italy, it was understood that quantum–like features could be present in classical systems. There remained however the big problem that the exponential–like estimates for the thermalization rates were not universal, because they involved, through some constants entering the stretched exponentials, the particular molecular parameters occurring in any given model, such as the parameters V_0, σ and m mentioned above. So one should explain why, in the curves giving the specific heat versus temperature, in general a thermodynamic behavior is observed, described by pure exponentials involving no molecular potentials at all, as in Planck's law.

c) Planck's law: a dynamical implementation of Einstein's fluctuation formula. A clue was found quite recently (see [27]) by understanding how Einstein interpreted Planck's law in terms of energy fluctuations, and by implementing Einstein's fluctuation law in terms of a dynamical fluctuation formula established in the framework of dynamical systems theory. Let us recall that Planck's law, giving the mean energy U for a system of N oscillators of frequency ω at inverse temperature β, has the form

$$U(\omega,\beta) = N\,(\,\frac{\epsilon}{e^{\beta\epsilon}-1}\,) \tag{1}$$

where $\epsilon = \hbar\omega$. Now, Planck's law turns out to be a solution of the differential equation

$$\frac{\mathrm{d}U}{\mathrm{d}\beta} = -(\epsilon U + \frac{U^2}{N}) \ , \tag{2}$$

with a suitable choice of the integration constant. This is indeed the way (apart from some notational differences) Planck himself introduced such a law in his first work (see [28]). In fact he made an interpolation between the right hand sides of the equations $\frac{\mathrm{d}U}{\mathrm{d}\beta} = -\frac{U^2}{N}$ and $\frac{\mathrm{d}U}{\mathrm{d}\beta} = -\epsilon U$, because by integration they lead to the equipartition law $U = N/\beta$ and to Wien's law $U = C\exp(-\beta\epsilon)$ (with a suitable constant C), which were known to fit well the data at low and high frequencies respectively. What Einstein did (see [29], [30]) was to split the equation (2) into two equations, namely

$$\frac{\mathrm{d}U}{\mathrm{d}\beta} = -\sigma_E^2 \tag{3}$$

and

$$\sigma_E^2 = \epsilon U + \frac{U^2}{N} \ . \tag{4}$$

where σ_E^2 is the energy variance. Indeed he conceived of (3) as being just a general thermodynamic relation, while (4), which expresses a special functional relation between energy variance and mean energy, might have some dynamical foundation. In his very words ([30]): these two relations "*exhaust the thermodynamic content of Planck's*" formula; and: "*a mechanics compatible with the energy fluctuation $\sigma_E^2 = \epsilon U + U^2/N$ must then necessarily lead to Planck's*" formula. For further details see [15].

Now, during some researches on the collisions between a particle (mimicking a heat reservoir) and a spring (mimicking a crystal), it was found that, for high frequencies and small energies of the spring, the energy exchange of the spring in a single collision is given by what we like to call the Benettin–Jeans formula, namely (see [31] and [32])

$$\delta e = \eta^2 + 2\eta\sqrt{e_0}\ \cos\varphi_0 \ , \tag{5}$$

where e_0 and φ_0 are the initial energy and phase of the oscillator, while η is an extremely small factor decreasing with frequency as a stretched exponential, and containing the molecular parameters. Then, in the paper [27] the connection with Planck's law was found. Indeed if one considers a sequence of k such collisions and averages over the phases, by elementary manipulations one finds between mean energy and energy variance exactly Einstein's functional relation (4), with $\epsilon = 2a_0\omega$ where a_0 is the initial action per oscillator. Our conjecture is that such a dependence on the initial data should disappear, if one takes into account that the Benettin–Jeans formula holds only for small enough initial actions, so that an average should be taken over the initial actions smaller than a certain critical action A. This is an important open problem.

d) An experimentum crucis. Many more things might be added here, for example concerning analytical and numerical estimates for the border of stochasticity in the sense of Izrailev and Chirikov (see [33]). But we think we may now summarize the situation. In classical mechanics equipartition occurs (or rather should occur, because rigorous results are lacking) when the nonuniformity of the relaxation times to equilibrium is altogether neglected. So equipartition constitutes a sort of zero–th order approximation. On the other hand, the relaxation rates are found to depend on frequency in a quite nonuniform way, and for high frequencies and low temperatures one is in presence of freezing phenomena as in the familiar cases of glasses and supercooled liquids. When this is taken into account, an approximation for the actual energy distribution much better than equipartition is found to be apparently given by a Planck–like distribution. In this sense, Planck's law appears just as a first order approximation, in which the freezing is dealt with as if it were a real equilibrium; this seems to correspond to the approximation of quantum mechanics. Thus classical mechanics and quantum mechanics should substantially agree within a certain time, which might be the analog of what is sometimes called Egorov's time or Ehrenfest's time (see [34] and [35]). For larger times the two mechanics disagree: quantum mechanics deals with the system as if it had reached equilibrium, while classical mechanics should lead to a "final relaxation" to equipartition (but on non human time scales), with a corresponding arising of chaotic motions.

If this phenomenon is real, it should be observed experimentally at the low frequencies, where the relaxation times should still be on a human scale. In fact, since the relaxation time to equipartition is expected to increase with frequency as a stretched exponential, for any given observation time t there should exist a frequency $\bar{\omega}(t)$ playing the role of an equipartition front, followed by an exponential tail towards the high frequencies; such an equipartition front should then be observed to move towards the high frequencies as the observation time increases. This phenomenon was indeed predicted by Jeans (see [36]), and so we call it *the Jeans effect.* We do not have time to discuss here a very interesting critical and historical problem, namely that of understanding how did it happen that Jeans, apparently under the influence of the famous paper of Poincaré on the necessity of quantization (see [37]), came to repudiate (see [38] and [39]) the ideas he had kept (following Rayleigh and Boltzmann) up to the first Solvay conference.

Looking for the Jeans' effect here plays the role of a kind of "*experimentum crucis*": if quantum mechanics is just a first order approximation to classical mechanics, one should observe that the temperature at which freezing occurs (namely essentially the Debye temperature, where the specific heat exhibits a rather abrupt decay) depends on the observation time, moving towards the low temperatures as the observation time is increased. An indirect proof of this is obviously given by the very existence of sound dispersion (the sound speed, and so the specific heat, depends on frequency, i.e. on the observation time), while a direct experimental proof of a time dependence of the specific heat has been found for supercooled liquids and for glasses, exhibiting in an impressive way exactly the feature described above (see Fig. 1 of [40]). Our conjecture is that such a phenomenon should be observed also in standard crystals, such as those considered by Einstein in his famous paper on specific heats (see Fig 1 of ref. [41]). In extremely concrete terms: the phenomenon observed in Fig. 1 of ref. [40] concerning supercooled liquids (i.e. the existence of a different curve c_V versus T for each observation time) should occur also for pure crystals, contrary to the common belief that one has there to do with a real equilibrium.

We do not have time here to discuss the relations between the stochasticity threshold and the zero–point energy, in the way suggested originally by Cercignani (see [42]), or to illustrate how such an idea leads to a modern reinterpretation of a very impressive deduction of Planck's law in terms of energy thresholds given by Nernst in the year 1916 (see [43]).

3 Quantum–like features of classical electrodynamics

The problem we will consider now is the fundamental one of classical electrodynamics, namely the interaction of a charged particle with the electromagnetic field, as described by the Maxwell–Lorentz system, the selfinteraction of the particle with the field being taken into due account. Dealing with such a problem in a rigorous way, in the spirit of the theory of dynamical systems, is quite a hard job. In our opinion there are good indications that such a job will be greatly rewarding.

Two limit cases exist which are essentially trivial. The first one is when the particle's motion (and thus the current too) is assigned, so that one remains with the linear problem of Maxwell's equations with a given current, namely

$$\partial_\mu F^{\mu\nu} = j^\mu \, , \tag{6}$$

in the standard notations, $F^{\mu\nu}$ and j^μ denoting electromagnetic field and current respectively (the homogeneous Maxwell equations being understood). Such an equation is easily resolved even for the case of a point particle, by considering the fields as distributions. The other trivial case is when the field is assigned, and one is reduced to the purely mechanical problem of a particle subjected to a special (i.e. the Lorentz) force, with equation

$$ma^\mu = F^{\mu\nu} j_\nu \ , \tag{7}$$

where m and a^μ denote the mass and the four–acceleration of the particle. Things are however completely different for the full coupled system. The most severe problem here arises in the case of a point particle, because the mathematical expression for the Lorentz force then makes no sense, due to the infinity (at the particle position) of the field created by the particle itself, which cannot be cared in any trivial way.

We will return to this point below. In the meantime we would like to stress that interesting features are disclosed even if one considers an extremely simplified model, namely the nonrelativistic model of a rigid fat particle in the dipole (i.e. linearized) approximation, the rotational degrees of freedom being altogether neglected.

a) The nonrelativistic model of a fat rigid particle; wholeness of particle and field, and particle diffraction. In dealing with such a simplified model, a first remarkable qualitative feature already shows up, namely the inseparability of particle and field; this endows the particle with intrinsic field properties leading it to undergo, for example, diffraction. In order to fully appreciate this, one should perhaps start up by becoming familiar with a simple exercise (see [4]), namely to realize that in free space a charged particle can perform uniform rectilinear motion only if some definite initial data are assigned to the field, explicitly adapted to the mechanical initial data x_0 and v_0 (position and velocity) of the particle. For example, with a vanishing initial field the particle is found to decelerate, by radiating a field as if it were trying to build up a field that would let it perform a uniform motion. Instead, if a certain suitable initial field is assigned, then the various Fourier modes do cooperate in producing mutual compensations such that the net force on the particle vanishes (while otherwise such a force does not vanish at all). In fact, the suitable initial field is, as one would imagine, nothing but the appropriate Lorentz transform of the Coulomb field created by the particle at rest. However, the existence of a special field producing uniform motion is not at all trivial from a dynamical point of view, and the first scientist that understood it, namely Abraham in the year 1903 (see [44]), even qualified such a property as proving the "*compatibility of electrodynamics with the inertia principle*". In any case, a charged particle needs a field to go straight. On the other hand, it is well known that in presence of an obstacle the field alone undergoes diffraction, due to the appropriate boundary conditions; thus in the same way it is obvious that a charged particle, due to the intimate relation with the field just described, also undergoes diffraction. This can be proved in a very easy way in some approximation, although up to now we were unable to obtain clear quantitative estimates. This seems to be an interesting open problem.

b) A second level of wholeness: the point particle problem, mass renormalization, runaway solutions and the Dirac principle. The most relevant quantum–like feature of classical electrodynamics is however manifested when the point particle problem is considered. We will try to show in a moment that this manifestation occurs through the appearing of the runaway solutions which, in turn, are a consequence of the need for mass renormalization, i.e. of the divergence of the electromagnetic mass, or ultimately of the divergence of the Coulomb force on the particle itself. As far as our personal experience is concerned, this is a quite delicate point, on which an agreement with a large part of the community of theoretical physicists is not easily found. Indeed the common opinion is that problems of this type could be dealt with only within the quantum formalism. Following Dirac (see [45] and [46]) and Feynman (see [47]), we believe instead that a quantum description would meet essentially with the same difficulties of a classical description, and that it makes sense to start up from the latter.

In order to go to the heart of the problem, let us first describe heuristically, following Feynman, how the problem of mass renormalization arises. The main point is the fact that everything goes as if there were attached to the particle a mass, known as the electromagnetic mass m_{em}, although such a mass does not appear explicitly in the equations defining the model, i.e. in the Maxwell–Lorentz system. This too is a consequence of the intimate relation between particle and field mentioned above: a particle in uniform motion drags along with it a field, and so also the correspondig energy and momentum of the field. For example, for a sphere of radius R and velocity v one has in the nonrelativistic approximation a momentum, due to the field, of modulus $p_{em} = m_{em}v$, with $m_{em} = (2/3)e^2/R$. It is thus clear that in a first approximation the particle behaves as if it had an effective mass $m_0 + m_{em}(R)$, where we have now denoted by m_0 instead of m the "bare or mechanical" mass entering Newton's equation (7). On the other hand, in the point limit $R \to 0$ the electromagnetic mass diverges, $m_{em} \to +\infty$, so that apparently there are just two possibilities in taking the point limit: either one keeps the bare mass m_0 fixed, in which case the effective mass tends to $+\infty$ (trivial dynamics; no finite force is able to accelerate the particle), or one introduces the prescription $m_0 = m_0(R) \to -\infty$ in such a way that the effective mass $m_0(R) + m_{em}(R)$ remains finite, say equal to a value m playing the role of a phenomenological mass. A detailed analysis, restricted to the nonrelativistic case in the dipole approximation (see [6] and [7]), shows that such a heuristic conclusion is correct, in the most rigorous way. Indeed it can be proven that a nontrivial dynamics is obtained if and only if mass renormalization is introduced as above; moreover the Cauchy data for particle and field should not be independent, but related in some definite way. Furthermore the initial particle acceleration turns out to be a well defined function of the initial data of the field (the relevance of this will be shown in a moment). Finally, one obtains for the motion of a particle under the action of an external force F_{ext} the third order equation

$$\tau \dot{a} = a - F_{ext}(x)/m \tag{8}$$

involving the characteristic time $\tau = (2/3)\, e^2/(mc^3)$. This is apparently a very strange equation, requiring as initial data the particle acceleration a_0

in addition to the standard data of position and velocity x_0, v_0 required for Newton's equation. But, as mentioned above, a_0 turns out to be a function of the global set of initial data for the Maxwell–Lorentz system describing particle and field, so that there is no mystery here, if one takes the attitude that one is always dealing with the complete system particle plus field. This is a very important point, so often misunderstood.

An analogous theorem for the full relativistic problem is still lacking. What we have available is a prescription, given by Dirac in the year 1938 (see [45], [46]), which leads to a relativistic analog of (8), namely

$$\tau(\dot{a}^\mu - a^\nu a_\nu \dot{x}^\mu) = a^\mu - F^{\mu\nu}_{ext}\dot{x}_\nu/m\,, \tag{9}$$

where now the dot denotes derivative with respect to proper time, but a proof of its necessity in the sense described above has not yet been found. We will refer jointly to equations (8) and (9) as the Abraham–Lorentz–Dirac (ALD) equations.

Concerning such equations, we have first of all to make clear that they constitute an extension of the ordinary model of classical physics: they are not theorems within the classical framework, being rather new prescriptions, almost freely chosen in going to the point limit. The theorem available (at least in the nonrelativistic linearized model) only says that the chosen prescription is the only possible one (apart from that leading to a trivial dynamics) that can be induced from the macroscopic equations. It is just in this new theory, which is obtained by extension of the classical one, that new unexpected and very interesting features show up. Before describing them, we would like however to point out an analogy. We refer to the case in which the first continuum limit, leading to a partial differential equation, was obtained in history of science, namely the case of d'Alembert equation. While d'Alembert got his equation in the year 1750 in the familiar way by analogy with the momentum equation for a finite system of particles, in the year 1759 Lagrange obtained it through a limit from a discretized system (a lattice field theory, in modern parlance), namely a linearized FPU system. The equations contain three parameters, namely the mass m of the particles, the discretization step a and the constant k entering the expression for the potential of the linear springs. Going to the continuum limit $a \to 0$, one obviously has to require $m \to 0$ in such a way that the density $\rho = m/a$ be fixed, but one also has to require that $k \to +\infty$ in such a way that the quantity ka, namely the tension, remains finite; otherwise one would obtain a trivial limit, with a vanishing tension, i.e. with vanishing sound speed. This example makes clear that in getting the limit some prescriptions have to be assigned, which are additional with respect to the framework defining the original discrete system; and the same seems to occur with the ALD equation.

Now the runaway solutions come into the arena. Apparently, although a third order equation of the form (8) had been considered even before Abraham and Lorentz, precisely by Planck, the first to explicitly point out the existence of runaway solutions was apparently Dirac in his 1938 paper for the case of the relativistic free particle. Such a phenomenon is however more easily observed in the case of the nonrelativistic free particle. In such a case, equation (8) reduces to a closed equation for the acceleration, namely $\tau \dot{a} = a$, with general solution $a(t) = a_0 \exp(t/\tau)$. So the free particle experiences an absurd exponential acceleration, unless one chooses the initial condition $a(0) = 0$, namely $a_0 = 0$, which gives the expected uniform rectilinear motion $a(t) = 0$. It is easy to see that runaway solutions are generic. What to do with them? Most physicists certainly interpret the very fact of their existence as intimating that the theory is nonsense, although they might later realize with some surprise that an analogous situation is met in quantum physics (see [48] and [49]).

Dirac had instead a quite opposite reaction. Inspired by the example of the free particle, he remarked that the theory should be complemented by a further prescription, namely that of restricting one's attention to the "physical motions", i.e. by definition those that do not present a runaway character. For example, for a particle subjected to an external force vanishing at infinity, in the case of scattering one should require that the particle "finally" behaves as a free particle, i.e. that $a(t) \to 0$ for $t \to +\infty$. In Dirac's words: "*we must restrict ourselves to those solutions for which the velocity is constant during the final period when the electron is left alone*"; and furthermore: "*We must merely impose the condition that these solutions are the ones that occur in Nature*". So we make the further assumption that the phase space is a certain submanifold of the original phase space, a kind of center manifold which we like to call the "*physical or Dirac manifold*". This is the deep new feature, because "*We now have a striking departure from the usual idea of mechanics. We must obtain solutions of our equations of motion for which the initial position and velocity of the electron are prescribed, together with its final acceleration, instead of solutions with all initial data prescribed.*" And this leads to unexpected consequences. Essentially, this is due to the fact that, in order that something happens in the future (the acceleration has to vanish after the action of the force), a suitable acceleration (with a corresponding energy radiation) has to exist before the particle meets with the external force. Thus "*It would appear here that we have a contradiction with elementary ideas of causality*" ... because ... "*a signal can be sent from A to B faster than light. This is a fundamental departure from the ordinary ideas of relativity...* (although) ... *our whole theory is Lorentz invariant.*" It is just for this reason that Dirac had previously stated, quite emphatically, that "*This will lead to the* **most beautiful feature** *of the theory*".

The opinion that by going to the point limit in classical electrodynamics (or equivalently by removing some previously introduced regularizing cutoffs) unexpected new features might show up, including some nonlocality properties appropriate to violate Bell's inequalities, was repeatedly put forward by Nelson (see [50] and [51]) with great emphasis. Our attitude is exactly the same. The only difference is that in our opinion no more job is needed to understand what is the strange relevant mechanism that should show up in the limit, because the job has already been done by Dirac. So we keep exactly Dirac's point of view recalled above: the new feature, the most beautiful feature appearing in the point limit is simply what comes out of Dirac's prescription that the phase space be restricted to the "physical" submanifold of nonrunaway solutions (*the Dirac principle*, as we call it), which leads to a fundamental departure form the ordinary ideas of relativity, apparently in contradiction with elementary ideas of causality, though in the framework of a Lorentz invariant theory. In Dirac's words (we are freely translating here from his original French paper [46]) "*The fundamental hypothesis of the theory of relativity is actually the invariance of all physical laws with respect to Lorentz transformations. ..... The hypothesis according to which a signal never can propagate faster than light is a secondary hypothesis, independent of the previous one.*"

c) Some quantum–like effects. The analytical property of the ALD equation which allows for its solutions to exhibit interesting new features is the fact of being a singular perturbation of Newton's equation, inasmuch as it reduces to the latter when the "small" parameter τ vanishes, but with a reduction of its order. Correspondingly, its solutions are represented by asymptotic series (see [10]). It turns out that the solutions can be divided into two classes, which we call the mechanical and the nonmechanical ones respectively. The former are small perturbations of solutions of the Newton equation, while the latter are not, being qualitatively completely different. In the case of scattering from a nucleus it is found (see [10]) that the mechanical solutions are characterized by having an initial angular momentum larger than a a certain action of the order $\hbar$, precisely $6Z^{2/3}\, e^2/c$, where Z is the atomic number. This seems already to be rather interesting.

The relevant problem is that of understanding what happens with the nonmechanical solutions. This problem was solved for the collision of a particle with a barrier (see [9] and [52]), in which case too nonmechanical solutions are found to exist only beyond a certain threshold. The distinguishing feature is the following one: while for the mechanical solutions the particle is either transmitted or reflected according to the initial mechanical state x_0, v_0 (actually, according to the corresponding value of the mechanical energy), for the nonmechanical solutions it turns out that the initial mechanical state does not uniquely define the initial acceleration a_0 leading to a nonrunaway solution. There are instead several possible initial accelerations (even an unlimited number of them), some of them leading to transmission and the other ones to reflection. By the way, the possibility of such a "nonuniqueness" property was first conceived in a clear mathematical way by Hale and Stokes (see [53]), while Dirac himself explicitly made the incorrect statement that uniqueness should always hold (see [46], page 21). In the case of the collision with a barrier, this nonuniqueness phenomenon happens to occur for initial energies in a small strip about the top of the barrier. Moreover it turns out that, as the initial position x_0 recedes from the barrier, the different allowed accelerations, corresponding to the same mechanical datum, have a mutual distance tending exponentially to zero. So the initial acceleration (or the initial field, for the reason explained above) really plays the role of a hidden parameter, in the sense indicated by Bell, inasmuch as it is macroscopically uncontrollable (see [54]). In such a way it is clear that one has here an analog of the tunnel effect, since the property that the particle be transmitted or reflected depends on the value of a hidden parameter which cannot be controlled, so that it has necessarily to be described by some probability distribution.

A further relevant property is that one meets here with some nonlocal effect. This is due to the fact that, for a fixed initial mechanical datum x_0, v_0, the set of allowed values for the hidden parameter a_0 (in particular the cardinality of such a set) depends on the height of the barrier, so that the probability distribution too is defined in a probability space which depends on the height of the barrier. This seems to be an analog of a key quantum feature: if one has to perform a measurement of an observable of a certain object, "*as a result of the interaction between the object and the measuring instrument, the object is entangled with the instrument*" (see [55]). Indeed, if in our example one performs the measurement consisting in observing whether the particle is transmitted or reflected by the barrier (which amounts to considering a suitable dichotomic variable in the standard way), then different experiments correspond to different heights of the barrier; in such a case, having fixed the initial mechanical state, for each different experiment one has a different probability space for the hidden variable describing the state of the particle.

Think now of two such experiments for two particles coming out of some point and going in opposite directions towards two barriers, each having a certain height chosen among three possible ones. Due to the nonlocality property described above, corresponding to the fact that the relevant probability space depends on the settings of the barriers, it is completely obvious that one can find initial probability distributions for which Bell's inequalities will be violated. For further details see [11] (see also the appendix to the present paper for a correction).

d) Further relevant features. Another important qualitative feature of classical electrodynamics is the possibility of describing pair creation or annihilation, contrary to the common opinion that this should only be possible within quantum field theory. In fact the classical description was already conceived by Stueckelberg and Feynman (see [56]), in terms of curves in space–time presenting angular points. In the paper [8] it was shown in addition that curves of such a type actually occur as solutions of the relativistic ALD equation for an external force presenting a singularity.

An extremely relevant problem is the stability of the atom, which is concerned with the motion of an electron about a nucleus. According to the common opinion, the electron should fall on the nucleus by losing energy by radiation, in agreement with Larmor formula. The situation is however different, if solutions of the ALD equation are considered. Indeed, at least in the nonrelativistic case, it turns out that there are no solutions falling on the nucleus either in a finite or an infinite time. This was shown for the one–dimensional case in a classical paper by Eliezer [57], and the result was recently extended (see [58]) to the three–dimensional case. Whether bounded nonrunaway solutions exist is a very interesting open problem. For some recent results concerning two–electron systems, as in the Helium atom, see [59].

4 Conclusions

In the present paper we have reviewed some works of interest for the relations between classical and quantum physics, where it was shown that the former presents some relevant quantum–like features. It has also been pointed out, in connection with the FPU problem, that quantum mechanics might, under certain aspects, appear as a first order approximation to classical mechanics.

In general, the point of view we are taking seems to be very similar to the one recently illustrated by 't Hooft(see [60]), who apparently is looking for a deterministic hidden variable theory presenting suitable nonlocal properties in order to explain quantum mechanics. This amounts, in his words "*.. to accept* both *quantum mechanics with its usual interpretation* and *to assume that there is a deterministic physical theory lying underneath it.*" The main difference is that, while such an author is looking for some new theory, we are instead pointing out that classical physics itself, particularly when it is extended to describe charged point particles, might already do the job, or at least some part of it. Moreover it turns out that, under the impetus of the modern theory of dynamical systems, classical physics is revealing so many beautiful and unexpected features, that we believe it to be a duty for the scientific community to be able to state in rigorous mathematical terms which actually are its predictions, independently of whether it will be able to explain quantum mechanics or not.

Acknowledgment

The work of one of the authors (A.C.) was made possible by a grant from Fondazione Cariplo per la Ricerca Scientifica.

Appendix: Correction of an error of ours concerning Bell's inequalities

In the paper [11] we report Bell's inequalities in Nelson's version, which makes reference (see [43], page 445) to a certain inequality (3), namely

$$Pr_{\mu\nu}\{\alpha_\mu \beta\nu = -1\} < 1/2 \ , \mu \neq \nu. \tag{10}$$

Nelson himself states however that the inequality should be expected to hold with 1/3 in place of 1/2. And indeed this turns out to be the case, as is immediately seen if one of the last lines of Nelson's proof is corrected, by remarking that the minimum of the function

$$\frac{1}{6} \sum_{\mu \neq \nu} p_\mu p_\nu + (1 - p_\mu)(1 - p_\nu) \qquad (\mu, \nu = 1, 2, 3) \tag{11}$$

($0 \leq p_\mu \leq 1$) is 1/3 and not 1/2 as stated there. In our paper we were concerned with a discussion of factorized states and we wanted to prove that it is impossible to violate Bell's inequality with states of such a type. Exactly the same error of Nelson was made, but the main statement, namely the inequality of page 496, is easily seen to continue to hold, again with 1/3 in place of 1/2.

We thank S. Goldstein for kindly pointing out such an error to us.

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