Algant exercises for the lecture "Complex manifolds" (3FCU)

December 26, 2019

Exercise 1 (Quotients). *Let X be a complex manifold and let G be a group of (holomorphic) automorphisms of X. Assume that the action of G on X is*

(i) free, *i.e. if $g(x) = x$ for some $g \in G$ and $x \in X$ then g is the identity, and*

(ii) properly discontinuous, *i.e. for any compact subset $K \subset X$ the set $\{g \in G : g(K) \cap K \neq \emptyset\}$ is finite.*

Show that the quotient X/G has a natural structure of complex manifold such that the quotient map $\pi : X \to X/G$ is holomorphic and locally a biholomorphism.

Exercise 2 (Hopf manifolds). *Let $0 < \lambda < 1$ be a real number and consider the action of $\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ given by*

$$m \cdot (z_1, \dots, m_n) = (\lambda^m z_1, \dots, \lambda^m z_n).$$

(a) Show that the quotient is diffeomorphic to $S^{2n-1} \times S^1$, so it's compact (Hint: note that $\mathbb{C}^n \setminus \{0\}$ is diffeomorphic to $S^{2n-1} \times \mathbb{R}$).

(b) Show that the quotient X admit a complex structure (you can use Exercise 1).

(c) Show that for $n \geq 2$, X does not admit any Kähler metric. What happens for $n = 1$?

Exercise 3 (Projective bundles). *Let E be a holomorphic vector bundle of rank $r + 1$ over a complex manifold X. The projectivisation $\mathbb{P}(E)$ of E is defined as the quotient of E minus the zero section by the natural action of $\mathbb{C}^*$. Let $\pi : \mathbb{P}(E) \to X$ the induced map.*

(a) Show that $\mathbb{P}(E)$ admits a natural complex structure and π is a holomorphic map. Each fibre of π is isomorphic to $\mathbb{P}^r$.

(b) Define the taoutological bundle on $\mathbb{P}(E)$ as

$$\mathcal{O}_{\mathbb{P}(E)}(-1) := \{(x, y) \in \mathbb{P}(E) \times \pi^* E : y \in \mathbb{C}x\}.$$

Show that $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is a holomorphic subbundle of $\pi^* E$ and that its restriction to each fibre is isomorphic the the tautological bundle of $\mathbb{P}^r$.

(c) Prove that if X is a compact Kähler manifold, then $\mathbb{P}(E)$ is also a compact Kähler manifold.

Exercise 4. *Compute the Hodge diamonds of the following manifolds.*

(a) A compact complex curve of genus g.

(b) The complex projective space $\mathbb{P}^r$.

(c) A complex torus $\mathbb{C}^n/\Lambda$.