COMPLEX GEOMETRY, 2018-2019
ALGANT HOMEWORK

Consign at least one (and three is quite sufficient) of the exercises at least 24 hours before the oral exam. Some exercises need the results from the 'Complex Manifolds' course.

1.1. Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, fix an integer $k \geq 0$ and let $D := k\infty \in \mathrm{Div}(X)$. (You might want to do this exercise (first or only) for the case $k = 3$).

Recall that the Riemann-Roch space $L(D)$ has basis $1, z, \ldots, z^k$ for $k \geq 0$. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^k$ be the map defined by the basis of $L(D)$.

Let $S := \mathbb{C}[x_0, \ldots, x_k]$ be the ring of polynomials in $k+1$-variables and let $S_d \subset S$ be the subspace of polynomials that are homogeneous of degree d. In particular, $\dim S_d = \binom{d+k}{d}$.

a) Show that the linear map

$$\pi_d : S_d \longrightarrow L(dD), \qquad F \longmapsto F(1, z, \ldots, z^k) ,$$

is surjective for any $d \geq 1$.

b) Let M be the following $2 \times k$ matrix with coefficients in S_1:

$$M := \begin{pmatrix} x_0 & x_1 & x_2 & \ldots & x_{k-1} \\ x_1 & x_2 & x_3 & \ldots & x_d \end{pmatrix} .$$

Show that in a point $p \in \phi(\mathbb{P}^1)$ the rank of of $M = M(x_0, \ldots, x_k)$ is at most one.

c) Show that the 2×2 minors of M, which are in S_2, span the kernel of π_2.

d) Using explicit elements in the kernel of π_2, show that the image $\phi(\mathbb{P}^1)$ is an intersection of quadrics.

1.2. Let E be a Riemann surface of genus 1 and let $P \in E$ be a point.

a) Given a basis $1, f_2$ of $L(2P)$ and $1, f_2, f_3$ of $L(3P)$, find a basis of $L(kP)$ for all $k > 0$.

b) Show that, for any $k > 2$, the multiplication map

$$L(kP) \otimes L(kP) \longrightarrow L(2kP), \qquad f \otimes g \longmapsto fg$$

is surjective.

c) Show that in case $k = 4$ the image of the map $\phi_k : E \to \mathbb{P}^3$ given by a basis of $L(kP)$ is the intersection of two quadrics.

d) Choose a basis $1, f, g$ of the subspace $L(3P) \subset L(4P)$ such that $g^2 = f^3 + af + b$, for some $a, b \in \mathbb{C}$. Give, with respect to a suitable basis of $L(4P)$, two quadratic polynomials in 4 variables defining the image $\phi(E)$ of E.

e) (Maybe too difficult!) Show that for any k, the image $\phi_k(E)$ is an intersection of quadrics.

1.3. Let $X = X^0 \cup \{P_\infty\}$ be the compact Riemann surface of genus 2 defined by the algebraic curve $X^0 : y^2 = f(x)$ in $\mathbb{C}^2$ and the (unique) point P_∞ over $x = \infty \in \mathbb{P}^1$, where f is a polynomial of degree 5 with distinct roots.

a) Using the meromorphic functions x, y on X, give a basis of $L(5P_\infty)$.
b) Find equations which define $\phi_{5P_\infty}(X) \subset \mathbb{P}^3$ in case $f = x^5 + 1$.
c) Try to generalize this to the divisor $(2g+1)P_\infty$, where P_∞ is the point over infinity on the hyperelliptic curve X defined by $y^2 = x^{2g+1} + 1$.
d) Recall that $K = (2g-2)P_\infty$ is a canonical divisor on X. Show that the multiplication map $L(K) \otimes L(K) \to L(2K)$ is not surjective if $g \geq 3$.
e) Try to show that if $g \geq 3$ then X is not trigonal, that is, there is no map $\phi : X \to \mathbb{P}^1$ of degree three. (You might need the results of Exercise 1.1).

1.4. Consider the subset X of $\mathbb{P}^4$ defined as

$$X := \{(x : y : z : u : v) \in \mathbb{P}^4 : \ u^2 = x^2 + z^2, \quad uv = y^2, \quad v^2 = x^2 - z^2\}.$$

a) Show that X is a Riemann surface.
b) Show that

$$\phi : X \longrightarrow Y := \{(x : y : z) \in \mathbb{P}^2 : y^4 = x^4 - z^4\} \qquad (x : \ldots : v) \longmapsto (x : y : z)$$

is a holomorphic map, of degree two, between Riemann surfaces.
c) Show that X has genus 5 and that X is canonically embedded in $\mathbb{P}^4$.
d) Consider the following points $P_1, \ldots, P_4 \in X$:

$$(1:1:0:1:1), \quad (1:1:0:-1:-1), \quad (1:-1:0:1:1), \quad (1:-1:0:-1:-1).$$

Determine $\dim L(P_1 + P_2 + P_3 + P_4)$. Can you find a rational function on $\mathbb{P}^4$ whose restriction to X is a non-constant function in $L(P_1 + P_2 + P_3 + P_4)$?
e) Let $\phi : X \to \mathbb{P}^2$ be the map induced by $(x : \ldots : v) \mapsto (y : u : v)$. Show that the image of X is a smooth conic C (so $C \cong \mathbb{P}^1$) and find the degree of the map $\phi : X \to C$.
If $D = Q_1 + Q_2 + Q_3 + Q_4$ is a fiber of ϕ, show that $\dim L(D) = 2$ and that $2D \equiv K$, the canonical class of X.

1.5. Give a non-constant holomorphic map $\phi : X \to Y$ between compact Riemann surfaces, let $\phi^* : \Omega^1(Y) \to \Omega^1(X)$ be the pull-back map on the vector spaces of holomorphic differentials.

a) Show that ϕ^* is injective.
b) Let $\phi_X : X \to \mathbb{P}^{g_X - 1}$ and $\phi_Y : Y \to \mathbb{P}^{g_Y - 1}$ be the canonical maps of X and Y. Show that there is a linear projection $\mathbb{P}^{g_X-1} - - \to \mathbb{P}^{g_Y-1}$ such that the image of $\phi_X(X)$ is $\phi_Y(Y)$ (so in suitable projective coordinates on $\mathbb{P}^{g_X-1}$, the map is given by $(x_0 : \ldots : x_n) \mapsto (x_0 : \ldots : x_m)$).
c) Show that if X is hyperelliptic then also Y is hyperelliptic.
d) More generally, if $D = P_1 + \ldots + P_k \in \mathrm{Div}(X)$ with $k < g_X$, can you give a relation between $l(D)$ and $l(\phi_*(D))$ where $\phi_*(D) = \phi_Y(P_1) + \ldots + \phi_Y(P_k) \in \mathrm{Div}(Y)$?

1.6. Let X be a Riemann surface, let $D \in \mathrm{Div}(X)$ and assume that $D = D_1 + D_2$ and

$$s+1 := \dim L(D_1) \geq 2, \qquad t+1 := \dim L(D_2) \geq 2, \qquad r+1 := \dim L(D) \ .$$

We denote by $f_0, \dots, f_r$, $g_0, \dots, g_s$, $h_0, \dots, h_t$ bases of $L(D)$, $L(D_1)$ and $L(D_2)$ respectively.

a) Show that there are complex numbers a_{ijk}, for $0 \leq i \leq s$, $0 \leq j \leq t$ and $0 \leq k \leq r$, such that

$$g_i h_j \ = \ \sum_{k=0}^{r} a_{ijk} f_k \ .$$

b) Show that for every point $P \in X - S$, where S is the finite set where at least one of the f_i, g_i, h_i has a pole, the $t \times s$ matrix $M(P)$ with coefficients $g_i(P)h_j(P)$ has rank at most one.

c) Let $\phi = \phi_D$ be the map defined by the basis f_i of $L(D)$:

$$\phi : X \longrightarrow \mathbb{P}^r, \qquad P \longmapsto (x_0 : \ldots : x_r) \ := \ (f_0(P) : \ldots : f_r(P)) \ .$$

Show that $\phi(X)$ is contained in the quadrics defined by the 2×2-minors of the $(s+1) \times (t+1)$ matrix of linear forms with coefficients

$$M \ := \ (M_{ij}), \qquad M_{ij} \ := \ \sum_{k=0}^{r} a_{ijk} x_k$$

where $x_0, \dots, x_k$ are the homogeneous coordinates on $\mathbb{P}^r$.

d) Show that the $2 \times k$ matrix M in Exercise 1.1.b is obtained by this construction, so identify the D_i and the bases f_i, g_i, h_i.

e) Let E be an elliptic curve and let $D = P_1 + \ldots + P_r$ be an effective divisor of degree $r \geq 4$. Notice that for any $P \in E$ one has $D = D_1 + D_2$ with $D_1 = P_1 + P$ and $D_2 = P_2 + \ldots + P_r - P$ and conclude that $\phi_D(E)$ is contained in quadrics.

For fixed $P_0 \in E$, try to construct, for some suitable choice of P, quadrics containing $\phi_{4P_0}(E)$ with the construction above (cf. Exercise 1.2).

1.7. Let $X = X^0 \cup \{P_\infty\}$ be the compact Riemann surface of genus 2 defined by the algebraic curve $X^0 : y^2 = f(x)$ in $\mathbb{C}^2$ and the (unique) point P_∞ over $x = \infty \in \mathbb{P}^1$, where f is a polynomial of degree 5 with distinct roots. Let $\iota : X \to X$ be the covering involution for the 2:1 map $x : X \to \mathbb{P}^1$, so $\iota(x,y) = (x,-y)$ for $(x,y) \in X^0$ and $\iota P_\infty = P_\infty$. Let K be a canonical divisor on X.

a) Show that if $[D] \in \mathrm{Pic}^0(X)$ then $[D] = [P_1 + P_2 - K]$ for some $P_i \in X$, that is $D \equiv P_1 + P_2 - K$, and the divisor $P_1 + P_2$ is unique except if $[D] = 0$.

b) Show that $2P_\infty$ is a canonical divisor and more generally that $P + \iota P$ is a canonical divisor for any $P \in X$. For any divisor $\sum n_i P_i$ we define $\iota D := \sum n_i \iota P_i$. Show that $D + \iota D = mK$ where $m = \deg(D)$.

c) Given $[D], [D']$ in $Pic^0(X)$, we write $D = P + Q - K$ and $D' = P' + Q' - K$ and we want to find $D'' = P'' + Q'' - K$ such that $[D] + [D'] = [D'']$ in $\mathrm{Pic}^0(X)$.
Show that $\iota(P'') + \iota(Q'') \equiv 3K - (P + Q + P' + Q')$.

d) Show that $\dim L(3K) = 5$ and find a basis of $L(3K)$ when $K = 2P_\infty$.

e) Let $y = a_0 + a_1x + a_2x^2 + a_3x^3$ be the (in general unique) cubic curve C in $\mathbb{C}^2$ such that P, Q, P', Q' are on this curve. Show that $\iota P'', \iota Q''$ are the remaining points of intersection of C with X^0.

f) You could try to generalize this description of the addition on $\mathrm{Pic}^0(X)$ to a hyperelliptic curve of genus $g > 2$, "replacing K by gP_∞". More precisely:
   1) show that $[D] \in \mathrm{Pic}^0(X)$ is linearly equivalent to $P_1 + \ldots + P_g - gP_\infty$ for certain $P_i \in X$ (in general the P_i will be unique).
   2) Show that $P + \iota P \equiv 2P_\infty$ for any $P \in X$ and $D + \iota D = 2mP_\infty$ where $m = \deg(D)$.
   3) If $D = D_+ - gP_\infty$, $D' = D'_+ - gP_\infty$ and $D'' = D''_+ - gP_\infty$, with effective divisors D_+, D'_+, D''_+ of degree g, and $[D] + [D'] = [D'']$ then show that $\iota(D''_+) \equiv 3gP_\infty - (D_+ + D'_+)$.
   4) Find a basis of $L(3gP_\infty)$.
   5) Find a curve $C \subset \mathbb{C}^2$, which contains the points in D_+, D'_+, whose remaining intersection points with X^0 are the points of $\iota(D''_+)$.

1.8. Let $X \subset \mathbb{P}^2$ be a Riemann surface which is defined as $F_4 = 0$ for a polynomial F_4 of degree 4, so X has genus 3. Let $P_0 \in X$ be a point.

a) Show that if $[D] \in \mathrm{Pic}^0(X)$ then $[D] = [P_1 + P_2 + P_3 - 3P_0]$ for some $P_i \in X$, that is $D \equiv P_1 + P_2 + P_3 - 3P_0$.

b) Show that the P_i in 1.8.a are unique unless there is a line $L \subset \mathbb{P}^2$ such that $P_i \in L \cap X$ for $i = 1, 2, 3$. In that case, describe all the effective degree 3 divisors $Q_1 + Q_2 + Q_3$ such that $P_1 + P_2 + P_3 \equiv Q_1 + Q_2 + Q_3$.

c) Given 4 points $P_1, \ldots, P_4 \in X$, in view of 1.8.a, there must be 3 points Q_1, Q_2, Q_3 such that

$$P_1 + P_2 + P_3 + P_4 - 4P_0 \equiv Q_1 + Q_2 + Q_3 - 3P_0 .$$

Describe how to find these points. (Hint: Take any conic $G = 0$ passing through $P_1, \ldots, P_4$ and show that there exists a (in general unique) conic $H = 0$ whose intersection with X contains the other $4 = 4 \cdot 2 - 4$ points of intersection of $X \cap (G = 0)$ as well as P_0).

d) You could try to generalize 1.8.c to plane curves of degree d. Such a Riemann surface has genus $g = (d-1)(d-2)/2$. Given general points $P_0, P_1, \ldots, P_{g+1}$, indicate how one could find $Q_1, \ldots, Q_g$ such that $P_1 + \ldots + P_{g+1} \equiv Q_1 + \ldots + Q_g + P_0$.

1.9. Let X be a compact Riemann surface and let $D = \sum_{i=1}^k n_i P_i$ be a divisor on X. Let $z_i : U_i \to \mathbb{C}$ be a local coordinate centered on P_i (so $z_i(P_i) = 0$) and assume that $U_i \cap U_j = \emptyset$ for $i \neq j$. Let $U_0 := X - \{P_1, \ldots, P_k\}$, let $z_0 := 1$, a constant function and put $n_0 := 1$.

We define a holomorphic line bundle L_D on X by the open covering $X = \cup_{i=0}^k U_i$ and the transition functions $g_{ij} := z_i^{n_i}/z_j^{n_j} : U_i \cap U_j \to \mathbb{C} - \{0\}$.

a) Show that $s_D \leftrightarrow \{s_i : U_i \to \mathbb{C}, s_i(x) := z_i^{n_i}(x)\}$ defines a meromorphic section of L_D.

b) Show that the map between the space of global sections of L_D and the Riemann-Roch space $L(D)$ is an isomorphism:

$$\Gamma(X, L_D) \longrightarrow L(D), \qquad t \longmapsto f_t \leftrightarrow \{f_{t,i} := t_i/s_i : U_i \to \mathbb{C}, \} ,$$

where $t \leftrightarrow \{t_i : U_i \to \mathbb{C}\}$, in particular the $f_{t,i}$ are meromorphic functions and $f_{t,i} = f_{t,j}$ on $U_i \cap U_j$ for all i, j.

c) Show that $L_{D+D'} \cong L_D \otimes L_{D'}$ for any $D, D' \in \mathrm{Div}(X)$.

d) Show that $L_D \cong X \times \mathbb{C}$, the trivial bundle if and only if $D \equiv 0$. Conclude that $\mathrm{Div}(X)/P(X)$ is isomorphic to a subgroup of the group of line bundles modulo isomorphism, with tensor product.

e) Show that the sheaf $\mathcal{F}_D$ of holomorphic sections of the line bundle L_D is isomorphic to the sheaf $\mathcal{O}_X(D)$, where now $D = \sum n_p P$:

$$\mathcal{F}_D(U) := \{t : U \to L_{D|U}\}, \qquad \mathcal{O}_X(D) := \{f \text{ meromorphic on } U, \quad \mathrm{ord}_p(f) + n_p \geq 0 \quad \forall p \in U\}.$$