## COMPLEX MANIFOLDS, 2013-2014 <br> ALGANT HOMEWORK

1.1. Let $T:=\mathbb{C} / \Lambda$ be a complex torus and let $\pi: \mathbb{C} \rightarrow T$ be the quotient map. Let $\wp$ be the Weierstrass $\wp$-function for the lattice $\Lambda$ and let $g_{2}, g_{3} \in \mathbb{C}$ be such that $\left(\wp^{\prime}\right)^{2}=4 \wp^{3}-g_{2} \wp-g_{3}$.
(a) Show that the subset $C \subset \mathbb{P}^{3}$, an intersection of the two quadrics,

$$
C=\left\{(x: y: z: t): \in \mathbb{P}^{3}: y^{2}=4 x t-g_{2} x z-g_{3} z^{2}, \quad x^{2}=z t\right\}
$$

is submanifold of $\mathbb{P}^{3}$.
(b) Show that the map

$$
\psi: T \longrightarrow \mathbb{P}^{3}, \quad t=\pi(z) \longmapsto\left(\wp(z): \wp^{\prime}(z): 1: \wp^{2}(z)\right)
$$

for $t \neq 0$ and $\psi(0)=(0: 0: 0: 1)$ is a holomorphic map and that $\psi(T)$ is isomorphic to $T$.
(c) Show that $C=\psi(T)$, hence that

$$
C \cong T .
$$

1.2. (a) Let $A=\left(a_{i j}\right)$ be an invertible $(n+1) \times(n+1)$ matrix with complex coefficients. Show that the map

$$
\alpha: \mathbb{P}^{n} \longrightarrow \mathbb{P}^{n}, \quad\left(x_{0}: \ldots: x_{n}\right) \longmapsto\left(y_{0}: \ldots: y_{n}\right), \quad y_{i}:=\sum_{j=1}^{n} a_{i j} x_{j}
$$

(so $\alpha$ is the map induced by $A: \mathbb{C}^{n+1}-\{0\} \rightarrow \mathbb{C}^{n+1}-\{0\}$ ) is a biholomomorphic map.
(b) Let $\lambda \in \mathbb{C}, \lambda \neq 0$. Show that

$$
\beta: \mathbb{P}^{2} \longrightarrow \mathbb{P}^{2}, \quad(x: y: z) \longmapsto(u: v: w):=\left(\lambda^{2} x: \lambda^{3} y: z\right),
$$

is a biholomorphic map and that the elliptic curves $E, E^{\prime}$ with (affine) equations

$$
y^{2}=4 x^{3}-g_{2} x-g_{3}, \quad v^{2}=4 u^{3}-\lambda^{4} g_{2} u-\lambda^{6} g_{3}
$$

respectively, are isomorphic.
(c) Show that the curves in $\mathbb{P}^{2}$ defined by

$$
x^{3}+y^{3}+z^{3}=0 \quad y^{2}=4 x^{3}-g_{3}
$$

are isomorphic, for any $g_{3} \in \mathbb{C}, g_{3} \neq 0$, and that these curves are also isomorphic to the complex torus $\mathbb{C} / \Lambda$ where $\Lambda=\left\{n+m \omega: n, m \in \mathbf{Z}, \omega^{3}=1, \omega \neq 1\right\}$. (Hint: substitute $x=u+v$, $y=u-v$ in the Fermat equation and use affine coordinates with $u=1$ ).
1.3. Let $E$ be the elliptic curve in $\mathbb{P}^{2}$ defined by the (affine) equation

$$
y^{2}=4 x^{3}-g_{3}, \quad\left(g_{3} \neq 0\right)
$$

Let $\mathcal{O}:=(0: 1: 0)$ be the neutral element in the group law on $E$.
(a) Show that the points $P_{ \pm}$with affine coordinates $(x, y)=\left(0, \pm \sqrt{-g_{3}}\right)$ are points of order three.
(b) Let $g_{3}$ be choosen in such a way that the map $F: \mathbb{C} \rightarrow \mathbb{P}^{2}, z \mapsto\left(\wp(z): \wp^{\prime}(z): 1\right)$, where $\wp$ is the Weierstrass $\wp$-function for the lattice $\Lambda=\left\{n+m \omega: n, m \in \mathbf{Z}, \omega^{3}=1, \omega \neq 1\right\}$ as in Exercise 1.2 has image $E$. Show that

$$
\left(\wp(\omega z): \wp^{\prime}(\omega z): 1\right)=\left(\omega \wp(z): \wp^{\prime}(z): 1\right)
$$

for all $z \in \mathbb{C}$. Conclude that the image of $(1-\omega) / 3 \in \mathbb{C}$ under $F$ is $P_{+}$or $P_{-}$.
1.4. Let $X$ be a complex manifold and let

$$
0 \longrightarrow E \longrightarrow F \longrightarrow F / E \longrightarrow 0
$$

be an exact sequence of vector bundles on $X$. Let $L$ be a line bundle on $X$.
(a) Show that there is an sequence of vector bundles

$$
0 \longrightarrow E \otimes L \longrightarrow F \otimes L \longrightarrow(F / E) \otimes L \longrightarrow 0
$$

(b) Recall that for $k \in \mathbf{Z}$ we have the line bundle $L(k)$ on $\mathbb{P}^{1}$. In particular, $L(0) \cong \mathbb{P}^{1} \times \mathbb{C}$ is the trivial bundle and $L(-1)$ is the tautological bundle, which, by definition, is a subbundle of $L(0)^{2}$.

Show that there is an exact sequence

$$
0 \longrightarrow L(-1) \longrightarrow L(0)^{2} \longrightarrow L(1) \longrightarrow 0
$$

(c) Deduce that there is an exact sequence of vector bundles on $\mathbb{P}^{1}$ :

$$
0 \longrightarrow L(-2) \longrightarrow L(-1)^{2} \longrightarrow L(0) \longrightarrow 0
$$

and show that $H^{1}\left(\mathbb{P}^{1}, L(-2)\right)$ is non-zero.
1.5. A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}^{\prime}$ such that for every open $U \subset X, \mathcal{F}^{\prime}(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps of the sheaf $\mathcal{F}^{\prime}$ are induced by those of $\mathcal{F}$.
(a) Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on $X$ and let $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of sheaves. Show that

$$
\mathcal{F}^{\prime}(U):=\operatorname{ker}\left(\alpha_{U}: \mathcal{F}(U) \rightarrow \mathcal{G}(U)\right)
$$

defines a subsheaf on $X$.
(b) Let $\mathcal{F}^{\prime}$ be a subsheaf of a sheaf $\mathcal{F}$. Let $\mathcal{G}$ be the presheaf defined by $\mathcal{G}(U):=\mathcal{F}(U) / \mathcal{F}^{\prime}(U)$, with restriction maps induced by those of $\mathcal{F}$. Show that

$$
\mathcal{G}_{a} \cong \mathcal{F}_{a} / \mathcal{F}_{a}^{\prime} \quad \text { for all } \quad a \in X .
$$

(c) Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space $X$ and let $\mathcal{F}^{+}$be the sheaf generated by this presheaf.

Show that the natural homomorphism of sheaves $\tau: \mathcal{F} \rightarrow \mathcal{F}^{+}$induces an isomorphism $\tau_{a}: \mathcal{F}_{a} \rightarrow \mathcal{F}_{a}^{+}$on the stalks for all $a \in X$.
1.6. Let $X$ be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on $X$.
(a) For an open subset $U \subseteq X$ and a section $s \in \mathcal{F}(U)$ define

$$
\operatorname{Supp}(s)=\left\{a \in U: s_{a} \neq 0\right\}
$$

where $s_{a}$ is the germ of $s$ in the stalk $\mathcal{F}_{a}$. Prove that $\operatorname{Supp}(s)$ is a closed subset of $U$.
(b) Let $Z \subseteq X$ be a closed subset. Define $\Gamma_{Z}(X, \mathcal{F})$ to be the subgroup of $\mathcal{F}(X)$ consisting of all sections whose support is contained in $Z$. Show that the presheaf

$$
V \mapsto \Gamma_{V \cap Z}\left(V,\left.\mathcal{F}\right|_{V}\right)
$$

is a sheaf.
1.7. Consider the sheaves $\mathcal{O}_{\mathbb{P}^{1}}(d)$ on $\mathbb{P}^{1}$, where $\mathcal{O}_{\mathbb{P}^{1}}(d)(U)$ are the holomorphic functions on $\pi^{-1}(U)$ which are homogeneous of degree $d$ and where $\pi: \mathbb{C}^{2}-\{0\} \rightarrow \mathbb{P}^{1}$ is the quotient map. One can show that the sheaf $\mathcal{O}_{\mathbb{P}^{1}}(d)$ is isomorphic to the sheaf of global sections of the line bundle $L(d)$ on $\mathbb{P}^{1}$. The homogeous coordinates on $\mathbb{P}^{1}$ are $\left(x_{0}: x_{1}\right)$ so $x_{0}, x_{1} \in \mathcal{O}_{\mathbb{P}^{1}}(1)\left(\mathbb{P}^{1}\right)$.
(a) Show that

$$
\varphi: \mathcal{O}_{\mathbb{P}^{1}}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^{1}}(d+1), \quad \varphi_{U}(f):=x_{0} f, \quad\left(f \in \mathcal{O}_{\mathbb{P}^{1}}(d)(U)\right),
$$

is an injective homomorphism of sheaves.
(b) Describe the stalks of the corresponding quotient sheaf $\mathcal{Q}:=\mathcal{O}_{\mathbb{P}^{1}}(d+1) / \mathcal{O}_{\mathbb{P}^{1}}(d)$ and conclude that $\mathcal{Q}$ is the skyscraper sheaf, with group $\mathbb{C}$, concentrated in the point $p:=(0: 1) \in \mathbb{P}^{1}$.
(c) Using the long exact cohomology sequence associated to the exact sequence of sheaves on $\mathbb{P}^{1}$ :

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^{1}}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^{1}}(d+1) \longrightarrow \mathcal{Q} \longrightarrow 0
$$

show that $H^{1}\left(\mathbb{P}^{1}, \mathcal{O}_{\mathbb{P}^{1}}(-2)\right) \neq 0$ (cf. Execercise 1.4 for another proof).
(d) Show that $H^{1}\left(\mathbb{P}^{1}, \mathcal{O}_{\mathbb{P}^{1}}(d)\right) \neq 0$ for all $d \leq-2$.

