

COMPLEX MANIFOLDS

LECTURE NOTES BASED ON THE COURSE BY LAMBERTUS VAN GEEMEN

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Chapter 1

Some preliminaries about \mathbb{C}^n

We assume that the reader has some familiarity with the notion of a holomorphic function in one complex variable. We extend that notion with the following

Definition 1.1. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$, $U \subseteq \mathbb{C}^n$ open with $a \in U$, and let $z = (z_1, \dots, z_n)$ be the coordinates in \mathbb{C}^n . f is *holomorphic* in $a = (a_1, \dots, a_n) \in U$ if f has a convergent power series expansion:

$$f(z) = \sum_{k_1, \dots, k_n=0}^{+\infty} a_{k_1, \dots, k_n} (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}$$

This means, in particular, that f is holomorphic in each variable. Moreover, we define

$$\mathcal{O}_{\mathbb{C}^n}(U) := \{f : U \rightarrow \mathbb{C} \mid f \text{ is holomorphic}\}$$

A map $F = (F_1, \dots, F_m) : U \rightarrow \mathbb{C}^m$ is holomorphic if each F_j is holomorphic.

Let f be a holomorphic function. One can write $f(z) = g(z) + ih(z)$, with $g, h : U \rightarrow \mathbb{R}$ smooth. The condition for f to be holomorphic on U is equivalent to the *Cauchy-Riemann conditions*:

$$\frac{\partial g}{\partial x_j}(a) = \frac{\partial h}{\partial y_j}(a) \quad \text{and} \quad \frac{\partial g}{\partial y_j}(a) = -\frac{\partial h}{\partial x_j}(a)$$

for $j = 1, \dots, n$, where $z_j = x_j + iy_j$.

Definition 1.2. Let $V \subseteq \mathbb{C}^n$ be open, let $F = (F_1, \dots, F_m) : V \rightarrow \mathbb{C}^m$ be a holomorphic map. The *complex Jacobian matrix* of F is

$$J_{\mathbb{C}}F := \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial z_1} & \cdots & \frac{\partial F_m}{\partial z_n} \end{pmatrix} = \left(\frac{\partial F_j}{\partial z_k} \right)$$

Now let $F_j = G_j + iH_j$ with $G_j, H_j : V \rightarrow \mathbb{R}$ smooth \mathbb{R} -valued functions. Let $\tilde{F} : V \rightarrow \mathbb{R}^{2n}$ defined as $\tilde{F}(z) = (G_1(z), \dots, G_m(z), H_1(z), \dots, H_m(z))$. The *real Jacobian matrix* of F is

$$J_{\mathbb{R}}F := J_{\mathbb{R}}\tilde{F} = \left(\begin{array}{c|c} \frac{\partial G_j}{\partial x_k} & \frac{\partial G_j}{\partial y_k} \\ \hline \frac{\partial H_j}{\partial x_k} & \frac{\partial H_j}{\partial y_k} \end{array} \right)$$

Remark.

If $F : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is holomorphic, then $\frac{\partial G_j}{\partial x_k} = \frac{\partial H_j}{\partial y_k}$, $\frac{\partial G_j}{\partial y_k} = -\frac{\partial H_j}{\partial x_k}$ that means

$$J_{\mathbb{R}}F = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \quad \text{with } A = \frac{\partial G_j}{\partial x_k}, \quad B = \frac{\partial H_j}{\partial x_k}$$

Moreover

$$\begin{aligned} \frac{\partial F_j}{\partial z_k} &= \frac{1}{2} \left(\frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right) (G_j + iH_j) = \frac{1}{2} \left(\frac{\partial G_j}{\partial x_k} + \frac{\partial H_j}{\partial y_k} + i \left(\frac{\partial H_j}{\partial x_k} - \frac{\partial G_j}{\partial y_k} \right) \right) = \\ &= \frac{\partial G_j}{\partial x_k} + i \frac{\partial H_j}{\partial x_k} = A_{jk} + iB_{jk} \Rightarrow J_{\mathbb{C}}F = A + iB. \end{aligned}$$

LEMMA 1.1. Let $M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in M_{2n}(\mathbb{R})$, and let $J = \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix}$. Then

$$JMJ^{-1} = M \iff P = S, Q = -R \iff M = \begin{pmatrix} P & -R \\ R & P \end{pmatrix}$$

Proof.

$$JMJ^{-1} = M \iff JM = MJ \iff \begin{pmatrix} -R & -S \\ P & Q \end{pmatrix} = \begin{pmatrix} Q & -P \\ S & -R \end{pmatrix}$$

□

Combining the above lemma and the previous remark, we can characterize an holomorphic function $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ (same dimension!) analyzing its real Jacobian matrix:

PROPOSITION 1.2. A function $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is holomorphic if and only if $J(J_{\mathbb{R}}F)J^{-1} = J_{\mathbb{R}}F$, with $J = \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix}$.

Remark.

It is worth to notice that J is the matrix representing the multiplication by i from \mathbb{C}^n to itself, as we shall see in the following chapter. Thus, one can also state: a function $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is holomorphic if and only if its real Jacobian matrix is self-conjugate under the conjugation action of the multiplication by i (or simply, its real Jacobian matrix commutes with J).

PROPOSITION 1.3. Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ holomorphic. Then $\det(J_{\mathbb{R}}F) \geq 0$.

Proof. Consider the matrix N defined as

$$N = \begin{pmatrix} \text{Id}_n & i \cdot \text{Id}_n \\ \text{Id}_n & -i \cdot \text{Id}_n \end{pmatrix} \in M_{2n}(\mathbb{C}), \quad N^{-1} = \frac{1}{2} \begin{pmatrix} \text{Id}_n & \text{Id}_n \\ -i \cdot \text{Id}_n & i \cdot \text{Id}_n \end{pmatrix}$$

Notice that

$$\begin{aligned} NJ_{\mathbb{R}}FN^{-1} &= \frac{1}{2} \begin{pmatrix} \text{Id}_n & i \cdot \text{Id}_n \\ \text{Id}_n & -i \cdot \text{Id}_n \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} \text{Id}_n & \text{Id}_n \\ -i \cdot \text{Id}_n & i \cdot \text{Id}_n \end{pmatrix} = \\ &= \frac{1}{2} \begin{pmatrix} A + iB & -B + iA \\ A - iB & -B - iA \end{pmatrix} \begin{pmatrix} \text{Id}_n & \text{Id}_n \\ -i \cdot \text{Id}_n & i \cdot \text{Id}_n \end{pmatrix} = \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix} = \begin{pmatrix} J_{\mathbb{C}}F & 0 \\ 0 & \overline{J_{\mathbb{C}}F} \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned}\det(J_{\mathbb{R}}F) &= \det(N)\det(N^{-1})\det(J_{\mathbb{R}}F) = \det(NJ_{\mathbb{R}}FN^{-1}) = \\ &= \det(J_{\mathbb{C}}F)\det(\overline{J_{\mathbb{C}}F}) = \det(J_{\mathbb{C}}F)\overline{\det(J_{\mathbb{C}}F)} = |\det(J_{\mathbb{C}}F)|^2 \geq 0\end{aligned}$$

□

Recall that an holomorphic function in one variable is a conformal mapping from \mathbb{R}^2 to itself, that is, it preserves orientations of angles. The latter proposition shows that, when dealing with an holomorphic function of several variables, the "orientation preserving" property translates to a really strict condition on the determinant of the real Jacobian of the function. As we will see in the next chapter, this condition is related in some sense with the notion of orientation (it will imply the orientability of complex manifolds, seen as differentiable manifolds).

THEOREM 1.4 (MAXIMUM PRINCIPLE). Let $g : V \rightarrow \mathbb{C}$ be holomorphic, $V \subseteq \mathbb{C}$ open, connected. Assume there is a $v \in V$ such that $|g(v)| \geq |g(z)| \forall z \in V$ ($|g|$ takes its maximum on V). Then g is constant, so $g(z) = g(v) \forall z \in V$.

This fundamental result about one-variable holomorphic function has many consequences in complex analysis; we will only use it once in the following chapter to see that a holomorphic function on a compact complex manifold is nothing but a constant function (see theorem 2.3).

There exists also a "holomorphic version" of the Dini theorem (local invertibility of maps with invertible Jacobian):

PROPOSITION 1.5. Let $V \subseteq \mathbb{C}^n$ open, $F : V \rightarrow \mathbb{C}^n$ holomorphic. Assume that $J_{\mathbb{C}}F$ has rank n in $a \in V$ (i.e. $J_{\mathbb{C}}F(a)$ has non-zero determinant). Then there is a neighborhood W of a and an holomorphic inverse $G : F(W) \rightarrow W$ such that $F \circ G = \text{id}_{F(W)}$, $G \circ F = \text{id}_W$.

Proof. As $\det(J_{\mathbb{R}}F)$ has rank $2n$, $\det(J_{\mathbb{R}}F) = |\det(J_{\mathbb{C}}F)|^2 \neq 0$. So, by the Dini theorem there exist a neighborhood W of a such that it is possible to find an inverse G for the map F regarded as a map from \mathbb{R}^{2n} to \mathbb{R}^{2n} . We are going to show that G is already the map we need, that is, G is holomorphic; or equivalently, $J(J_{\mathbb{R}}G)J^{-1} = J_{\mathbb{R}}G$, J as in lemma 1.1.

We know that $G \circ F = \text{id}_W \Rightarrow J_{\mathbb{R}}G \cdot J_{\mathbb{R}}F = \text{Id}$; moreover, since F is holomorphic $J(J_{\mathbb{R}}F)J^{-1} = J_{\mathbb{R}}F$. Then $J(J_{\mathbb{R}}G)^{-1}J^{-1} = J_{\mathbb{R}}G^{-1} \Rightarrow J(J_{\mathbb{R}}G)J^{-1} = (J(J_{\mathbb{R}}G)^{-1}J^{-1})^{-1} = (J_{\mathbb{R}}G^{-1})^{-1} = J_{\mathbb{R}}G$. □

Definition 1.3. A function F is *biholomorphic* on $W \subseteq \mathbb{C}^n$ if there exists an holomorphic inverse $G : F(W) \rightarrow W$ (as in the previous proposition).

Chapter 2

Basic theory of complex manifolds

Throughout this chapter, we will assume that the reader has some familiarity with the theory of differentiable manifolds, since many times we will refer to (and use) the definitions and the results related to that theory.

2.1 Complex charts and atlases

Let X be a topological manifold of dimension $2n$, that is, X is a Hausdorff topological space such that each point of X admits an open neighborhood U which is homeomorphic to an open subset V of \mathbb{R}^{2n} . Such an homeomorphism $x : U \rightarrow V$ is called *coordinate neighborhood*. In this course, we do not require X to be second countable (as it happened for differentiable manifolds).

Definition 2.1. A *local complex chart* (U, z) of X is an open subset $U \subseteq X$ and an homeomorphism $z : U \rightarrow V := z(U) \subset \mathbb{C}^n (\equiv \mathbb{R}^{2n})$.

Two local complex charts $(U_\alpha, z_\alpha), (U_\beta, z_\beta)$ are *compatible* if the map $f_{\beta\alpha} := z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \rightarrow z_\beta(U_\alpha \cap U_\beta)$ is holomorphic. The map $f_{\beta\alpha}$ is called *transition function* or *coordinate change*. (We note that $f_{\alpha\beta}$ is holomorphic, too).

Definition 2.2. A *holomorphic atlas* (or *complex analytical atlas*) of X is a collection $A = \{(U_\alpha, z_\alpha)\}$ of local complex charts, such that $X = \cup_\alpha U_\alpha$ and such that all transition functions $f_{\alpha\beta}$ are biholomorphic, for each α, β . (In this way, each pair of charts is compatible).

A *complex analytic structure* on X is a maximal holomorphic atlas $A = \{(U_\alpha, z_\alpha)\}_{\alpha \in I}$. Maximal means: if (U, z) is a local complex chart and (U, z) is compatible with (U_α, z_α) $\forall \alpha \in I$, then $(U, z) \in A$.

A *complex (analytic) manifold* is a topological manifold together with a complex analytic structure.

Remark.

A holomorphic atlas $B = \{(U_\beta, z_\beta)\}_{\beta \in J}$ determines a (unique) maximal atlas A with $B \subset A$ and hence it determines a complex manifold. (The atlas is given by $A = \{(U, z) \mid (U, z) \text{ compatible with } (U_\beta, z_\beta) \forall \beta \in J\}$).

Example(THE PROJECTIVE SPACE): As usual, we define $\mathbb{P}^n(\mathbb{C}) = \mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\sim$ where $u \sim v \Leftrightarrow u = tv$ for some $t \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. Let

$$\begin{aligned} \pi : \mathbb{C}^{n+1} - \{0\} &\rightarrow \mathbb{P}^n \\ (u_0, \dots, u_n) &\mapsto (u_0 : \dots : u_n) \end{aligned}$$

be the quotient map. \mathbb{P}^n has the quotient topology: $U \subseteq \mathbb{P}^n$ is open if $\pi^{-1}(U)$ is open in $\mathbb{C}^{n+1} - \{0\}$.

The open sets $U_j := \{p = (u_0 : \dots : u_n) \mid u_j \neq 0\}$ of \mathbb{P}^n together with the local complex charts

$$\begin{aligned} z_j : U_j &\rightarrow \mathbb{C}^n \\ p = (u_0 : \dots : u_n) &\mapsto \left(\frac{u_0}{u_j}, \dots, \frac{\widehat{u_j}}{u_j}, \dots, \frac{u_n}{u_j} \right) \end{aligned}$$

are an holomorphic atlas of \mathbb{P}^n . Indeed, $\mathbb{P}^n = \cup_j U_j$. The inverse for z_j is $z_j^{-1} : (t_1, \dots, t_n) \mapsto (t_1 : \dots : t_j : 1 : \dots : t_n)$. We verify the compatibility between z_0 and z_1 (the others are similar):

$$z_0 \circ z_1^{-1} : (t_1, \dots, t_n) \mapsto (t_1 : 1 : t_2 : \dots : t_n) \mapsto \left(\frac{1}{t_1}, \dots, \frac{t_n}{t_1} \right)$$

which is an holomorphic map on $z_1(U_0 \cap U_1)$.

We notice that \mathbb{P}^n is compact: let $S^{2n+1} = \{u \in \mathbb{C}^{n+1} \mid \|u\| = \sqrt{\sum |u_j|^2} = 1\}$ as usual. S^{2n+1} is compact (closed and bounded) and the map π restricted to S^{2n+1} is surjective. In fact, if $p = \pi(u) \in \mathbb{P}^n$, there exists a $t \in \mathbb{C}^\times$ such that $\|tu\| = 1$; then $tu \in S^{2n+1}$ and $\pi(tu) = \pi(u) = p$. At this point, it is sufficient to notice that π is continuous to state that \mathbb{P}^n is compact.

Given a complex manifold X , we can think about X without its complex structure, that is: if $\dim_{\mathbb{C}} X = n$, X defines a differentiable manifold X_0 with $\dim_{\mathbb{R}} X_0 = 2n$, where a complex chart (U, z) gives rise to a real chart (U, \tilde{z}) via the identification

$$z = (z_1, \dots, z_n) \leftrightarrow \tilde{z} = (x_1, x_2, \dots, x_n, y_1, \dots, y_n) \quad z_j = x_j + iy_j, x_j, y_j : U \rightarrow \mathbb{R}$$

One can easily check that if $(U_\alpha, z_\alpha), (U_\beta, z_\beta)$ are compatible, then $(U_\alpha, \tilde{z}_\alpha), (U_\beta, \tilde{z}_\beta)$ are compatible, too.

PROPOSITION 2.1. Consider a complex manifold X as a differentiable manifold X_0 with the coordinates inherited from the complex structure on X . Then X_0 is orientable.

Proof. Any transition map $F := z_\beta \circ z_\alpha^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ on X is holomorphic, and so is the inverse. As we've seen at the beginning of the chapter, $\det(J_{\mathbb{R}}F) = |\det(J_{\mathbb{C}}F)|^2 > 0$ (it is not zero since F has an inverse). It is easy to show that $J_{\mathbb{R}}F$ is nothing else than the Jacobian matrix of the transition map \tilde{F} on X_0 . Then, each transition map of X_0 has Jacobian with positive determinant, i.e X_0 is equipped with a positive atlas, and is positively oriented. \square

A simple consequence of this proposition is: not every differentiable manifold X_0 can be seen as the underlying differentiable manifold of a complex manifold X .

2.2 Holomorphic functions

Definition 2.3. Let $U \subseteq X$ be open, $f : U \rightarrow \mathbb{C}$ be a function. Then f is *holomorphic* on U if, taken (U_α, z_α) such that $U \cap U_\alpha \neq \emptyset$, the function

$$f \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U) \rightarrow \mathbb{C}$$

is holomorphic. This definition does not depend on the choice of the coordinate (U_α, z_α) . In addition, we define

$$\mathcal{O}_X(U) := \{f : U \rightarrow \mathbb{C} \mid f \text{ is holomorphic}\}$$

Remark.

Let $(U, z = (z_1, \dots, z_n))$ be a local complex chart on X . Let $a \in U$ with $z(a) = 0$, and let $f : U \rightarrow \mathbb{C}$ be holomorphic, Then

$$(f \circ z^{-1})(u) = \sum_{k_1, \dots, k_n=0}^{\infty} a_{k_1, \dots, k_n} u_1^{k_1} \cdots u_n^{k_n} \quad \text{where } x \in U, z(x) = u$$

This means

$$\begin{aligned} f(x) &= (f \circ z^{-1})(z(x)) = \sum_{k_1, \dots, k_n=0}^{+\infty} a_{k_1, \dots, k_n} z_1^{k_1}(x) \cdots z_n^{k_n}(x) \\ &\Rightarrow f = \sum_{k_1, \dots, k_n=0}^{+\infty} a_{k_1, \dots, k_n} z_1^{k_1} \cdots z_n^{k_n}. \end{aligned}$$

Definition 2.4. A map $\phi : X^n \rightarrow Y^m$ between complex manifolds is *holomorphic* if

$$w_\beta \circ \phi \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap \phi^{-1}(V_\beta)) \rightarrow \mathbb{C}^m$$

is holomorphic for all charts (U_α, z_α) of X , (V_β, w_β) of Y . It is sufficient to verify that the above map is holomorphic for any $(U_\alpha, z_\alpha), (V_\beta, w_\beta)$ in one atlas of X, Y respectively.

Example: The projection map $\pi : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$ is holomorphic. To check this, we use the atlases $\{(\mathbb{C}^{n+1} - \{0\}, \text{id}_{\mathbb{C}^{n+1} - \{0\}})\}$ for $\mathbb{C}^{n+1} - \{0\}$ and $\{(U_j, z_j)\}_{j=1, \dots, n}$ defined on \mathbb{P}^n as in the example of the previous section. We will check the definition only for $j = 0$.

$$(z_0 \circ \pi \circ \text{id}_{\mathbb{C}^{n+1} - \{0\}})(u_0, \dots, u_n) = z_0(u_0 : \dots : u_n) = \left(\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0} \right)$$

This map is holomorphic on $\pi^{-1}(U_0)$.

PROPOSITION 2.2. Let $\phi : X^n \rightarrow Y^m$ be a holomorphic map between complex manifolds. Let $(U, z), (V, w)$ be local complex charts of X, Y respectively such that $\phi(U) \subseteq V$. The map $F := w \circ \phi \circ z^{-1}$ is holomorphic; assume that $J_{\mathbb{C}}F(a)$ has constant rank $k \forall a \in U$ (i.e., with the usual terminology, ϕ has constant rank on U). Then for any $a \in U$ there exists a neighborhood W of a , local complex charts $(U', z'), (V', w')$ with $a \in U' \subseteq W$ such that $\phi(U') \subseteq V', z'(a) = 0, w'(\phi(a)) = 0$ and $F' := w' \circ \phi \circ (z')^{-1} : (u_1, \dots, u_n) \mapsto (u_1, \dots, u_k, 0, \dots, 0)$.

Proof. Similar to the proof for differentiable manifolds, using Prop. 1.5, too. \square

THEOREM 2.3. Let X be a (connected) compact complex manifold, let $f : X \rightarrow \mathbb{C}$ be a holomorphic function. Then f is constant.

Proof. $|f| : X \rightarrow \mathbb{R}$ is a continuous function, X is compact $\Rightarrow \{|f(x)| : x \in X\}$ is compact, hence bounded. Thus, there is an $x_0 \in X$ such that $|f(x_0)| = M$ is maximal. Let $a = f(x_0) \in \mathbb{C}$. Obviously, $f^{-1}(a)$ is closed in X (it is a pre-image of a point); if we are able to show that $f^{-1}(a)$ is open, too, then $f^{-1}(a) = X$, that implies $f(x) = a \forall x \in X$.

Let $x \in f^{-1}(a)$, (U, z) be a chart with $z(x) = 0$. Then $F : f \circ z^{-1} : z(U) \rightarrow \mathbb{C}$ is holomorphic on the open subset $z(U) \subseteq \mathbb{C}^n$; $F(0) = f(x) = a$ and $|F|$ has a maximum in $z = 0$. Let $\varepsilon > 0$ such that $B_\varepsilon := \{y \in \mathbb{C}^n : \|y\| < \varepsilon\} \subseteq z(U)$. For $y \in B_\varepsilon$, the function $g(t) := F(ty)$ is holomorphic on $\{t \in \mathbb{C} : \|ty\| < \varepsilon\}$ and $|g|$ takes its maximum in $t = 0$. By the "maximum principle" (theorem 1.4), g is constant $\Rightarrow a = g(0) = g(1) = F(y)$, that means $F(y) = a \forall y \in B_\varepsilon$. Hence $f \equiv a$ on $z^{-1}(B_\varepsilon)$, an open subset of X containing x . Hence $f^{-1}(a)$ is open. \square

This result is somewhat surprising and disappointing: the condition of compactness for X , which usually makes life a lot easier when dealing with a manifold, does not allow us to consider holomorphic functions on X , since all of them are constant. As we will see in the following chapter, some interesting results will be achieved looking at meromorphic functions on complex tori (which are compact complex manifolds).

2.3 The complex tangent space and cotangent space

Consider a complex manifold X and its underlying differentiable manifold X_0 . Let $a \in X_0$ and let \mathcal{C}_a^∞ be the \mathbb{R} -vectorspace of germs of smooth \mathbb{R} -valued functions in a . $T_a X_0 := \{v : \mathcal{C}_a^\infty \rightarrow \mathbb{R} \mid v \text{ is } \mathbb{R}\text{-linear and a derivation}\}$ is the tangent space to X_0 in a . If $a \in U$ where $(U, \tilde{z} = (x_1, x_2, \dots, x_n, y_1, \dots, y_n))$ is the real chart associated to a complex chart (U, z) , we have a \mathbb{R} -basis of $T_a X_0 (\cong \mathbb{R}^{2n})$:

$$\left. \frac{\partial}{\partial x_1} \right|_a, \dots, \left. \frac{\partial}{\partial x_n} \right|_a, \left. \frac{\partial}{\partial y_1} \right|_a, \dots, \left. \frac{\partial}{\partial y_n} \right|_a$$

where

$$\left. \frac{\partial}{\partial x_j} \right|_a (f) = \frac{\partial(f \circ \tilde{z}^{-1})}{\partial t_j}(\tilde{z}(a)) \quad \left. \frac{\partial}{\partial y_j} \right|_a (f) = \frac{\partial(f \circ \tilde{z}^{-1})}{\partial t_{n+j}}(\tilde{z}(a))$$

Let A_a^0 be the \mathbb{C} -vector space of germs of smooth \mathbb{C} -valued functions in a . $A_a^0 = \mathcal{C}_a^\infty \oplus i\mathcal{C}_a^\infty$, since $f = g + ih$, $g, h \in \mathcal{C}_a^\infty$ for each $f \in A_a^0$. Any $v \in T_a X_0$ defines by \mathbb{C} -linear extension a derivation on A_a^0 , that is, $g + ih \mapsto v(g) + iv(h)$. Let $(T_a X_0)_{\mathbb{C}} = T_a X_0 \otimes_{\mathbb{R}} \mathbb{C} = T_a X_0 \oplus iT_a X_0$ be the complexification of $T_a X_0$; then $\dim_{\mathbb{C}}(T_a X_0)_{\mathbb{C}} = \dim_{\mathbb{R}}(T_a X_0) = 2n$. We point out that elements of $(T_a X_0)_{\mathbb{C}}$ are $v + iw$, $v, w \in T_a X_0$ and clearly $(v + iw)(f) = v(f) + iw(f) = v(g + ih) + iw(g + ih) = v(g) - w(h) + i(v(h) + w(g))$.

Notice that the dimension of $(T_a X_0)_{\mathbb{C}}$, that seems to be the correct tangent space to X , is twice the dimension of the complex manifold. It is clear that we have to cut out some part of it, in order to have something similar to the usual notion of tangent space for differential manifolds.

Definition 2.5. A *complex structure* on a \mathbb{R} -vector space V is a \mathbb{R} -linear endomorphism J such that $J^2 = -\text{Id}$.

The multiplication by i from \mathbb{C}^n to itself induces a complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\cdot i} & \mathbb{C}^n \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{R}^{2n} & \xrightarrow{J} & \mathbb{R}^{2n} \end{array} \quad \begin{array}{ccc} (\dots, x_j + iy_j, \dots) & \longmapsto & (\dots, ix_j - y_j, \dots) \\ \downarrow \cong & & \downarrow \cong \\ (\dots, x_j, \dots, y_j, \dots) & & (\dots, -y_j, \dots, x_j, \dots) \end{array}$$

So

$$J \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \Rightarrow J = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix} \Rightarrow J^2 = \begin{pmatrix} -\text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}$$

For example, if $n = 1$ we get $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; its eigenvalues are $\pm i$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ \mp i \end{pmatrix}$. Furthermore, one easily verifies that in the general case the eigenvalues are still $\pm i$, and the corresponding eigenspaces are, respectively:

$$E_i = \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ -i \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -i \end{pmatrix} \right\rangle, \quad E_{-i} = \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ i \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ i \end{pmatrix} \right\rangle$$

($\dim E_{\pm i} = n$; the number of zeroes between 1 and $\pm i$ is $n - 1$ in each vector). Moreover $\mathbb{C}^{2n} = (\mathbb{R}^{2n})_{\mathbb{C}} = E_i \oplus E_{-i}$.

Since we can identify $T_a X_0$ with \mathbb{R}^{2n} using the basis $\left. \frac{\partial}{\partial x_1} \right|_a, \dots, \left. \frac{\partial}{\partial y_n} \right|_a$, we immediately obtain a map J_a in $T_a X_0$ that is the equivalent of J in \mathbb{R}^{2n} : to be precise,

$$J_a : \begin{cases} \left. \frac{\partial}{\partial x_j} \right|_a \mapsto \left. \frac{\partial}{\partial y_j} \right|_a \\ \left. \frac{\partial}{\partial y_j} \right|_a \mapsto -\left. \frac{\partial}{\partial x_j} \right|_a \end{cases}$$

By \mathbb{C} -linear extension of J_a on $(T_a X_0)_{\mathbb{C}}$, we get the following eigenspaces:

$$T_a^{1,0} X := \{w \in (T_a X_0)_{\mathbb{C}} \mid J_a w = iw\} \quad \text{The holomorphic tangent space}$$

$$T_a^{0,1} X := \{w \in (T_a X_0)_{\mathbb{C}} \mid J_a w = -iw\} \quad \text{The anti-holomorphic tangent space}$$

Note that $(T_a X_0)_{\mathbb{C}} = T_a^{1,0} X \oplus T_a^{0,1} X$ and $\dim_{\mathbb{C}}(T_a^{1,0} X) = \dim_{\mathbb{C}}(T_a^{0,1} X) = n$. Analogously to the situation in \mathbb{R}^{2n} , the vectors $\left\{s_j = \left. \frac{\partial}{\partial x_j} \right|_a - i \left. \frac{\partial}{\partial y_j} \right|_a\right\}$ and $\left\{\bar{s}_j = \left. \frac{\partial}{\partial x_j} \right|_a + i \left. \frac{\partial}{\partial y_j} \right|_a\right\}$ are basis of $T_a^{1,0} X$ and $T_a^{0,1} X$, respectively. Then if we define

$$\begin{aligned} \left. \frac{\partial}{\partial z_j} \right|_a &:= \frac{1}{2} \left(\left. \frac{\partial}{\partial x_j} \right|_a - i \left. \frac{\partial}{\partial y_j} \right|_a \right), & j \in \{1, \dots, n\} \\ \left. \frac{\partial}{\partial \bar{z}_j} \right|_a &:= \frac{1}{2} \left(\left. \frac{\partial}{\partial x_j} \right|_a + i \left. \frac{\partial}{\partial y_j} \right|_a \right), & j \in \{1, \dots, n\} \end{aligned}$$

these vectors are again basis of $T_a^{1,0} X$ and $T_a^{0,1} X$, respectively (the reason why we prefer these basis to $\{s_j\}$ and $\{\bar{s}_j\}$ is only due to having analogies with the usual relation in \mathbb{C}^n).

If $f = g + ih \in A_a^0$, then

$$2 \left. \frac{\partial}{\partial \bar{z}_j} \right|_a f = \left(\left. \frac{\partial}{\partial x_j} \right|_a + i \left. \frac{\partial}{\partial y_j} \right|_a \right) (g + ih) = \left(\frac{\partial g}{\partial x_j}(a) - \frac{\partial h}{\partial y_j}(a) \right) + i \left(\frac{\partial h}{\partial x_j}(a) + \frac{\partial g}{\partial y_j}(a) \right)$$

This means (dropping the a dependence, to lighten notation)

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \Leftrightarrow \frac{\partial g}{\partial x_j} = \frac{\partial h}{\partial y_j}, \frac{\partial h}{\partial x_j} = -\frac{\partial g}{\partial y_j}$$

that are the Cauchy-Riemann equations. Then, if f is the germ of a holomorphic function, $\frac{\partial f}{\partial \bar{z}_j} = 0$ for each j : thus $w(f) = 0$ for all $w \in T_a^{0,1} X$. So, to study holomorphic functions we only need $T_a^{1,0} X$, that from now on will be considered as the "correct" tangent space to X in a .

Now we can define $T_a^{1,0} X, T_a^{0,1} X$ intrinsically:

Definition 2.6.

$$T_a^{0,1} X := \{w \in (T_a X_0)_{\mathbb{C}} \mid w(f) = 0 \quad \forall f \in A_a^0, f \text{ holomorphic}\}$$

$$T_a^{1,0} X := \{\bar{w} = w - iw_2 \mid w = w_1 + iw_2 \in T_a^{0,1} X_0\}$$

We say that $T_a^{1,0} X$ is the *tangent space* to X in a .

Remark.

One checks that

$$\frac{\partial}{\partial z_j}(z_k) = \frac{\partial}{\partial t_j}(x_k + iy_k) = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

so $\forall w \in T_a^{1,0}X$ there is a holomorphic function f such that $w(f) \neq 0$.

Consider $T_a^*X_0 := \text{Hom}_{\mathbb{R}}(T_aX_0, \mathbb{R})$. For $f \in \mathcal{C}_a^\infty$ we have its differential $df = (df)_a \in T_a^*X_0$, defined by the relation $(df)(v) := v(f) \in \mathbb{R}$. We know that $T_a^*X_0 = \langle dx_1, \dots, dy_n \rangle$, where x_1, \dots, y_n are the coordinates of the real chart derived from a complex chart (U, z) . Define $(T_a^*X_0)_{\mathbb{C}} = T_a^*X_0 \otimes_{\mathbb{R}} \mathbb{C} = T_a^*x_0 \oplus iT_a^*x_0$: this space is equal to $(T_a^*X)^{1,0} \oplus (T_a^*X)^{0,1} = \langle \dots, dz_j \dots \rangle \oplus \langle \dots, d\bar{z}_j \dots \rangle$ where $dz_j = dx_j + idy_j$, $d\bar{z}_j = dx_j - idy_j$. Moreover

$$dz_j \left(\frac{\partial}{\partial z_k} \right) = \delta_{jk}, \quad dz_j \left(\frac{\partial}{\partial \bar{z}_k} \right) = 0 = d\bar{z}_j \left(\frac{\partial}{\partial z_k} \right), \quad d\bar{z}_j \left(\frac{\partial}{\partial \bar{z}_k} \right) = \delta_{jk}$$

As an example, we check the first relation:

$$dz_k \left(\frac{\partial}{\partial z_j} \right) = \frac{\partial}{\partial z_j} z_k = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) (x_k + iy_k) = \frac{1}{2} (\delta_{jk} + 0 + 0 + (-i)i\delta_{jk}) = \delta_{jk}$$

This means, in particular, $(T_a^*X)^{1,0} = (T_a^{1,0}X)^*$.

Now consider A_a^0 as defined previously. We have the map

$$d = d_a : A_a^0 \rightarrow (T_a^*X_0)_{\mathbb{C}}$$

$$f \mapsto df := (df)_a = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)(dx_j)_a + \frac{\partial f}{\partial y_j}(a)(dy_j)_a$$

or, in the new basis,

$$(df)_a = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(a)(dz_j)_a + \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j}(a)(d\bar{z}_j)_a = \partial f + \bar{\partial} f$$

So we have $d = \partial + \bar{\partial}$, and $\bar{\partial}f = 0$ if f is holomorphic. Since we want to work with holomorphic functions on X and we want $(df)_a$ to be an element of our cotangent space, the following definition arises naturally:

Definition 2.7. The *cotangent space* of X in a is $(T_a^*X)^{1,0} = (T_a^{1,0}X)^*$.

2.4 Differential forms

Let $V \subset X$ be an open subset. Define $\mathcal{E}^r(V) := \{\text{smooth, real valued } r\text{-forms on } V\}$ for $0 \leq r \leq 2n$. Take a local coordinate U . On $U \cap V$, for $\omega \in \mathcal{E}^r(V)$, we can write $\omega = \sum_{\#I=r} f_I dx_I$ where $I = \{i_1, \dots, i_r\}$, $i_1 < \dots < i_r$, $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_r}$, $f_I : U \cap V \rightarrow \mathbb{R}$ smooth (note that now $dx_{j+n} := dy_j$).

Let $A^r(V) := \mathcal{E}^r(V)_{\mathbb{C}} = \{\text{smooth, complex valued } r\text{-forms on } V\} = \{\eta = \eta_1 + i\eta_2 \mid \eta_1, \eta_2 \in \mathcal{E}^r(V)\}$. Note that $\omega(a) \in \bigwedge^r T_a^* X_0$ if $\omega \in \mathcal{E}^r(V)$, while $\eta(a) \in \bigwedge^r (T_a^* X_0)_{\mathbb{C}}$ if $\eta \in A^r(V)$. Moreover

$$\bigwedge^r (T_a^* X_0)_{\mathbb{C}} = \bigoplus_{\substack{p,q \geq 0 \\ p+q=r}} \left(\bigwedge^r T_a^* X \right)^{p,q}$$

where $(\bigwedge^r T_a^* X)^{p,q} := \langle \dots, dz_I \wedge d\bar{z}_J, \dots \rangle$ with I, J multi-indices, $\#I = p$, $\#J = q$, so that $I := \{i_1, \dots, i_p\}, i_1 < \dots < i_p, J := \{j_1, \dots, j_q\}, j_1 < \dots < j_q, p + q = r$ and $dz_I \wedge d\bar{z}_J = dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$ (thus $\dim_{\mathbb{C}} (\bigwedge^r T_a^* X)^{p,q} = \binom{n}{p} \binom{n}{q}$). This induces a decomposition

$$A^r(V) = \bigoplus_{\substack{p,q \geq 0 \\ p+q=r}} A^{p,q}(V)$$

where $A^{p,q}(V) = \{\eta \in A^r(V) \mid \eta|_{U \cap V} = \sum_{\substack{\#J=q \\ \#I=p}} f_{I,J} dz_I \wedge d\bar{z}_J\}$.

Definition 2.8. $\Omega^r(V) := \{\omega \in A^{r,0}(V) \mid \omega = \sum_{\#I=r} f_I dz_I, f_I \text{ holomorphic}\}$.

For real r -forms, we have the exterior derivative $d : \mathcal{E}^r(V) \rightarrow \mathcal{E}^{r+1}(V)$. If we extend it \mathbb{C} -linearly, we get $d : A^r(V) \rightarrow A^{r+1}(V)$ (simply: $d(\eta_1 + i\eta_2) = d\eta_1 + id\eta_2$).

Recall that $df = \partial f + \bar{\partial} f$ for $f \in A^0(X)$, and

$$\partial f = \sum_j \frac{\partial f}{\partial z_j} dz_j \in A^{1,0}(X) \quad \bar{\partial} f = \sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \in A^{0,1}(X)$$

Similarly, if $\eta \in A^{p,q}(X)$, we have

$$\begin{aligned} d\eta &= d\left(\sum_{\substack{\#J=q \\ \#I=p}} f_{I,J} dz_I \wedge d\bar{z}_J\right) = \sum_{\substack{\#J=q \\ \#I=p}} df_{I,J} \wedge dz_I \wedge d\bar{z}_J = \\ &= \sum_{\substack{\#J=q \\ \#I=p}} \partial f_{I,J} \wedge dz_I \wedge d\bar{z}_J + \sum_{\substack{\#J=q \\ \#I=p}} \bar{\partial} f_{I,J} \wedge dz_I \wedge d\bar{z}_J \in A^{p+1,q}(X) \oplus A^{p,q+1}(X) \end{aligned}$$

Hence $d = \partial + \bar{\partial} : A^{p,q} \rightarrow A^{p+1,q} \oplus A^{p,q+1}$. Now, since $d^2 = 0$ on \mathcal{E}^r and then on A^r ,

$$0 = (\partial + \bar{\partial})^2 = \underbrace{\partial^2}_{\in A^{p+2,q}} + \underbrace{\partial \circ \bar{\partial} + \bar{\partial} \circ \partial}_{\in A^{p+1,q+1}} + \underbrace{\bar{\partial}^2}_{\in A^{p,q+2}} \Rightarrow \begin{cases} \partial^2 = \bar{\partial}^2 = 0 \\ \bar{\partial} \circ \partial = -\partial \circ \bar{\partial} \end{cases}$$

Consider the projection $\pi^{p,q} : A^r = \bigoplus_{\substack{s,t \geq 0 \\ s+t=r}} A^{s,t} \rightarrow A^{p,q}$. Then $\partial = \pi^{p+1,q} \circ d : A^{p,q} \rightarrow A^{p+1,q}$

and $\bar{\partial} = \pi^{p,q+1} \circ d$.

PROPOSITION 2.4. Let $f : X \rightarrow Y$ be a holomorphic map between complex manifolds. Then

1. $f^* \pi_Y^{p,q} = \pi_X^{p,q} \circ f^*$ (that implies $f^* A_Y^{p,q} \subseteq A_X^{p,q}$);
2. $f^* \partial_Y = \partial_X f^*$ (notice the similarities with $f^* d = d f^*$ for the real case).

Proof. 1. Let $g : Y \rightarrow \mathbb{C}$ be a smooth function. Recall that $f^*(dg) = d(g \circ f)$. Let $(V, w = (w_1, \dots, w_m))$ be a local complex chart on Y . Each w_j is holomorphic on V , hence also $w_j \circ f$ is holomorphic: this means $\bar{\partial}(w_j \circ f) = 0$. Thus

$$f^*(dw_j) = d(w_j \circ f) = \partial(w_j \circ f) + \underbrace{\bar{\partial}(w_j \circ f)}_{=0} \in A^{1,0}(X)$$

so $f^* A^{1,0}(Y) \subseteq A^{1,0}(X)$; similarly, $f^* A^{0,1}(Y) \subseteq A^{0,1}(X)$ (just notice that $\bar{w}_j \circ f$ is antiholomorphic, and then $\partial(\bar{w}_j \circ f) = 0$).

Now let $\eta = \sum_{I,J} g_{I,J} dw_I \wedge d\bar{w}_J \in A^r(Y)$. We have

$$f^*(\pi_Y^{p,q} \eta) = f^* \left(\sum_{\substack{\#J=q \\ \#I=p}} g_{I,J} dw_I \wedge d\bar{w}_J \right) = \sum_{\substack{\#J=q \\ \#I=p}} (g_{I,J} \circ f) f^*(dw_{i_1}) \wedge \dots \wedge f^*(d\bar{w}_{j_q})$$

and

$$\begin{aligned} \pi_X^{p,q}(f^* \eta) &= \pi_X^{p,q} \left(\sum_{I,J} (g_{I,J} \circ f) f^*(dw_I) \wedge f^*(d\bar{w}_J) \right) = \\ &= \sum_{\substack{\#J=q \\ \#I=p}} (g_{I,J} \circ f) f^*(dw_{i_1}) \wedge \dots \wedge f^*(d\bar{w}_{j_q}) \end{aligned}$$

This shows the equality we wanted to prove.

2. Let $\omega \in A^{p,q}(Y)$. Then

$$\partial_X(f^* \omega) = (\pi_X^{p+1,q} \circ d)(f^* \omega) = \pi_X^{p+1,q}((d \circ f^*)(\omega)) = \pi_X^{p+1,q}((f^* \circ d)(\omega))$$

By the above result

$$(\pi_X^{p+1,q} \circ f^*)(d\omega) = (f^* \circ \pi_Y^{p+1,q})(d\omega) = f^*(\partial_Y \omega)$$

showing what needed. □

2.5 Complex submanifolds

Definition 2.9. A *complex submanifold* of a complex manifold X is a subset $Y \subseteq X$ such that for each $a \in Y$ there exists a local complex chart $(U, z = (z_1, \dots, z_n))$ of X , called "preferred chart", with $z(a) = 0$ and $z(U \cap Y) = \{u \in z(U) \subseteq \mathbb{C}^n \mid u_{k+1} = \dots = u_n = 0\}$. A complex submanifold is a complex manifold itself, of dimension k : if (U, z) is a preferred chart, one obtains a complex chart of Y by $(U \cap Y, z|_{U \cap Y})$ where now $z|_{U \cap Y} : U \cap Y \rightarrow \mathbb{C}^k$. The compatibility for those charts follows from the compatibility of the charts on X .

The following theorems will give us two ways to obtain complex submanifolds. In the first case, we will show that the pre-image of a point via a "sufficiently regular" map is a submanifold. The second theorem shows that under certain (strong) condition for $\phi : X \rightarrow Y$, the image $\phi(X)$ is an *embedded* submanifold of Y - intuitively, it means that $\phi(X)$ is contained in Y in some "non-singular" way.

THEOREM 2.5. Let $\phi : X^n \rightarrow Y^m$ be a holomorphic map between complex manifolds, with $n > m$. Let $b \in \phi(X) \subseteq Y$ such that the rank of ϕ is maximal ($\text{rk}(\phi) = m$) $\forall a \in \phi^{-1}(b)$. Then $\phi^{-1}(b)$ is a complex submanifold of X of dimension $n - m$.

Proof. Since ϕ has maximal rank on $\phi^{-1}(b)$, it also has maximal rank on a neighborhood of $\phi^{-1}(b)$ in X ("det $\neq 0$ " is an open condition). By Proposition 2.2, there are local charts $(U, z), (V, w)$ with $a \in U$ such that $\phi(U) \subseteq V, z(a) = 0, w(\phi(a)) = w(b) = 0$ and

$$w \circ \phi \circ z^{-1} : (u_1, \dots, u_n) \mapsto (u_1, \dots, u_m)$$

Hence $z(U \cap \phi^{-1}(b)) = \{u \in z(U) \mid z^{-1}(u) \in \phi^{-1}(b) \Leftrightarrow \phi(z^{-1}(u)) = b \Leftrightarrow w(\phi(z^{-1}(u))) = w(b) = 0\}$. Since $w(b) = (u_1, \dots, u_m)$, we obtain $z(U \cap \phi^{-1}(b)) = \{u = (u_1, \dots, u_n) \in z(U) \mid u_1 = \dots = u_m = 0\}$. Then, if $u \in z(U \cap \phi^{-1}(b))$ one has $u = (0, \dots, 0, u_{m+1}, \dots, u_n)$. After a permutation of the coordinates in \mathbb{C}^n , this clearly shows that z is a preferred chart for $\phi^{-1}(b)$ in a . \square

THEOREM 2.6. Let $\phi : Y \rightarrow X$ be an injective holomorphic map between complex manifolds with $\dim Y = m \leq n = \dim X$, such that ϕ has maximal rank m on all Y . If Y is compact, then $\phi(Y)$ is a submanifold of X and $\phi : Y \rightarrow \phi(Y)$ is a biholomorphic map (we say that $\phi(Y)$ is *isomorphic* to Y and that ϕ is an *embedding*).

Proof. First, we show that the continuous, bijective map $\phi : Y \rightarrow \phi(Y)$ is a homeomorphism. It suffices to show that ϕ is open.

Let $W \subset Y$ be open, then $Y \setminus W$ is closed in Y . Since Y is compact, $Y \setminus W$ is compact, too. ϕ is continuous, so $\phi(Y \setminus W)$ is compact in $\phi(Y) \subseteq X$ and since X is Hausdorff $\phi(Y \setminus W)$ is closed in $\phi(Y)$. Therefore its complement in $\phi(Y)$, which is $\phi(W)$, is open in $\phi(Y)$. This shows that ϕ is open.

Given $a \in Y$, there are local charts $(U, z), (V, w)$ of X, Y respectively with $a \in V, \phi(V) \subseteq U, z \circ \phi \circ w^{-1} : (u_1, \dots, u_m) \mapsto (u_1, \dots, u_m, 0, \dots, 0)$ (because $\text{rk}(\phi) = m$). As ϕ is a homeomorphism and V is open in Y , $\phi(V)$ is open in $\phi(Y)$: then there is an open set U' of X such that $\phi(V) = \phi(Y) \cap U'$. Since $\phi(V) \subseteq U, \phi(V) = \phi(Y) \cap (U' \cap U)$. Hence

$$z(\phi(Y) \cap (U \cap U')) = z(\phi(V)) = \{u \in z(U \cap U') \mid u = (u_1, \dots, u_m, 0, \dots, 0)\}$$

This shows that $\phi(Y)$ has a preferred chart, that is, $\phi(Y)$ is a submanifold. \square

2.6 Submanifolds of \mathbb{P}^n

In this section we will look closely to some examples of submanifolds; each of them will be contained in some \mathbb{P}^n . The main tools to find such manifolds are the two theorems seen in the previous sections. Moreover, we will see another way to find submanifolds in \mathbb{P}^n , that is, as zero locus of some homogeneous polynomials.

FERMAT HYPERSURFACE

Let $F : \mathbb{C}^n \rightarrow \mathbb{C}, (z_1, \dots, z_n) \mapsto 1 + z_1^k + \dots + z_n^k$ with $k \neq 0$. The rank of F is the rank of the $1 \times n$ matrix $J_{\mathbb{C}}F = (kz_1^{k-1}, \dots, kz_n^{k-1})$, hence $J_{\mathbb{C}}F$ has maximal rank (rank 1) in each point except for $z = 0$. Since $F(0) = 1$, $F^{-1}(0)$ does not contain 0. By theorem 2.5, $Z = \{z \in \mathbb{C} \mid 1 + z_1^k + \dots + z_n^k = 0\}$ is a complex manifold of dimension $n - 1$.

Now consider \mathbb{P}^n and the usual projection $\pi : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$. Let $Z = \{\pi(z) = (z_0 : \dots : z_n) \in \mathbb{P}^n \mid z_0^k + \dots + z_n^k = 0\}$. Z is a complex manifold, because Z has an open cover $Z \cap U_j = \{(z_0 : \dots : z_n) \in Z \mid z_j \neq 0\}$ and each $Z_j \cap U_j$ is basically the same as Z in the previous example (after a composition with charts). Then Z is a complex manifold of \mathbb{P}^n , called *Fermat hypersurface*. Z is compact (it is closed in the compact manifold \mathbb{P}^n).

THE VERONESE MAP

The *Veronese map* is a function $\phi_d : \mathbb{P}^n \rightarrow \mathbb{P}^m$, where $m = \binom{n+d}{d} - 1$: the map sends $(x_0 : \dots : x_n)$ in points of \mathbb{P}^m which coordinates are all possible monomials with variables x_0, \dots, x_n , of degree d (thus the expression for m).

The following examples will all consider a really special case, that is, $n = 1$. In this case, $\phi_d(\mathbb{P}^1)$ is called *rational normal curve*.

- Consider the Veronese map with $n = 1, d = 3$:

$$\begin{aligned} \phi_3 &=: \varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2 \\ (s : t) &\mapsto (s^3 : s^2t : st^2 : t^3) \end{aligned}$$

This map makes sense, since it is easily verified that $\varphi(s : t) \neq (0 : 0 : 0 : 0)$ (that point is not in \mathbb{P}^3 ! We say that φ has no *base locus*). Moreover, φ is well defined:

$$(s : t) = (\lambda s : \lambda t) \quad \forall \lambda \in \mathbb{C}^\times \Rightarrow \varphi(\lambda s : \lambda t) = \lambda^3 \varphi(s : t) = \varphi(s : t)$$

We show now that φ is holomorphic. Recall that $\mathbb{P}^1 = U_0 \cup U_1$ and $\mathbb{P}^3 = \bigcup_{j=0}^3 \tilde{U}_j$, according to the usual atlas on \mathbb{P}^n , and notice that $\varphi(U_0) \subseteq \tilde{U}_0, \varphi(U_1) \subseteq \tilde{U}_3$. If $F := \tilde{z}_0 \circ \varphi \circ z_0^{-1} : z_0(U_0) \subseteq \mathbb{C} \rightarrow \mathbb{C}^3$,

$$F(u) = \tilde{z}_0(\varphi(1 : u)) = \tilde{z}_0(1 : u : u^2 : u^3) = (u, u^2, u^3) \quad \text{for } u \in U_0$$

that is a holomorphic map. In a similar fashion, $G := \tilde{z}_3 \circ \varphi \circ z_1^{-1}$ is holomorphic; then φ is holomorphic everywhere, since $\mathbb{P}^1 = U_0 \cup U_1$. Note also that φ is injective, and both $JF = \left(\frac{\partial F_j}{\partial u} \right) = (1, 2u, 3u^2)$ and JG have rank 1. Moreover, \mathbb{P}^1 is compact, so by theorem 2.6 we get that $\varphi(\mathbb{P}^1)$ is a submanifold of \mathbb{P}^3 and $\mathbb{P}^1 \cong \varphi(\mathbb{P}^1)$.

- Now consider the Veronese map with $n = 1, d = 2$:

$$\begin{aligned}\phi_2 &=: \varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^3 \\ (s : t) &\mapsto (s^2 : st : t^2)\end{aligned}$$

Just doing the same steps as the previous example, it is easy to show that this map is well defined and holomorphic. On $U_0 \subseteq \mathbb{P}^1$, $\varphi(1 : t) = (1 : t : t^2)$, so φ is injective on U_0 ; one similarly checks that it is injective also on U_1 , and then ϕ is injective on the whole \mathbb{P}^1 . Moreover φ has rank 1, thus $\mathbb{P}^1 \cong \varphi(\mathbb{P}^1)$.

THE SEGRE MAP

The general form of the *Segre map* is

$$\begin{aligned}\sigma : \mathbb{P}^n \times \mathbb{P}^m &\rightarrow \mathbb{P}^{(n+1)(m+1)-1} \\ ((x_0 : \dots : x_n), (y_0 : \dots : y_m)) &\mapsto (x_0 y_0 : x_0 y_1 : \dots : x_i y_{j-1} : x_i y_j : \dots : x_n y_m)\end{aligned}$$

We will consider the particular case $n = m = 1$:

$$\begin{aligned}\phi : \mathbb{P}^1 \times \mathbb{P}^1 &\rightarrow \mathbb{P}^3 \\ ((s : t), (u : v)) &\mapsto (su : sv : tu : tv)\end{aligned}$$

ϕ has no base locus: suppose it has, then $(su : sv : tu : tv) = (0 : 0 : 0 : 0)$ means $su = sv = 0$, but $(u : v) \neq (0 : 0) \Rightarrow s = 0$. Similarly we obtain $t = 0$, but since $(s : t) \neq (0 : 0)$ we have a contradiction.

ϕ is well defined (easy check) and it is holomorphic: as an example, consider the usual charts $U_0 \times U'_0$ of $\mathbb{P}^1 \times \mathbb{P}^1$ (where $s = u = 1$). One has $\phi((1 : t), (1 : v)) = (1 : v : t : tv)$, so $\phi(U_0 \times U'_0) \subseteq \tilde{U}_0$ with \tilde{U}_0 chart of \mathbb{P}^3 . Then

$$F := \tilde{z}_0 \circ \phi \circ \{z_0^{-1}, (z'_0)^{-1}\} : (a, b) \mapsto (a, b, ab)$$

is clearly holomorphic, so ϕ is holomorphic on $U_0 \times U'_0$. In a similar way one can show ϕ is holomorphic on the whole $\mathbb{P}^1 \times \mathbb{P}^1$.

ϕ is injective and has maximal rank. We work again on $U_0 \times U'_0$. The expression for ϕ in this set (shown above) clearly implies injectivity, while $J_{\mathbb{C}}F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ t & v \end{pmatrix} \Rightarrow \text{rk} F = \text{rk}(\phi|_{U_0 \times U'_0}) = 2$, so ϕ has maximal rank in this set. Again, similar calculations show that ϕ is injective and has rank 2 on every $U_i \times U'_j$, $i, j = 0, 1$.

Since $\mathbb{P}^1 \times \mathbb{P}^1$ is compact, $\phi(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Remark.

We want to underline that $\mathbb{P}^1 \times \mathbb{P}^1 \not\cong \mathbb{P}^2$ (even if they have the same dimension 2). They are not even diffeomorphic: suppose they are. Then, there exists a diffeomorphism $\alpha : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^2$ that induces an isomorphism $\alpha^* : H_{DR}^2(\mathbb{P}^1 \times \mathbb{P}^1) \rightarrow H_{DR}^2(\mathbb{P}^2)$ on the

de Rham cohomology groups. It is known that $H_{DR}^{2k}(\mathbb{P}^n(\mathbb{C})) = \mathbb{R}$ if $k = 0, \dots, n$; by the Künneth formula we obtain

$$\begin{aligned} H_{DR}^2(\mathbb{P}^1 \times \mathbb{P}^1) &\cong H_{DR}^0(\mathbb{P}^1) \otimes H_{DR}^2(\mathbb{P}^1) \oplus H_{DR}^1(\mathbb{P}^1) \otimes H_{DR}^1(\mathbb{P}^1) \oplus H_{DR}^2(\mathbb{P}^1) \otimes H_{DR}^0(\mathbb{P}^1) \cong \\ &\cong (\mathbb{R} \otimes \mathbb{R}) \oplus 0 \oplus (\mathbb{R} \otimes \mathbb{R}) \cong \mathbb{R}^2 \not\cong \mathbb{R} \cong H_{DR}^2(\mathbb{P}^2) \end{aligned}$$

which leads us to a contradiction.

2.6.1 Complete intersections

Consider a homogeneous polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ of degree d , that is, $f(tx) = f(tx_0, \dots, tx_n) = t^d f(x_0, \dots, x_n) \forall t \in \mathbb{C}$. Deriving with respect to t ,

$$\sum_{i=0}^n \frac{\partial f}{\partial x_i}(tx) x_i = dt^{d-1} f(x_0, \dots, x_n) \xrightarrow{t=1} \sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = d \cdot f$$

The last equation is called *Euler's Relation*.

PROPOSITION 2.7. Let $f_1, \dots, f_m \in \mathbb{C}[x_0, \dots, x_n]$ be homogeneous polynomials of degree $d_j, j \in \{1, \dots, m\}$. Let $Y = \{x \in \mathbb{P}^n \mid f_1(x) = \dots = f_m(x) = 0\}$ (a *projective algebraic set*) with $n > m$. Then, if $\text{rk}\left(\frac{\partial f_j}{\partial x_k}(x)\right) = m \forall x \in Y, k \in \{0, \dots, n\}$ and $j \in \{1, \dots, m\}$, Y is a complex submanifold of \mathbb{P}^n , which is compact, of dimension $n - m$. Y is called a *complete intersection* of degree (d_1, \dots, d_m) in \mathbb{P}^n .

Proof. It suffices to know that $Y \cap U_i$ is a complex submanifold of each $U_i \subseteq \mathbb{P}^n, \forall i$. As an example, we do the case $i = 0$ (the others are similar).

Let $F = (f_1, \dots, f_m) : U_0 \rightarrow \mathbb{C}^m$. Then $F^{-1}(0) = Y \cap U_0$. If we show that $\text{rk}(F) = m \forall x \in F^{-1}(0)$, then we are done: this is equivalent to show that $G := F \circ z_0^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}^m$ has rank m ((U_0, z_0) is the chart in which we are working). By Euler's Relation, for any given j we have $\sum_{k=0}^n x_k \frac{\partial f_j}{\partial x_k}(x) = d_j \cdot f_j(x)$; then if $x \in Y \cap U_0$ we have

$$1 \cdot \frac{\partial f_j}{\partial x_0}(x) + x_1 \frac{\partial f_j}{\partial x_1}(x) + \dots + x_n \frac{\partial f_j}{\partial x_n}(x) = d_j \cdot 0 = 0$$

or, equivalently, $\begin{pmatrix} \frac{\partial f_1}{\partial x_0} \\ \vdots \\ \frac{\partial f_m}{\partial x_0} \end{pmatrix}$ is a linear combination of $\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}, \dots, \begin{pmatrix} \frac{\partial f_1}{\partial x_m} \\ \vdots \\ \frac{\partial f_m}{\partial x_m} \end{pmatrix}$. This means

that $\text{rk}\left(\frac{\partial f_j}{\partial x_k}(x)\right)$ with $k \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ is the same as $\text{rk}\left(\frac{\partial f_j}{\partial x_k}(x)\right)$ with $k \in \{0, \dots, n\}$ and $j \in \{1, \dots, m\}$.

We have

$$\frac{\partial G_j}{\partial z_k} = \frac{\partial(f_j \circ z_0^{-1})}{\partial z_k} = \frac{\partial f_j}{\partial x_k} \quad k = 1, \dots, n$$

and then

$$\text{rk}\left(\frac{\partial G_j}{\partial z_k}(x)\right)_{\substack{k=1, \dots, n \\ j=1, \dots, m}} = \text{rk}\left(\frac{\partial f_j}{\partial x_k}(x)\right)_{\substack{k=1, \dots, n \\ j=1, \dots, m}} = \text{rk}\left(\frac{\partial f_j}{\partial x_k}(x)\right)_{\substack{k=0, \dots, n \\ j=1, \dots, m}} = m.$$

where the last equality was our assumption. \square

Example: Let $P : x_0x_2 - x_1^2$. P is a homogeneous polynomial of degree 2. Consider $Y := \{x \in \mathbb{P}^2 \mid P(x) = 0\}$. We want to show that Y is a complex submanifold of \mathbb{P}^2 of dimension 1 as a complete intersection of degree 2 in \mathbb{P}^2 : that is, our goal is to show $\text{rk}\left(\frac{\partial P}{\partial x_k}\right) = 1$. We have

$$v = \left(\frac{\partial P}{\partial x_0}, \frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2} \right) = (x_2 - x_1^2, x_0x_2 - 2x_1, x_0 - x_1^2)$$

Take $x \in Y$. Suppose $x_1 = 0$: then $v = (x_2, x_0x_2, x_0)$ but since $x_0x_2 = x_1^2 = 0$, $v = (x_2, 0, x_0)$. Clearly x_2, x_0 can't be zero at the same time, so $v \neq (0, 0, 0)$. Now suppose $x_1 \neq 0$: then we can write $x = (y_0 : 1 : y_2)$. Then $y_0y_2 = 1 \Rightarrow v = (y_2 - 1, -1, y_0 - 1) \neq (0, 0, 0)$. So $\text{rk}(v) = 1$.

To be honest, we have already seen the submanifold Y , but it was in another form. We want to prove that $Y = \phi_2(\mathbb{P}^1)$, where ϕ_2 is the Veronese map with $n = 1, d = 2$ (we have already talked about this map).

$\phi_2(\mathbb{P}^1) \subseteq Y$: Just substitute the coordinates of a generic point $x = (s^2 : st : t^2) \in \phi_2(\mathbb{P}^1)$ inside P to get $P(x) = 0$.

$\phi_2(\mathbb{P}^1) \supseteq Y$: Consider $x = (x_0 : x_1 : x_2) \in Y$. Suppose $x_0 \neq 0$. Then we can assume $(x_0 : x_1 : x_2) = (1 : y_1 : y_2)$, and by the definition of Y we have $y_2 = y_1^2$. Thus $x = (1 : y_1 : y_1^2) = \phi_2(1 : y_1)$.

Suppose now $x_0 = 0$. Then we get $x_1 = 0$, so $x = (0 : 0 : 1) = \phi_2(0 : 1)$.

Remark.

We gather here some useful observations:

- It is possible to show that any smooth conic in \mathbb{P}^2 (Y in the previous example is one of those) is isomorphic to \mathbb{P}^1 . On the other hand, not every smooth cubic in \mathbb{P}^2 is isomorphic to \mathbb{P}^1 .
- It turns out that a rational normal curve can be expressed as a complete intersection only if $d \leq 2$. Sometimes it is still possible to find a set of equations that describes $\phi_d(\mathbb{P}^1)$, but the number of these equations will not be equal to the codimension of the manifold in \mathbb{P}^m . As an example consider the following set of equation in \mathbb{P}^3 :

$$\begin{cases} Q_1 : x_0x_2 - x_1^2 = 0 \\ Q_2 : x_0x_3 - x_1x_2 = 0 \\ Q_3 : x_1x_3 - x_2^2 = 0 \end{cases}$$

We want to check that $Q := \{x \in \mathbb{P}^3 \mid Q_1(x) = Q_2(x) = Q_3(x) = 0\} = \varphi(\mathbb{P}^1)$, where $\varphi := \phi_3$ (see the first example in the Veronese map section).

$\varphi(\mathbb{P}^1) \subseteq Q$: Just substitute the coordinates of a generic point $(s^3 : s^2t : st^2 : t^3) \in \varphi(\mathbb{P}^1)$ inside Q_1, Q_2, Q_3 and observe that the three polynomials are automatically zero.

$\varphi(\mathbb{P}^1) \supseteq Q$: Consider $x = (x_0 : x_1 : x_2 : x_3) \in Q$. Suppose $x_0 \neq 0$. Then we can assume $(x_0 : x_1 : x_2 : x_3) = (1 : y_1 : y_2 : y_3)$, and the equations defining Q give us $y_2 = y_1^2$ and $y_3 = y_1^3$. Thus $x = (1 : y_1 : y_1^2 : y_1^3) = \varphi(1 : y_1)$.

Suppose now $x_0 = 0$. Then we get $x_1 = x_2 = 0$, so $x = (0 : 0 : 0 : 1) = \varphi(0 : 1)$.

Notice that, as previously stated, we have 3 equations defining Q but $\varphi(\mathbb{P}^1)$ has codimension 2 in \mathbb{P}^3 .

- With some calculations (which are really similar to what we have already done) one can show that $Y := \{x \in \mathbb{P}^3 \mid x_0x_3 - x_1x_2 = 0\} \cong \phi(\mathbb{P}^1 \times \mathbb{P}^1)$ where ϕ is the Segre map with $n = m = 1$. Moreover, any smooth quadric in \mathbb{P}^3 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Chapter 3

The Weierstrass \wp -function; complex tori and cubics in \mathbb{P}^2

3.1 Complex tori

Definition 3.1. Let $w_1, \dots, w_{2n} \in \mathbb{C}^n$ be linearly independent vectors (considering \mathbb{C}^n as a \mathbb{R} -vector space), that is, $\mathbb{C}^n = \mathbb{R}w_1 \oplus \dots \oplus \mathbb{R}w_{2n}$. Consider the lattice $\Lambda := \{z \in \mathbb{C}^n \mid z = k_1 w_1 + \dots + k_{2n} w_{2n}, k_j \in \mathbb{Z}\}$. The quotient \mathbb{C}^n / Λ is called *complex torus*.

The lattice Λ is an additive subgroup of \mathbb{C}^n , isomorphic to \mathbb{Z}^{2n} . Thus T is isomorphic to $\mathbb{R}^{2n} / \mathbb{Z}^{2n} = \left(\mathbb{R} / \mathbb{Z}\right)^{2n}$ as a group; the usual identification $\mathbb{R} / \mathbb{Z} \simeq S^1$ shows why we see T as a torus. Furthermore, notice that T can be seen as \mathbb{C}^n / \sim where $z \sim w \Leftrightarrow z - w \in \Lambda$. T is a topological space with the quotient topology and it is Hausdorff. The projection $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^n / \Lambda = T$ is an open map. Indeed, take $V \subseteq \mathbb{C}^n$ open and consider $\pi(V)$: then $\pi(V)$ is open if $\pi^{-1}(\pi(V))$ (the saturation of V) is open, but

$$\pi^{-1}(\pi(V)) = \bigsqcup_{\lambda \in \Lambda} (V + \lambda) \quad \text{which is open (infinite union of translated open sets).}$$

Since $T = \pi(\bar{\Pi})$, where $\bar{\Pi} = \{t_1 w_1 + \dots + t_{2n} w_{2n} \mid 0 \leq t_j \leq 1\}$ is a compact set in \mathbb{C}^n , and π is a continuous map, then T is compact.

Let's find the local complex charts for T . For $x \in T$, choose any $z \in \mathbb{C}^n$ such that $\pi(z) = x$. Choose a neighborhood V of z in \mathbb{C}^n such that $\pi_V := \pi|_V : V \rightarrow \pi(V)$ is a bijection (for example, $V = \{z + t_1 w_1 + \dots + t_{2n} w_{2n} \mid |t_j| < \frac{1}{2}\}$). Then if $z, z' \in V$, $z - z' \notin \Lambda$ unless $z = z'$. Since π is continuous and open, π_V is a homeomorphism. Then we can choose $(\pi(V), \pi_V^{-1})$ as local complex charts for x .

If $(\pi_W^{-1} \circ (\pi_V^{-1})^{-1})(z) = z'$ then $\pi_W(z) = \pi_V(z') \Rightarrow \pi(z) = \pi(z')$, that means $z = z' + \lambda$ for some $\lambda \in \Lambda$. Then $\pi_W^{-1} \circ (\pi_V^{-1})^{-1}$ is just a translation, for each choice of V, W : thus, the transition maps on T with this atlas are holomorphic, or in other words, T is a complex manifold.

Locally, $\pi : \mathbb{C}^n \rightarrow T$ is holomorphic. In fact, restricting π to those sets V in which it is a bijection, then $\pi_V^{-1} \circ \pi_V \circ \text{id}_{\mathbb{C}^n} = \text{id}_{\mathbb{C}^n}$ which is clearly holomorphic.

If f is holomorphic on $\pi(V) \subseteq T$, V as above, then $f \circ \pi : \pi^{-1}(V) = \bigsqcup_{\lambda \in \Lambda} (V + \lambda) \rightarrow \mathbb{C}^n$ is holomorphic as it is a composition of holomorphic function, and it is periodic of period λ for each choice of $\lambda \in \Lambda$, since π is periodic of this period. Conversely, any $F : \pi^{-1}(V) \rightarrow \mathbb{C}^n$ holomorphic and Λ -periodic defines a holomorphic function f on T by the relation $f(\pi(z)) = F(z)$.

3.2 Elliptic functions

Consider the one-dimensional torus $T = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$. T is compact, so any holomorphic function on T is constant; or equivalently, any holomorphic function $F : \mathbb{C} \rightarrow \mathbb{C}$ such that $F(z) = F(z + w_1) = F(z + w_2)$ is constant. It is then clear that we must analyze functions on T that have some kind of singularities, in order to have something interesting to study.

Definition 3.2. A function f on a complex manifold X is *meromorphic* if for any $x \in X$ there is a connected neighborhood V of x and there are holomorphic functions g, h on V , $h \not\equiv 0$, such that $f = \frac{g}{h}$ on $\tilde{V} := \{v \in V \mid h(v) \neq 0\}$. (We observe that this definition makes sense, since h is holomorphic and it can't be zero on the whole V without being the zero function).

Definition 3.3. Let Λ be a lattice. An *elliptic function* (depending from Λ) is a meromorphic function on \mathbb{C} such that $F(z + \lambda) = F(z) \forall z \in \mathbb{C}, \lambda \in \Lambda$.

From what we observed above, there is a bijective correspondence between elliptic functions on \mathbb{C} that are Λ -periodic, and meromorphic functions on $T = \mathbb{C}/\Lambda$.

Let $F = \frac{G}{H}$ be a meromorphic function on $V \subseteq \mathbb{C}$. Then for $a \in V$, we have the following convergent power series:

$$\begin{aligned} G(z) &= (z - a)^k (b_0 + b_1(z - a) + \dots) & k \in \mathbb{Z}, k \geq 0, b_0 \neq 0 \\ H(z) &= (z - a)^l (c_0 + c_1(z - a) + \dots) & l \in \mathbb{Z}, l \geq 0, c_0 \neq 0 \end{aligned}$$

Then $F(z) = (z - a)^n (d_0 + d_1(z - a) + \dots)$, $n = k - l$, $d_0 = \frac{b_0}{c_0} \neq 0$ or shortly $F(z) = \sum_{k=n}^{+\infty} a_k (z - a)^k$ with $a_n = d_0$.

Definition 3.4. Let $f(z) = \sum_{k=n}^{+\infty} a_k (z - a)^k$ with $a_n \neq 0$. The *order* of f in a (denoted as $\text{ord}_a(f)$) is the number n . If $n > 0$, we say that f has a *zero of order* n ; if $n < 0$ f has a *pole of order* $|n|$.

The *residue* of f in a is $\text{Res}_a(f) := a_{-1}$.

Remark.

If f is an elliptic function, $\text{Res}_a(f) = \text{Res}_{a+\lambda}(f)$ and $\text{ord}_a(f) = \text{ord}_{a+\lambda}(f)$.

Moreover, since the zeroes and the poles of a meromorphic function are isolated and T is compact, then a meromorphic function on T has only finitely many zeroes and poles (or equivalently, an elliptic function has a finite number of poles in $\overline{\Pi}$).

Let $\Pi := \{sw_1 + tw_2 \in \mathbb{C} \mid 0 \leq s, t < 1\}$. Then π is bijective, if we restrict to Π . For any $\alpha \in \mathbb{C}$, also $\pi : \alpha + \Pi \rightarrow T$ is a bijection.

PROPOSITION 3.1. Let F be an elliptic function. Let $\alpha \in \mathbb{C}$ such that F has no poles and no zeroes on $\partial(\alpha + \Pi) =: C$. Then

1. $\sum_{a \in \alpha + \Pi} \text{Res}_a(F) = 0;$
2. $\sum_{a \in \alpha + \Pi} \text{ord}_a(F) = 0.$

Proof. 1. By Cauchy's residue theorem (cfr. Complex Analysis course)

$$\sum_{a \in \alpha + \Pi} \text{Res}_a(F) = \frac{1}{2\pi i} \int_C F(z) dz = \frac{1}{2\pi i} \sum_{i=1}^4 \int_{S_i} F(z) dz$$

where each S_i is a segment of C . We have $z \in S_1 \Leftrightarrow z = \alpha + tw_1, t \in [0, 1]$ and $z \in S_3 \Leftrightarrow z = \alpha + tw_1 + w_2, t \in [0, 1]$. But since F is elliptic, $F(\alpha + tw_1 + w_2) = F(\alpha + tw_1), t \in [0, 1]$, so F takes the same values on S_1 and S_3 . This implies $\int_{S_1} F(z) dz + \int_{S_3} F(z) dz = 0$, because S_1 and S_3 are taken with opposite directions. The same happens for the integrals on S_2 and S_4 , and consequently $\int_C F(z) dz = 0$. This proves the assertion.

2. By the Argument Principle (cfr. Complex Analysis course)

$$\sum_{a \in \alpha + \Pi} \text{ord}_a(F) = \frac{1}{2\pi i} \int_C \frac{F'(z)}{F(z)} dz$$

Then, simply noticing that F' is again elliptic of the same period of F (just try to compute it) and that $\frac{F'}{F}$ is again elliptic of the same period, we can apply the previous result to obtain what we need. □

COROLLARY. There are no meromorphic functions on T that have only one pole of order 1 on T .

Proof. Suppose there exist such an f . Then the corresponding elliptic function $F = f \circ \pi$ has only one pole of order 1 in $a \in \Pi$, so $F(z) = a_{-1} \left(\frac{1}{z-a} \right) + \sum_{k=0}^{+\infty} a_k (z-a)^k$ with $a_{-1} \neq 0$. But then, for a suitable α , $\sum_{a \in \alpha + \Pi} \text{Res}_a(F) = a_{-1}$ leading to a contradiction with point 1 in the previous proposition. □

Definition 3.5. Let F be an elliptic function with zeroes of order m_1, \dots, m_k in Π and poles of order n_1, \dots, n_l in Π . We define $\deg(f) = \deg(F) = \sum m_i$ where f is the meromorphic function on T corresponding to F . (Note that, by previous proposition (point 2), $\sum m_i = \sum n_i$).

3.3 The Weierstrass \wp -function

Definition 3.6. Let $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$ be a lattice in \mathbb{C} . The function

$$\wp(z) := \frac{1}{z^2} + \sum_{\substack{w \in \Lambda \\ w \neq 0}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

is called *Weierstrass \wp -function*. \wp is meromorphic in \mathbb{C} , holomorphic on $\mathbb{C} \setminus \Lambda$ with poles of order 2 in each $w \in \Lambda$.

PROPOSITION 3.2.

1. $\wp(-z) = \wp(z)$ (\wp is even).
2. \wp is an elliptic function (depending from Λ).
3. $\deg(\wp) = 2$.
4. \wp is surjective.

Proof. 1. $\frac{1}{(-z-w)^2} - \frac{1}{w^2} = \frac{1}{(z-(-w))^2} - \frac{1}{(-w)^2}$, but $\Lambda = -\Lambda \Rightarrow \wp(z) = \wp(-z)$.

2. $\wp'(z) = -\frac{2}{z^3} + \sum_{w \neq 0} \frac{-2}{(z-w)^3} = -2 \sum_w \frac{1}{(z-w)^3}$. Hence $\wp'(z+w) = \wp'(z)$ for each $w \in \Lambda$ (as $w + \Lambda = \Lambda$). \wp is meromorphic, so \wp' is meromorphic, too: thus \wp' is elliptic function. This implies $\wp(z+w_1) = \wp(z) + c_1$ for some $c_1 \in \mathbb{C}$. If we choose $z = \frac{-w_1}{2}$, then $\wp(\frac{w_1}{2}) = \wp(\frac{-w_1}{2}) + c_1$; but \wp is even, so $c_1 = 0$. Similarly $\wp(z+w_2) = \wp(z)$ and then \wp is elliptic.

3. The degree of \wp is 2 since \wp has only one pole in Π of order 2 (namely, 0).
4. Clearly, the function $\wp - c$ for any choice of $c \in \mathbb{C}$ has the same poles of \wp in Π , hence the same zeroes of \wp in Π . This means $\exists z_0 \in \Pi$ such that $\wp(z_0) = c$, so \wp is surjective. □

LEMMA 3.3.

1. The function \wp' has degree 3, and it has three distinct zeroes in Π , each of multiplicity one: those are $\frac{w_1}{2}, \frac{w_2}{2}, \frac{w_3}{2}$ where $w_3 = w_1 + w_2$.
2. The function $z \mapsto \wp(z) - c$, for each choice of $c \in \mathbb{C}$ has a double zero in $z_0 \in \Pi$ if and only if $c \in \{u_1, u_2, u_3\}$ where $u_i = \wp(\frac{w_i}{2})$. Moreover, the u_i are three distinct complex numbers.

Proof. 1. As seen before, \wp' has only poles in $w \in \Lambda$ of order 3. The only pole of \wp' in Π is 0, so $\deg(\wp') = 3$.

Since \wp is even, \wp' is odd, so $\wp'(-z) = -\wp'(z)$. Moreover, \wp' is elliptic: then

$$\wp' \left(\frac{w_i}{2} \right) = -\wp' \left(-\frac{w_i}{2} \right) = -\wp' \left(-\frac{w_i}{2} + w_i \right) = -\wp' \left(\frac{w_i}{2} \right)$$

Hence $\wp'(\frac{w_i}{2}) = 0$ for $i = 1, 2, 3$. As $\deg(\wp') = 3$, these must be all zeroes of \wp' and they must have multiplicity one.

2. $\wp(z) - c$ has a double zero in $z_0 \in \Pi \Leftrightarrow \wp(z_0) = c$ and the derivative in $z = z_0$ is zero $\Leftrightarrow \wp'(z_0) = 0$, that is, $z_0 = \frac{w_i}{2}, i = 1, 2, 3$ and $c = \wp(\frac{w_i}{2}) = u_i$.
If $u_i = u_j$ and $i \neq j$, then $z \mapsto \wp(z) - u_j$ has a double zero in both $\frac{w_i}{2}$ and $\frac{w_j}{2}$. But this means $\deg(\wp(z) - u_i) \geq 4$, in contrast with the fact that $\wp(z) - u_i$ has only one pole of order 2 in Π .

□

Definition 3.7. We call *Eisenstein series* the number $G_n(\Lambda) := \sum_{\substack{w \in \Lambda \\ w \neq 0}} \frac{1}{w^n}$.

$G_n(\Lambda)$ is defined for $n \geq 3$. Notice that $G_m(\Lambda) = 0$ if m is odd, because $\Lambda = -\Lambda$ and $G_m(-\Lambda) = (-1)^m G_m(\Lambda)$.

THEOREM 3.4. For all $z \in \mathbb{C}$, we have:

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \quad g_2 = 60G_4(\Lambda), \quad g_3 = 140G_6(\Lambda)$$

Proof. Recall that $1 + t + t^2 + \dots = \frac{1}{1-t}$. Deriving, we obtain

$$\frac{1}{(1-t)^2} = 0 + 1 + 2t + 3t^2 + \dots \quad \Rightarrow \quad \frac{1}{(1-t)^2} = \sum_{k=0}^{+\infty} (k+1)t^k$$

Hence

$$\frac{1}{(z-w)^2} - \frac{1}{w^2} = \frac{1}{w^2} \left(\frac{1}{(\frac{z}{w} - 1)^2} - 1 \right) = \frac{1}{w^2} \left(1 + 2\left(\frac{z}{w}\right) + \dots - 1 \right) = \sum_{k=1}^{+\infty} \frac{(k+1)}{w^{k+2}} z^k$$

This means

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum_{k=1}^{+\infty} (k+1) \left(\sum_{\substack{w \in \Lambda \\ w \neq 0}} \frac{1}{w^{k+2}} \right) z^k = \frac{1}{z^2} + \sum_{k=1}^{+\infty} (k+1) G_{k+2}(\Lambda) z^k = \\ &= \frac{1}{z^2} + 3G_4(\Lambda)z^2 + 5G_6(\Lambda)z^4 + \dots \end{aligned}$$

and

$$\wp'(z) = -\frac{2}{z^3} + 6G_4(\Lambda)z + 20G_6(\Lambda)z^3 + \dots \quad \Rightarrow \quad (\wp'(z))^2 = \frac{4}{z^6} - 24G_4(\Lambda)\frac{1}{z^2} + \dots$$

With a simple computation, one can show that $h(z) := (\wp'(z))^2 - (4\wp(z)^3 - g_2\wp(z) - g_3)$ has no pole in $0 \in \Pi$ (each term with z^n and $n < 0$ vanishes) and $h(0) = 0$. As \wp, \wp' do not have poles on $\pi - \{0\}$, h is holomorphic on Π and periodic (since \wp, \wp' are). But a periodic holomorphic function on \mathbb{C} is equivalent to a holomorphic function on the compact manifold T , which must be constant; the fact that $h(0) = 0$ implies $h \equiv 0$ as we wanted to show. □

THEOREM 3.5. The polynomial $P(x) = 4x^2 - g_2x - g_3$ has three distinct zeroes if and only if $\Delta := g_2^3 - 27g_3^2 \neq 0$ (Δ is the discriminant of P).

Proof. P has a double zero $\Leftrightarrow P, P'$ have a common zero. $P'(x) = 12x^2 - g_2 \Rightarrow$ the zeroes of P' are $x_{\pm} = \pm\sqrt{\frac{g_2}{12}}$. Now

$$P(x_{\pm}) = x_{\pm}(4x_{\pm}^2 - g_2) - g_3 = \pm\sqrt{\frac{g_2}{12}}\left(4\frac{g_2}{12} - g_2\right) - g_3 = \mp\frac{2}{3}g_2\sqrt{\frac{g_2}{12}} - g_3$$

Hence $P(x_{\pm}) = 0 \Leftrightarrow g_2^3 = \frac{4}{9}g_3^2 \Leftrightarrow g_2^3 - 27g_3^2 = 0$. \square

Remark.

Notice that $(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 = 4\prod_{i=1}^3(\wp(z) - a_i)$ for some $a_i \in \mathbb{C}$ which are the zeroes of $P(x) = 4x^3 - g_2x - g_3$. As $\wp'(z) = 0$ for $z \in \Pi$ only if $z = \frac{w_i}{2}, i = 1, 2, 3$ and $\wp(\frac{w_i}{2}) = u_i$, we get $(\wp'(z))^2 = 4\prod_{i=1}^3(\wp(z) - u_i)$. Recall that the u_i are distinct, so $P(x)$ has three distinct zeroes and $\Delta \neq 0$ (for any lattice Λ !)

Example: Let $\Lambda = \mathbb{Z}[i] = \{n + mi : m, n \in \mathbb{Z}, i^2 = -1\}$. Notice that $i\Lambda = \Lambda$, thus $G_6(\Lambda) = G_6(i\Lambda) = \sum_{w \neq 0} \frac{1}{(iw)^6} = -\sum_{w \neq 0} \frac{1}{w^6} = -G_6(\Lambda) \Rightarrow G_6(\Lambda) = 0$, and then $g_3 = 0$, too. $\Delta \neq 0$, so $g_2 \neq 0$. So we have the equation $(\wp'(z))^2 = 4\wp^3 - g_2\wp$.

In general, $G_m(c\Lambda) = g^{-m}G_m(\Lambda)$ (and if $i\Lambda = \Lambda$, then $i(c\Lambda) = c(i\Lambda) = c\Lambda$). Hence, choosing a suitable $c \in \mathbb{C}$, we get that for $\Lambda = c\mathbb{Z}[i]$ the equation $(\wp')^2 = 4\wp - d\wp$ is true, for any choice of $d \neq 0$. For example, $d = 4$ gives $\left(\frac{\wp'}{2}\right)^2 = \wp^3 - \wp$.

Moreover,

$$\wp(iz) = \frac{1}{(iz)^2} + \sum_{\substack{w \in \Lambda \\ w \neq 0}} \left(\frac{1}{(iz - w)^2} - \frac{1}{w^2} \right) = -\frac{1}{z^2} + \sum_{\substack{w \in \Lambda \\ w \neq 0}} \left(\frac{1}{(iz - iw)^2} - \frac{1}{(iw)^2} \right) = -\wp(z)$$

and then $i\wp'(iz) = -\wp'(z) \Rightarrow \wp'(iz) = i\wp'(z)$.

Remark.

Similarly, one can consider $\Lambda = \mathbb{Z}[\rho], \rho = e^{\frac{2\pi i}{3}}$ (hexagonal lattice). In this case $g_2 = 0, g_3 \neq 0$ and $\wp(\rho z) = \rho\wp(z), \wp'(\rho z) = \wp'(z)$.

3.4 Tori and cubic curves

THEOREM 3.6. Consider

$$\begin{aligned} \phi : T = \mathbb{C}/\Lambda &\rightarrow \mathbb{P}^2 \\ t = \pi(z) &\mapsto \begin{cases} (\wp(z) : \wp'(z) : 1) & \text{if } z \notin \Lambda \\ (0 : 1 : 0) & \text{if } z \in \Lambda \end{cases} \end{aligned}$$

Then ϕ is holomorphic with image $\phi(T) = E := \{(x : y : z) \in \mathbb{P}^2 \mid -y^2z + 4x^3 - g_2xz^2 - g_3z^3 = 0\}$. Moreover, ϕ is injective and has maximal rank 1 on T .

Proof. First we show that ϕ is holomorphic on $T - \{0\}$.

Let $t \in T - \{0\}$. Then $\phi(t) \in U_2 \subseteq \mathbb{P}^2$.

$$\begin{array}{ccccc} T - \{0\} \supseteq & V & \xrightarrow{\phi} & \phi(V) \subseteq U_2 \ni & (x : y : z) \\ & \downarrow (\pi_V)^{-1} & & \downarrow z_2 & \downarrow \\ \mathbb{C} \supseteq & \tilde{V} & & \mathbb{C}^2 & \ni \left(\frac{x}{z}, \frac{y}{z} \right) \end{array}$$

We must check that $F := z_2 \circ \phi \circ ((\pi_V)^{-1})^{-1} = z_2 \circ \phi \circ \pi$ is holomorphic on \tilde{V} ; this is true since

$$F(z) = z_2(\phi(\pi(z))) = z_2((\wp(z) : \wp'(z) : 1)) = (\wp(z), \wp'(z)).$$

ϕ has maximal rank in $\mathbb{C} \setminus \Lambda$ if and only if $J_{\mathbb{C}}F(z) = \begin{pmatrix} \wp'(z) \\ \wp''(z) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (for $z \in \mathbb{C} \setminus \Lambda$). Suppose not: then there exists a z such that $\wp'(z) = 0$, hence $z = \frac{w_i}{2}$. As we have seen, these are simple zeroes of \wp' , hence $\wp''(\frac{w_i}{2}) \neq 0$, a contradiction with our assumption. Hence $J_{\mathbb{C}}F(z) \neq 0 \forall z \in \mathbb{C} \setminus \Lambda$.

Now we check that ϕ is holomorphic in a neighborhood of 0. $\phi(\pi(0)) = (0 : 1 : 0) \in U_1$; let \tilde{V} be a neighborhood of $0 \in \mathbb{C}$ and let $z \in \tilde{V}$. Then

$$F(z) := (z_1 \circ \phi \circ ((\pi_V)^{-1})^{-1})(z) = (z_1 \circ \phi \circ \pi)(z) = z_1(\wp(z) : \wp'(z) : 1) = \left(\frac{\wp(z)}{\wp'(z)}, \frac{1}{\wp'(z)} \right)$$

(We can divide by $\wp'(z)$ since we are near 0). Recall that $\wp(z) = \frac{1}{z^2}(1 + a_1z + \dots)$ and $\wp'(z) = \frac{-2}{z^3}(1 + b_1z + \dots)$, then

$$\frac{\wp(z)}{\wp'(z)} = -\frac{z}{2}(1 + c_1z + \dots) \quad \frac{1}{\wp'(z)} = -\frac{z^3}{2}(1 + d_1z + \dots)$$

This shows that F is holomorphic in 0, so ϕ is holomorphic.

ϕ has maximal rank in $t = 0$ if and only if $\begin{pmatrix} \left(\frac{\wp}{\wp'}\right)'(z) \\ \left(\frac{1}{\wp'}\right)'(z) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in $z = 0$. A simple calculation shows

$$\begin{pmatrix} \left(\frac{\wp}{\wp'}\right)'(z) \\ \left(\frac{1}{\wp'}\right)'(z) \end{pmatrix}_{z=0} = \begin{pmatrix} -\frac{1}{2} + \dots \\ -\frac{3z^2}{2} + \dots \end{pmatrix}_{z=0} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$$

so ϕ has maximal rank in 0.

This final considerations allow us to say that ϕ is holomorphic and has maximal rank on the whole T . We still have to prove that $\phi(T) = E$ and that ϕ is injective.

$\phi(T) \subseteq E$ is clear, since $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. We prove $\phi(T) \supseteq E$: let $P = (x : y : z) \in E$. If $z = 0$, then by the definition of E , $x = 0$, so $P = (0 : 1 : 0)$ and then $P = \phi(0) \in \phi(T)$. Now we assume $z \neq 0$, that is $P = (x : y : 1)$. \wp is surjective, so for any $x \in \mathbb{C}$, there exists a $z \in \mathbb{C}$ such that $\wp(z) = x$. As $P \in E$, we get

$$y^2 = 4x^3 - g_2x - g_3 = 4\wp(z)^3 - g_2\wp(z) - g_3 = (\wp'(z))^2 \Rightarrow y = \pm \wp'(z)$$

If $y = \wp'(z)$, $P = (x : y : 1) = \phi(\pi(z))$; if $y = -\wp'(z)$, $x = \wp(z) = \wp(-z)$ and $y = -\wp'(z) = \wp'(-z)$, hence $P = \phi(\pi(-z))$.

ϕ is also injective. Let $z_1, z_2 \in \Pi$ such that $\phi(\pi(z_1)) = \phi(\pi(z_2))$. If $z_1 = 0$, then $\phi(\pi(z_1)) = (0 : 1 : 0) = \phi(\pi(z_2)) \Rightarrow z_2 = 0$. If $z_1, z_2 \neq 0$, then

$$\begin{cases} \wp(z_1) = \wp(z_2) & \Rightarrow z_1 = z_2 \vee z_1 = -z_2 + w \text{ for a suitable } w \in \Lambda \\ \wp'(z_1) = \wp'(z_2) \end{cases}$$

If $z_1 = z_2$ we are done. If $z_1 = -z_2 + w$, $\wp'(z_2) = \wp'(z_1) = \wp'(-z_2 + w) = -\wp'(z_2) \Rightarrow \wp'(z_2) = 0 \rightarrow z_2 = \frac{w_i}{2}$, but also $z_1 = \frac{w_j}{2}$ because $\wp'(z_1) = \wp'(z_2)$. As $\wp(z_1) = \wp(z_2)$, we have $\wp(\frac{w_j}{2}) = \wp(\frac{w_i}{2})$ that is $u_i = u_j \Rightarrow i = j$ (the u_i are distinct). \square

Remark.

Since T is compact, we get that $\phi : T \rightarrow E$ is biholomorphic, so $T \cong E$.

Such cubic curves E in \mathbb{P}^2 are called *elliptic curves*.

3.4.1 Addition law on cubic curves

We have seen that $\phi : T \rightarrow E$ is a biholomorphism. T is a group, hence E inherits a group structure. More precisely: given $P, Q \in E$, and $P = \phi(\pi(z_1)), Q = \phi(\pi(z_2))$, then we define $P + Q := \phi(\pi(z_1) + \pi(z_2)) = \phi(\pi(z_1 + z_2))$. The neutral element is $\phi(\pi(0)) = (0 : 1 : 0) =: O$.

Let us consider a special case: $z_2 = -z_1$. Then

$$\begin{cases} P = \phi(\pi(z_1)) = (\wp(z_1) : \wp'(z_1) : 1) = (a : b : 1) \\ Q = \phi(\pi(z_2)) = (\wp(-z_1) : \wp'(-z_1) : 1) = (\wp(z_1) : -\wp'(z_1) : 1) = (a : -b : 1) \end{cases}$$

So, if $P = (a : b : 1)$ then $Q = -P = (a : -b : 1)$. Notice that the line $l = \langle P, Q \rangle$ has equation $X = aZ$ and $l \cap E = \{P, -P, O\}$.

THEOREM 3.7. Let F be an elliptic function (with respect to Λ). Let $\alpha \in \mathbb{C}$ such that F has no poles and zeroes on C , the boundary of $\alpha + \Pi$. Then

$$\sum_{a \in \alpha + \Pi} \text{ord}_a(F) \cdot a \in \Lambda$$

Proof. We know (by the Argument Principle) that $\frac{F'}{F} = \frac{n}{z-a} + \dots$ where $n = \text{ord}_a(F)$. Notice that $z \frac{F'}{F} = (a + (z-a)) \frac{F'}{F} = \frac{an}{z-a} + \dots$ hence $\text{Res}_a \left(z \frac{F'}{F} \right) = an = \text{ord}_a(F) \cdot a$. So, the theorem follows if we are able to show that

$$\frac{1}{2\pi i} \int_C z \frac{F'}{F} dz \in \Lambda.$$

We divide the integral on the parallelogram C into four segments S_i , $i \in \{1, 2, 3, 4\}$, with $S_3 = S_1 + w_2$ and $S_4 = S_2 - w_1$.

$$I := \int_{S_1} z \frac{F'(z)}{F(z)} dz + \int_{S_3} u \frac{F'(u)}{F(u)} dz = \int_{S_1} z \frac{F'(z)}{F(z)} dz + \int_{S_1} -(z + w_2) \frac{F'(z + w_2)}{F(z + w_2)} dz$$

Since F, F' are elliptic, $\frac{F'(z+w_2)}{F(z+w_2)} = \frac{F'(z)}{F(z)}$ and then

$$\begin{aligned} I &= -w_2 \int_{S_1} \frac{F'(z)}{F(z)} dz = \left[d \log F = \frac{1}{F} dF = \frac{F'}{F} dz \right] = -w_2 \log F \Big|_{z=\alpha}^{z=\alpha+w_1} = \\ &= -w_2 (\log(F(\alpha + w_1)) - \log(F(\alpha))) = -w_2 2\pi i k, \quad \text{for some } k \in \mathbb{Z}. \end{aligned}$$

A similar proof shows $\int_{S_2} + \int_{S_4} = w_1 2\pi i l$, $l \in \mathbb{Z}$ yielding

$$\frac{1}{2\pi i} \int_C z \frac{F'}{F} dz = l w_1 + (-k) w_2 \in \Lambda.$$

□

COROLLARY. Any line $M : aX + bY + cZ = 0$ in \mathbb{P}^2 cuts the curve E in 3 points $P = \phi(\pi(z_1)), Q = \phi(\pi(z_2)), R = \phi(\pi(z_3))$ such that $z_1 + z_2 + z_3 \in \Lambda$ ($\Rightarrow \pi(z_1 + z_2 + z_3) = 0 \in T$, so $P + Q + R = O$ in E).

Proof. Let $b = 0$ (then $O = (0 : 1 : 0) \in M$). Then M is $aX + cZ = 0$: if $a = 0$, we get $M : Z = 0$ and clearly $M \cap E = \{O, O, O\}$ that is ok. If $a \neq 0$, then $M : X = -\left(\frac{c}{a}\right) Z$ and we already know we get $M \cap E = \{P, -P, O\}$, with $P = (x : y : 1)$ and $x = -\frac{c}{a}$. This is ok again, since $P = \phi(\pi(z))$, $-P = \phi(\pi(-z))$, $O = \phi(\pi(0))$ and $z + (-z) + 0 = 0$.

Now let $b \neq 0$. Then $M \cap E = \phi(\{\pi(z) \in T - \{0\} \mid a\wp(z) + b\wp'(z) + c\})$. In general, a function $f(z) = m\wp(z) + n\wp'(z) + q$ with $n \neq 0$ is elliptic, has a pole of order 3 in 0 (since $\wp'(z)$ has) and is holomorphic on $\mathbb{C} \setminus \Lambda$, so has degree 3. The elliptic function $a\wp(z) + b\wp'(z) + c$ has degree 3, so has 3 zeroes in $\alpha + \Pi$, say z_1, z_2, z_3 , and a pole of order 3 in $w \in (\alpha + \Pi) \cap \Lambda$. By previous theorem,

$$1 \cdot z_1 + 1 \cdot z_2 + 1 \cdot z_3 + 3 \cdot w \in \Lambda \quad \Rightarrow \quad z_1 + z_2 + z_3 \in \Lambda$$

□

So, given $P, Q \in E$, let $M = \langle P, Q \rangle$ be the line spanned by P and Q . By the above corollary, $E \cap M = \{P, Q, R\}$ and moreover $P + Q + R = O$.

Definition 3.8. Given $P, Q \in E$ elliptic curve, we define $P + Q := -R$, where R is the point on E as found in the above construction. (To find "explicitly" the point $-R$, note that $\langle R, O \rangle$ cuts E in $\{R, O, -(R + O) = -R\}$).

COROLLARY. The addition law $+$ defined above endows an elliptic curve E with a group structure.

The proof of this corollary is quite straightforward (simply check the group definition, using previous results). It is still possible to prove it without using elliptic functions, but in that case, proving associativity of $+$ is more complicated.

If $g_2, g_3 \in \mathbb{Q}$, then $E(\mathbb{Q}) := \{(x, y) \in \mathbb{Q}^2 \mid y^2 = 4x^3 - g_2x - g_3\} \cup \{O\}$ is an abelian group. More generally, the following theorem holds:

THEOREM 3.8 (MORDELL'S THEOREM). $E(\mathbb{Q}) \cong \mathbb{Z}^N \oplus T$ with T finite (T is the "torsion" group). The integer N is called *rank* of E .

One of the main problems is: which are the possible values for N ? Until now, mathematicians have found cubics with rank ≤ 30 , but we have no clue whether N is even bounded. The *Birch-Swinnerton-Dyer conjecture* (currently, one of the millennium prize problems), if proven true, would give a way to determine N .

Exercise: Find $P \in E$ such that $3P = O$ (flexes of E).

In general, the points $P, P, -2P \in E$ stay on the same line, since their sum is O (this line is the tangent line to E in P because it has a double zero in P). But $3P = O$ means $P = -2P$, that is: in our case, the line has a flex in P . We notice

$$\{\text{flexes on } E\} = \{P \in E \mid 3P = O\} = \{z \in \Pi \mid 3z \in \Lambda\}$$

If $z = sw_1 + tw_2, 0 \leq s, t < 1$ the above condition is equivalent to $3z = (3s)w_1 + (3t)w_2 \in \Lambda \Rightarrow s, t \in \{0, \frac{1}{3}, \frac{2}{3}\}$, so a cubic E has nine flexes in total.

Remark.

Any smooth cubic curve C in \mathbb{P}^2 has nine flexes; choosing coordinates such that $(0 : 1 : 0)$ is a flex and $z = 0$ is the flex line, then the equation of C becomes

$$y^2 + a_1xy + a_3y = x^3 + a_2 + a_4x + a_6 \quad a_i \in \mathbb{C}$$

It is possible to show that, in suitable coordinates, C has equation

$$y^2 = 4x^3 - g_2x - g_3$$

There is a theorem which asserts that if $\Delta = g_2^3 - 27g_3^2 \neq 0$ for $g_2, g_3 \in \mathbb{C}$ (this condition is equivalent to C smooth), then there exists a lattice Λ such that $g_2 = 60G_4(\Lambda)$ and $g_3 = 140G_6(\Lambda)$. This leads us to a conclusion:

Any smooth cubic in \mathbb{P}^2 is isomorphic to a $T = \mathbb{C}/\Lambda$.

3.4.2 Isomorphisms between tori

Let T_1, T_2 be tori, $T_i \cong \mathbb{C}/\Lambda_i$ and let $T_i \cong E_i$, cubic curves with equation $E_i : y^2 = 4x^3 - a_ix - b_i$ ($a_i = g_2(\Lambda_i), b_i = g_3(\Lambda_i)$). Let $\psi : T_1 \rightarrow T_2$ be an isomorphism; up to translation, we can assume $\psi(0) = 0$.

Let f_i be the meromorphic function on T_i such that $f_i \circ \pi_i$ is the \wp -function for Λ_i (here $\pi_i : \mathbb{C} \rightarrow \mathbb{C}/\Lambda_i$ is the quotient map). Similarly, let g_i be the map such that $g_i \circ \pi_i$ is \wp' (relatively to Λ_i). Then we have

$$g_i^2 = 4f_i^3 - a_if_i - b_i \quad \text{on } T_i$$

and f_i, g_i are holomorphic on $T_i - \{0\}$ with a pole of order 2 (3 for g_i) in 0. Hence $f_2 \circ \psi, g_2 \circ \psi$ are holomorphic on $T_1 - \{0\}$ and have a pole of order 2 (3 for $g_2 \circ \psi$) in

$0 \in T_1$. Thus, there is an $a \in \mathbb{C}, a \neq 0$ such that $f_2 \circ \psi - af_1$ has a pole of order ≤ 1 in $0 \in T_1$ and is holomorphic on $T_1 - \{0\}$. Hence, by corollary of proposition 3.1, this function is constant, so $f_2 \circ \psi = af_1 + b, b \in \mathbb{C}$. Similarly, $\exists c, d, e \in \mathbb{C}, c \neq 0$ such that $g_2 \circ \psi = cg_1 + df_1 + e$.

$$\begin{array}{ccc} T_1 & \xrightarrow{\psi} & T_2 \\ \downarrow \phi_1 & & \downarrow \phi_2 \\ E_1 & \xrightarrow{h} & E_2 \end{array}$$

We obtained that $E_1 \ni (x : y : 1) = (f_1(t) : g_1(t) : 1), t \in T_1$ gets mapped through h in $(f_2(\psi(t)) : g_2(\psi(t)) : 1) = (af_1(t) + b : g_1(t) + df_1(t) + e : 1)$, i.e. $h : (x : y : 1) \mapsto (ax + b : dx + cy + e : 1)$. So h is a linear map!

On T_2 , we have

$$g_2^2 = 4f_2^3 - a_2f_2 - b_2 \quad \xrightarrow{\psi} \quad (cg_1 + df_1 + e)^2 = 4(af_1 + b)^3 - a_2(af_1 + b) - b_2$$

$(cg_1 + df_1 + e)^2 = \text{"even part"} + \overbrace{2cdf_1g_1 + 2ce g_1}^{\text{"odd part"}}$, but the right hand side in the previous equation is an even function. Since the equality must be true for all $t \in T_1$, the "odd part" is identically zero for all $t \in T_1$; moreover, f_1g_1 has a pole of order 5, g_1 a pole of order 3, but they can't cancel. Then $cd = 0, ce = 0$, but $c \neq 0 \Rightarrow d = e = 0$. Thus

$$\begin{aligned} (cg_1)^2 &= 4(af_1 + b)^3 - a_2(af_1 + b) - b_2 && \text{(from what we have said before)} \\ c^2 \cdot g_1^2 &= (4f_1^3 - a_1f_1 - b_1) \cdot c^2 && \text{(differential equation for } \wp \text{ in } T_1) \\ \Rightarrow 0 &= 4(a^3 - c^2)f_1^3 + 12a^2bf_1^2 + \dots && \text{(subtract the two equations)} \end{aligned}$$

Again, the equality must hold for all $t \in T_1$, but f_1^3 and f_1^2 have poles of order 6 and 4, respectively. Then $a^3 = c^2, a^2b = 0; a \neq 0$, so $b = 0, a^3 = c^2$. Let $\lambda \in \mathbb{C}$ such that $\lambda^2 = a$, then $c^2 = \lambda^6 \Rightarrow c = \lambda^3$ or $c = (-\lambda)^3$. In both cases $a = \mu^2, c = \mu^3$, where $\mu = \pm\lambda$. Thus

$$f_2 \circ \psi = \mu^2 f_1 \quad g_2 \circ \psi = \mu^3 g_1$$

This means: if there exists an isomorphism between E_1 and E_2 , this must be of the form

$$\begin{aligned} h : E_1 &\rightarrow E_2 \\ (x : y : 1) &\mapsto (\mu^2 x : \mu^3 y : 1) \end{aligned}$$

The relation $y^2 = 4x^3 - a_1x - b_1$ holds for any $(x : y : 1) \in E_1$, but we have also

$$(\mu^3 y)^2 = 4(\mu^2 x)^3 - a_2 \mu^2 x - b_2 \Rightarrow \mu^6 y^2 = 4\mu^6 x^3 - a_2 \mu^2 x - b_2 \Rightarrow y^2 = 4x^3 - \frac{a_2}{\mu^4} x - \frac{b_2}{\mu^6}$$

Thus we get:

COROLLARY. Let $E_i : y^2 = 4x^3 - a_i x - b_i$ be cubic curves. Then $E_1 \cong E_2$ if and only if $a_1 = \frac{a_2}{\mu^4}$ and $b_1 = \frac{b_2}{\mu^6}$, for some $\mu \in \mathbb{C}, \mu \neq 0$.

There is a numerical invariant for isomorphic elliptic curves. It is indeed possible to prove that $E_1 \cong E_2$ if and only if $j(E_1) = j(E_2)$, where $j(y^2 = 4x^3 - g_2x - g_3) := 1728 \frac{g_2^3}{g_2^3 - 27g_3^2}$. Moreover, $\forall c \in \mathbb{C}$, there always exists a lattice Λ such that $j(\frac{T}{\Lambda}) = c$.