KÄHLER GEOMETRY AND HODGE THEORY

ANDREAS HÖRING

The aim of these lecture notes is to give an introduction to analytic geometry, that is the geometry of complex manifolds, with a focus on the Hodge theory of compact Kähler manifolds. It starts with an introduction to complex manifolds and the objects naturally attached to them (differential forms, cohomology theories, connections...). In Section 2 we will also study the positivity of holomorphic line bundles and make the relation with intersection theory of curves on compact complex surfaces. In Section 3 the analytic results established in the Appendix A (by O. Biquard) are used to prove the existence of the Hodge decomposition on compact Kähler manifolds. Finally in Section 4 we prove the Kodaira vanishing and embedding theorems which establish the link with complex algebraic geometry.

Among the numerous books on this subject, we especially recommend the ones by Jean-Pierre Demailly [Dem96], Claire Voisin [Voi02] and Raymond Wells [Wel80]. Indeed our presentation usually follows closely one of these texts.

Contents

Date: 16th May 2013.

1. Complex manifolds

In this chapter we will see that complex manifolds are differentiable manifolds whose transition functions are holomorphic and we will adapt the notions of tangent bundle and differential forms to this new context. In analogy to the definition of the de Rham cohomology in differential geometry, we will use the calculus of (p, q)-forms and the differential operator $\overline{\partial}$ to define the Dolbeault cohomology groups of a complex manifold. The subject of the Hodge Theorem 3.36 is to relate this cohomology theory to de Rham cohomology, but this needs serious technical preparation and will be the subject of the following sections.

Throughout the whole text, we assume that the reader is familiar with the basic notions of differential geometry as explained in [Biq08]. We will use the term differentiable as a synonym for smooth or C^∞[1]. Let $U \subset \mathbb{C}^n$ be an open subset and $f : U \to \mathbb{C}$ be any complex-valued function. We say that f is differentiable if for some $\mathbb{R}$-linear identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and $\mathbb{C} \simeq \mathbb{R}^2$, the composition $f : U \subset \mathbb{R}^{2n} \to \mathbb{C} \simeq \mathbb{R}^2$ is differentiable. It is straightforward to see that this definition does not depend on the choice of the identifications.

1.A. **Holomorphic functions in several variables.** In this section we recall very briefly the notions from holomorphic function theory of several variables. A reader who is not so familiar with this subject may want to consult [Voi02, Ch.1]. For a much more ample introduction to the function theory of several complex variables, [Gun90, KK83, LT97] are standard references.

1.1. **Definition.** Let $U \subset \mathbb{C}^n$ be an open subset, and let $f : U \to \mathbb{C}$ be a (complex-valued) differentiable function. We say that f is holomorphic in the point $a \in U$ if for all $j \in \{1, \ldots, n\}$ the function of one variable

$$z_j \mapsto f(a_1, \ldots, a_{j-1}, z_j, a_{j+1}, \ldots, a_n)$$

is holomorphic in a_j.

1.2. **Exercise.** Let $U \subset \mathbb{C}^n$ be an open subset, and let $f : U \to \mathbb{C}$ be a differentiable function.

a) Denote by $z_1, \ldots, z_n$ the standard coordinates on U, and by x_j (resp. y_j) their real and imaginary parts. Show that f is holomorphic in $a \in U$ if and only if

$$\frac{\partial f}{\partial \overline{z_j}}(a) := \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)(a) = 0 \qquad \forall\ j = 1, \ldots, n.$$

b) For $a \in U$, consider the $\mathbb{R}$-linear application given by the differential

$$df_a : \mathbb{C}^n \to \mathbb{C}.$$

Show that the function f is holomorphic in a if and only if df_a is $\mathbb{C}$-linear. □

[1] In many cases, the C^∞-condition is actually much more than what we will need. For the sake of simplicity we will make this assumption throughout the whole text.

1.3. **Definition.** Let $a \in \mathbb{C}^n$ be a point. The polydisc around a with multiradius $R \in (\mathbb{R}^+)^n$ is the set

$$D(a, R) := \{z \in \mathbb{C}^n \mid |z_j - a_j| < R_j \text{ for all } j \in \{1, \dots, n\}\}.$$

If $R = (1, \dots, 1)$ and $a = 0$, we abbreviate $D(0, R)$ by $\mathbb{D}^n$ and call $\mathbb{D}^n$ the unit disc in $\mathbb{C}^n$.

1.4. **Theorem.** [Voi02, Thm.1.17] Let $U \subset \mathbb{C}^n$ be an open subset, and let $f : U \to \mathbb{C}$ be a differentiable function. The function f is holomorphic in every point $z_0 \in U$ if and only if it satisfies one of the following conditions:

(1) For every point $a \in U$ there exists a polydisc $D \subset U$ such that the power series
$$f(a + z) = \sum_I \alpha_I z^I,$$
converges for every $a + z \in D$.

(2) If $D = D(a, r)$ is a polydisc contained in U, then for every $z \in D$
$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_j - a_j| = r_j} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \ldots \wedge \frac{d\zeta_n}{\zeta_n - z_n}$$

1.5. **Exercise.** (Maximum principle) Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. If $|f|$ admits a local maximum in a point $z_0 \in U$, there exists a polydisc D around z_0 such that $f|_D$ is constant. □

The notion of holomorphic function immediately generalises to the case of a map with values in $\mathbb{C}^m$.

1.6. **Definition.** Let $U \subset \mathbb{C}^n$ be an open subset, and let $f : U \to \mathbb{C}^m$ be a differentiable map. We say that f is holomorphic in the point $z_0 \in U$ if $f_1, \dots, f_m$ are holomorphic in z_0 for every $j = 1, \dots, m$.

A holomorphic map $f : U \to \mathbb{C}^n$ is locally biholomorphic in the point $z_0 \in U$ if there exists a neighbourhood $V \subset U$ of z_0 such that $f|_V$ is bijective onto $f(V)$ and $f|_V^{-1}$ is holomorphic. It is biholomorphic if it is bijective on its image and locally biholomorphic in every point.

1.7. **Definition.** Let $U \subset \mathbb{C}^n$ be an open subset, and let $f : U \to \mathbb{C}^m$ be a holomorphic map. The Jacobian matrix of f at a point $a \in U$ is the matrix

$$J_f(a) = \left(\frac{\partial f_k}{\partial z_j}(a)\right)_{1 \leqslant k \leqslant m, 1 \leqslant j \leqslant n}.$$

As for differentiable maps, a holomorphic map whose Jacobian matrix has locally constant rank admits locally a simple representation:

1.8. **Theorem.** (Rank theorem, [KK83, Thm.8.7]) Let $U \subset \mathbb{C}^n$ be an open subset, let $f : U \to \mathbb{C}^m$ be a holomorphic map, and let $z_0 \in U$ be a point such that $J_f(z)$ has constant rank k in a neighbourhood of z_0. Then there exist open neighbourhoods

$z_0 \in V \subset U$ and $f(z_0) \in W \subset \mathbb{C}^m$ and biholomorphic mappings $\phi : \mathbb{D}^n \to V$ and $\psi : W \to \mathbb{D}^m$ such that $\phi(0) = z_0$, $\psi(f(z_0)) = 0$ and

$$\psi \circ f \circ \phi : \mathbb{D}^n \to \mathbb{D}^m$$

is given by

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0).$$

1.9. **Exercise.** Let $f : U \subset \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map. Show that f is locally biholomorphic in the point $z_0 \in U$ if and only if

$$\det J_f(z_0) \neq 0.$$

$\square$

1.10. **Exercise.** (Cauchy-Riemann equations) Let $U \subset \mathbb{C}^n$ be an open subset, and let $f = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$ be a differentiable function such that

$$f_j(z) = f_j(z_1, \ldots, z_n).$$

Set

$$x_k = \mathrm{Re}(z_k), y_k = \mathrm{Im}(z_k) \text{ and } u_j = \mathrm{Re}(f_j), v_j = \mathrm{Im}(f_j)$$

for all $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$. Show that f is holomorphic if and only if

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}, \ \frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k}$$

for all $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$. $\square$

1.11. **Exercise.** Let $V \subset \mathbb{C}^n$ be a simply connected open subset. Let $\phi : V \to \mathbb{R}$ be a differentiable pluriharmonic function, i.e. a function such that for every $a, b \in \mathbb{C}^n$ the restriction of ϕ to the line $V \cap \{a + b\zeta \mid \zeta \in \mathbb{C}\}$ is harmonic. Then there exists a holomorphic function $f : V \to \mathbb{C}$ such that $\mathrm{Re}(f) = \phi$. $\square$

1.12. **Theorem.** (Hartog's theorem) Let Δ be a polydisc of dimension at least two, and let $f : \Delta \setminus 0 \to \mathbb{C}$ be a holomorphic function. Then there exists a unique holomorphic function $\bar{f} : \Delta \to \mathbb{C}$ such that $\bar{f}|_{\Delta \setminus 0} = f$.

1.B. **Complex manifolds.**

1.13. **Definition.** A complex manifold of dimension n is a connected Hausdorff topological space X such that there exists a countable covering $(U_i)_{i \in I}$ by open sets and homeomorphisms $\phi_i : U_i \to V_i$ onto open sets $V_i \subset \mathbb{C}^n$ such that for all $(i, j) \in I \times I$, the transition functions

$$\phi_j \circ \phi_i^{-1}|_{\phi_i(U_i \cap U_j)} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

are biholomorphic. We call the collection $(U_i, \phi_i)_{i \in I}$ a complex atlas of the manifold. A complex manifold is compact if the underlying topological space is compact.

As in the case of differential geometry [Biq08, Ch.1.2], we say that two atlas are equivalent if their union is still an atlas. This defines an equivalence relation on the set of complex atlas on X.

1.14. **Definition.** A complex structure on X is the data of an equivalence class of a complex atlas on X.

1.15. **Remark.** Note that in contrast to differentiable manifolds, it is in general not possible to choose the whole affine space is a coordinate chart: just take $X = \mathbb{D}$ the unit disc, then by Liouville's theorem there is no non-constant holomorphic map $\mathbb{C} \to \mathbb{D}$. We say that a complex manifold X is (Brody-)hyperbolic if it does not admit non-constant holomorphic maps $f : \mathbb{C} \to X$. Deciding whether certain manifolds are hyperbolic is a very active (and difficult!) research subject.

1.16. **Examples.**

1. Let $U \subset \mathbb{C}^n$ be an open set. Then U is a complex manifold, an atlas is given by one chart.

2. More generally let X be a complex manifold of dimension n, and let $U \subset X$ be a connected open set. Then U has an induced structure of complex manifold of dimension n.

3. Let $\Lambda \subset \mathbb{C}^n$ be a lattice of rank $2n$. Then the quotient group $X := \mathbb{C}^n/\Lambda$ endowed with the quotient topology has a unique holomorphic structure induced by the standard holomorphic structure on $\mathbb{C}^n$ (cf. Exercise 1.17). We call X a complex torus[2].

4. Let V be a complex vector space of dimension $n+1$ and let $\mathbb{P}(V)$ be the set of complex lines in V passing through the origin, i.e. the set of complex subvector spaces of dimension one. If $v \in V \setminus 0$ is a point, then $\mathbb{C}v$, the complex vector space generated by v is an element of $\mathbb{P}(V)$ which we denote by $[v]$. Furthermore if $v' = \lambda v$ for some $\lambda \in \mathbb{C}^*$, then $[v'] = [v]$. Vice versa if $l \in \mathbb{P}(V)$, there exists a $v \in V \setminus 0$ such that $l = [v]$ and v is unique up to multiplication by an element $\lambda \in \mathbb{C}^*$. Therefore we have a surjective map

$$\pi : V \setminus 0 \to \mathbb{P}(V), v \mapsto [v]$$

and we endow $\mathbb{P}(V)$ with the quotient topology defined by π and the standard topology on V.

Let $V \simeq \mathbb{C}^{n+1}$ be a $\mathbb{C}$-linear isomorphism, then we can write $v = (v_0, \ldots, v_n)$ and we call

$$[v_0 : \ldots : v_n]$$

homogeneous coordinates of $[v] \in \mathbb{P}(V)$. As in the case of the real projective space, we can then define a structure of complex manifold on $\mathbb{P}(V)$ as follows: for every $i \in \{0, \ldots, n\}$, set

$$U_i := \{[v] \in \mathbb{P}(V) \mid v_i \neq 0\}$$

and

$$\phi_i : U_i \to \mathbb{C}^n,\ [v] \ \mapsto \ (\frac{v_0}{v_i}, \ldots, \frac{\widehat{v_i}}{v_i}, \ldots, \frac{v_n}{v_i}).$$

[2]In the sequel whenever we write $X := \mathbb{C}^n/\Lambda$, we implicitly assume that the lattice has maximal rank. Equivalently X is always supposed to be compact.

With this definition we have

$$\phi_i(U_i \cap U_j) = \{(z_1, \ldots, z_n) \mid z_j \neq 0\},$$

so the transition functions $\phi_j \circ \phi_i^{-1}|_{\phi_i(U_i \cap U_j)} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ given by

$$(z_1, \ldots, z_n) \mapsto [z_1 : \ldots : z_{i-1} : 1 : z_i : \ldots : z_n] \mapsto (\frac{z_1}{z_j}, \ldots, \frac{\widehat{z_j}}{z_j}, \ldots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_i}{z_j}, \ldots, \frac{z_n}{z_j})$$

are well-defined and biholomorphic. One checks easily that the complex structure on $\mathbb{P}(V)$ does not depend on the choice of the isomorphism $V \simeq \mathbb{C}^{n+1}$.

Very often we will denote by $\mathbb{P}^n$ the projective space $\mathbb{P}(\mathbb{C}^{n+1})$.

1.17**. Exercise.** Let X be a complex manifold, and let Γ be a subgroup of the group of automorphisms of X. We say that Γ acts properly discontinuous on X if for any two compact subsets $K_1, K_2 \subset X$, we have

$$\gamma(K_1) \cap K_2 \neq \emptyset$$

for at most finitely many $\gamma \in \Gamma$. The group acts without fixed points if

$$\gamma(x) \neq x \qquad \forall\ \gamma \in \Gamma.$$

Suppose that Γ acts properly discontinuous and without fixed points on X, and denote by X/Γ the set of equivalence classes under this action. Show that X/Γ admits a unique complex structure such that the natural map $\pi : X \to X/\Gamma$ is holomorphic and locally biholomorphic. $\square$

1.18**. Exercise.** Show that as a differentiable manifold, we have

$$\mathbb{P}^n \simeq S^{2n+1}/S^1,$$

where $S^{2n+1} \subset \mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$ denotes the unit sphere and $S^1 \subset \mathbb{C}$ acts on $\mathbb{C}^{n+1}$ by scalar multiplication

$$S^1 \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, (\lambda, x) \mapsto \lambda \cdot x.$$

In particular the topological space $\mathbb{P}^n(\mathbb{C})$ is compact. $\square$

1.19**. Exercise.** (Grassmannian) Let V be a complex vector space of dimension n. For an integer $0 < r < n$, we define the Grassmannian as the set

$$G_r(V) := \{S \subset V \text{subspace of dimension } r\}.$$

Fix a Hermitian product on V and denote by $U_V \subset GL(V, \mathbb{C})$ the unitary group with respect to this metric. Show that we have a surjective map

$$U_V \to G_r(V).$$

We endow $G_r(V)$ with the quotient topology induced by the surjection $U_V \to G_r(V)$. Show that $G_r(V)$ is a compact topological space.

We define an atlas on $G_r(V)$ as follows: for any $T_i \subset V$ a subspace of dimension $n - r$, set

$$U_i := \{S \subset V \text{of dimension } r \mid S \cap T_i = 0\}.$$

Choose an arbitrary $S_i \in U_i$, then we can define

$$\phi_i : U_i \to \mathrm{Hom}(S_i, T_i) \simeq \mathbb{C}^{r(n-r)}$$

by associating to $S \in U_i$ the unique linear map $f \in \mathrm{Hom}(S_i, T_i)$ such that

$$S \subset V = S_i \oplus T_i$$

is the graph of f. Show that the maps $\phi_i = \phi_i(S_i, T_i)$ define a complex atlas on $G_r(V)$. $\square$

1.20. **Exercise.** (Hopf varieties) Let λ be a real number such that $0 < \lambda < 1$. We define a group action

$$\mathbb{Z} \times (\mathbb{C}^n \setminus 0) \to (\mathbb{C}^n \setminus 0), (m, z) \mapsto \lambda^m z,$$

and denote by H the quotient $(\mathbb{C}^n \setminus 0)/\mathbb{Z}$. Show that H admits the structure of a complex manifold and is diffeomorphic to $S^{2n-1} \times S^1$. Hint: note that $\mathbb{C}^n \setminus 0$ is diffeomorphic to $S^{2n-1} \times \mathbb{R}^+$. $\square$

1.21. **Definition.** Let X (resp. Y) be a complex manifold of dimension n (resp. m) and denote by $(U_i, \phi_i : U_i \to V_i)_{i \in I}$ (resp. $(M_j, \psi_j : M_j \to N_j)_{j \in J}$) the corresponding atlas. A holomorphic map from X to Y is a continuous map $f : X \to Y$ such that for every $(i, j) \in I \times J$, the map

$$\phi_j \circ f \circ \phi_i^{-1} : V_i \subset \mathbb{C}^n \to N_j \subset \mathbb{C}^m$$

is holomorphic.

A holomorphic function on a complex manifold X is a holomorphic map $f : X \to \mathbb{C}$.

1.22. **Example.** The Hopf varieties (cf. Exercise 1.20) admit a holomorphic map $f : H \to \mathbb{P}^{n-1}$ defined as follows: by definition

$$H = (\mathbb{C}^n \setminus 0)/\mathbb{Z} \qquad \text{and} \qquad \mathbb{P}^{n-1} = (\mathbb{C}^n \setminus 0)/\mathbb{C}^*$$

and it is straightforward to see that the projection $\pi : \mathbb{C}^n \setminus 0 \to \mathbb{P}^{n-1}$ factors through the projection $\pi : \mathbb{C}^n \setminus 0 \to H$.

1.23. **Exercise.** Show that the fibres of f are elliptic curves. $\square$

1.24. **Exercise.** Let V be a complex vector space of dimension n, and fix an integer $0 < r < n$. Show that there exists a natural biholomorphism between the Grassmannians (cf. Exercise 1.19)

$$G_r(V) \to G_{n-r}(V^*),$$

where V^* is the dual space of V. $\square$

1.25. **Exercise.** Show that a holomorphic function on a compact complex manifold is constant. $\square$

1.26. **Remark.** We define the category of complex manifolds as the topological spaces that locally look like open sets in some $\mathbb{C}^n$ and the holomorphic functions as the holomorphic maps to $\mathbb{C}$. While this approach is very close to the corresponding definitions in differential geometry, an equivalent approach that is closer to the spirit of modern algebraic geometry is to define a complex manifold as a ringed space $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is the structure sheaf (cf. Definition 1.57), i.e. the sheaf of rings whose sections we define to be the holomorphic ones. For more details on this point of view, cf. [Wel80, Ch.1].

1.27. **Definition.** A holomorphic map $f : X \to Y$ is a submersion (resp. immersion) if for every $x \in X$, there exists a coordinate neighbourhood of x such that the Jacobian of f has the maximal rank $\dim Y$ (resp. $\dim X$). A holomorphic map $f : X \to Y$ is an embedding if it is an immersion and f is a homeomorphism from X onto $f(X)$.

1.28. **Remark.** It is an easy exercise to check that the rank of the Jacobian does not depend on the choice of the coordinate charts. Note also that a *proper* holomorphic map $f : X \to Y$ is an embedding if and only if it is injective and immersive.

1.29. **Definition.** Let X be a complex manifold of dimension n, and let $Y \subset X$ be a closed connected subset. Then Y is a submanifold of X of codimension k if for each point $x \in Y$, there exist an open neighbourhood $U \subset X$ and a holomorphic submersion $f : U \to \mathbb{D}^k$ such that $U \cap Y = f^{-1}(0)$.

1.30. **Example.** Let X and Y be complex manifolds of dimension n and m respectively. Let $f : X \to Y$ be a holomorphic map, and $y \in Y$ such that the Jacobian J_f has rank m for every $x \in f^{-1}(y)$. Then the fibre $f^{-1}(y)$ is a submanifold of dimension $n - m$.

1.31. **Exercise.**

a) Show that a submanifold of a complex manifold is a complex manifold.

b) Show that the image of an embedding $f : X \to Y$ is a submanifold of Y. □

1.32. **Exercise.** Let X be a compact complex submanifold of $\mathbb{C}^n$. Show that X has dimension zero (Hint: cf. Exercise 1.25). □

1.33. **Exercise.** (1-dimensional complex tori)

Let $\Lambda \subset \mathbb{C}$ be a lattice, and let $X := \mathbb{C}/\Lambda$ be the associated complex torus.

a) Show that X is diffeomorphic to $S^1 \times S^1$.

b) Let $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be a biholomorphic map such that $\varphi(0) = 0$. Show that there exists a unique $\alpha \in \mathbb{C}^*$ such that $\alpha\Lambda = \Lambda'$ and such that the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "z \mapsto \alpha z"] \arrow[d, "\pi"] & \mathbb{C} \arrow[d, "\pi'"] \\ \mathbb{C}/\Lambda \arrow[r, "\varphi"] & \mathbb{C}/\Lambda' \end{tikzcd}$$

commutes. Hint: recall (or prove) that the group of biholomorphic automorphisms of $\mathbb{C}$ is $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto \alpha z + \beta \mid \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}\}$.

c) Show that X is biholomorphic to a torus of the form $X(\tau) := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ where $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$.

d) Let $\mathbb{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$ be the Poincaré upper half plane. We define a group action

$$\mathrm{SL}(2,\mathbb{Z}) \times \mathbb{H} \to \mathbb{H}, (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) \;\mapsto\; \frac{a\tau + b}{c\tau + d}.$$

Show that the biholomorphic equivalence classes of complex tori of dimension 1 have a natural bijection with $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$.

Remark: the set $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ has a natural complex structure. „The J-invariant “ defines a biholomorphism $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}$ (cf. e.g. [Eke06]). □

1.34. **Definition.** A projective manifold is a submanifold $X \subset \mathbb{P}^N$ such that there exist homogeneous polynomials $f_1, \ldots, f_k \in \mathbb{C}[X_0, \ldots, X_N]$ of degree $d_1, \ldots, d_k$ such that

$$X = \{x \in \mathbb{P}^N \mid f_1(x) = \ldots = f_k(x) = 0\}.$$

Let $f_1, \ldots, f_k \in \mathbb{C}[X_0, \ldots, X_N]$ be homogeneous polynomials. We will establish a sufficient (but not necessary!) condition for the closed set

$$X = \{x \in \mathbb{P}^N \mid f_1(x) = \ldots = f_k(x) = 0\}$$

to be a submanifold of $\mathbb{P}^N$.

Let $\pi : \mathbb{C}^{N+1} \setminus 0 \to \mathbb{P}^N$ be the projection map, we call $\pi^{-1}(X)$ the affine cone over X. It is straightforward to see that

$$\pi^{-1}(X) = \{x \in (\mathbb{C}^{N+1} \setminus 0) \mid f_1(x) = \ldots = f_k(x) = 0\},$$

where we consider the f_j as polynomials on $\mathbb{C}^{N+1}$. Suppose now that for every $x \in \pi^{-1}(X)$, the Jacobian matrix

$$J = \left(\frac{\partial f_j}{\partial z_l}\right)_{1 \leqslant j \leqslant k, 0 \leqslant l \leqslant N}$$

has rank k. Then $\pi^{-1}(X)$ is a submanifold of $(\mathbb{C}^{N+1} \setminus 0)$ of dimension $N + 1 - k$. A straightforward computation shows that X is a submanifold of $\mathbb{P}^N$ of dimension $N - k$.

1.35. **Definition.** A projective submanifold $X \subset \mathbb{P}^n$ of dimension m defined by $n - m$ homogeneous polynomials of degree $d_1, \ldots, d_{n-m}$ such that the Jacobian has rank $n - m$ in every point is called a complete intersection.

1.36. **Exercise.** Let $f_1, \ldots, f_k \in \mathbb{C}[X_0, \ldots, X_N]$ be homogeneous polynomials of degree $d_1, \ldots, d_k$ and set

$$X := \{x \in \mathbb{P}^N \mid f_1(x) = \ldots = f_k(x) = 0\}.$$

Show that X is a submanifold of codimension m if and only if the Jacobian matrix

$$J = \left(\frac{\partial f_j}{\partial z_l}\right)_{1 \leqslant j \leqslant k, 0 \leqslant l \leqslant N}$$

has rank m for every point in the affine cone $\pi^{-1}(X)$. □

1.37. **Exercise.** Let $f \in \mathbb{C}[X_0, \ldots, X_N]$ be a homogeneous irreducible polynomial and set

$$X := \{x \in \mathbb{P}^N \mid f(x) = 0\}.$$

Then the hypersurface X is smooth, i.e. a submanifold, if and only if

$$\{x \in \mathbb{P}^N \mid \frac{\partial f}{\partial X_0}(x) = \ldots = \frac{\partial f}{\partial X_N}(x) = 0\}$$

is empty. Show that this criterion is not true for homogeneous polynomials that are reducible.

Hint: this exercise is more difficult than it seems at first glance. You will need the Jacobian criterion for smoothness, e.g. [Fis76, 2.15]. □

1.38. **Exercise.** Show that

$$C := \{[X : Y : Z : T] \in \mathbb{P}^3 \mid XT - YZ = Y^2 - XZ = Z^2 - YT = 0\}$$

is a submanifold of dimension one of $\mathbb{P}^3$. Can you find two homogenuous polynomials f_1, f_2 such that, as a set,

$$C = \{[X : Y : Z : T] \in \mathbb{P}^3 \mid f_1([X : Y : Z : T]) = f_2([X : Y : Z : T]) = 0\} \ ?$$

What is the rank of the Jacobian matrix? □

1.39. **Exercise.** Let V be a complex vector space of dimension n, and fix an integer $0 < k < n$. Let $G_k(V)$ be the Grassmannian defined in Exercise 1.19. We define a map

$$\psi : G_k(V) \to \mathbb{P}(\bigwedge^k V)$$

as follows: let $U \subset V$ be a subspace of dimension k and let $u_1, \ldots, u_k$ be a basis of U. The multivector

$$u_1 \wedge \ldots \wedge u_k$$

gives a point in $\mathbb{P}(\bigwedge^k V)$.

a) Show that ψ is well-defined, i.e. does not depend on the choice of the basis. Show that ψ defines an embedding, the Plücker embedding.

b) Show that $G_k(V)$ is a projective manifold.

Hint: show that $\operatorname{im} \psi$ can be identified with the set of multivectors $w \in \bigwedge^k V$ that are decomposable, i.e. there exists vectors $v_1, \ldots, v_k \in V$ such that

$$w = v_1 \wedge \ldots \wedge v_k.$$

For every $w \in \bigwedge^k V$ consider the linear map

$$\phi_w : V \to \bigwedge^{k+1} V, v \mapsto v \wedge w.$$

and prove that w is decomposable if and only if $\operatorname{rk} \phi_w \leqslant n - k$.

c) Set $V := \mathbb{C}^4$, and let $e_1, \ldots, e_4$ be the canonical basis. Every 2-vector $w \in \bigwedge^2 \mathbb{C}^4$ has a unique decomposition

$$w = X_0 e_1 \wedge e_2 + X_1 e_1 \wedge e_3 + X_2 e_1 \wedge e_4 + X_3 e_2 \wedge e_3 + X_4 e_2 \wedge e_4 + X_5 e_3 \wedge e_4.$$

Show that for the homogeneous coordinates $[X_0 : \ldots : X_5]$ on $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$, the Plücker embedding of $G(2, \mathbb{C}^4)$ in $\mathbb{P}(\bigwedge^2 \mathbb{C}^4) \simeq \mathbb{P}^5$ has the equation

$$X_0X_5 - X_1X_4 + X_2X_3 = 0.$$

□

The statement *b*) of the preceding exercise has the following vast generalisation.

1.40**. Theorem.** (Chow's theorem) Let X be a compact complex manifold that admits an embedding $X \hookrightarrow \mathbb{P}^N$ into some projective space. Then X is algebraic, i.e. defined by a finite number of homogeneous polynomials.

1.C. **Vector bundles.** We will now define two different notions of vector bundles. Complex vector bundles are just differentiable vector bundles with complex values while holomorphic bundles have holomorphic transition functions.

1.41**. Definition.** Let X be a differentiable manifold. A complex vector bundle of rank r over X is a differentiable manifold E together with a surjective map $\pi : E \to X$ such that

1. for every $x \in X$, the fibre $E_x := \pi^{-1}(x)$ is isomorphic to $\mathbb{C}^r$
2. for every $x \in X$, there exists an open neighbourhood U of x and a diffeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that
$$\pi|_{\pi^{-1}(U)} = p_1 \circ h$$
and for all $x \in U$,
$$p_2 \circ h : E_x \to \mathbb{C}^r$$
is a $\mathbb{C}$-vector space isomorphism[3]. We call (U, h) a local trivialisation of the vector bundle E.

We call E the total space of the vector bundle and X the base space.

Let (U_α, h_α) and (U_β, h_β) be two local trivialisations of E, then the map

$$h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

induces a differentiable map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}, r)$$

where $g_{\alpha\beta}(x)$ is the $\mathbb{C}$-linear isomorphism $h_\alpha^x \circ (h_\beta^x)^{-1} : \mathbb{C}^r \to \mathbb{C}^r$. The functions $g_{\alpha\beta}$ are called the transition functions of the vector bundle E.

1.42**. Exercise.** Show that the transition functions satisfy the cocycle relations

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = Id$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ and

$$g_{\alpha\alpha} = Id$$

[3]Throughout the whole text we will denote by $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ the natural projection on the first and second factor of a product $X \times Y$.

on U_α. Vice versa given an open covering U_α and a collection of functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}, r)$ that satisfy these relations, there exists a unique complex vector bundle E with these transition functions. $\square$

1.43. **Example.** Let X be a differentiable manifold, then its tangent bundle T_X is a real vector bundle of rank $\dim_{\mathbb{R}} X =: n$ over X. The complexified vector bundle $T_X \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector bundle of rank n over X. More precisely, let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}, n)$ be the transition functions of T_X. Using the inclusion $GL(\mathbb{R}, n) \subset GL(\mathbb{C}, n)$, we define $T_X \otimes_{\mathbb{R}} \mathbb{C}$ as the complex vector bundle given by the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}, n)$. We will study this example in detail in Subsection 1.D.

1.44. **Definition.** Let X be a complex manifold. Let $\pi : E \to X$ be a complex vector bundle given by transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}, r)$. The vector bundle is holomorphic if the transition functions $g_{\alpha\beta}$ are holomorphic.

1.45. **Exercise.** Let $\pi : E \to X$ be a holomorphic vector bundle over a complex manifold X. Show that the total space E is a complex manifold. $\square$

The trivial bundle $X \times \mathbb{C}^r$ is of course a holomorphic vector bundle. More interesting and very important in the following is the tangent bundle.

1.46. **Definition.** Let X be a complex manifold of dimension n and denote by $(U_i, \phi_i : U_i \to V_i)_{i \in I}$ the corresponding atlas. We define the holomorphic tangent bundle T_X as the vector bundle of rank n that is trivial over U_i for every $i \in I$ and the transition morphisms

$$U_\alpha \cap U_\beta \times \mathbb{C}^n \subset U_\beta \times \mathbb{C}^n \to U_\alpha \cap U_\beta \times \mathbb{C}^n \subset U_\alpha \times \mathbb{C}^n$$

are given by

$$(u, v) \mapsto (u, J_{\phi_\alpha \circ \phi_\beta^{-1}, u}(v)),$$

where $J_{\phi_\alpha \circ \phi_\beta^{-1}, u}$ is the Jacobian matrix of $\phi_\alpha \circ \phi_\beta^{-1}$ at the point u.

1.47. **Remark.** As in the case of real differential geometry, one can define the tangent bundle in terms of equivalence classes of paths through a point [Biq08, Ch.1.3].

Yet another way of seeing the tangent bundle is in terms of derivations (cf. also [Wel80, p.15f]): for any open set $U \subset X$, let $\mathscr{O}_X(U)$ be the $\mathbb{C}$-algebra of holomorphic functions $f : U \to \mathbb{C}$. For a point $x \in X$ we set

$$\mathscr{O}_{X,x} := \varinjlim_{x \in U \subset X \text{open}} \mathscr{O}_X(U),$$

the $\mathbb{C}$-algebra of germs of holomorphic functions. A derivation of the algebra $\mathscr{O}_{X,x}$ is a $\mathbb{C}$-linear map $D : \mathscr{O}_{X,x} \to \mathbb{C}$ that satisfies the Leibniz rule

$$D(fg) = D(f) \cdot g(x) + f(x) \cdot D(g) \qquad \forall\ f, g \in \mathscr{O}_{X,x}.$$

The tangent space of X at x is the space of all derivations of $\mathscr{O}_{X,x}$.

1.48. **Exercise.** Let V be a complex vector space of dimension n and fix an integer $0 < r < n$. The Grassmannian $G_r(V)$ parametrises the subspaces of dimension r of V, and we denote by $[U] \in G_r(V)$ the point corresponding to $U \subset V$. In this spirit we define the total space of the tautological vector bundle $U_r(V)$ as

$$\{([U], x) \in G_r(V) \times V \mid x \in U\} \subset G_r(V) \times V.$$

The projection on the first factor gives a map $\pi : U_r(V) \to G_r(V)$. Show that this defines a holomorphic vector bundle of rank r. □

1.49. **Examples.** A very useful tool for constructing vector bundles is to take well-known constructions from linear algebra and use Exercise 1.42 to show that these constructions "glue" together to a vector bundle. More precisely let X be a complex manifold, and let E and F be complex (resp. holomorphic) vector bundles over X. Then one can define the following complex (resp. holomorphic) vector bundles over X.

- $E \oplus F$, the direct sum.
- $E \otimes F$, the tensor product.
- $\mathscr{H}om(E, F)$, the vector bundle of fibrewise $\mathbb{C}$-linear maps from E to F.
- $E^* := \mathscr{H}om(E, \mathbb{C})$, the fibrewise $\mathbb{C}$-linear maps from E to $\mathbb{C}$.
- $\bigwedge^k E$, the k-th exterior algebra of E, in particular the determinant bundle $\det E := \bigwedge^{\operatorname{rk} E} E$.
- $S^k E$, the k-th symmetric product of E.

Let us show for two examples how these constructions work: let U_α be an open covering of X that trivialises both E and F, and let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}, \operatorname{rk} E)$ and $h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}, \operatorname{rk} F)$ be the transition functions for E and F respectively. Then $E \oplus F$ is the vector bundle of rank $\operatorname{rk} E + \operatorname{rk} F$ with transition functions

$$f_{\alpha\beta} := \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}.$$

It is straightforward to see that $f_{\alpha\beta}$ satisfies the cocycle relations.

Analogously $E^* := \mathscr{H}om(E, \mathbb{C})$ is the vector bundle of rank $\operatorname{rk} E$ with transition functions

$$f_{\beta\alpha} := g^t_{\alpha\beta} \in GL(\mathbb{C}, \operatorname{rk} E).$$

Then we have

$$\begin{aligned} f_{\alpha\beta} \circ f_{\beta\gamma} \circ f_{\gamma\alpha} &= g^t_{\beta\alpha} \circ g^t_{\gamma\beta} \circ g^t_{\alpha\gamma} \\ &= (g^{-1}_{\alpha\beta})^t \circ (g^{-1}_{\beta\gamma})^t \circ (g^{-1}_{\gamma\alpha})^t \\ &= (g^t_{\gamma\alpha} \circ g^t_{\beta\gamma} \circ g^t_{\alpha\beta})^{-1} \\ &= ((g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha})^t)^{-1} = Id \end{aligned}$$

1.50. **Definition.** Let X be a complex manifold, and denote by T_X its tangent bundle. We call

$$\Omega_X := T^*_X$$

the cotangent bundle,

$$K_X := \det \Omega_X$$

the canonical bundle and

$$K_X^* = \det T_X$$

the anticanonical bundle of X.

1.51. **Definition.** Let X be a complex manifold and let $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ be complex (resp. holomorphic) vector bundles of rank r over X. We say that E_1 is isomorphic to E_2 if there exists a diffeomorphism (resp. biholomorphism) $\phi : E_1 \to E_2$ that is fibrewise $\mathbb{C}$-linear such that

$$\pi_1 = \pi_2 \circ \phi.$$

1.52. **Exercise.** Let X be a complex manifold. In analogy to Exercise 1.42 it is immediate to see that if U_α is an open covering of X and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ a collection of holomorphic functions such that

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\circ\alpha} = Id$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ and

$$g_{\alpha\alpha} = Id$$

on U_α, there exists a holomorphic line bundle (vector bundle of rank one) L with these transition functions. Show that L is isomorphic to the trivial line bundle $X \times \mathbb{C}$ if and only if up to taking a refinement of the covering, there exist holomorphic functions $s_\alpha : U_\alpha \to \mathbb{C}^*$ such that

$$g_{\alpha\beta} = \frac{s_\beta}{s_\alpha}$$

on $U_\alpha \cap U_\beta$.

Show that the set of isomorphism classes of holomorphic line bundles on X has a natural group structure. We will denote this group by $\mathrm{Pic}(X)$, the Picard group of X. $\square$

1.53. **Exercise.** Let $f : X \to Y$ be a holomorphic map between complex manifolds, and let $\pi : E \to Y$ be a complex (resp. holomorphic) vector bundle of rank r. We define the pull-back f^*E as the closed set

$$f^*E = \{(x, v) \in X \times E \mid f(x) = \pi(v)\} \subset X \times E.$$

Let $\pi' : f^*E \to X$ be the map induced on f^*E by the projection $p_1 : X \times E \to X$. Show that $\pi' : f^*E \to X$ is a complex (resp. holomorphic) vector bundle of rank r. If $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}, r)$ are the transition functions of E, what are the transition functions of f^*E ? $\square$

This exercise shows in particular that if $U \subset X$ is a submanifold and E a vector bundle over X, then the restriction $E|_U$ given by restricting the transition functions to U defines a vector bundle.

1.54. **Definition.** Let $\pi : E \to X$ be a complex vector bundle over a differentiable manifold E. A (global) section of E is a differentiable map $s : X \to E$ such that $\pi \circ s = Id$.

If X is a complex manifold and E is a holomorphic vector bundle, a (global) section of E is a holomorphic map $s : X \to E$ such that $\pi \circ s = Id$.

1.55. **Remarks.**

1. The definition of a section makes sense, since the total space of a holomorphic vector bundle is a complex manifold.

2. Complex vector bundles have many sections, since we have local bump functions at our disposal. For holomorphic vector bundles, the situation is very different (see Exercise 1.60 below).

3. The set of global sections of a complex or holomorphic vector bundle has a natural $\mathbb{C}$-vector space structure given by fibrewise addition. If E is a complex vector bundle over X, we will denote by

$$C^\infty(X, E)$$

the space of differentiable sections. If E is a holomorphic vector bundle over X, we will denote by

$$\Gamma(X, E)$$

the space of holomorphic sections.

4. Let E be a complex (resp. holomorphic) vector bundle of rank r over X, and let $x \in X$ be a point. A local frame (resp. local holomorphic frame) of E around x is given by an open neighbourhood $x \in U \subset X$ and sections $s_1, \ldots, s_r \in C^\infty(U, E)$ (resp. $s_1, \ldots, s_r \in \Gamma(U, E)$) such that for all $x \in U$, the vectors $s_1(x), \ldots, s_r(x)$ are a basis of E_x.

As an example, let T_X be the tangent bundle. Let $x \in U \subset X$ be a coordinate neighbourhood with local holomorphic coordinates $z_1, \ldots, z_n$. The description of the tangent bundle in terms of derivation (Remark 1.47) shows that the partial derivations $\frac{\partial}{\partial z_j}$ form a holomorphic frame for $T_X|_U$.

1.56. **Exercise.** Let $\pi : E \to X$ be a complex (resp. holomorphic) vector bundle of rank r over X, and let $x \in X$ be a point. Let $s_1, \ldots, s_r \in C^\infty(U, E)$ (resp. $s_1, \ldots, s_r \in \Gamma(U, E)$) be a local frame (resp. local holomorphic frame). Show that the frame induces a trivialisation $h : \pi^{-1}(U) \to U \times \mathbb{C}^r$. $\square$

1.57. **Definition.** Let X be a topological space. A sheaf of abelian groups $\mathscr{F}$ on X consists of the data

a) for every open set $U \subset X$, an abelian group $\mathscr{F}(U)$ and

b) for every inclusion $V \subset U$ of open sets, a morphism of abelian groups

$$r_{UV} : \mathscr{F}(U) \to \mathscr{F}(V),$$

that satisfies the following conditions:

(1) $\mathscr{F}(\emptyset) = 0$.
(2) r_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$.
(3) If $W \subset V \subset U$ are three open subsets, then $r_{UW} = r_{VW} \circ r_{UV}$.
(4) If U is an open subset and V_i is an open covering of U, and $s \in \mathscr{F}(U)$ such that $r_{UV_i}(s) = 0$ for all i, then $s = 0$.
(5) If U is an open subset and V_i is an open covering of U, and $s_i \in \mathscr{F}(V_i)$ are sections such that

$$r_{V_i(V_i \cap V_j)}(s_i) = r_{V_j(V_i \cap V_j)}(s_j)$$

for all i, j, then there exists a unique $s \in \mathscr{F}(U)$ such that $r_{UV_i}(s) = s_i$.

1.58. **Exercise.** a) Let X be a complex manifold, and let E be a holomorphic vector bundle over X. For every $U \subset X$ an open set, let $\Gamma(U, E)$ be the vector space of sections of $E|_U$. Furthermore if $V \subset U$ is another open set, we define $r_{UV} : \Gamma(U, E) \to \Gamma(V, E)$ by restricting a section of $E|_U$ to the open subset V.

Show that this defines a sheaf of abelian groups on X which we will call the sheaf of sections $\mathscr{O}_X(E)$. In particular if we take E to be the trivial bundle, this shows that the holomorphic functions form a sheaf of abelian groups (in fact of a sheaf of rings), the structure sheaf $\mathscr{O}_X$.

Note that the same statement (and proof) holds for the space of sections $C^\infty(U, E)$ of a complex vector bundle E.

b) Let X be a complex manifold. We say that a sheaf of abelian groups $\mathscr{F}$ on X is invertible if there exists an open covering $(U_\alpha)_{\alpha \in A}$ such that $\mathscr{F}|_{U_\alpha} \simeq \mathscr{O}_{U_\alpha}$. Show that we have a bijection between invertible sheaves and holomorphic line bundles on X. $\square$

We come now to the most important line bundle in algebraic geometry: the tautological line bundle over the projective space.

1.59. **Example.** Recall that $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^*$ can be seen as the set of lines in $\mathbb{C}^{n+1}$ passing through the origin and we denote by $[l] \in \mathbb{P}^n$ the point corresponding to $l \subset \mathbb{C}^{n+1}$. In this spirit we define the total space of the tautological line bundle $\mathscr{O}_{\mathbb{P}^n}(-1)$ as

$$\{([l], x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid x \in l\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}.$$

The projection on the first factor $p_1 : \mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathbb{P}^n$ gives a map $\pi : \mathscr{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$ and it is clear that $\pi^{-1}(l)$ is exactly the line $l \subset \mathbb{C}^{n+1}$.

Let $U_i = \{[l] \in \mathbb{P}^n \mid l_i \neq 0\}$ be the standard open set, then we define a section $s_i \in \Gamma(U_i, \mathscr{O}_{\mathbb{P}^n}(-1))$ by

$$[l_0 : \ldots : l_n] \ \mapsto \ (\frac{l_0}{l_i}, \ldots, \frac{l_n}{l_i}).$$

Since the i-th component of s_i is equal to 1, the section s_i does not vanish in any point. Therefore we can use s_i to define the local trivialisation

$$h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}, \ ([l], x) \ \mapsto \ ([l], \lambda_i),$$

where λ_i is the unique complex number such that $x = \lambda_i s_i([l])$.

On the open set $U_i \cap U_j$, we have

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{C} \to (U_i \cap U_j) \times \mathbb{C}, ([l], \lambda_j) \mapsto ([l], \lambda_j s_j([l])) \mapsto ([l], \lambda_i),$$

where λ_j is the unique complex number such that $\lambda_i s_i([l]) = \lambda_j s_j([l])$. Looking at the i-th coordinate, we see that

$$\lambda_i = \lambda_j \frac{l_i}{l_j}.$$

Thus the transition function $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ is given by

$$g_{ij} = \frac{l_i}{l_j}.$$

Using the constructions of vector bundles in Example 1.49, we define for all $k \in \mathbb{N}$

$$\mathscr{O}_{\mathbb{P}^n}(-k) := \mathscr{O}_{\mathbb{P}^n}(-1)^{\otimes k}$$

and

$$\mathscr{O}_{\mathbb{P}^n}(k) := \mathscr{O}_{\mathbb{P}^n}(-k)^*.$$

Set furthermore $\mathscr{O}_{\mathbb{P}^n}(0)$ for the trivial line bundle, then for all $k \in \mathbb{Z}$ the transition functions of the line bundle $\mathscr{O}_{\mathbb{P}^n}(k)$ are

$$g_{ij} = (\frac{l_j}{l_i})^k.$$

1.60. **Exercise.** Let $\mathscr{O}_{\mathbb{P}^1}(k)$ be the line bundles on $\mathbb{P}^1$ defined in Example 1.59. Show that

$$\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(k)) = \begin{cases} 0 & \text{if } k < 0, \\ \mathbb{C} & \text{if } k = 0, \\ \text{homog. polynomials of two variables of degree } k & \text{if } k > 0. \end{cases}$$

Generalise the statement to the line bundles $\mathscr{O}_{\mathbb{P}^n}(k)$ on $\mathbb{P}^n$. $\square$

1.61. **Definition.** Let $\pi : E \to X$ be a complex (resp. holomorphic) vector bundle of rank r over a complex manifold. A submanifold $F \subset E$ is a subbundle of rank m if

(1) $F \cap E_x$ is a subvector space of dimension m for every $x \in X$,
(2) $\pi|_F : F \to X$ has the structure of complex (resp. holomorphic) vector bundle induced by E, i.e. there exist local trivialisations U_i for E and F such that the transition functions of F are the restriction of the transition function of E to the corresponding subspaces.

1.62. **Examples.**

1. The tautological bundle $\mathscr{O}_{\mathbb{P}^n}(-1)$ is a subbundle of the trivial vector bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$.

2. Let $Y \subset X$ be a submanifold of a complex manifold X. Then the tangent bundle T_Y is a subbundle of the restricted tangent bundle $T_X|_Y$.

1.63**. Exercise.** (a bit harder, but interesting) Let X be a complex manifold, and let E be a holomorphic subbundle of rank r of the trivial vector bundle $X \times \mathbb{C}^n$. Show that there exists a unique holomorphic map $f : X \to G_r(\mathbb{C}^n)$ such that $E = f^* U_r(\mathbb{C}^n)$, where $U_r(\mathbb{C}^n)$ is the tautological bundle (Exercise 1.48). Hint: set-theoretically, the definition of f is clear. For $x \in X$, the image $f(x)$ is the point corresponding to the subspace $E_x \subset \mathbb{C}^n$. $\square$

1.64**. Exercise.** Let X be a complex manifold, and let $\pi : E \to X$ and $\psi : F \to X$ be holomorphic vector bundles over X. A morphism of vector bundles of rank k is a holomorphic map $\phi : E \to F$ such that $\pi = \psi \circ \phi$ and for every $x \in X$, the induced map

$$\phi_x := \phi|_{E_x} : E_x \to F_x$$

is $\mathbb{C}$-linear of rank k. Show that $\operatorname{im} \phi$ is a holomorphic subbundle of rank k of F. We set

$$\ker \phi := \{e \in E \mid \phi_{\pi(e)}(e) = 0\}.$$

Show that $\ker \phi$ is a holomorphic subbundle of rank $\operatorname{rk} E - k$ of E. $\square$

1.65**. Exercise.** Let X be a complex manifold of dimension n, and let S, E and Q be holomorphic vector bundles over X. Let $\phi : S \to E$ and $\psi : E \to Q$ be morphisms of vector bundles. We say that the sequence

$$S \xrightarrow{\phi} E \xrightarrow{\psi} Q$$

is exact at E if $\operatorname{im} \phi = \ker \psi$.

a) Let

$$0 \to S \xrightarrow{\phi} E \xrightarrow{\psi} Q \to 0.$$

be an exact sequence of vector bundles, i.e. a sequence that is exact at S, E and Q. Show that we have an induced isomorphism

$$\det E \simeq \det S \otimes \det Q$$

b) Let $L \to X$ be a holomorphic line bundle, and let $\sigma \in \Gamma(X, L)$ be a non-zero section. We set

$$D := \{x \in X \mid \sigma(x) = 0\}$$

and suppose that D is smooth. Show that there exists a well-defined global section

$$d\sigma \in \Gamma(D, (\Omega_X \otimes L)|_D),$$

such that locally (i.e. in a trivialising neighbourhood U of L) we have $d\sigma|_{U \cap D} = ds$ where s is a holomorphic function on U corresponding to the section σ.

Suppose now that $d\sigma(x) \neq 0$ for all $x \in D$. Show that we have an exact sequence on D

$$0 \to T_D \to T_X|_D \to L|_D \to 0,$$

where $T_D \to T_X|_D$ is the natural inclusion between of the tangent bundles. Deduce the adjunction formula

$$K_D \simeq (K_X \otimes L)|_D.$$

c) Show that on $X = \mathbb{P}^n$ we have an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0. \tag{1.1}$$

This sequence is called the Euler sequence. Deduce that

$$K^*_{\mathbb{P}^n} \simeq \mathscr{O}_{\mathbb{P}^n}(n+1).$$

d) Let $H \subset \mathbb{P}^n$ be a submanifold defined by a homogeneous polynomial of degree d. Show that we have an exact sequence on H

$$0 \to T_H \to T_{\mathbb{P}^n}|_H \to \mathscr{O}_{\mathbb{P}^n}(d)|_H \to 0,$$

where $T_H \to T_{\mathbb{P}^n}|_H$ is the natural inclusion of tangent bundles. Deduce that

$$K^*_H \simeq \mathscr{O}_{\mathbb{P}^n}(n+1-d)|_H.$$

Generalise to the case of a complete intersection (cf. Definition 1.35).

e) Show that the twisted cubic (cf. Exercise 1.38) is not a complete intersection. □

1.D. **The complexified (co-)tangent bundle.** We will now start the systematic investigation of the relation between the differentiable and the complex structure of a complex manifold. The main tool will be the decomposition of the complexified (co-)tangent bundle into holomorphic and anti-holomorphic parts. We will illustrate the concept on the example of vector spaces, then generalise to the situation of vector bundles.

Let V be a real vector space of real dimension $2n$, and let $J : V \to V$ be a $\mathbb{R}$-linear isomorphism such that $J^2 = -Id$. We call J a complex structure on V. Indeed, J induces a structure of complex vector space where the scalar multiplication is defined by

$$(\alpha + i\beta)v := \alpha v + \beta J(v) \qquad \forall\ \alpha, \beta \in \mathbb{R}.$$

Vice versa if V is a complex vector space of dimension n, then it can be considered as a real vector space of dimension $2n$ and the multiplication by i defines an $\mathbb{R}$-linear endomorphism of V that is a complex structure.

Let V be a real vector space of real dimension $2n$, and let J be a complex structure on V. Consider the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ of V, then V has complex dimension $2n$. We extend J to a $\mathbb{C}$-linear map on $V \otimes_{\mathbb{R}} \mathbb{C}$ by setting

$$J(v \otimes \alpha) = J(v) \otimes \alpha.$$

It is clear that the extended morphism still satisfies $J^2 = -Id$, so the endomorphism J is diagonalisable and has two eigenvalues $\{i, -i\}$. We denote by $V^{1,0}$ (resp. $V^{0,1}$) the eigenspace corresponding to i (resp. $-i$). Thus we get a canonical identification

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}.$$

Furthermore we can define a conjugation on $V \otimes_{\mathbb{R}} \mathbb{C}$ by setting

$$\overline{v \otimes \alpha} = v \otimes \overline{\alpha} \qquad \forall\ v \in V, \alpha \in \mathbb{C}.$$

With this definition, we obtain an equality of subspaces

$$V^{0,1} = \overline{V^{1,0}}.$$

1.66**. Example.** Let $\mathbb{C}^n$ the complex vector space of n-tuples $(z_1, \ldots, z_n)$, and let $z_j = a_j + ib_j$ be its decomposition in real and imaginary parts: this gives an identification of $\mathbb{C}^n$ with the real vector space of $2n$-tuples $(a_1, b_1, \ldots, a_n, b_n)$. The scalar multiplication by i in $\mathbb{C}^n$ induces a linear map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$(a_1, b_1, \ldots, a_n, b_n) \mapsto (-b_1, a_1, \ldots, -b_n, a_n).$$

We call J the standard complex structure on $\mathbb{R}^{2n}$.

Denote now by $x_1, y_1, \ldots, x_n, y_n$ the canonical basis on the complexified vector space $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. The extended morphism J is then given by

$$x_j \mapsto y_j, \ y_j \mapsto -x_j.$$

Therefore

$$x_j - iy_j \qquad \forall\, j \in \{1, \ldots, n\}$$

forms a basis of the i-eigenspace and

$$x_j + iy_j \qquad \forall\, j \in \{1, \ldots, n\}$$

forms a basis of the $-i$-eigenspace. Since $\overline{x_j} = x_j$ and $\overline{y_j} = y_j$, we have

$$\overline{x_j - iy_j} = x_j + iy_j,$$

so the $-i$-eigenspace is the conjugate of the i-eigenspace.

We will now define the analogue of a complex structure in the case of differentiable manifolds.

1.67**. Definition.** Let X be a differentiable manifold of dimension $2n$. An almost complex structure on X is a differentiable vector bundle isomorphism $J : T_X \to T_X$ such that $J^2 = -Id$.

1.68**. Remark.** In general a differentiable manifold of even dimension does not admit an almost complex structure [Wel80, p.31].

1.69**. Proposition.** A complex manifold X induces an almost complex structure on its underlying differentiable manifold, that is it defines a differentiable vector bundle isomorphism $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ such that $J^2 = -Id$.

The key point of this proposition is as follows: the tangent space T_X of a complex manifold is a holomorphic vector bundle of rank $\dim X$, in particular $T_{X,x}$ is a complex vector space of dimension n. Let $T_{X,\mathbb{R}}$ be the tangent bundle of the underlying differentiable manifold, then $T_{X,\mathbb{R},x}$ is a real vector space of dimension $2n$. The complex structure on $T_{X,\mathbb{R},x}$ will be defined by constructing a canonical isomorphism between $T_{X,\mathbb{R},x}$ and the real vector space underlying $T_{X,x}$.

Proof. We follow the proof in [Wel80] and proceed in two steps : first we define for every $x \in X$ a complex structure on $T_{X,\mathbb{R},x}$. Then we show that the complex structure does not depend on the choices made in the definition. It will be immediate from the construction that the vector bundle isomorphism $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ is differentiable.

Step 1. Fix a point $x \in X$ and let $\phi : U \to V \subset \mathbb{C}^n$ be a coordinate neighbourhood such that $\phi(x) = 0$. Denote by $z_1, \dots, z_n$ the local holomorphic coordinates around x, and by

$$x_1 = \mathrm{Re}(z_1), y_1 = \mathrm{Im}(z_1), \dots, x_n = \mathrm{Re}(z_n), y_n = \mathrm{Im}(z_n)$$

the local differentiable coordinates induced by them. Then the holomorphic vector fields $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ (resp. the differentiable vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$) give a local frame of T_X (resp. $T_{X,\mathbb{R}}$), so they define a biholomorphism

$$T_X|_U \simeq U \times \mathbb{C}^n$$

and a diffeomorphism

$$T_{X,\mathbb{R}}|_U \simeq U \times \mathbb{R}^{2n}$$

The construction of the coordinates gives an isomorphism of (real) vector bundles

$$U \times \mathbb{C}^n \simeq U \times \mathbb{R}^{2n}.$$

Thus we have an isomorphism of differentiable vector bundles

$$T_X|_U \simeq U \times \mathbb{C}^n \simeq T_{X,\mathbb{R}}|_U$$

which defines a complex structure on $T_{X,\mathbb{R}}|_U$.

Step 2. In order to see that the definition does not depend on the choice of the holomorphic coordinates, let $f : V \to V$ be a biholomorphism such that $f(0) = 0$. Let $\zeta_1, \dots, \zeta_n$ be the local holomorphic coordinates around x such that

$$\zeta_j = f_j(z_1, \dots, z_n),$$

and

$$\xi_1 = \mathrm{Re}(\zeta_1), \eta_1 = \mathrm{Im}(\zeta_1), \dots, \xi_n = \mathrm{Re}(\zeta_n), \eta_n = \mathrm{Im}(\zeta_n)$$

the local differentiable coordinates induced by them. The diffeomorphism f can then be expressed in these local coordinates by

$$\xi_j = u_j(x_1, \dots, x_n, y_1, \dots, y_n) \qquad \eta_j = v_j(x_1, \dots, x_n, y_1, \dots, y_n)$$

where u_j and v_j are the real and imaginary parts of f_j. By definition of the tangent bundle, the real Jacobian of this map is a transition function between the corresponding trivialisations of the tangent bundle. Since for both trivialisations, the complex structure is defined by the standard complex structure, we only have to check if the transition function commutes with the operator J. The real Jacobian is a $n \times n$ matrix of 2×2-blocks

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}.$$

Since f is holomorphic the Cauchy-Riemann equations (1.10) hold, so

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_j}{\partial y_k} & \frac{\partial u_j}{\partial y_k} \\ -\frac{\partial u_j}{\partial y_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}.$$

Thus the Jacobian is a $n \times n$ matrix of 2×2-blocks of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Since the operator J is the standard complex structure, its matrix is a $n \times n$ matrix of 2×2-blocks of the form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

along the diagonal and zero elsewhere. It is straightforward to see that the two matrices commute. □

1.70**. Remark.** A much harder question to answer is which almost complex structures arise from a structure of complex manifolds. This question is answered by the Newlander-Nirenberg theorem: an almost complex structure comes from a complex structure if and only if the almost complex structure is integrable in terms of Lie brackets. For a proof of this statement in the case when X is a real-analytic manifold, cf. [Voi02, p.56].

As in the case of vector spaces, the existence of a complex structure on $T_{X,\mathbb{R}}$ induces a canonical decomposition of the complexified bundle: let X be a complex manifold, and set

$$T_{X,\mathbb{C}} := T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

for the complexification of the real tangent bundle (it is a complex vector bundle of rank $2n$ over X, cf. Example 1.43). We extend the complex structure $J : T_{X,\mathbb{R}} \rightarrow T_{X,\mathbb{R}}$ to a $\mathbb{C}$-linear isomorphism

$$J \otimes_{\mathbb{R}} Id_{\mathbb{C}} : T_{X,\mathbb{C}} \rightarrow T_{X,\mathbb{C}}$$

which we still denote by J and which satisfies $J^2 = -Id$. We denote by $T_X^{1,0}$ (resp. $T_X^{0,1}$) the vector bundle of $+i$-eigenspaces (resp. $-i$-eigenspaces) for J. These are complex vector bundles of rank n and we have

$$T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}.$$

We extend the conjugation on $\mathbb{C}$ to a conjugation on $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ by tensoring with $Id_{T_{X,\mathbb{R}}}$. By the definition of $T_X^{1,0}$ and $T_X^{0,1}$ we get

$$T_X^{0,1} = \overline{T_X^{1,0}}.$$

Let T_X be the holomorphic tangent bundle of X, then we have a natural inclusion

$$T_X \hookrightarrow T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

which can locally be defined as follows: let $z_1, \ldots, z_n$ be local holomorphic coordinates and $x_j = \mathrm{Re}(z_j)$ and $y_j = \mathrm{Im}(z_j)$ the local coordinates induced by $z_1, \ldots, z_n$ on the underlying differentiable manifold. Then we have

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) \qquad \forall\, j \in \{1, \ldots, n\}.$$

Since in these local coordinates J maps $\frac{\partial}{\partial x_j}$ to $\frac{\partial}{\partial y_j}$ and $\frac{\partial}{\partial y_j}$ to $-\frac{\partial}{\partial x_j}$, the subbundle $T_X \hookrightarrow T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ identifies to the subbundle of i-eigenspaces $T_X^{1,0}$. Although this identification shows that $T_X^{1,0}$ naturally carries a structure of holomorphic vector bundle, we will consider it in the following as a mere complex vector bundle. In particular a section of $T_X^{1,0}$ is only supposed to be a differentiable section. Note furthermore that the holomorphic coordinates $z_1, \ldots, z_n$ induce a local (anti-holomorphic) frame of the complex vector bundle $T_X^{0,1}$ given by

$$\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}).$$

By duality, the decomposition

$$T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}.$$

induces a decomposition of $\Omega_{X,\mathbb{C}} := T^*_{X,\mathbb{C}}$ into

$$\Omega_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}, \tag{1.2}$$

where $\Omega_X^{1,0} := (T_X^{1,0})^*, \Omega_X^{0,1} = (T_X^{0,1})^*$. Thus we get a decomposition of the complex-valued differentiable 1-forms into what we will call forms of type $(1,0)$ and $(0,1)$. More generally we get

$$\Omega^k_{X,\mathbb{C}} := \bigwedge^k \Omega_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega_X^{p,q},$$

where

$$\Omega_X^{p,q} := \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}$$

is the vector bundle of (p,q)-forms on X.

1.71. **Definition.** A k-form of type (p,q) with $p+q=k$ is a differentiable section of the subbundle $\Omega_X^{p,q} \subset \Omega^k_{X,\mathbb{C}}$.

The formal definition of the vector bundles $\Omega_X^{p,q}$ can be easily understood in local coordinates: fix a point $x \in X$, and let $z_1, \ldots, z_n$ be local holomorphic coordinates around x. Let $x_1, y_1, \ldots, x_n, y_n$ be the corresponding local differentiable coordinates, then

$$dx_1, dy_1, \ldots, dx_n, dy_n$$

are a local frame of $\Omega_{X,\mathbb{C}}$. Let $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}})$ be a 1-form, then we have locally

$$\alpha = \sum_{j=1}^n \alpha_j dx_j + \beta_j dy_j.$$

We have seen before that $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial \overline{z_1}}, \ldots, \frac{\partial}{\partial \overline{z_n}}$ are a local frame of $T_X^{1,0}$ and $T_X^{0,1}$, so we can define the dual frames $dz_1, \ldots, dz_n$ and $d\overline{z_1}, \ldots, d\overline{z_n}$. Note that

$$dz_j = dx_j + idy_j$$

and

$$d\overline{z_j} = dx_j - idy_j$$

for all $j \in \{1, \ldots, n\}$. Thus we get a local decomposition of α in its components of type $(1,0)$ and $(0,1)$

$$\alpha = \sum_{j=1}^{n} \gamma_j dz_j + \delta_j d\overline{z_j}.$$

where $\gamma_j = \alpha_j - i\beta_j$ and $\delta_j = \alpha_j + i\beta_j$. More generally, let $\alpha \in C^\infty(X, \Omega_X^{p,q})$ be a form of type (p,q), then we have in local coordinates

$$\alpha = \sum_{|J|=p, |K|=q} \alpha_{J,K} dz_J \wedge d\overline{z_K},$$

where $\alpha_{J,K}$ are differentiable functions and we use the usual multi-index notation.

1.72. **Exercise.** Let $f : X \to Y$ be a holomorphic map between complex manifolds.

a) Show that the pull-back

$$f^* : C^\infty(Y, \Omega^k_{Y,\mathbb{C}}) \to C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

respects the decomposition in forms of type (p,q), i.e. if ω has type (p,q), then also $f^*\omega$.

b) Let $T_f : T_{X,\mathbb{C}} \to f^*T_{Y,\mathbb{C}}$ be the tangent map. Show that $T_f(T_X^{1,0}) \subset f^*T_Y^{1,0}$ and $T_f(T_X^{0,1}) \subset f^*T_Y^{0,1}$. $\square$

1.73. **Exercise.** Let $z_1, \ldots, z_n$ and $x_j = \mathrm{Re}(z_j), y_j = \mathrm{Im}(z_j)$ be the canonical complex and real coordinates on $\mathbb{C}^n$. We denote by

$$d\lambda = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$$

the standard volume form on $\mathbb{R}^{2n}$. Show that we have

$$d\lambda = \frac{i}{2} dz_1 \wedge d\overline{z_1} \wedge \ldots \wedge \frac{i}{2} dz_n \wedge d\overline{z_n}$$

and that for every $\varphi \in \mathrm{End}(\mathbb{C}^n)$

$$\varphi^* d\lambda = |\det_{\mathbb{C}} \varphi|^2 d\lambda$$

and $\det_{\mathbb{R}} \varphi = |\det_{\mathbb{C}} \varphi|^2$, where $\det_{\mathbb{R}} \varphi$ denotes the determinant of φ seen as an element of $\mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$.

Deduce that a complex variety always admits a canonical orientation. (Hint: show that if $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ is a local frame of T_X, the local frame $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$ is an oriented basis of $T_{X,\mathbb{R}}$.)

Deduce that we have a canonical isomorphism

$$e : H_c^{2n}(X, \mathbb{C}) \to \mathbb{C}, [\alpha] \mapsto \int_X \alpha.$$

$\square$

1.E. **Exterior differentials.** If X is a differentiable manifold, we can consider for every $k \in \mathbb{N}$ the exterior differential

$$d : C^\infty(X, \Omega_X^k) \to C^\infty(X, \Omega_X^{k+1}),$$

which satisfies the Leibniz rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \qquad \forall\ \alpha \in C^\infty(X, \Omega_X^k), \beta \in C^\infty(X, \Omega_X^l)$$

Let now X be a complex manifold, then $d \otimes \mathrm{Id}_\mathbb{C}$ defines an exterior differential on the complexified cotangent bundle $\Omega_{X,\mathbb{C}} = \Omega_{X,\mathbb{R}} \otimes_\mathbb{R} \mathbb{C}$ which for simplicity of notation we will denote by

$$d : C^\infty(X, \Omega_{X,\mathbb{C}}^k) \to C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1}).$$

If $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$ is a form of type (p,q) then we can decompose $d\alpha$ according to the decomposition

$$C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1}) = \bigoplus_{p'+q'=k+1} C^\infty(X, \Omega_X^{p',q'}),$$

into

$$d\alpha = \sum_{p'+q'=k+1} \beta^{p',q'}$$

and it is natural to ask how this decomposition looks like. We start by considering the case where $\alpha : X \to \mathbb{C}$ is a complex-valued differentiable function on X (i.e. a section of $C^\infty(X, \Omega_{X,\mathbb{C}}^0)$). Then

$$d\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}) = C^\infty(X, \Omega_X^{1,0}) \oplus C^\infty(X, \Omega_X^{0,1})$$

and we define $\partial\alpha$ (resp. $\overline{\partial}\alpha$) to be the $(1,0)$-part (resp. $(0,1)$-part).

Fix now a point $x \in X$, and let $z_1, \ldots, z_n$ be local holomorphic coordinates around x. Let $x_1, y_1, \ldots, x_n, y_n$ be the corresponding local differentiable coordinates, then

$$\begin{aligned} d\alpha &= \sum_{j=1}^n \frac{\partial\alpha}{\partial x_j} dx_j + \frac{\partial\alpha}{\partial y_j} dy_j \\ &= \sum_{j=1}^n \frac{1}{2}(\frac{\partial\alpha}{\partial x_j} - i\frac{\partial\alpha}{\partial y_j}) dz_j + \sum_{j=1}^n \frac{1}{2}(\frac{\partial\alpha}{\partial x_j} + i\frac{\partial\alpha}{\partial y_j}) d\overline{z_j} \\ &= \sum_{j=1}^n \frac{\partial\alpha}{\partial z_j} dz_j + \frac{\partial\alpha}{\partial \overline{z_j}} d\overline{z_j}, \end{aligned}$$

thus we get the local expressions

$$\partial\alpha = \sum_{j=1}^n \frac{\partial\alpha}{\partial z_j} dz_j$$

and

$$\overline{\partial}\alpha = \sum_{j=1}^n \frac{\partial\alpha}{\partial \overline{z_j}} d\overline{z_j}.$$

More generally, let $\alpha \in C^\infty(X, \Omega_X^{p,q})$ be a complex-valued differentiable form of type (p,q) given in local coordinates

$$\alpha = \sum_{|J|=p,|K|=q} \alpha_{J,K} dz_J \wedge d\overline{z}_K,$$

where $\alpha_{J,K}$ are differentiable functions. Since $d(dz_J \wedge d\overline{z}_K) = 0$ for all multiindices J, K, we get by the Leibniz rule

$$\begin{aligned} d\alpha &= \sum_{|J|=p,|K|=q} d(\alpha_{J,K}) dz_J \wedge d\overline{z}_K \\ &= \sum_{|J|=p,|K|=q} \partial(\alpha_{J,K}) dz_J \wedge d\overline{z}_K + \sum_{|J|=p,|K|=q} \overline{\partial}(\alpha_{J,K}) dz_J \wedge d\overline{z}_K \\ &= \sum_{|J|=p,|K|=q} \sum_{l=1}^n \frac{\partial \alpha_{J,K}}{\partial z_l} dz_l \wedge dz_J \wedge d\overline{z}_K + \sum_{|J|=p,|K|=q} \frac{\partial \alpha_{J,K}}{\partial \overline{z_l}} d\overline{z_l} \wedge dz_J \wedge d\overline{z}_K. \end{aligned}$$

We see that $d\alpha$ decomposes uniquely into a sum of two forms, one of type $(p+1,q)$, the other of type $(p,q+1)$, so

$$d\alpha \in C^\infty(X, \Omega_X^{p+1,q}) \oplus C^\infty(X, \Omega_X^{p,q+1}).$$

1.74. **Definition.** Let X be a complex manifold, and let $\alpha \in C^\infty(X, \Omega_X^{p,q})$ be a differentiable form of type (p,q). Then we define $\overline{\partial}\alpha$ (resp. $\partial\alpha$) as the component of type $(p,q+1)$ (resp. $(p+1,q)$) of $d\alpha$.

More generally, if $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$ is a complex-valued differentiable form of degree k, let $\alpha = \sum_{p+q=k} \alpha^{p,q}$ be its unique decomposition in forms of type (p,q). Then we set

$$\overline{\partial}\alpha := \sum_{p+q=k} \overline{\partial}\alpha^{p,q}, \qquad \partial\alpha := \sum_{p+q=k} \partial\, \alpha^{p,q}.$$

Note that by definition, we have

$$\partial\alpha = \overline{\overline{\partial}\overline{\alpha}}.$$

We will now show some properties of these operators.

1.75. **Lemma.** Let X be a complex manifold, and let $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$ and $\beta \in C^\infty(X, \Omega_{X,\mathbb{C}}^{k'})$. Then the operators $\overline{\partial}$ and ∂ satisfy the Leibniz rule, that is

$$\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^k \alpha \wedge \overline{\partial}\beta$$

and

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^k \alpha \wedge \partial\beta.$$

Proof. Note first that by the additivity of $\overline{\partial}$ it is sufficient to show the equality under the additional assumption that α has type (p,q) and β has type (p',q'). By the usual Leibniz rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta,$$

thus

$$\begin{aligned} d(\alpha\wedge\beta) &= d\alpha\wedge\beta+(-1)^k\alpha\wedge d\beta \\ &= \overline{\partial}\alpha\wedge\beta+\partial\alpha\wedge\beta+(-1)^k\alpha\wedge\overline{\partial}\beta+(-1)^k\alpha\wedge\partial\beta \\ &= (\overline{\partial}\alpha\wedge\beta+(-1)^k\alpha\wedge\overline{\partial}\beta)+(\partial\alpha\wedge\beta+(-1)^k\alpha\wedge\partial\beta) \end{aligned}$$

gives a decomposition in forms of type $(p+p', q+q'+1)$ and $(p+p'+1, q+q')$. By definition $\overline{\partial}(\alpha\wedge\beta)$ (resp. $\partial(\alpha\wedge\beta)$)is the component of type $(p+p', q+q'+1)$ (resp. $(p+p'+1, q+q')$). □

1.76. **Lemma.** Let X be a complex manifold, then we have the following relations:

$$\overline{\partial}^2=0, \qquad \overline{\partial}\partial+\partial\overline{\partial}=0, \qquad \partial^2=0.$$

Proof. Note again that by the additivity of the differential operators it is sufficient to check the equality for forms α of type (p,q). By definition, we have $d=\partial+\overline{\partial}$, so $d^2=0$ implies

$$0=d^2\alpha=\overline{\partial}^2\alpha+(\overline{\partial}\partial+\partial\overline{\partial})\alpha+\partial^2\alpha.$$

Since $\overline{\partial}^2\alpha$ (resp. $(\overline{\partial}\partial+\partial\overline{\partial})\alpha$, resp. $\partial^2\alpha$) have type $(p,q+2)$ (resp. $(p+1,q+1)$, resp. $(p,q+2)$), all three forms are zero. □

The preceding lemma shows that for each $p\in\{0,\dots,n\}$ we can define a cohomological complex of $\mathbb{C}$-vector spaces

$$0\to C^\infty(X,\Omega_X^{p,0})\xrightarrow{\overline{\partial}} C^\infty(X,\Omega_X^{p,1})\xrightarrow{\overline{\partial}}\dots\xrightarrow{\overline{\partial}} C^\infty(X,\Omega_X^{p,n})\to 0.$$

For every $q\in\{0,\dots,n\}$, we define the cocycles

$$Z^{p,q}(X):=\{\alpha\in C^\infty(X,\Omega_X^{p,q})\mid\overline{\partial}\alpha=0\}$$

and coboundaries

$$B^{p,q}(X):=\{\alpha\in C^\infty(X,\Omega_X^{p,q})\mid\exists\beta\in C^\infty(X,\Omega_X^{p,q-1}),\overline{\partial}\beta=\alpha\}.$$

The Dolbeault cohomology groups of X are the cohomology associated to the complexes above, i.e.

$$H^{p,q}(X):=H^q_{\overline{\partial}}(C^\infty(X,\Omega_X^{p,\bullet})):=Z^{p,q}(X)/B^{p,q}(X).$$

If the vector spaces $H^{p,q}(X)$ have finite dimension[4], their dimensions

$$h^{p,q}:=\dim H^{p,q}(X)$$

are called the Hodge numbers of X.

One of the main objectives of these lectures is to get a better understanding of these Dolbeault cohomology groups. We will see later that if X is a compact complex manifold, then the Dolbeault cohomology groups are $\mathbb{C}$-vector spaces of finite dimension. Since any form of type (p,q) is also a complex-valued k-form, one

[4]This holds as soon as X is a compact complex manifold, cf. Corollary 3.28.

should also ask for the relation with the de Rham cohomology groups $H^k(X, \mathbb{C})$. More precisely, we can ask if the decomposition of global sections

$$C^\infty(X, \Omega^k_{X,\mathbb{C}}) = \bigoplus_{p+q=k} C^\infty(X, \Omega^{p,q}_X)$$

translates into a decomposition of cohomology groups

$$(*) \qquad H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

We will see very soon that in general this is not the case, but the Hodge decomposition Theorem 3.36 will tell us that the decomposition $(*)$ holds on the rather large class of Kähler manifolds that we will introduce in Section 3.

Before we start to examine these much more involved problems, let us take a look at the cohomology groups $H^{p,0}(X)$ for $p \in \{0, \ldots, n\}$. Since there are no forms of type $(p, -1)$ we have $B^{p,0} = 0$, so

$$H^{p,0}(X) = Z^{p,0}(X) = \{\alpha \in C^\infty(X, \Omega^{p,0}_X) \mid \overline{\partial}\alpha = 0\}.$$

Let $\alpha \in H^{p,0}(X)$, then we can identify it to a global section of $\Omega^{p,0}_X$. Fix now a point $x \in X$ and holomorphic coordinates $z_1, \ldots, z_n$ then

$$\alpha = \sum_{|J|=p} \alpha_J dz_J,$$

where the α_J are differentiable functions. By definition

$$0 = \overline{\partial}\alpha = \sum_{|J|=p} \sum_{k=1}^n \frac{\partial \alpha_J}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_J.$$

Since the monomials $d\overline{z}_k \wedge dz_J$ define a local frame of $\Omega^{p,1}_X$, it follows that for every J and every $k \in \{1, \ldots, n\}$

$$\frac{\partial \alpha_J}{\partial \overline{z}_k} = 0,$$

so the functions α_J are holomorphic. Recall now that we can identify $\Omega^{p,0}_X$ to the holomorphic vector bundle Ω^p_X. Then we have just shown that α is a global holomorphic section of Ω^p_X, so we get

1.77. **Proposition.**

$$H^{p,0}(X) = \Gamma(X, \Omega^p_X) \qquad \forall\, p \in \{0, \ldots, n\}.$$

1.78. **Exercise.** Let $f : X \to Y$ be a holomorphic map between complex manifolds. Show that the pull-back f^* induces functorial linear maps

$$f^* : H^{p,q}(Y) \to H^{p,q}(X).$$

Hint: cf. Exercise 1.72. $\square$

1.79**. Exercise.** Let X be a complex manifold, and let E be a *holomorphic* vector bundle of rank r over X. In this exercise, we will extend the definitions and statements about Dolbeault cohomology to the case of $(0,q)$-forms with values in E[5].

Let $\alpha \in C^\infty(X, \Omega_X^{0,q} \otimes E)$, and let $U \subset X$ be a trivialising subset of E, i.e. let $e_1, \ldots, e_r$ be a local holomorphic frame for E on U. Then we can write in a neighbourhood

$$\alpha|_U = \sum_{j=1}^r \alpha_j \otimes e_j,$$

where $\alpha_j \in C^\infty(U, \Omega_X^{0,q})$. We set

$$\overline{\partial}_E \alpha|_U := \sum_{j=1}^r \overline{\partial}(\alpha_j) \otimes e_j.$$

Show that if $V \subset X$ is another trivialising subset of E and $e'_1, \ldots, e'_r$ a local holomorphic frame for E on V, then

$$(\overline{\partial}_E \alpha|_U)|_{U\cap V} = (\overline{\partial}_E \alpha|_V)|_{U\cap V}.$$

Since the sections of a complex vector bundle form a sheaf (cf. Exercise 1.58), we can use the local expressions to define

$$\overline{\partial}_E : C^\infty(X, \Omega_X^{0,q} \otimes E) \to C^\infty(X, \Omega_X^{0,q+1} \otimes E).$$

Show that $\overline{\partial}_E$ satisfies the Leibniz rule

$$\overline{\partial}_E(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^q \alpha \wedge \overline{\partial}_E \beta$$

for all $\alpha \in C^\infty(X, \Omega_X^{0,q})$ and $\beta \in C^\infty(X, \Omega_X^{0,q'} \otimes E)$.

Show furthermore that $\overline{\partial}_E \circ \overline{\partial}_E = 0$, so we get a complex of $\mathbb{C}$-vector spaces

$$0 \to C^\infty(X, E) \xrightarrow{\overline{\partial}_E} C^\infty(X, \Omega_X^{0,1} \otimes E) \xrightarrow{\overline{\partial}_E} \ldots \xrightarrow{\overline{\partial}_E} C^\infty(X, \Omega_X^{0,n} \otimes E) \to 0$$

and we define the Dolbeault cohomology groups of E as the cohomology associated to the complex above, i.e.

$$H^q(X, E) := H^q_{\overline{\partial}_E}(C^\infty(X, \Omega_X^{0,\bullet} \otimes E)).$$

Show that

$$H^0(X, E) = \Gamma(X, E).$$

□

[5]Note that since Ω_X^p is a holomorphic vector bundle, we also get statements for (p,q)-forms with values in E, we simply define $H^{p,q}(X,E) := H^q(X, \Omega_X^p \otimes E)$.

1.F. **Dolbeault lemma and comparison theorems.** The goal of this section is to indicate how the isomorphism $H^{p,0}(X) = \Gamma(X, \Omega_X^p)$ can be generalised to all the Dolbeault cohomoloy groups $H^{p,q}(X)$. This can be done by the comparison theorems. The first step towards the comparison theorems is the Dolbeault-Grothendieck lemma that computes the Dolbeault cohomology of polydiscs (cf. Definition 1.3) and is the analogue of the Poincaré lemma for the de Rham cohomology.

1.80**. Theorem.** Let $X := D(z_0, R_0) \subset \mathbb{C}^n$ be a polydisc. Then we have

$$H^{p,q}(X) = 0 \qquad \forall\ p \geqslant 0, q \geqslant 1,$$

i.e. for every differentiable form u of type (p,q) such that $\overline{\partial} u = 0$, there exists a differentiable form v of type $(p, q-1)$ such that $\overline{\partial} v = u$.

The proof needs some auxiliary statements, starting with the following generalisation of Cauchy's theorem in one variable.

1.81**. Theorem.** Let $U \subset \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$ be a differentiable complex-valued function. Let $\overline{D} \subset U$ be the closure of a disc contained in U. Then for every $w \in D$, we have

$$f(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-w} dz - \int_{\overline{D}} \frac{i}{2\pi(z-w)} \frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z}.$$

Proof. For simplicity's sake we suppose that $w = 0$ and D is the the unit disc $\mathbb{D}$, the general case being analogous. The function $z \mapsto \frac{1}{z}$ is locally integrable at $z = 0$, so

$$\int_{\mathbb{D}} \frac{i}{2\pi z} \frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z} = \lim_{\varepsilon \to 0} \int_{\mathbb{D} \setminus D(0,\varepsilon)} \frac{i}{2\pi z} \frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z}$$

Since $d = \partial + \overline{\partial}$, we have

$$d[\frac{1}{2\pi i} f(z) \frac{dz}{z}] = \frac{i}{2\pi z} \frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z}.$$

Thus for $0 < \varepsilon < 1$ by Stokes' theorem

$$\int_{\mathbb{D} \setminus D(0,\varepsilon)} \frac{i}{2\pi z} \frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z} = \frac{1}{2\pi i} \int_{S^1} \frac{f(z)}{z} dz - \frac{1}{2\pi i} \int_{S^\varepsilon} \frac{f(z)}{z} dz,$$

where S^ε is the circle of radius ε around 0. A path integration shows that the second integral converges to $f(0)$ for $\varepsilon \to 0$. $\square$

As usual denote by $\mathscr{D}(\mathbb{C})$ the space of differentiable complex-valued function with compact support in $\mathbb{C}$, and by $\mathscr{D}'(\mathbb{C})$ its dual, the distributions on $\mathbb{C}$.

1.82**. Corollary.** Let δ_0 be the Dirac measure at 0. Then in $\mathscr{D}'(\mathbb{C})$ we have an equality

$$\frac{\partial}{\partial \overline{z}} \frac{1}{\pi z} = \delta_0.$$

Proof. Let $\varphi \in \mathscr{D}(\mathbb{C})$ be a test function and denote by $K \subset \mathbb{C}$ its support. Let D be a disc around 0 such that $K \subset D$, then $\varphi(z) = 0$ for every $z \in \partial D$. By definition

$$\langle \frac{\partial}{\partial \overline{z}} \frac{1}{\pi z}, \varphi \rangle = \langle \frac{1}{\pi z}, -\frac{\partial \varphi}{\partial \overline{z}} \rangle = -\int_{\mathbb{D}} \frac{1}{\pi z} \frac{\partial \varphi}{\partial \overline{z}} \frac{i}{2} dz \wedge d\overline{z}$$

By Theorem 1.81 this integral is equal to $\varphi(0) = \langle \delta_0, \varphi \rangle$. □

As a consequence we can locally resolve the $\overline{\partial}$-equation in $\mathbb{C}$:

1.83. **Theorem.** Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a differentiable complex-valued function. Then for every disc $D \subset U$ there exists a differentiable function g such that

$$\frac{\partial g}{\partial \overline{z}} = f|_D.$$

If furthermore f depends holomorphically on some parameters $z_1, \ldots, z_n$, then g depends holomorphically on the parameters $z_1, \ldots, z_n$.

Proof. By the preceding corollary $\frac{1}{\pi z}$ is a fundamental solution, so the convolution $g = \frac{1}{\pi z} \star f$ works since

$$\frac{\partial g}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}} \frac{1}{\pi z} \star f = \delta_0 \star f = f.$$

The second statement follows from inverting differentiation and the integration used to define the convolution. □

Remark. For an essentially equivalent but more down to earth presentation of the last proof cf. [Voi02, Thm.1.28].

Proof of Theorem 1.80. We suppose without loss of generality that $z_0 = 0$ and $R_0 = (1, \ldots, 1)$, and denote by $z_1, \ldots, z_n$ the holomorphic coordinates on $\mathbb{C}^n$. Since we are in $\mathbb{C}^n$, the vector bundle $\Omega_X^{p,q}$ is trivial, so we can write

$$u = \sum_{|J|=p, |K|=q} u_{J,K} dz_J \wedge d\overline{z}_K = \sum_{|J|=p} dz_J \wedge \left(\sum_{|K|=q} u_{J,K} d\overline{z}_K \right).$$

Since $\overline{\partial} \left(\sum_{|K|=q} u_{J,K} d\overline{z}_K \right)$ has type $(0, q+1)$, the equality

$$0 = \overline{\partial} u = (-1)^p \sum_{|J|=p} dz_J \wedge \overline{\partial} \left(\sum_{|K|=q} u_{J,K} d\overline{z}_K \right)$$

shows that

$$\overline{\partial} \left(\sum_{|K|=q} u_{J,K} d\overline{z}_K \right) = 0 \qquad \forall\ |J| = p.$$

Hence it is sufficient to show for every J that there exists a v_J such that $\overline{\partial} v_J = \left(\sum_{|K|=q} u_{J,K} d\overline{z}_K\right)$. We are thus reduced to treat the case where $p=0$ and will in the following consider only this situation, i.e. we suppose that

$$u = \sum_{|K|=q} u_K d\overline{z}_K$$

with $\overline{\partial} u = 0$.

We will now argue by induction on the integer $0 \leqslant l \leqslant n$ such that the monomials $d\overline{z}_K$ appearing with non-zero coefficient in u involve only $d\overline{z}_1, \ldots, d\overline{z}_l$. More formally speaking if $u_K \neq 0$, then $K \subset \{1, \ldots, l\}$. If $l < q$, the $(0,q)$-form u is zero, so the statement is trivially true. For the induction step, suppose that the statement holds for $l-1$ and write

$$u = \sum_{|K|=q, k<l} u_K d\overline{z}_K + \sum_{|K|=q-1, k<l} w_K d\overline{z}_K \wedge d\overline{z}_l.$$

Then we have

$$0 = \overline{\partial} u = \sum_{|K|=q, k<l} \sum_{m=1}^{n} \frac{\partial u_K}{\partial \overline{z}_m} d\overline{z}_m \wedge d\overline{z}_K + \sum_{|K|=q-1, k<l} \sum_{m=1}^{n} \frac{\partial w_K}{\partial \overline{z}_m} d\overline{z_m} \wedge d\overline{z}_K \wedge d\overline{z}_l.$$

Since only the forms $\frac{\partial w_K}{\partial \overline{z}_m} d\overline{z_m} \wedge d\overline{z}_K \wedge d\overline{z}_l$ for $m > l$ have two factors $d\overline{z}_l$ with $m \geqslant l$, we get that

$$\frac{\partial w_K}{\partial \overline{z}_m} = 0 \qquad \forall\ m > l,\ |K| = q-1.$$

This shows that the differentiable functions $w_K(z_1, \ldots, z_n)$ are in fact holomorphic with respect to the variables $z_{l+1}, \ldots, z_n$. For every multi-index $|K| = q-1$, we can resolve by Theorem 1.83 the equation

$$\frac{\partial h_K}{\partial \overline{z}_l} = w_K$$

and a solution h_K is holomorphic with respect to the variables $z_{l+1}, \ldots, z_n$. Set now

$$h := \sum_{|K|=q-1} h_K d\overline{z}_K,$$

then

$$\overline{\partial} h = \sum_{|K|=q-1} \sum_{m=1}^{n} \frac{\partial h_K}{\partial \overline{z}_m} d\overline{z}_m \wedge d\overline{z}_K = \sum_{|K|=q-1} \sum_{m=1}^{l-1} \frac{\partial h_K}{\partial \overline{z}_m} d\overline{z}_m \wedge d\overline{z}_K + \sum_{|K|=q-1} w_K d\overline{z}_l \wedge d\overline{z}_K$$

since $\frac{\partial h_K}{\partial \overline{z}_l} = w_K$ and $\frac{\partial h_K}{\partial \overline{z}_m} = 0$ for $m > l$. Therefore we get a $(0,q)$-form

$$u' := u - (-1)^{q-1} \overline{\partial} h = \sum_{|K|=q, k<l} u_K d\overline{z}_K - (-1)^{q-1} \sum_{|K|=q-1} \sum_{m=1}^{l-1} \frac{\partial h_K}{\partial \overline{z}_m} d\overline{z}_m \wedge d\overline{z}_K,$$

such that the monomials $d\overline{z}_K$ appearing in u' involve only $d\overline{z}_1, \ldots, d\overline{z}_{l-1}$. Furthermore $\overline{\partial} u' = \overline{\partial}(u - (-1)^{q-1} \overline{\partial} h) = 0$, so u' is $\overline{\partial}$-closed. By the induction hypothesis, there exists a $(0, q-1)$-form v' such that $\overline{\partial} v' = u'$. Therefore

$$v := v' + (-1)^{q-1} h$$

satisfies $\overline{\partial} v = u$. $\square$

Let us recall briefly the basics of Čech cohomology (cf. [Har77, III, Ch. 4]): let X be a topological space, and let $\mathscr{U} = (U_i)_{i\in I}$ be an open (countable) covering of X. We fix an ordering on the index set I. For any finite set of indices $i_0, \ldots, i_p \in I$, we denote the intersection $U_{i_0} \cap \ldots \cap U_{i_p}$ by $U_{i_0,\ldots,i_p}$.

Let $\mathscr{F}$ be a sheaf of abelian groups on X. We define a complex of abelian groups $C^\bullet(\mathscr{U}, \mathscr{F})$ as follows. For each $p \geqslant 0$, let

$$C^p(\mathscr{U}, \mathscr{F}) = \Pi_{i_0 < \ldots < i_p} \mathscr{F}(U_{i_0,\ldots,i_p}).$$

Let now $\alpha \in C^p(\mathscr{U}, \mathscr{F})$ be given by

$$\alpha_{i_0,\ldots,i_p} \in \mathscr{F}(U_{i_0,\ldots,i_p}).$$

Then we define the coboundary map $\delta : C^p(\mathscr{U}, \mathscr{F}) \to C^{p+1}(\mathscr{U}, \mathscr{F})$ by

$$(\delta\alpha)_{i_0,\ldots,i_p,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}} |_{U_{i_0,\ldots,i_{p+1}}}$$

1.84. **Exercise.** Show that $\delta : C^\bullet \to C^{\bullet+1}$ defines a cohomological complex, i.e. show that $\delta \circ \delta = 0$. $\square$

1.85. **Definition.** Let X be a topological space, and let $\mathscr{U}$ be an open covering of X. Let $\mathscr{F}$ be a sheaf of abelian groups on X, then the q-th Čech cohomology group of $\mathscr{F}$ with respect to the covering $\mathscr{U}$ is defined as the q-th cohomology associated to the complex above, i.e.

$$\check{\mathrm{H}}^q(\mathscr{U}, \mathscr{F}) := H^q_\delta(C^\bullet(\mathscr{U}, \mathscr{F})).$$

We define the q-th Čech cohomology group of $\mathscr{F}$ by taking an inductive limit[6]

$$\check{\mathrm{H}}^q(X, \mathscr{F}) := \varinjlim_{\mathscr{U}} \check{\mathrm{H}}^q(\mathscr{U}, \mathscr{F}),$$

where two classes α and α' for some covering $\mathscr{U}$ and $\mathscr{U}'$ are identified if they map to the same class in a common refinement of the coverings.

Let X now be a complex manifold, and let E be a holomorphic vector bundle over X. We have seen in Exercise 1.58 that the sections of E form a sheaf of abelian groups which we denote by $\mathscr{O}_X(E)$. Let now $\mathscr{U}$ be an open covering of X, then we can consider the Čech cohomology groups

$$\check{\mathrm{H}}^q(\mathscr{U}, \mathscr{O}_X(E))$$

and

$$\check{\mathrm{H}}^q(X, \mathscr{O}_X(E)) := \varinjlim_{\mathscr{U}} \check{\mathrm{H}}^q(\mathscr{U}, \mathscr{O}_X(E)).$$

and ask about their relation with the Dolbeault cohomology groups $H^q(X, E)$ defined in Exercise 1.79.

[6]There are some technical facts that one has to check for the definition to make sense, cf. [For91, Ch.12] for a precise and readable presentation.

1.86. **Exercise.** Construct an example of a complex manifold X, an open covering $\mathscr{U}$ and a holomorphic vector bundle E such that

$$\check{\mathrm{H}}^q(\mathscr{U}, \mathscr{O}_X(E)) \not\simeq H^q(X, E)$$

□

We have seen in the preceding paragraph, that for every open set $U \subset X$

$$\Omega_X(U) = \ker(\overline{\partial} : \Omega_X^{p,0}(U) \to \Omega_X^{p,1}(U)).$$

Therefore the Dolbeault-Grothendieck Theorem 1.80 shows that for every $p \geqslant 0$, the sequence of sheaves

$$(*) \qquad 0 \to \Omega_X^p \xrightarrow{i} \Omega_X^{p,0} \xrightarrow{\overline{\partial}} \Omega_X^{p,1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Omega_X^{p,n} \to 0$$

is exact: indeed exactness can be checked locally and the Dolbeault-Grothendieck lemma shows that on a polydisc $U \subset X$, we have

$$\ker\left(\overline{\partial} : C^\infty(U, \Omega_X^{p,q}) \to C^\infty(U, \Omega_X^{p,q+1})\right) = \operatorname{im}\left(\overline{\partial} : C^\infty(U, \Omega_X^{p,q-1}) \to C^\infty(U, \Omega_X^{p,q})\right).$$

The de Rham-Weil isomorphism [Wel80, Ch.II,Thm.3.13] shows that the Čech cohomology of Ω_X^p is the cohomology of the corresponding complex of global sections[7]:

$$0 \to C^\infty(X, \Omega_X^{p,0}) \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,1}) \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,n}) \to 0.$$

By definition the Dolbeault cohomology groups are exactly the cohomology of this complex, thus we obtain the following comparison theorem.

1.87. **Theorem.** Let X be a complex manifold, then

$$\check{\mathrm{H}}^q(X, \Omega_X^p) \simeq H^{p,q}(X).$$

Using the Dolbeault-Grothendieck lemma for forms with values in a holomorphic vector bundle (Exercise 1.89), the result generalises to

$$\check{\mathrm{H}}^q(X, \Omega_X^p \otimes \mathscr{O}_X(E)) \simeq H^{p,q}(X, E).$$

One of the advantages of Čech cohomology is that it is more tractable, since we have the following theorem of Leray.

[7] *Advanced technical remark.* On page 50 we will give an explicit description of the map between the cohomology groups in the simplest case. Showing that this map is an isomorphism is a bit more complicated, one strategy of proof is the following: in a first step one shows that the Čech cohomology groups are isomorphic to the cohomology groups of the sheaf Ω_X^p in terms of the right derived functor of the global section functor. Then the de Rham-Weil theorem shows that the cohomology in terms of the right derived functor can be computed by taking a resolution by acyclic sheaves. In our case, the de Rham-Weil theorem applies since the complex vector bundles $\Omega_X^{p,q}$ can be seen as sheaves of modules over the sheaf of rings $C_X^\infty$of differentiable functions on X. Since C_X^∞ is a fine sheaf, the sheaves $\Omega_X^{p,q}$ are fine [Wel80, Ch.II,Defn.3.3]. In particular they are acylic, that is

$$\check{\mathrm{H}}^s(X, \Omega_X^{p,q}) = 0 \qquad \forall\, s > 0.$$

Therefore $(*)$ defines a resolution of the vector bundle Ω_X^p by acyclic sheaves.

1.88**. Theorem.** Let X be a complex manifold, and let E be a holomorphic vector bundle over X. Let now $\mathscr{U}$ be an open covering of X such that for any $U_{i_0,\dots,i_p}$ we have

$$H^k(U_{i_0,\dots,i_p}, E) = 0 \qquad \forall\ k > 0.$$

Then the morphism

$$\check{\mathrm{H}}^q(\mathscr{U}, \Omega^p_X \otimes \mathscr{O}_X(E)) \to \check{\mathrm{H}}^q(X, \Omega^p_X \otimes \mathscr{O}_X(E))$$

is an isomorphism.

Using the classical Poincaré lemma, one shows that

$$0 \to \mathbb{C} \xrightarrow{i} \mathscr{C}^\infty \xrightarrow{d} \Omega_{X,\mathbb{C}} \xrightarrow{d} \dots \xrightarrow{d} \Omega^{2n}_{X,\mathbb{C}} \to 0$$

is exact, where C^∞ is the sheaf of differentiable functions. Arguing as above we get an isomorphism between the de Rham cohomology with complex coefficients and the Čech cohomology of the sheaf of locally constant functions $\mathbb{C}$:

$$H^q(X, \mathbb{C}) \simeq \check{\mathrm{H}}^q(X, \mathbb{C}). \tag{1.3}$$

1.89**. Exercise.** Let X be a complex manifold, and let E be a *holomorphic* vector bundle of rank r over X. Show that an analogue of the Dolbeault-Grothendieck lemma holds for holomorphic q-forms with values in E: if $\alpha \in C^\infty(X, \Omega^{0,q}_X \otimes E)$ such that $\overline{\partial}_E \alpha = 0$, there exists for every point $x \in X$ a neighbourhood U and $\beta \in C^\infty(U, \Omega^{0,q-1}_X \otimes E)$ such that

$$\overline{\partial}_E \beta = \alpha|_U.$$

□

1.90**. Exercise.** Let X be a compact complex manifold, and let $f : X \to \mathbb{C}$ be a differentiable function such that $\partial\overline{\partial} f = 0$. Show that f is constant.

Suppose that $\omega \in H^{1,0}(X)$ such that there exists a differentiable function f such that $\omega = \partial f$. Show that $\omega = 0$. □

1.91**. Exercise.** Let X be a complex manifold. Show that the Picard group $\mathrm{Pic}(X)$ (cf. Exercise 1.52) is isomorphic to the Čech cohomology group $\check{\mathrm{H}}^1(X, \mathscr{O}^*_X)$. □

1.92**. Exercise.** Use the comparison theorem to compute the cohomology of the line bundle $K_{\mathbb{P}^1} \simeq \mathscr{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1$. □

2. Connections, curvature and Hermitian metrics

In this slightly technical section we will introduce the tools we will need in order to get a deeper understanding of complex manifolds. After a brief resume of the general theory of Hermitian differential geometry, we will specify to the case of holomorphic line and vector bundles.

2.A. Hermitian geometry of complex vector bundles.

The concepts of connections, curvature and Hermitian metrics for complex differentiable vector bundles are essentially the same as for real differentiable vector bundles. We will therefore only give a very short exposition and refer to [Biq08, Ch.3] for details and explanations.

2.1. **Definition.** Let X be a differentiable manifold, and let $\pi : E \to X$ be a complex vector bundle over X. A connection on E is a $\mathbb{C}$-linear differential operator

$$D : C^\infty(X, E) \to C^\infty(X, \Omega_{X,\mathbb{C}} \otimes E)$$

that satisfies the Leibniz rule

$$D(f\sigma) = df \cdot \sigma + f \cdot D\sigma \qquad \forall\ f \in C^\infty(X), \sigma \in C^\infty(X, E).$$

It is easy to see that a connection defines in fact a $\mathbb{C}$-linear differential operator

$$D : C^\infty(X, \bigwedge^k \Omega_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \bigwedge^{k+1} \Omega_{X,\mathbb{C}} \otimes E)$$

for every $k \in \mathbb{N}$ that satisfies the Leibniz rule

$$D(\tau \wedge \sigma) = d\tau \wedge \sigma + (-1)^k \tau \wedge D\sigma \qquad \forall\, \tau \in C^\infty(X, \bigwedge^k \Omega_{X,\mathbb{C}}), \sigma \in C^\infty(X, \bigwedge^l \Omega_{X,\mathbb{C}} \otimes E).$$

Indeed if we fix $\sigma \in C^\infty(X, \bigwedge^k \Omega_{X,\mathbb{C}} \otimes E)$, a point $x_0 \in X$, and a local frame $e_1, \ldots, e_r$ of the vector bundle E in a neighbourhood U of x_0, then we can write

$$\sigma = \sum_{j=1}^r s_j \otimes e_j,$$

where the s_j are k-forms defined on U. Since we want D to be a linear operator that defines the Leibniz rule, the only possible definition for $D\sigma$ is

$$D\sigma := \sum_{j=1}^r \left[ds_j \otimes e_j + (-1)^k s_j \otimes De_j \right].$$

2.2. **Exercise.** Check that the definition of $D\sigma$ does not depend on the choice of the frame $e_1, \ldots, e_r$. Show that D satisfies the Leibniz rule for every $k \in \mathbb{N}$. $\square$

Suppose now that $k = 0$, then the preceding computation shows that in order to compute

$$(*) \qquad D\sigma = \sum_{j=1}^r \left[ds_j \otimes e_j + s_j \otimes De_j \right],$$

we only have to know De_j for every $j = 1, \ldots, r$. Since

$$De_j \in C^\infty(U, \Omega_{X,\mathbb{C}} \otimes E),$$

we can express De_j in the frame $e_1, \ldots, e_r$

$$De_j = \sum_{i=1}^{r} a_{i,j} \otimes e_i,$$

where the $a_{i,j}$ are 1-forms defined on U. Thus the connection D is locally given by a matrix

$$A := (a_{i,j})_{1 \leqslant i,j \leqslant r}\,.$$

The frame $e_1, \ldots, e_r$ defines a trivialisation $\pi^{-1}(U) \overset{\theta}{\simeq} U \times \mathbb{C}^r$, so we can identify the section σ with a r-tuple

$$\sigma \overset{\theta}{=} s = (s_1, \ldots, s_r).$$

Thus we can rewrite $(*)$ in this trivialisation as

$$D\sigma \overset{\theta}{=} ds + As.$$

Vice versa it is clear that the choice of a matrix A of 1-forms on the open set U defines locally an operator D that satisfies the Leibniz rule.

Let us now see what happens under a change of frame, i.e. let $e'_1, \ldots, e'_r$ be another local frame of the vector bundle E defined in a neighbourhood U' of x_0. On the intersection $U \cap U'$ we have two different trivialisations

$$\pi^{-1}(U \cap U') \overset{\theta}{\simeq} (U \cap U') \times \mathbb{C}^r \text{ and } \pi^{-1}(U \cap U') \overset{\theta'}{\simeq} (U \cap U') \times \mathbb{C}^r,$$

inducing two representations

$$D\sigma \overset{\theta}{=} ds + As \text{ and } D\sigma \overset{\theta'}{=} ds' + A's'.$$

Let $g : (U \cap U') \times \mathbb{C}^r \to (U \cap U') \times \mathbb{C}^r$ be the transition function from the second to the first trivialisation, then

$$s = gs' \qquad \text{and} \qquad D(s) = gD(s').$$

Then we have

$$ds = d(gs') = dgs' + gds' = g(g^{-1}dg\ s' + ds'),$$

thus

$$ds + As = g(g^{-1}dg\ s' + ds') + Ags' = g\left(ds' + (g^{-1}dg + g^{-1}Ag)s'\right).$$

This implies that we have the following transition relation for the matrices defining the connection

$$A' = g^{-1}dg + g^{-1}Ag. \tag{2.4}$$

In particular the connection on E is certainly not a C^∞-linear operator.

Let us now see what happens if we apply the connection twice. Since $D\sigma \stackrel{\theta}{=} ds + As$, the Leibniz rule implies

$$\begin{aligned} D^2\sigma &\stackrel{\theta}{=} D(ds + As) = d(ds + As) + A(ds + As) \\ &= d^2 s + dAs - Ads + Ads + A \wedge As \\ &= (dA + A \wedge A)s. \end{aligned}$$

Using the transition relation for the matrix A established above, one sees that $dA + A \wedge A$ defines a globally defined 2-form with values in the bundle $\mathrm{End}\, E$. So we have shown

2.3. **Proposition.** There exists a section $\Theta_D \in C^\infty(X, \bigwedge^2 \Omega_{X,\mathbb{C}} \otimes \mathrm{End}\, E)$ such that for every $\sigma \in C^\infty(X, \bigwedge^k \Omega_{X,\mathbb{C}} \otimes E)$

$$D^2(\sigma) = \Theta_D \wedge \sigma$$

If the connection is locally given by a (r, r)-matrix of 1-forms A, then

$$\Theta_D = dA + A \wedge A.$$

2.4. **Remark.** Since A is a matrix of 1-forms, the product $A \wedge A$ is in general not zero. This will be the case if E is a line bundle, a case that we will study much more in detail at the end of this section.

2.5. **Definition.** Let X be a differentiable manifold, and let $\pi : E \to X$ be a complex vector bundle over X. A Hermitian metric h on E is an assignment of a Hermitian inner product $< \bullet, \bullet >$ to each fibre E_x of E such that for any open set $U \subset X$ and any $\zeta, \eta \in C^\infty(U, E)$ the function

$$< \zeta, \eta >: U \to \mathbb{C}, \ x \mapsto < \zeta(x), \eta(x) >$$

is differentiable. A complex vector bundle E equipped with a Hermitian metric h is called a Hermitian vector bundle (E, h).

2.6. **Remark.** In these notes we follow the convention that a Hermitian product is $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second variable.

Fix a point $x_0 \in X$, and let $e_1, \ldots, e_r$ be a local frame for E in a neighbourhood U of x_0 so that we get a trivialisation $\pi^{-1}(U) \stackrel{\theta}{\simeq} U \times \mathbb{C}^r$. The (r, r)-matrix of differentiable functions $H = (h_{\lambda,\mu})_{1 \leqslant \lambda,\mu \leqslant r}$ defined by

$$h_{\lambda,\mu}(x) := < e_\lambda(x), e_\mu(x) > \tag{2.5}$$

represents the Hermitian metric with respect to the chosen frame. More precisely if we identify $\zeta, \eta \in C^\infty(U, E)$ to r-tuples $(\zeta_1, \ldots, \zeta_r)$ and $(\eta_1, \ldots, \eta_r)$ then

$$h_x(\zeta, \eta) = \zeta^t \ H \ \overline{\eta} = \sum_{1 \leqslant \lambda,\mu \leqslant r} \zeta_\lambda h_{\lambda,\mu}(x) \overline{\eta_\mu}.$$

If $e'_1, \ldots, e'_r$ is another local frame for E on a trivialising subset U' defining a trivialisation $\pi^{-1}(U') \stackrel{\theta'}{\simeq} U' \times \mathbb{C}^r$ so that we get a transition function $g : (U \cap U') \times$

$\mathbb{C}^r \to (U \cap U') \times \mathbb{C}^r$ from the second to the first trivialisation, it is not hard to see that

$$H' = g^t \circ H \circ \overline{g}. \tag{2.6}$$

2.7. **Proposition.** Every complex vector bundle $\pi : E \to X$ admits a Hermitian metric.

Proof. Let U_α be a locally finite covering of X with local frames $e_1^\alpha, \ldots, e_r^\alpha$ for E. We define a Hermitian metric on $E|_{U_\alpha}$ by

$$< \zeta, \eta >_x^\alpha := \sum_\lambda \zeta_\lambda \overline{\eta_\lambda}.$$

Let ρ_α be a differentiable partition of unity subordinate to the covering, then we set

$$< \zeta, \eta > := \sum_\alpha \rho_\alpha < \zeta, \eta >_x^\alpha .$$

It is clear that for every $x \in X$ this defines a Hermitian inner product on E_x. Furthermore if $\zeta, \eta \in C^\infty(U, E)$, the function

$$x \mapsto < \zeta, \eta > = \sum_\alpha \rho_\alpha < \zeta, \eta >_x^\alpha = \sum_\alpha \rho_\alpha \sum_\lambda \zeta_\lambda \overline{\eta_\lambda}$$

is differentiable. □

A Hermitian metric on E defines bilinear mappings

$$C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E) \times C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{p+q}_{X,\mathbb{C}}), (\sigma, \tau) \mapsto \{\sigma, \tau\} \tag{2.7}$$

which are locally described as follows : fix a point $x \in X$ and let $e_1, \ldots, e_r$ be a local frame for E in a neighbourhood $x \in U$ so that we get a trivialisation $\pi^{-1}(U) \overset{\theta}{\simeq} U \times \mathbb{C}^r$. Then locally $\sigma = \sum_{j=1}^r \sigma_j \otimes e_j$ and $\tau = \sum_{j=1}^r \tau_j \otimes e_j$, where the σ_λ (resp. τ_λ) are p-forms (resp. q-forms). We set

$$\{\sigma, \tau\} = \sigma^t \ H \ \overline{\tau},$$

where H is the matrix 2.5. Using the Gram-Schmidt procedure, we can replace the local frame $e_1, \ldots, e_r$ by an orthonormal local frame, i.e. a frame such that

$$< e_j, e_k > = \delta_{j,k}.$$

We call the corresponding trivialisation isometric. Since in such a trivialisation, the matrix H representing the Hermitian metric h is the identity, we have

$$\{\sigma, \tau\} = \sum_{j=1}^r \sigma_j \wedge \overline{\tau_j} =: \sigma^t \wedge \overline{\tau}.$$

2.8. **Definition.** Let X be a differentiable manifold, and let (E, h) be a Hermitian vector bundle over X. We say that a connection D on E is Hermitian (or compatible with h) if the Leibniz rule

$$d\{\sigma, \tau\} = \{D\sigma, \tau\} + (-1)^p \{\sigma, D\tau\}$$

holds for every $\sigma \in C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E), \tau \in C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E)$.

2.9. **Lemma.** Let $x \in X$ be a point, and let $e_1, \ldots, e_r$ be a local frame for E in a neighbourhood U of x defining an isometric trivialisation $\pi^{-1}(U) \overset{\theta}{\simeq} U \times \mathbb{C}^r$. Let A be the (r,r)-matrix of 1-forms defining the connection D with respect to the trivialisation. Then D is Hermitian if and only if

$$\overline{A}^t = -A,$$

that is the matrix A is anti-autodual.

Proof. By what preceeds, we have

$$\{\sigma, \tau\} = \sigma^t \wedge \overline{\tau},$$

so

$$d\{\sigma, \tau\} = (d\sigma)^t \wedge \overline{\tau} + (-1)^p \sigma^t \wedge d\overline{\tau}.$$

Since in the trivialisation $D. = d. + A \wedge .$, we have

$$\{D\sigma, \tau\} = (d\sigma + A \wedge \sigma)^t \wedge \overline{\tau} = d\sigma^t \wedge \overline{\tau} + (-1)^p \sigma^t \wedge A^t \wedge \overline{\tau}$$

and

$$\{\sigma, D\tau\} = \sigma^t \wedge (\overline{d\tau + A \wedge \tau}) = \sigma^t \wedge d\overline{\tau} + \sigma^t \wedge \overline{A} \wedge \overline{\tau}.$$

Therefore

$$\{D\sigma, \tau\} + (-1)^p \{\sigma, D\tau\} - d\{\sigma, \tau\} = (-1)^p \sigma^t \wedge (A^t + \overline{A}) \wedge \overline{\tau}$$

is zero for arbitrary σ, τ if and only if $A^t + \overline{A} = 0$. □

2.10. **Remark.** A similar computation shows that if $e_1, \ldots, e_r$ is any local frame for E in a neighbourhood $x \in U$, then

$$dH = A^t H + H\overline{A}. \tag{2.8}$$

Given a connection D on a complex vector bundle, we can define the adjoint connection D^{adj} to be the connection given locally by the matrix $-\overline{A}^t$. With this definition it follows from the computation above that

$$d\{\sigma, \tau\} = \{D\sigma, \tau\} + (-1)^p \{\sigma, D^{adj}\tau\}$$

and D is Hermitian if and only if $D = D^{adj}$. We can now produce a Hermitian connection by taking $\frac{1}{2}(D + D^{adj})$ defined locally by $\frac{1}{2}(A - \overline{A}^t)$. This proves:

2.11. **Proposition.** Let X be a differentiable manifold, and let (E, h) be a Hermitian vector bundle over X. Then there exists a Hermitian connection D.

2.B. **Holomorphic vector bundles.** Let now X be a complex manifold, and let $\pi : E \to X$ be a complex vector bundle over X. Recall that by Formula (1.2) the complex structure induces a decomposition

$$\Omega_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}.$$

Let

$$D : C^\infty(X, E) \to C^\infty(X, \Omega_{X,\mathbb{C}} \otimes E)$$

be a connection on E, then we can define the $(1,0)$-part (resp. $(0,1)$-part) by composing D with the projection

$$C^\infty(X, \Omega_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega_X^{1,0} \otimes E)$$

resp. with

$$C^\infty(X, \Omega_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega_X^{0,1} \otimes E).$$

Let $e_1, \ldots, e_r$ be a local frame for E on some open set $U \subset X$ defining a trivialisation $\pi^{-1}(U) \overset{\theta}{\simeq} U \times \mathbb{C}^r$. Locally the connection D is given by a (r,r)-matrix A of 1-forms such that if we identify a section $\sigma \in C^\infty(U, E)$ to a r-tuple $s = (s_1, \ldots, s_r)$, we have

$$D\sigma \overset{\theta}{=} ds + A \wedge s.$$

Since we are working on a complex manifold, we have $ds = \partial s + \overline{\partial} s$, furthermore the matrix A has a unique decomposition $A = A^{1,0} + A^{0,1}$, where $A^{1,0}$ (resp. $A^{0,1}$) is a (r,r)-matrix of $(1,0)$-forms (resp. $(0,1)$-forms). Thus the local presentation of the $(1,0)$-part (resp. $(0,1)$-part) is given by

$$D^{1,0}\sigma \overset{\theta}{=} \partial s + A^{1,0} \wedge s$$

respectively by

$$D^{0,1}\sigma \overset{\theta}{=} \overline{\partial} s + A^{0,1} \wedge s.$$

With these local descriptions, it is clear that the differential operators

$$D^{1,0} : C^\infty(X, E) \to C^\infty(X, \Omega_X^{1,0} \otimes E)$$

and

$$D^{0,1} : C^\infty(X, E) \to C^\infty(X, \Omega_X^{0,1} \otimes E)$$

satisfy the Leibniz rule

$$D^{1,0}(f \wedge \sigma) = \partial f \wedge \sigma + f \wedge D^{1,0}\sigma \qquad \forall\ f \in C^\infty(X), \sigma \in C^\infty(X, E).$$

and

$$D^{0,1}(f \wedge \sigma) = \overline{\partial} f \wedge \sigma + f \wedge D^{0,1}\sigma \qquad \forall\ f \in C^\infty(X), \sigma \in C^\infty(X, E).$$

respectively.

Suppose now that E is a *holomorphic* vector bundle on X. We have defined in Exercise 1.79 a differential operator

$$\overline{\partial}_E : C^\infty(X, E) \to C^\infty(X, \Omega_X^{0,1} \otimes E)$$

that satisfies $\overline{\partial}_E^2 = 0$ and the Leibniz rule

$$\overline{\partial}_E(f \wedge \sigma) = \overline{\partial} f \wedge \sigma + f \wedge \overline{\partial}_E \sigma \qquad \forall\ f \in C^\infty(X), \sigma \in C^\infty(X, E).$$

We will now see that $\overline{\partial}_E$ is the $(0,1)$-part of a unique connection that is compatible with the metric.

2.12. **Theorem.** Let X be a complex manifold, and let (E,h) be a Hermitian holomorphic vector bundle of rank r on X. Then there exists a unique Hermitian connection D_E on E such that

$$D_E^{0,1} = \overline{\partial}_E.$$

We call D_E the canonical connection or Chern connection of (E,h) and the corresponding curvature tensor the Chern curvature of E (or (E,h)).

Proof. We will start by showing that if the connection D_E exists, then it is unique. This proof will also give us an idea on how to construct the Chern connection.

Let $e_1, \ldots, e_r$ be a local *holomorphic* frame for E in some open subset $U \subset X$ defining a holomorphic trivialisation $\pi^{-1}(U) \stackrel{\theta}{\simeq} U \times \mathbb{C}^r$. Locally the connection D_E is given by a (r,r)-matrix of 1-forms such that if a section $\sigma \in C^\infty(U,E)$ identifies to a r-tuple $s = (s_1, \ldots, s_r)$, we have

$$D_E \sigma \stackrel{\theta}{=} ds + A \wedge s$$

and

$$D_E^{0,1} \sigma \stackrel{\theta}{=} \overline{\partial} s + A^{0,1} \wedge s.$$

If $\sigma \in \Gamma(U,E)$, then we have $D_E^{0,1}\sigma = \overline{\partial}_E \sigma = 0$ by Exercise 1.79. Furthermore $\overline{\partial} s = 0$, so we see that $A^{0,1} = 0$. Therefore A is a (r,r)-matrix of $(1,0)$-forms.

Let now $H = (h_{\lambda,\mu})_{1 \leqslant \lambda,\mu \leqslant r}$ be the (r,r)-matrix of differentiable functions defined by

$$h_{\lambda,\mu}(x) := < e_\lambda(x), e_\mu(x) > .$$

Since D_E is compatible with h, we have by Formula (2.8)

$$dH = A^t \ H + H \ \overline{A}.$$

Comparing the $(1,0)$- and $(0,1)$-parts, we get

$$\overline{\partial} H = H \ \overline{A}.$$

Since $e_1, \ldots, e_r$ is a local frame and the Hermitian metric is nondegenerate, the matrix H is invertible, thus

$$A = \overline{H}^{-1} \partial \overline{H}. \tag{2.9}$$

This shows that the Hermitian metric determines the connection matrix A, so the Chern connection is unique.

Vice versa we can use Formula (2.9) to define the canonical connection, once we have shown that the definition is compatible with a holomorphic change of frame. Let $e_1', \ldots, e_r'$ be another local *holomorphic* frame for E in an open subset $U' \subset X$ defining a holomorphic trivialisation $\pi^{-1}(U') \stackrel{\theta}{\simeq} U' \times \mathbb{C}^r$ so that we get a transition function $g : U \cap U' \to GL(\mathbb{C}, r)$ from the second to the first trivialisation. Let H'

be the matrix representing h with respect to the new frame, then by Formula (2.6) we have

$$H' = g^t\ H\ \overline{g},$$

so

$$g\ (\overline{H'})^{-1} = \overline{H}^{-1}\ \overline{g^t}^{-1}$$

and

$$\begin{aligned} g\ A' &= g\ (\overline{H'})^{-1}\ \partial\overline{H'} \\ &= \overline{H}^{-1}\ \overline{g^t}^{-1}\ \partial(\overline{g^t}\ \overline{H}\ g) \\ &= \overline{H}^{-1}\ \overline{g^t}^{-1}\ \left(\partial\overline{g^t}\ \overline{H}\ g + \overline{g^t}\ \partial\overline{H}\ g + \overline{g^t}\ \overline{H}\ \partial g\right). \end{aligned}$$

Yet g is a *holomorphic* change of frame, so

$$\partial\overline{g^t} = \overline{\overline{\partial}g^t} = 0 \text{ and } \partial g = dg.$$

Thus the expression above simplifies to

$$g\ A' = \overline{H}^{-1}\ (\partial\overline{H})\ g + dg = dg + A\ g.$$

By Formula (2.4) this is a necessary and sufficient condition for the matrices to define a connection. □

2.13**. Corollary.** Let X be a complex manifold, and let (E, h) be a Hermitian holomorphic vector bundle on X. Let D_E be the Chern connection on E and Θ_E its curvature tensor. Let A be a matrix representing the Chern connection with respect to some local *holomorphic* frame. Then

(1) A is of type $(1,0)$ and $\partial A = -A \wedge A$.
(2) Locally $\Theta_E = \overline{\partial}A$, thus Θ is of type $(1,1)$.
(3) $\overline{\partial}\Theta_E = 0$.

Proof. Let $e_1, \dots, e_r$ be a local holomorphic frame, and let H be the matrix representing h with respect to this local frame. Then we have by Formula (2.9)

$$A = \overline{H}^{-1}\partial\overline{H}.$$

Since $\partial^2\overline{H} = 0$ and $\partial\overline{H}^{-1} = -\overline{H}^{-1}\partial\overline{H}\ \overline{H}^{-1}$, we have

$$\partial A = \partial(\overline{H}^{-1}\partial\overline{H}) = -(\overline{H}^{-1}\partial\overline{H}) \wedge (\overline{H}^{-1}\partial\overline{H}).$$

For the second statement recall that for any complex vector bundle and any connection $\Theta = dA + A\wedge A$, so by the first statement $\Theta = dA + A\wedge A = dA - \partial A = \overline{\partial}A$. Since A is of type $(1,0)$, it is clear that $\Theta = \overline{\partial}A$ is of type $(1,1)$. The third item is immediate from $\overline{\partial}^2 A = 0$. □

Let X be a complex manifold, and let (E, h) be a Hermitian holomorphic vector bundle on X. Let $S \hookrightarrow E$ be a holomorphic subbundle of E and we define h_S to be the Hermitian metric on S given by restricting h. Furthermore let

$$S_x^\perp := \{e \in E_x \mid h(e_x, s_x) = 0\} \qquad \forall\ x \in X.$$

Now $S^\perp$ is a complex vector bundle such that $E \simeq S \oplus S^\perp$ as complex vector bundles. For all $k \in \mathbb{N}$, we denote by $p_S : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes S)$ the projection induced by the isomorphism

$$C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \simeq C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes S) \oplus C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes S^\perp).$$

2.14. **Corollary.** Let D_E and D_S be the Chern connections on (E, h) and (S, h_S). Then we have

$$D_S = p_S \circ D_E.$$

Proof. We will show that $p_S \circ D_E$ satisfies the properties of the Chern connection on (S, h_S) and conclude by uniqueness. Let $\sigma \in C^\infty(X, S)$, then

$$(p_S \circ D_E)^{0,1}(\sigma) = p_S(D_E^{0,1}\sigma) = p_S(\overline{\partial}_E \sigma) = \overline{\partial}_S \sigma.$$

Furthermore for every $\sigma, \tau \in C^\infty(X, S) \subset C^\infty(X, E)$, we have

$$d\{\sigma, \tau\}_{h_S} = d\{\sigma, \tau\}_h = \{D_E\sigma, \tau\}_h + \{\sigma, D_E\tau\}_h$$

Yet since τ and σ have values in $S \subset E$, we have

$$\{D_E\sigma, \tau\}_h = \{p_S(D_E\sigma), \tau\}_{h_S}, \ \{\sigma, D_E\tau\}_h = \{\sigma, p_S(D_E\tau)\}_{h_S}.$$

□

2.15. **Exercise.** Let X be a complex manifold, and let (E, h) and (E', h') be Hermitian holomorphic vector bundles on X. Show that

$$D_{(E\otimes E', h\otimes h')} = D_E \otimes Id_{E'} + Id_E \otimes D_{E'},$$

cf. also [Dem96, Ch.V]. □

So far all the computations we made were for arbitrary local holomorphic frames. It would of course be very helpful if we could choose a local holomorphic frame that is isometric, i.e. the matrix representing locally the Hermitian metric is the identity. In general this is not possible, since the computations in the Gram-Schmidt orthonormalisation algorithm are not holomorphic. The following lemma shows that nevertheless we can find a holomorphic frame that gives an approximation to the first order of an isometric frame. A similar statement about normal coordinates on Kähler manifolds (Theorem 3.15) will be the technical corner stone of the next section.

2.16. **Lemma.** Let X be a complex manifold, and let (E, h) be a Hermitian holomorphic vector bundle on X. Fix a point $x \in X$, then there exists a local *holomorphic* frame such that

(1) $H(z) = Id + O(|z|^2)$.
(2) $i\Theta_E(0) = -i\partial\overline{\partial H}(0)$.

Proof. The second statement follows immediately from the first and Formula (2.9).

For the proof of the first statement, we will make two changes of frame. So start by choosing any local holomorphic frame $e_1, \ldots, e_r$, and let H be the matrix respresenting the metric h with respect to this frame. The matrix $H(0)$ is a positive definite Hermitian matrix, so by linear algebra there exists a matrix A such that

$$A^t H(0)\overline{A} = Id.$$

Since A is nonsingular, the vectors $A(e_1), \ldots, A(e_r)$ form a local frame that is orthonormal in the point 0. Therefore if H' is the matrix representing h with respect to that frame, then

$$H' = Id + O(|z|).$$

We will now make a second change of frame, this one will be of the form $Id+B$ where $B = B(z)$ is a matrix of holomorphic linear forms. The matrix H'' representing h with respect to that frame will then be of the form

$$H'' = (Id + B^t)H'(Id + \overline{B})$$

and by Taylor's formula we will have $H'' = Id + O(|z|^2)$ if and only if $dH''(0) = 0$. Since

$$dH'' = dH' + d(Id + B^t)H' + H'd(Id + \overline{B}) + O(|z|)$$

and $H'(0) = Id$, we have

$$dH''(0) = dH'(0) + dB^t(0) + d\overline{B}(0) = \partial H'(0) + dB^t(0) + \overline{\partial} H'(0) + d\overline{B}(0).$$

Thus if we set

$$B_{j,k} := -\sum_{l=1}^{n} \frac{\partial H'_{k,j}}{\partial z_l}(0) z_l,$$

then $dB^t(0) = \partial B^t(0) = -\partial H(0)$ and $-\overline{\partial} H'(0) = \overline{\partial B}(0) = d\overline{B}(0)$. Therefore if H'' is the matrix representing h with respect to the frame $(Id + B)(A(e_1)), \ldots, (Id + B)(A(e_r))$ the new frame satisfies $dH''(0) = 0$. □

2.C. **The first Chern class and holomorphic line bundles.** Let $\pi : L \to X$ be a complex line bundle over a differentiable manifold, and let D be a connection on L. By Proposition 2.3, the curvature of the connection D is given by

$$\Theta_D \in C^\infty(X, \bigwedge^2 \Omega_{X,\mathbb{C}} \otimes \operatorname{End} L).$$

Since for a line bundle $\operatorname{End} L \simeq L^* \otimes L \simeq X \times \mathbb{C}$, we see that Θ_D is in fact a two-form. Fix now a local frame e_1 for L defining a trivialisation $\pi^{-1}(U) \overset{\theta}{\simeq} U \times \mathbb{C}$, then D is represented by a $(1,1)$-matrix of 1-forms A, so again by Proposition 2.3

$$\Theta_D \overset{\theta}{=} dA + A \wedge A = dA,$$

since the product of a 1-form with itself is zero. This implies immediately

$$d\Theta_D = 0,$$

that is the curvature tensor is a closed two-form, and we denote by $[\Theta_D] \in H^2(X, \mathbb{C})$ the corresponding de Rham cohomology class. Let now D' be another connection on L, that is given with respect to the frame e_1 by a 1-form A'. For every $\sigma \in C^\infty(X, \bigwedge^k \Omega_X \otimes L)$ we have locally

$$D(\sigma) - D'(\sigma) \stackrel{\theta}{=} (ds + As) - (ds + A's) = (A - A')s,$$

where s represents σ with respect to the frame e_1. Using the transition relation (2.4) for the matrices A and A', we see that $A - A'$ glues to a global form, i.e.

$$D(\sigma) - D'(\sigma) \stackrel{\theta}{=} B \wedge \sigma,$$

where $B \in C^\infty(X, \Omega_{X,\mathbb{C}})$. Therefore

$$\Theta_D - \Theta_{D'} = dB$$

is a coboundary, so we have an equality of cohomology classes in $H^2(X, \mathbb{C})$

$$[\Theta_D] = [\Theta_{D'}].$$

We resume our considerations in the following

2.17. **Proposition.** Let $\pi : L \to X$ be a complex line bundle defined over a differentiable manifold, and let D be a connection on L. Then the curvature Θ_D defines an element

$$c_1(L) := [\frac{i}{2\pi}\Theta_D] \in H^2(X, \mathbb{C})$$

that does not depend on the choice of D. We call $c_1(L)$ the first Chern class of L.

The following lemma gives some more precise information on the first Chern class.

2.18. **Lemma.** Let $\pi : L \to X$ be a complex Hermitian line bundle defined over a differentiable manifold. Let D be a Hermitian connection on L, and let be Θ_D the corresponding curvature form. Then

$$i\Theta_D \in C^\infty(X, \Omega^2_{X,\mathbb{R}}),$$

that is $i\Theta_D$ is a real differential form. In particular

$$c_1(L) = [\frac{i}{2\pi}\Theta_D] \in H^2(X, \mathbb{R}).$$

2.19. **Remark.** The first Chern class is only the easiest case of a more general theory of Chern classes for vector bundles, cf. [Biq08, Ch.3.6] or [Wel80, Ch.III,3].

Proof. Being real- or complex-valued is a local property, so fix a point $x \in X$, and let e_1 be a local isometric frame for L in a neighbourhood U of x defining a trivialisation $\pi^{-1}(U) \stackrel{\theta}{\simeq} U \times \mathbb{C}$. Locally the connection D is given by a 1-form such that if a section $\sigma \in C^\infty(U, E)$ identifies to a 1-tuple s, we have

$$D\sigma \stackrel{\theta}{=} ds + As.$$

Furthermore we have seen that the form representing Θ_D is given by

$$\Theta_D \stackrel{\theta}{=} dA.$$

Since the frame is isometric, the matrix A is anti-autodual $\overline{A} = -A$, so

$$\overline{i\Theta_D} = -i\overline{\Theta_D} = -i\overline{dA} = -id\overline{A} = i\Theta_D.$$

This shows that $i\Theta_D$ is invariant under complex conjugation, so it is a real form. □

The lemma explains why we add a constant factor i in the definition of the first Chern class. The constant factor $\frac{1}{2\pi}$ improves the situation even further: the inclusion $\mathbb{Z} \subset \mathbb{R}$ induces a morphism[8] of cohomology groups

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$$

and we have the following

2.20. **Lemma.** Let $\pi : L \to X$ be a complex Hermitian line bundle defined over a differentiable manifold. Let D be a Hermitian connection on L, and let be Θ_D the corresponding curvature form. Then

$$c_1(L) = [\frac{i}{2\pi}\Theta_D] \in H^2(X, \mathbb{Z}).$$

We omit the proof (cf. [Dem96, V, 9.2]), but will prove a converse at the end of this section.

2.21. **Exercise.** Let X be a differentiable manifold, and let V be a complex vector bundle. We define the first Chern class of V by[9]

$$c_1(V) := c_1(\det V).$$

Let now L be a complex line bundle, and let E be a complex vector bundle of rank r. Show that

$$c_1(E \otimes L) = c_1(E) + rc_1(L).$$

□

Let now X be a complex manifold, and let (L, h) be a Hermitian holomorphic line bundle over X. Let D_L be the Chern connection, and denote by Θ_L its curvature tensor. Fix a point $x \in X$, and let e_1 be a local holomorphic frame for L in a neighbourhood U of x defining a holomorphic trivialisation $\pi^{-1}(U) \overset{\theta}{\simeq} U \times \mathbb{C}$. Locally the Hermitian metric is given by a differentiable function

$$H(z) =< e_1(z), e_1(z) >=: ||e_1(z)||_h^2.$$

The function $H : U \to \mathbb{C}$ is real-valued and everywhere strictly positive, so we can take the logarithm to obtain a differentiable function $\varphi : U \to \mathbb{R}$ such that

$$H(z) = e^{-\varphi(z)}$$

and we call φ the weight of the metric with respect to the frame e_1. By Formula (2.9) the Chern connection is given by the $(1,0)$-form

$$A = \overline{H}^{-1}\partial\overline{H} = e^{\varphi(z)}\partial e^{-\varphi(z)} = -\partial\varphi(z),$$

so by Corollary 2.13

$$\Theta_L = \overline{\partial}A = -\overline{\partial}\partial\varphi(z) = \partial\overline{\partial}\varphi(z).$$

[8]In general this morphism is not injective since $H^2(X, \mathbb{Z})$ may have torsion elements.

[9]This definition is compatible with the definition given by the general theory of Chern classes of vector bundles, but you will not need this here.

It follows that the first Chern class $[\frac{i}{2\pi}\Theta_L]$ is represented by a real-valued $(1,1)$-form given locally by

$$\frac{i}{2\pi}\Theta_L(z) = \frac{i}{2\pi}\partial\overline{\partial}\varphi(z). \tag{2.10}$$

Since $\varphi(z) = -\log ||e_1(z)||_h^2$ and every holomorphic nonvanishing section $s : U \to L$ defines a local holomorphic frame for L, we see that

$$\frac{i}{2\pi}\Theta_L(z) = \frac{1}{2\pi i}\partial\overline{\partial}\log ||s(z)||_h^2. \tag{2.11}$$

We come to one of the fundamental definitions of these lectures.

2.22. **Definition.** Let X be a complex manifold of dimension n, and let L be a holomorphic line bundle over X. We say that L is positive if it admits a Hermitian metric h such that the curvature form Θ_L defines a Hermitian product on T_X. More precisely, if the metric h is given locally by a weight function φ such that

$$\Theta_L(z) = \partial\overline{\partial}\varphi(z) = \sum_{1 \leqslant j,k \leqslant n} \frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}} dz_j \wedge d\overline{z_k},$$

then the matrix $\left(\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}}\right)_{1 \leqslant j,k \leqslant n}$ is positive definite.

A related notion that is of fundamental importance in contemporary algebraic geometry is the notion of plurisubharmonic or psh functions.

2.23. **Definition.** (Lelong, Oka 1942) A function $\phi : U \to [-\infty, \infty[$ defined on some open set $U \subset \mathbb{C}^n$ is plurisubharmonic if

- it is upper semicontinuous;
- for every complex line $L \subset \mathbb{C}^n$, the restriction $\phi|_{U \cap L}$ is subharmonic, that is, for all $a \in U$ and $z \in \mathbb{C}^n$ such that $|z| < d(a, \mathbb{C}^n \setminus U)$, the function satisfies the mean value inequality
$$\phi(a) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \phi(a + e^{i\theta}z) d\theta.$$

2.24. **Exercise.** Let $\phi : U \to \mathbb{R}$ be a C^2-function. Show that ϕ is plurisubharmonic if and only if the matrix $\left(\frac{\partial^2\phi}{\partial z_j \partial \overline{z_k}}\right)_{1 \leqslant j,k \leqslant \dim X}$ is positive semidefinite. $\square$

2.25. **Exercise.** (Fubini-Study metric)

a) We set

$$f : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{R}, f(z) = \log(\sum_{j=0}^{n} |z_j|^2)$$

and

$$i\partial\overline{\partial}f(z) = i\sum_{1 \leqslant j,k \leqslant n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}} dz_j \wedge d\overline{z_k}.$$

Show that f is plurisubharmonic, i.e. the matrix $\left(\frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}\right)_{1 \leqslant j,k \leqslant \dim X}$ is positive semidefinite.

Show that $i\partial\overline{\partial}f$ induces a $(1,1)$-form ω on $\mathbb{P}^n$ such that $d\omega = 0$.

Show that f can be seen as a weight function of a metric h on $L := \mathscr{O}_{\mathbb{P}^n}(1)$ such that $\Theta_{L,h}$ is positive. □

2.26. **Exercise.**

a)) Let X be a complex manifold, and let L be a holomorphic line bundle on X. Show that L is positive if and only if $L^{\otimes m}$ is positive for some $m \in \mathbb{N}^*$.

b) Let X be a complex manifold and let L be a holomorphic line bundle on X. We suppose that L is globally generated, i.e. for every $x \in X$ there exists a global section $\sigma \in \Gamma(X, L)$ such that $\sigma(x) \neq 0$. Show that L admits a hermitian metric with weight function φ such that

$$\left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}}\right)_{1 \leqslant j,k \leqslant \dim X}$$

is positive semidefinite.

Bonus question (hard) : show that if we suppose only $\Gamma(X, L) \neq 0$, the line bundle L admits a singular metric with plurisubharmonic weight. □

2.27. **Exercise.** Let X be a compact complex manifold, and let L be a positive holomorphic line bundle on X. Let M be a holomorphic line bundle on X. Show that there exists a $N_0 \in \mathbb{N}$ such that

$$M \otimes L^{\otimes N}$$

is positive for every $N \geqslant N_0$. □

If one pushes the computations of Lemma 2.16 one step further, we get

2.28. **Proposition.** [Dem96, V.,12.10] Let X be a complex manifold, and let (E, h) be a Hermitian holomorphic vector bundle on X. Fix a point $x \in X$, and local coordinates $z_1, \ldots, z_n$ around x. Then there exists a local *holomorphic* frame $e_1, \ldots, e_r$ such that the matrix H representing h satisfies

$$h_{\lambda,\mu}(z) = \delta_{\lambda,\mu} - \sum_{j,k} c_{j,k,\lambda,\mu} z_j \overline{z_k} + O(|z|^3),$$

where the $c_{j,k,\lambda,\mu}$ are constants. In particular we have

$$i\Theta_E(x) = i \sum_{j,k,\lambda,\mu} c_{j,k,\lambda,\mu} dz_j \wedge d\overline{z_k} \otimes e_\lambda^* \otimes e_\mu.$$

Using the notation from the proposition, we can write down the Hermitian form on the vector space $(T_X \otimes E)_x$ associated to the curvature tensor $\Theta(E)(x)$. Let $\eta = \sum_{j,\lambda} \eta_{j,\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in (T_X \otimes E)_x$ then we set

$$\widetilde{\Theta}_E(x)(\eta, \eta) = \sum_{j,k,\lambda,\mu} c_{j,k,\lambda,\mu} \eta_{j,\lambda} \overline{\eta_{k,\mu}}.$$

2.29**. Definition.** Let X be a complex manifold, and let (E, h) be a Hermitian holomorphic vector bundle on X.

i) We say that (E, h) is positive in the sense of Nakano if for all $x \in X$, the Hermitian form $\widetilde{\Theta}_E(x)$ is positive definite on $(T_X \otimes E)_x$.

ii) We say that (E, h) is positive in the sense of Griffiths if for all $x \in X$, the Hermitian form $\widetilde{\Theta}_E(x)$ is positive definite on all the decomposable vectors in $(T_X \otimes E)_x$, that is

$$\widetilde{\Theta}_E(x)(\xi \otimes v, \xi \otimes v) > 0 \qquad \forall\, \xi \in T_{X,x}, v \in E_x.$$

If E is a line bundle, the definitions of Nakano and Griffiths positivity coincide and are just the same as a positive line bundle. For vector bundles of higher rank, this is no longer case: the definition of Nakano positivity is actually rather restrictive, but has the advantage that we get vanishing theorems (cf. Theorem 4.13). In general the less restrictive notion of Griffiths positivity is much more useful, since it has better functorial properties as shows Exercise 2.54.

We close this section by showing a converse statement to Lemma 2.20.

2.30**. Theorem.** (Lefschetz theorem on $(1,1)$-classes) Let X be a complex manifold, and let $\omega \in C^\infty(X, \Omega_X^{1,1})$ be a d-closed real $(1,1)$-form such that $[\omega] \in H^2(X, \mathbb{Z})$. Then there exists a holomorphic Hermitian line bundle (L, h) on X such that $\frac{i}{2\pi}\Theta_{L,h} = \omega$.

Before we can prove this theorem, we need the following explicit construction of the isomorphism in the comparison theorem for the de Rham cohomology (1.3):

let X be a differentiable manifold and let $Z^1 \subset \Omega_X$ be the sheaf of d-closed 1-forms. By the Poincaré lemma, we have an exact sequence

$$0 \to \mathbb{R} \to \mathscr{C}^\infty \xrightarrow{d} Z^1 \to 0,$$

where $\mathbb{R}$ is the sheaf of locally constant real-valued functions. Analogously, let $Z^2 \subset \Omega_X^2$ be the sheaf of d-closed 2-forms, then we have an exact sequence

$$0 \to Z^1 \to \Omega_X \xrightarrow{d} Z^2 \to 0.$$

Now by definition of the de Rham cohomology

$$H^2(X, \mathbb{R}) = \frac{Z^2(X)}{dC^\infty(X, \Omega_X)}.$$

Let $\mathscr{U} = (U_\alpha)_{\alpha \in A}$ be an open covering of X such that for all $\alpha \in A$, the morphisms

$$d : C^\infty(U_\alpha, \Omega_X) \to Z^2(U_\alpha)$$

and for all $\alpha, \beta \in A$ the morphisms

$$d : \mathscr{C}^\infty(U_\alpha \cap U_\beta) \to Z^1(U_\alpha \cap U_\beta)$$

are surjective.

If ω is a d-closed 2-form, then choose for every α an $A_\alpha \in C^\infty(U_\alpha, \Omega_X)$ such that $\omega|_{U_\alpha} = dA_\alpha$. On the intersection $U_\alpha \cap U_\beta$, the 1-form $A_\beta - A_\alpha$ is d-closed, and an easy computation shows that

$$(A_\beta - A_\alpha)_{\alpha\beta}$$

is a Čech 1-cocycle $C^1(\mathscr{U}, Z^1)$.

Choose for $\alpha, \beta \in A$ a function $f_{\alpha\beta} \in \mathscr{C}^\infty(U_\alpha \cap U_\beta)$ such that $A_\beta - A_\alpha = df_{\alpha\beta}$. On the intersection $U_\alpha \cap U_\beta \cap U_\gamma$, the differentiable function $f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}$ is d-closed, hence it is a locally constant function. An easy computation shows that

$$(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})_{\alpha\beta\gamma}$$

is a Čech 2-cocycle in $C^2(\mathscr{U}, \mathbb{R})$. The proof of the comparison theorem shows that this cocycle represents the image of $[\omega]$ under the isomorphism

$$H^2(X, \mathbb{R}) \simeq \check{\mathrm{H}}^2(X, \mathbb{R}).$$

2.31. **Exercise.** Let $D \subset \mathbb{C}^n$ be a polydisc and let $\omega \in C^\infty(D, \Omega^2_{D,\mathbb{R}})$ be a d-closed two-form of type $(1,1)$. Show that there exists a $\varphi \in C^\infty(D)$ such that

$$\omega = \partial\overline{\partial}\phi.$$

□

Proof of Theorem 2.30. Let $\mathscr{U} = U_\alpha$ be an open covering by polydiscs such that the intersections $U_\alpha \cap U_\beta$ are simply connected. The form ω is d-closed, so by Exercise 2.31 there exist differentiable functions ϕ_α on U_α such that

$$\frac{i}{2\pi}\partial\overline{\partial}\phi_\alpha = \omega|_{U_\alpha}.$$

Therefore for every α, β the function $\phi_\beta - \phi_\alpha$ is pluriharmonic on the intersection $U_\alpha \cap U_\beta$. By Exercise 1.11 there exist holomorphic functions $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that

$$2\,\mathrm{Re}(f_{\alpha\beta}) = \phi_\beta - \phi_\alpha.$$

We consider $f = f_{\alpha\beta}$ as a Čech 1-chain in $C^1(\mathscr{U}, \mathscr{O}_X)$ (cf. page 33 for the definition), then its Čech differential is

$$(\delta f)_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ and

$$2\,\mathrm{Re}(\delta f)_{\alpha\beta\gamma} = 0.$$

Since the $f_{\alpha\beta}$ are holomorphic, this shows $(\delta f)_{\alpha\beta\gamma} \in \Gamma(U_\alpha \cap U_\beta \cap U_\gamma, i\mathbb{R})$.

Consider now the real forms $A_\alpha = \frac{i}{4\pi}(\overline{\partial}\phi_\alpha - \partial\phi_\alpha)$. Since

$$\partial(\phi_\beta - \phi_\alpha) = \partial(f_{\alpha\beta} + \overline{f_{\alpha\beta}}) = \partial f_{\alpha\beta} = df_{\alpha\beta},$$

and analogously $\overline{\partial}(\phi_\beta - \phi_\alpha) = d\overline{f_{\alpha\beta}}$, we get

$$A_\beta - A_\alpha = \frac{i}{4\pi}d(\overline{f_{\alpha\beta}} - f_{\alpha\beta}) = \frac{1}{2\pi}d\,\mathrm{Im}\,f_{\alpha\beta}.$$

Since $\omega|_{U_\alpha} = dA_\alpha$, the explicit construction of the isomorphism $H^2(X, \mathbb{R}) \simeq \check{\mathrm{H}}^2(X, \mathbb{R})$ shows that the Čech cohomology class of

$$(\frac{1}{2\pi} \operatorname{Im}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma}$$

equals the Čech cohomology class corresponding to $[\omega]$. By hypothesis $[\omega]$ is the image of a Čech cocycle $(n_{\alpha\beta\gamma}) \in \mathrm{H}^2(X, \mathbb{Z})$, so

$$(\frac{1}{2\pi} \operatorname{Im}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma} = n_{\alpha\beta\gamma} + \delta(c_{\alpha\beta})$$

for some 1-chain $(c_{\alpha\beta})$ with values in $\mathbb{R}$. Set now $f'_{\alpha\beta} := f_{\alpha\beta} - 2\pi i c_{\alpha\beta}$, then the preceding computations show that

$$(f'_{\beta\gamma} - f'_{\alpha\gamma} + f'_{\alpha\beta}) \in \Gamma(U_{\alpha\beta\gamma}, 2\pi i \mathbb{Z}),$$

so $g_{\alpha\beta} := exp(-f'_{\alpha\beta})$ defines a 1-cocycle in $\mathscr{O}_X^*$. Since

$$\phi_\beta - \phi_\alpha = 2 \operatorname{Re} f'_{\alpha\beta} = -\log |g_{\alpha\beta}|^2,$$

the line bundle L corresponding to $g_{\alpha\beta}$ admits a global Hermitian metric defined in every trivialisation by $H_\alpha = \exp(-\phi_\alpha)$ and therefore by Formula (2.10)

$$\frac{i}{2\pi} \Theta_{L,h} = \frac{i}{2\pi} \partial\overline{\partial}\phi_\alpha = \omega$$

on U_α. □

2.D. Hypersurfaces and divisors.

2.32. **Definition.** Let X be a complex manifold. A hypersurface of X is a closed subset $D \subset X$ such that for all $x \in D$ there exists an open neighbourhood $x \in U \subset X$ and a non-zero holomorphic map $f : U \to \mathbb{D}$ such that

$$D \cap U = \{x \in U \mid f(x) = 0\}.$$

We say that $x \in D$ is a smooth point if we can choose a $f : U \to \mathbb{D}$ that is a submersion. We denote by $D_{\text{nons}} \subset D$ the union of smooth points and call it smooth or nonsingular locus of D. We say that the hypersurface is smooth if $D_{\text{nons}} = D$.

2.33. **Remark.** We do *not* suppose that a hypersurface is connected. In particular a smooth hypersurface is a disjoint union of submanifolds of codimension one.

The next exercise shows that hypersurfaces and holomorphic line bundles are closely related:

2.34. **Exercise.** Let X be a complex manifold, and let $D \subset X$ be a smooth hypersurface. We say that a holomorphic function $f_\alpha : U_\alpha \to \mathbb{C}$ is a local equation for D on an open subset $U_\alpha \subset X$ if f_α is submersive and

$$D \cap U_\alpha = \{x \in U_\alpha \mid f_\alpha(x) = 0\}.$$

a) Let $(U_\alpha)_{\alpha \in A}$ be an open covering of X such that there exist local equations $f_\alpha \in \mathscr{O}_{U_\alpha}$ for D.

We define meromorphic functions on $U_\alpha \cap U_\beta$ by

$$g_{\alpha\beta} := \frac{f_\alpha}{f_\beta}.$$

Show that $g_{\alpha\beta}$ extends to a holomorphic map on $U_\alpha \cap U_\beta$ that is non-zero for every $x \in U_\alpha \cap U_\beta$. Show that $(g_{\alpha\beta})_{\alpha,\beta\in A}$ is a Čech 1-cocyle in $\mathscr{O}_X^*$. We set

$$\mathscr{O}_X(D) \in \operatorname{Pic}(X)$$

for the corresponding holomorphic line bundle. Show that the isomorphism class of the line bundle does not depend on the choice of the covering or the local equations.

b) Let $\mathscr{I}_D \subset \mathscr{O}_X$ be the ideal sheaf of D in X, that is the sheaf defined for every open set $U \subset X$ by

$$\mathscr{I}_D(U) := \{s \in \mathscr{O}_X(U) \mid s(x) = 0 \ \forall \ x \in D \cap U\}.$$

Show that if U_α is a coordinate neighbourhood and f_α a local equation, then $\mathscr{I}_D(U_\alpha) = f_\alpha \mathscr{O}_X(U_\alpha)$.

Thus $\mathscr{I}_D$ is invertible (cf. Exercise 1.58) and we denote by the same letter the line bundle associated to it. Show that

$$\mathscr{I}_D = \mathscr{O}_X(D)^*.$$

Thus we have an exact sequence

$$0 \to \mathscr{O}_X(D)^* \to \mathscr{O}_X \to \mathscr{O}_D \to 0$$

c) Let $L \to X$ be a holomorphic line bundle, and let $\sigma \in \Gamma(X, L)$ be a section such that

$$D := \{x \in X \mid \sigma(x) = 0\}$$

is smooth and $d\sigma(x) \neq 0$ for all $x \in D$ (cf. Exercise 1.65). Show that

$$L \simeq \mathscr{O}_X(D).$$

□

We will now generalise these statements without the smoothness hypothesis. This needs some extra effort:

2.35. **Lemma.** Let X be a complex manifold and $D \subset D$ a hypersurface. Then the nonsingular locus D_{nons} is an open, dense subset of D.

Proof. The nonsingular locus is clearly open, so all we have to show is that if $x \in D$ is a point, then D_{nons} is dense in some neighbourhood of x. Taking a local coordinate neighbourhood we are reduced to consider the situation where X is a polydisc $\mathbb{D}^n$ and $x = 0$. Set

$$\mathscr{I}_D := \{g : \mathbb{D}^n \to \mathbb{D} \mid g(z) = 0 \qquad \forall \ z \in D\},$$

then $\mathscr{I}_D$ is an ideal in the ring of holomorphic function $\mathscr{O}_{\mathbb{D}^n}$ which is principal (cf. [GH78, p.18ff]), so there exists a holomorphic function $f : \mathbb{D}^n \to \mathbb{D}$ such that

$$\mathscr{I}_D = f\mathscr{O}_{\mathbb{D}^n}.$$

Let $z_1, \ldots, z_n$ be some linear coordinates on $\mathbb{D}^n$. If $f(z_1, 0, \ldots, 0) = 0$ for all $z_1 \in \mathbb{D}$, the line $\{z_2 = \ldots = z_n = 0\}$ is contained in D. Since D is a proper subset of $\mathbb{D}^n$ we can suppose (up to changing our coordinates) that this is not the case.

Let l be the order of vanishing of $f(z_1, 0, \ldots, 0)$ in $z_1 = 0$, then by the Weierstrass preparation theorem there exist (up to replacing $\mathbb{D}^n$ by a smaller disc and coordinate change) holomorphic functions

$$f_j : \mathbb{D}^n \to \mathbb{D} \qquad \forall\, j = 0, \ldots, l-1$$

such that

$$f = z_1^l + \sum_{0 \leqslant j < l} z_1^j f_j(z_2, \ldots, z_n).$$

If the restriction of $\frac{\partial f}{\partial z_1}$ to D is not zero we are done, since this implies that f is submersive on the open, dense set

$$D \cap \{z \in D \mid \frac{\partial f}{\partial z_1}(z) \neq 0\}.$$

Suppose now that $\frac{\partial f}{\partial z_1}(z) = 0$ for all $z \in D$, then $\frac{\partial f}{\partial z_1} \in \mathscr{I}_D$. Thus there exists a holomorphic function h such that

$$\frac{\partial f}{\partial z_1} = hf.$$

Yet

$$\deg_{z_1} \frac{\partial f}{\partial z_1} < \deg_{z_1} f,$$

so we get $\frac{\partial f}{\partial z_1} = 0$. We repeat this argument for the other variables, then the worst case would be

$$\frac{\partial f}{\partial z_j} = 0 \qquad \forall\, j = 1, \ldots, n.$$

Yet this implies that f is constant, a contradiction to $\emptyset \neq D \subsetneq \mathbb{D}^n$. $\square$

2.36. **Proposition.** Let X be a complex manifold and $D \subset X$ a hypersurface. Then there exists a holomorphic line bundle $\mathscr{O}_X(D)$ on X that has a global section $\sigma \in \Gamma(X, \mathscr{O}_X(D))$ such that

$$D = \{x \in X \mid \sigma(x) = 0\}.$$

Proof. Set $Z := D \setminus D_{\mathrm{nons}}$. Then $D_{\mathrm{nons}} \subset X \setminus Z$ is a smooth hypersurface, so by Exercise 2.34 there exists an open covering U_α of X and holomorphic functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \setminus Z \to \mathbb{C}^*$$

defining a line bundle $\mathscr{O}_{X \setminus Z}(D_{\mathrm{nons}})$ which has the stated property. By the preceding lemma the nonsingular locus is dense in D, so Z does not contain any hypersurface. Thus by Hartog's Theorem 1.12 the holomorphic functions $g_{\alpha\beta}$ extend to holomorphic functions on $U_\alpha \cap U_\beta$. The same holds for $\frac{1}{g_{\alpha\beta}}$, so we obtain holomorphic functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*.$$

It is easy to check that this function still verify the cocycle relation and the corresponding line bundle $\mathscr{O}_X(D)$ satisfies

$$\mathscr{O}_X(D)|_{X\setminus Z} = \mathscr{O}_{X\setminus Z}(D_{\mathrm{nons}}).$$

$\square$

2.37. **Definition.** Let X be a complex manifold and $D \subset X$ a hypersurface. We say that D is reducible if there exist hypersurfaces $D_1, D_2 \subset X$ such that $D_i \subsetneq D$ and

$$D = D_1 \cup D_2.$$

If this is not the case we say that D is irreducible.

Remark. A hypersurface is irreducible if and only if its nonsingular locus is connected [GH78, Ch.1.1]).

2.38. **Definition.** Let X be a complex manifold. A divisor on X is a finite formal sum

$$\sum_i a_i D_i \qquad a_i \in \mathbb{Z},$$

where the D_i are irreducible hypersurfaces in X. The holomorphic line bundle associated to D is defined by

$$\mathscr{O}_X(D) := \otimes_i \mathscr{O}_X(D_i)^{\otimes a_i}.$$

Furthermore we set

$$c_1(D) := c_1(\mathscr{O}_X(D)).$$

Now that we have seen that to every divisor we can associate a holomorphic line bundle, it is natural to ask if the inverse holds. In general this is not the case : for example there are complex tori that do not have any hypersurfaces (cf. Exercise 3.50). The following theorem will be an immediate consequence of the results of Section 4:

2.39. **Theorem.** Let X be a projective manifold, and let L be a holomorphic line bundle over X. Then there exist hypersurfaces $D_1, D_2 \subset X$ such that

$$L \simeq \mathscr{O}_X(D_1 - D_2).$$

2.40. **Remark.** If we admit Bertini's theorem [GH78, p.137] we can even suppose that D_1 and D_2 are smooth and intersect transversally.

We can now prove a fundamental result relating the first Chern class with the integration over a corresponding divisor: let X be a compact complex variety of dimension n, and let $M \subset X$ be a smooth hypersurface. If $\omega = d\eta$ is an exact form of degree $2n-2$, then the integral $\int_M \omega$ equals zero by Stokes' theorem. Thus the map

$$C^\infty(X, \Omega^{2n-2}_{X,\mathbb{R}}) \to \mathbb{R},\ \omega \mapsto \int_M \omega|_M$$

induces a linear form

$$[M] : H^{2n-2}(X, \mathbb{R}) \to \mathbb{R}.$$

Moreover by Poincaré duality $H^2(X,\mathbb{R}) = H^{2n-2}(X,\mathbb{R})^*$, so $[M]$ can be seen as a cohomology class in $H^2(X,\mathbb{R})$ which we call the Poincaré dual of M. By definition of the de Rham cohomology, the class $[M]$ can be represented by some closed real 2-form. The following theorem shows that a curvature tensor of the corresponding line bundle works.

2.41**. Theorem.** (Formula of Lelong-Poincaré) Let X be a compact complex variety of dimension n, and let $M \subset X$ be a smooth hypersurface[10]. Denote by $\mathscr{O}_X(M)$ the holomorphic line bundle corresponding to M and by $c_1(M)$ the first Chern class of $\mathscr{O}_X(M)$. Then we have

$$c_1(M) = [M],$$

that is if $\frac{i}{2\pi}\Theta$ is some curvature form representing $c_1(L)$ and $\psi \in C^\infty(X, \Omega^{2n-2}_{X,\mathbb{R}})$ is d-closed, then

$$\int_X \frac{i}{2\pi}\Theta \wedge \psi = \int_M \psi|_M.$$

Proof. We endow $\mathscr{O}_X(M)$ with some hermitian metric h and denote by Θ the corresponding Chern curvature tensor. Let $s \in \Gamma(X, \mathscr{O}_X(M))$ be a global section vanishing exactly along M. For small $\varepsilon > 0$ the open set

$$M(\varepsilon) := \{z \in X \mid ||s(z)||_h < \varepsilon\}.$$

is a tubular neighbourhood of M in X. By Formula (2.11) we have

$$\frac{i}{2\pi}\Theta(z) = \frac{1}{2\pi i}\partial\overline{\partial}\log||s(z)||_h^2 \qquad \forall\ z \in X \setminus M(\varepsilon).$$

Thus

$$\int_X \frac{i}{2\pi}\Theta \wedge \psi = \lim_{\varepsilon\to 0}\int_{X\setminus M(\varepsilon)} \frac{1}{2\pi i}\partial\overline{\partial}\log||s(z)||_h^2 \wedge \psi.$$

Since $\partial\overline{\partial} = d\overline{\partial}$ and ψ is d-closed, Stokes' theorem yields

$$\int_{X\setminus M(\varepsilon)} \frac{1}{2\pi i}\partial\overline{\partial}\log||s(z)||_h^2 \wedge \psi = \int_{\partial M(\varepsilon)} \frac{-1}{2\pi i}\overline{\partial}\log||s(z)||_h^2 \wedge \psi.$$

We want to compute the integral on the right hand side. If we take a finite covering of M by polydiscs $U_\alpha \subset X$, then $\partial M(\varepsilon) \subset \cup_\alpha U_\alpha$ for ε small enough. Taking a partition of unity subordinate to U_α we are reduced to computing

$$\int_{\partial M(\varepsilon)\cap U_\alpha} \frac{-1}{2\pi i}\overline{\partial}\log||s(z)||_h^2 \wedge \psi$$

for every α. Up to replacing the covering by smaller polydiscs and choosing appropriate coordinates $z_1, \ldots, z_n$ we can suppose that

$$M \cap U_\alpha = \{z_n = 0\}$$

and

$$(*) \qquad \partial M(\varepsilon) \cap U_\alpha = \{|z_n| = \varepsilon\} \simeq (M \cap U_\alpha) \times S^\varepsilon, .$$

[10]The statement is true without the smoothness hypothesis, but technically more difficult: the first non-trivial issue is to show that the integral $\int_{M_{\mathrm{nons}}} \omega|_{M_{\mathrm{nons}}}$ converges.

where S^ε is the circle of radius ε around $0 \in \mathbb{C}$. Now for every $z \in M(\varepsilon) \cap U_\alpha$ we have

$$||s(z)||_h^2 = |z_n|^2 h_\alpha,$$

where $h_\alpha : M(\varepsilon) \cap U_\alpha \to \mathbb{R}^+$ is positive and bounded from below by some $\delta > 0$. Thus in the complement of $M \cap U_\alpha$

$$\overline{\partial} \log ||s(z)||_h^2 = \frac{d\overline{z_n}}{\overline{z_n}} + \overline{\partial} \log h_\alpha.$$

Since $\overline{\partial} \log h_\alpha$ is bounded one sees easily that

$$\lim_{\varepsilon \to 0} \int_{\partial M(\varepsilon)} \frac{-1}{2\pi i} \overline{\partial} \log h_\alpha \wedge \psi = 0,$$

so we don't have to worry about this term. Since $\frac{d\overline{z_n}}{\overline{z_n}} = \overline{\frac{dz_n}{z_n}}$ and ψ is real, we have

$$\int_{\partial M(\varepsilon) \cap U_\alpha} \frac{-1}{2\pi i} \frac{d\overline{z_n}}{\overline{z_n}} \wedge \psi = \overline{\int_{\partial M(\varepsilon) \cap U_\alpha} \frac{1}{2\pi i} \frac{dz_n}{z_n} \wedge \psi}.$$

Set

$$d\lambda := dz_1 \wedge \ldots \wedge dz_{n-1} \wedge d\overline{z_1} \wedge \ldots d\overline{z_{n-1}},$$

then we can write the $2n-2$-form in our local coordinates as

$$\psi|_{U_\alpha} = \tilde{\psi}(z)\; d\lambda + \eta,$$

where each term of η contains dz_n or $d\overline{z_n}$. Note that

$$(**) \qquad \psi|_{U_\alpha \cap M} = \tilde{\psi}(z_1, \ldots, z_{n-1}, 0)\; d\lambda.$$

By $(*)$ we have $d\overline{z_n} = -\frac{\overline{z_n}}{z_n} dz_n$ on $\partial M(\varepsilon) \cap U_\alpha$, so

$$\left(\frac{dz_n}{z_n} \wedge \psi\right)|_{\partial M(\varepsilon) \cap U_\alpha} = \left(\frac{dz_n}{z_n} \wedge \tilde{\psi}\; d\lambda\right)|_{\partial M(\varepsilon) \cap U_\alpha}$$

Since $\partial M(\varepsilon) \cap U_\alpha \simeq (M \cap U_\alpha) \times S^\varepsilon$ Fubini's theorem gives

$$\int_{\partial M(\varepsilon) \cap U_\alpha} \frac{1}{2\pi i} \frac{dz_n}{z_n} \wedge \psi = \int_{M \cap U_\alpha} d\lambda \int_{S^\varepsilon} \frac{1}{2\pi i} \frac{\tilde{\psi}}{z_n} dz_n.$$

By the generalised Cauchy formula 1.81

$$\int_{S^\varepsilon} \frac{1}{2\pi i} \frac{\tilde{\psi}}{z_n} dz_n = \tilde{\psi}(z_1, \ldots, z_{n_1}, 0) + \int_{\mathbb{D}^\varepsilon} \frac{i}{2\pi z_n} \frac{\partial \tilde{\psi}}{\partial \overline{z_n}} dz_n \wedge d\overline{z_n}.$$

Since $\frac{1}{z_n}$ is L^1 around $z_n = 0$, the second term converges to 0 for $\varepsilon \to 0$. Thus

$$\lim_{\varepsilon \to 0} \int_{\partial M(\varepsilon) \cap U_\alpha} \frac{1}{2\pi i} \frac{dz_n}{z_n} \wedge \psi = \int_{M \cap U_\alpha} \tilde{\psi}(z_1, \ldots, z_{n-1}, 0) d\lambda$$

Thus by $(**)$

$$\lim_{\varepsilon \to 0} \int_{\partial M(\varepsilon) \cap U_\alpha} \frac{1}{2\pi i} \overline{\partial} \log ||s(z)||_h^2 \wedge \psi = \int_{M \cap U_\alpha} \psi.$$

□

2.42**. Remark.** If we are willing to work with currents, that is differential forms whose coefficients are not C^∞-functions but merely distributions, we can give a more conceptual proof of the theorem, at least for the local situation: let $\mathbb{D}^n$ be the unit disc and consider the hypersurface M defined by $z_n = 0$. As in the proof of the theorem we want to understand the distribution

$$\frac{i}{2\pi}\partial\overline{\partial}\log|z_n^2| = \frac{-i}{2\pi}\overline{\partial}\left(\frac{dz_n}{z_n}\right) = \frac{\partial}{\partial\overline{z_n}}\left(\frac{1}{\pi z_n}\right)\frac{i}{2}dz_n \wedge d\overline{z_n}.$$

Yet by Corollary 1.82 the distribution $\frac{\partial}{\partial\overline{z_n}}\left(\frac{1}{\pi z_n}\right)$ equals the Dirac mass centered on the hypersurface $z_n = 0$.

2.43**. Notation.** If L is a holomorphic line bundle over a compact complex manifold X of dimension n and $\psi \in C^\infty(X, \Omega^{2n-2}_{X,\mathbb{R}})$ is d-closed, we set

$$\int_X c_1(L) \wedge \psi := \int_X \frac{i}{2\pi}\Theta \wedge \psi$$

where Θ is some curvature tensor of L. Stokes' theorem shows that this is well-defined, i.e. does not depend on the choice of Θ.

The theorem has a number of important geometric consequences:

2.44**. Corollary.** Let X be a compact complex curve, and let $D = \sum_i a_i D_i$ be a divisor on X. Denote by $c_1(D)$ the first Chern class of the holomorphic line bundle $\mathscr{O}_X(D)$. Then

$$\sum_i a_i = \int_X c_1(D).$$

Proof. The statement is additive on both sides, so we reduce to the case where D is a point. In this case let $\psi = 1$ be the constant function with value one on X, then by Theorem 2.41

$$\int_X c_1(D) \wedge 1 = \int_D 1 = 1.$$

□

On surfaces the product of Chern classes counts the number of points in the intersection.

2.45**. Corollary.** Let X be a compact complex surface, and let $M_1, M_2 \subset X$ be two smooth curves meeting transversally, i.e.

$$T_{X,x} = T_{M_1,x} \oplus T_{M_2,x} \qquad \forall\, x \in M_1 \cap M_2.$$

Denote by $c_1(M_i)$ the first Chern class of the holomorphic line bundle $\mathscr{O}_X(M_i)$. Then

$$\#(M_1 \cap M_2) = \int_X c_1(M_1) \wedge c_1(M_2).$$

Proof. Let $\frac{i}{2\pi}\Theta_2$ be some Chern curvature tensor representing $c_1(D_2)$. By Theorem 2.41 we have

$$\int_X c_1(M_1) \wedge c_1(M_2) = \int_X c_1(M_1) \wedge \frac{i}{2\pi}\Theta_2 = \int_{M_1} \frac{i}{2\pi}\Theta_2|_{M_1}.$$

Since the intersection is transversal, we can choose a covering $U_\alpha \subset X$ by open sets, such that for every $z \in M_1 \cap M_2 \in U_\alpha$ we have local holomorphic coordinates z_1, z_2 such that

$$M_i \cap U_\alpha = \{z_i = 0\}.$$

Thus the restriction of the local equation of M_2 to M_1 is $z_2 = 0$, i.e. the equation of a reduced point. By the construction in Exercise 2.34 this shows that

$$\mathscr{O}_X(M_2)|_{M_1} = \mathscr{O}_{M_1}(M_1 \cap M_2),$$

so $\Theta_2|_{M_1}$ is a curvature form for the line bundle $\mathscr{O}_{M_1}(M_1 \cap M_2)$. Thus by Corollary 2.44

$$\int_{M_1} \frac{i}{2\pi}\Theta_2|_{M_1} = \#(M_1 \cap M_2).$$

□

The statement of the corollary holds for irreducible distinct curves not meeting transversally if we count the number of points in the intersection with the appropriate multiplicity. A first approximation to this more general situation is the following.

2.46**. Exercise.** Let X be a compact complex surface, and let $C \subset X$ be a smooth irreducible curve. Let D be a curve such that C is not an irreducible component of D. Show that

$$\#(C \cap D) \leqslant \int_X c_1(C) \wedge c_1(D).$$

□

2.47**. Theorem.** Let X be a projective surface and denote by $\mathrm{Pic}(X)$ its Picard group. Then there is a unique pairing $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$ denote by $L_1 \cdot L_2$ for any two holomorphic line bundles L_1, L_2, such that

(1) if C and D are nonsingular curves meeting transversally, then

$$\mathscr{O}_X(C) \cdot \mathscr{O}_X(D) = \#(C \cap D),$$

(2) it is symmetric: $L_1 \cdot L_2 = L_2 \cdot L_1$,
(3) is is additive: $(L_1 \otimes L_2) \cdot D = L_1 \cdot D + L_2 \cdot D$.

This pairing is given by

$$L_1 \cdot L_2 = \int_X c_1(L_1) \wedge c_1(L_2).$$

2.48**. Remark.** We will study this intersection product in more detail in Subsection 3.E.

Proof. We leave it to the reader to check that if L_1 and L_1' are isomorphic line bundles, then $c_1(L_1) = c_1(L_1')$ in $H^2(X, \mathbb{R})$. Thus the integral

$$\int_X c_1(L_1) \wedge c_1(L_2)$$

depends only on the isomorphism class of L_1 and we get a well-defined map

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{R}, (L_1, L_2) \mapsto \int_X c_1(L_1) \wedge c_1(L_2).$$

This map is clearly symmetric and additive and also satisfies the property 1) by Corollary 2.45. In order to see that it is integer-valued we use Theorem 2.39 and Remark 2.40: given two line bundles L_1 and L_2 we choose smooth hypersurfaces D_1, D_2, D_3, D_4 meeting transversally such that

$$L_1 \simeq \mathscr{O}_X(D_1 - D_2), L_2 \simeq \mathscr{O}_X(D_3 - D_4).$$

Then the properties $1) - 3)$ imply that $\int_X c_1(L_1) \wedge c_1(L_2) \in \mathbb{Z}$.

The uniqueness is shown in the same way, i.e. we use Theorem 2.39 and Remark 2.40 and the properties $2) + 3)$ to reduce to the case where $L_1 \simeq \mathscr{O}_X(D_1), L_2 \simeq \mathscr{O}_X(D_2)$ with D_1, D_2 smooth and meeting transversally and conclude by property 1). □

The theorem above also holds for X an arbitrary compact complex surface. However in this case we must use that the first Chern class $c_1(L_i)$ is an element of $H^2(X, \mathbb{Z})$. The intersection product $\int_X c_1(L_1) \wedge c_1(L_2)$ then identifies to the cup product in cohomology, in particular it takes values in $H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$.

If C and D are curves in X, we set

$$C \cdot D := \mathscr{O}_X(C) \cdot \mathscr{O}_X(D)$$

Note that this definition also makes sense when $C = D$, i.e. we can define the self-intersection number of a curve $C \subset X$ by

$$C^2 := \mathscr{O}_X(C) \cdot \mathscr{O}_X(C).$$

The next exercise gives a (very important) example where this number is strictly negative !

2.49. **Exercise.** Let $0 \in U \subset \mathbb{C}^n$ be an open neighbourhood of 0. The blow-up of U in 0 is the set

$$U' := \{((x_1, \ldots, x_n), (y_1 : \ldots : y_n)) \in U \times \mathbb{P}^{n-1} \mid x_i y_j = x_j y_i \quad \forall\, i, j \in \{1, \ldots, n\}\}.$$

a) Show that U' is a submanifold of dimension n of $U \times \mathbb{P}^{n-1}$.

b) Let $\pi : U' \to U$ be the map induced by the projection on the first factor. Show that $\pi^{-1}(0) \simeq \mathbb{P}^{n-1}$ and that $\pi|_{U' \setminus \pi^{-1}(0)}$ is a biholomorphism.

c) Let now X be a complex manifold of dimension n and $x_0 \in X$ a point. Let U_i be an open covering of X such that $x_0 \in U_1$ is a coordinate neighbourhood and such that $x_0 \notin U_i$ for $i \neq 1$. Let U_1' be the blow-up of U_1 in x_0. Show that $U_1' \cup \cup_{i \geqslant 2} U_i$ glues to a complex manifold X' that admits a holomorphic map $\pi : X' \to X$ such that $\pi^{-1}(x_0) \simeq \mathbb{P}^{n-1}$ and $\pi|_{X' \setminus \pi^{-1}(0)}$ is a biholomorphism. We call X' the blow-up of X in x_0 and $E := \pi^{-1}(x_0) \simeq \mathbb{P}^{n-1}$ the exceptional divisor.

d) Show that

$$K_{X'} \simeq \mu^* K_X \otimes \mathscr{O}_{X'}(E)^{\otimes n-1}.$$

Deduce that

$$\mathscr{O}_{X'}(E)|_E \simeq \mathscr{O}_{\mathbb{P}^{n-1}}(-1).$$

e) Suppose now that X is a compact complex surface. Show that

$$\int_{X'} c_1(\mathscr{O}_{X'}(E))^2 = -1.$$

Hint: use Theorem 2.41 □

The intersection product on a surface is an extremely useful tool. Let us mention the following statements which we will be able to prove at the end of Section 4.

2.50**. Theorem.** (Criterion of Nakai-Moishezon on surfaces) Let X be a projective surface, and let L be a holomorphic line bundle on X. Then L is positive if and only if $L^2 > 0$ and

$$L \cdot C > 0$$

for all curves $C \subset X$.

Let X be a projective surface and let M be a holomorphic line bundle on X. We define the holomorphic Euler characteristic

$$\chi(X, M) = h^0(X, M) - h^1(X, M) + h^2(X, M).$$

2.51**. Theorem.** (Riemann-Roch on surfaces) Let X be a projective surface, and let L be a holomorphic line bundle on X. Then we have

$$\chi(X, L) = \frac{1}{2} L^2 - \frac{1}{2} K_X \cdot L + \chi(X, \mathscr{O}_X).$$

2.E. **Extension of vector bundles.** Let X be a complex manifold of dimension n, and let

$$0 \to S \xrightarrow{\phi} E \xrightarrow{\psi} Q \to 0$$

be an exact sequence of holomorphic vector bundles (cf. Exercise 1.65). A holomorphic (resp. C^∞)-splitting of the exact sequence is a morphism of holomorphic (resp. complex) vector bundles $\tau : Q \to E$ such that $\psi \circ \tau = Id_Q$. We say that E is an extension of Q by S and the extension is trivial if there exists a holomorphic splitting.

While a C^∞-splitting always exists, an extension of holomorphic vector bundles is in general not trivial: for example the Euler sequence (Exercise 1.1) on $\mathbb{P}^1$

$$0 \to \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to T_{\mathbb{P}^1} \to 0$$

is not trivial, since this would imply $\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus T_{\mathbb{P}^1}$. Yet this would imply that $\mathscr{H}om(\mathscr{O}_{\mathbb{P}^1}(1), \mathscr{O}_{\mathbb{P}^1}) \simeq \mathscr{O}_{\mathbb{P}^1}(-1)$ has a global holomorphic section, a contradiction to Exercise 1.60. The following exercise shows how to measure the possible extensions of Q by S.

Let X be a complex manifold of dimension n, and let

$$0 \to S \xrightarrow{\phi} E \xrightarrow{\psi} Q \to 0$$

be an exact sequence of holomorphic vector bundles. Let h be a Hermitian metric on E, then we define h_S to be the Hermitian metric on S given by restricting h. Furthermore for $x \in X$, set

$$S_x^\perp := \{e \in E_x \mid h(e,s) = 0 \quad \forall\ s \in S_x\}.$$

It is easily seen that $S^\perp$ is a complex vector bundle that is a subbundle of E such that $\psi|_{S^\perp} : S^\perp \to Q$ is a C^∞-isomorphism. We define the quotient metric h_Q on Q to be the metric given by restricting h to $S^\perp$. Thus we get a C^∞-isomorphism of hermitian complex vector bundles

$$(*) \qquad (E,h) = (S,h_S) \oplus (S^\perp, h_{S^\perp}) \simeq (S,h_S) \oplus (Q,h_Q).$$

Note that in general this isomorphism is very far from inducing an isomorphism $E \simeq S \oplus Q$ of *holomorphic* vector bundles.

Let D_E, D_S, D_Q the Chern connections corresponding to the metrics h, h_S, h_Q. Using the isomorphism $(*)$, we define a hermitian connection on (E,h) by

$$\nabla_E := D_S \oplus D_Q.$$

The difference $D_E - \nabla_E$ is given by a 1-form

$$\Gamma \in C^\infty(X, \Omega_{X,\mathbb{C}} \otimes End(E)).$$

Let $e_1, \ldots, e_r$ be an isometric frame of (E,h) such that $e_1, \ldots, e_s$ is a frame for S. Since D_E and ∇_E are hermitian connections we see by Lemma 2.9 that $\overline{\Gamma}^t = -\Gamma$. Therefore we can write

$$\Gamma = \begin{pmatrix} \alpha & -\overline{\beta}^t \\ \beta & \delta \end{pmatrix},$$

where $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}} \otimes End(Q)), \delta \in C^\infty(X, \Omega_{X,\mathbb{C}} \otimes End(Q))$ such that $\overline{\alpha}^t = -\alpha, \overline{\delta}^t = -\delta$ and $\beta \in C^\infty(X, \Omega_{X,\mathbb{C}} \otimes \mathscr{H}om(S,Q))$. With this notation we have

$$D_E = \begin{pmatrix} D_S + \alpha & -\overline{\beta}^t \\ \beta & D_Q + \delta \end{pmatrix}.$$

Yet by Corollary 2.14 we have

$$D_S = p_S \circ D_E|_S = D_S + \alpha,$$

so $\alpha = 0$. Using the dual exact sequence one shows analogously that $\delta = 0$. Note furthermore that since β represents the linear map

$$D_E|_S - D_S : C^\infty(X,S) \to C^\infty(X, \Omega_{X,\mathbb{C}} \otimes Q)$$

and the matrices of D_E and D_S are of type $(1,0)$ by Corollary 2.13 we have

$$\beta \in C^\infty(X, \Omega^{1,0} \otimes \mathscr{H}om(S,Q)).$$

We call β the second fundamental form of S in E.

2.52. **Exercise.** Let X be a complex manifold of dimension n, and let

$$0 \to S \stackrel{\phi}{\to} E \stackrel{\psi}{\to} Q \to 0$$

be an exact sequence of holomorphic vector bundles. We say that two extensions are equivalent if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & & S & & E & & Q & & 0 \\ & & & Id_S & & & & Id_Q & \\ 0 & & S & & F & & Q & & 0 \end{array}$$

The goal of this exercise is to show that the elements of the cohomology group $\check{\mathrm{H}}^1(X, \mathscr{H}om(Q,S))$ have a natural bijection with the isomorphism classes of extensions of Q by S.

Note first that the exact sequence of sheaves

$$0 \to \mathscr{H}om(Q,S) \stackrel{\mathrm{Id}_{Q^*}\otimes\phi}{\to} \mathscr{H}om(Q,E) \stackrel{\mathrm{Id}_{Q^*}\otimes\psi}{\to} \mathscr{H}om(Q,Q) \to 0$$

induces a long exact sequence of Čech cohomology groups

$$\ldots \to \check{\mathrm{H}}^0(X, \mathscr{H}om(Q,E)) \to \check{\mathrm{H}}^0(X, \mathscr{H}om(Q,Q)) \stackrel{\delta}{\to} \check{\mathrm{H}}^1(X, \mathscr{H}om(Q,S)) \to \ldots$$

We denote $[E] := \delta(\mathrm{Id}_Q) \in \check{\mathrm{H}}^1(X, \mathscr{H}om(Q,S))$.

a) Show that $[E] = 0$ if and only if the exact sequence splits.

b) Let $e \in \check{\mathrm{H}}^1(X, \mathscr{H}om(Q,S))$. Show that there exists an exact sequence

$$0 \to S \stackrel{\phi}{\to} E \stackrel{\psi}{\to} Q \to 0$$

such that $[E] = e$.

Hint: in order to define the vector bundle E, let $(U_\alpha)_{\alpha\in A}$ be a Leray covering of X and let $(e_{\alpha\beta})_{\alpha,\beta\in A}$ be a Čech 1-cocycle that represents e. We set $E|_{U_\alpha} := S|_{U_\alpha} \oplus Q|_{U_\alpha}$ and define the transition functions by

$$g_{\alpha\beta}(s_\beta, q_\beta) = (s_\beta + e_{\alpha\beta}(q_\beta), q_\beta).$$

c) Conclude.

We will now do the analogue construction from the analytic point of view: fix h a Hermitian metric on E, and let

$$\beta \in C^\infty(X, \Omega^{1,0} \otimes \mathscr{H}om(S,Q)).$$

be the second fundamental form of S in E. Let β^* be its adjoint (i.e. the element of $C^\infty(X, \Omega^{0,1} \otimes \mathscr{H}om(Q,S))$ given in an isometric frame by $\overline{\beta}^t$).

d) Show that $\overline{\partial}\beta^* = 0$ and the Chern curvature of E is

$$\Theta(E) = \begin{pmatrix} \Theta(S) - \beta^* \wedge \beta & -D^{1,0}_{\mathscr{H}om(Q,S)}\beta^* \\ \overline{\partial}\beta & \Theta(Q) - \beta \wedge \beta^* \end{pmatrix}.$$

Since $\overline{\partial}\beta^* = 0$, the form gives a cohomology class $[\beta^*]$ in $H^1(X, \mathscr{H}om(Q,S))$.

e) Show that the class does not depend on the choice of the Hermitian metric on E.

f) Show that $[\beta^*]$ corresponds to $[E]$ under the de Rham-Weil isomorphism

$$\check{\mathrm{H}}^1(X, \mathscr{H}om(Q,S)) \simeq H^1(X, \mathscr{H}om(Q,S)).$$

Hint: cf. page 50 for an explicit description of the isomorphism. □

2.53. **Exercise.** Let X be a complex manifold of dimension n, and let

$$0 \to S \xrightarrow{\phi} E \xrightarrow{\psi} Q \to 0$$

be exact. Show that there exists a long exact sequence of $\mathbb{C}$-vector spaces (the long exact cohomology sequence)

$$\begin{array}{rl} 0 & \to \; H^0(X,S) \to H^0(X,E) \to H^0(X,Q) \\ & \xrightarrow{\delta} \; H^1(X,S) \to H^1(X,E) \to H^1(X,Q) \\ & \xrightarrow{\delta} \; H^2(X,S) \to H^2(X,E) \ldots \\ & \xrightarrow{\delta} \; H^n(X,S) \to H^n(X,E) \to H^n(X,Q) \to 0. \end{array}$$

Describe in particular the morphisms δ. □

2.54. **Exercise.**

a) Let X be a complex manifold and let (E_1, h_1) and (E_2, h_2) be Hermitian holomorphic vector bundles that are Griffiths positive. Show that $(E_1 \oplus E_2, h_1 \oplus h_2)$ is Griffiths positive.

b) Let X be a complex manifold and let (E, h) be a Hermitian vector bundle that is Griffiths positive. Let

$$0 \to S \xrightarrow{\phi} E \xrightarrow{\psi} Q \to 0$$

be an exact sequence of holomorphic vector bundles. Show that the Hermitian bundle (Q, h_Q) where h_Q is the induced quotient metric (cf. Exercise 2.52) is Griffiths positive.

Note that a subbundle of a Griffiths positive line bundle may be negative: the vector bundle $\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ is Griffiths positive since it is a direct sum of positive line bundles. Since

$$\mathscr{H}om(\mathscr{O}_{\mathbb{P}^1}(-1), \mathscr{O}_{\mathbb{P}^1}(1)) \simeq \mathscr{O}_{\mathbb{P}^1}(2),$$

we have

$$\Gamma(\mathbb{P}^1, \mathscr{H}om(\mathscr{O}_{\mathbb{P}^1}(-1), \mathscr{O}_{\mathbb{P}^1}(1))) \simeq \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2)).$$

We have seen in Exercise 1.60 that the global sections of $\mathscr{O}_{\mathbb{P}^1}(2)$ can be identified to the homogeneous polynomials of degree two. Choose two such polynomials s_1, s_2 that have no common zero, then

$$\mathscr{O}_{\mathbb{P}^1}(-1) \xrightarrow{(s_1,s_2)} \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$$

has rank one in every point, so $\mathscr{O}_{\mathbb{P}^1}(-1)$ is a subbundle of $\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. $\square$

2.55. **Exercise.** Let X be a complex manifold of dimension n. A $\mathbb{P}^r$-bundle over X is a complex manifold M together with a surjective holomorphic submersion map $\pi : M \to X$ such that

(1) for every $x \in X$, the fibre $M_x := \pi^{-1}(x)$ is isomorphic to $\mathbb{P}^r$
(2) for every $x \in X$, there exists an open neighbourhood U of x and a biholomorphism $h : \pi^{-1}(U) \to U \times \mathbb{P}^r$ such that

$$\pi|_{\pi^{-1}(U)} = p_1 \circ h$$

and for all $x \in U$,

$$p_2 \circ h : E_x \to \mathbb{P}^r$$

is an element of $PGL(\mathbb{C}, r)$.

Let (U_α, h_α) and (U_β, h_β) be two local trivialisations of M, then the map

$$h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{P}^r \to (U_\alpha \cap U_\beta) \times \mathbb{P}^r$$

induces a holomorphic map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to PGL(\mathbb{C}, r)$$

where $g_{\alpha\beta}(x) = h_\alpha^x \circ (h_\beta^x)^{-1} : \mathbb{P}^r \to \mathbb{P}^r$.

Let E be a holomorphic vector bundle of rank $r+1$ over X given by a collection of transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}, r+1)$. Using the natural map $GL(\mathbb{C}, r+1) \to PGL(\mathbb{C}, r)$, show that we can associate a $\mathbb{P}^r$-bundle $\phi : \mathbb{P}(E) \to X$, the projectivised bundle of E.

We define the tautological bundle on $\mathbb{P}(E)$ as

$$\mathscr{O}_{\mathbb{P}(E)}(-1) := \{(x, y) \in \mathbb{P}(E) \times \phi^* E \mid y \in \mathbb{C}x\}.$$

Show that $\mathscr{O}_{\mathbb{P}(E)}(-1)$ is a holomorphic subbundle of $\phi^* E$.

Show that a Hermitian metric on E induces a Hermitian metric h on $\mathscr{O}_{\mathbb{P}(E)}(1)$ such that $(\mathscr{O}_{\mathbb{P}(E)}(1)|_{\phi^{-1}(x)}, h|_{\phi^{-1}(x)})$ is a positive line bundle for every $x \in X$. $\square$

2.56. **Exercise.** Let X be a compact complex manifold and let E be a Griffiths positive vector bundle over X. Let $\mathbb{P}(E^*)$ be the projectivised vector bundle of E^*, and let $\mathscr{O}_{\mathbb{P}(E^*)}(-1)$ be its tautological bundle. Show that the dual bundle $\mathscr{O}_{\mathbb{P}(E^*)}(1)$ is positive.

A famous conjecture of Griffiths claims that if $\mathscr{O}_{\mathbb{P}(E)}(1)$ is positive, then E is Griffiths positive. Note that this conjecture is only known to be true if X is a curve! $\square$

3. Kähler manifolds and Hodge theory

In this section we introduce a special class of complex manifolds, the Kähler manifolds. We will see that every complex curve is Kähler but this is not true in higher dimension: the Hopf varieties are examples of non-Kähler manifolds. Roughly speaking, the Kähler condition assures a certain compatibility of the differentiable and the complex structure of the manifold. One consequence of this compatibility is the Hodge decomposition Theorem 3.36 that establishes a strong link between the de Rham and the Dolbeault cohomology groups of a compact Kähler manifold.

3.A. **Kähler manifolds.** In order to introduce Kähler manifolds, we proceed as in Subsection 1.D: we illustrate the basic concept in the case of a complex vector space, then globalise to the setting of a complex manifold.

Let V be a complex vector space of dimension n, and let $V_{\mathbb{R}}$ be the underlying real vector space of dimension $2n$. Recall that the multiplication by i induces a complex structure on $V_{\mathbb{R}}$. Furthermore this complex structure induces a decomposition of the complexified vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ in

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

In the same way, let $W_{\mathbb{R}} := \operatorname{Hom}(V_{\mathbb{R}}, \mathbb{R})$ be the dual space and $W_{\mathbb{C}} := W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification, then we get a decomposition

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}.$$

Let now $W^{1,1} = W^{1,0} \wedge W^{0,1} \subset \bigwedge^2 W_{\mathbb{C}}$ be the 2-forms of type $(1,1)$, and set

$$W_{\mathbb{R}}^{1,1} := W^{1,1} \cap \bigwedge^2 W_{\mathbb{R}},$$

where $\bigwedge^2 W_{\mathbb{R}} \subset \bigwedge^2 W_{\mathbb{C}}$ are the real 2-forms.

Recall that a Hermitian form on V is a map

$$h : V \times V \to \mathbb{C}$$

that is $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second variable and satisfies

$$h(u, v) = \overline{h(v, u)}.$$

The next lemma establishes the link between the real $(1,1)$-forms and Hermitian forms on V.

3.1. **Lemma.** Let V be a complex vector space of dimension n, and let J be the induced complex structure on the underlying real vector space $V_{\mathbb{R}}$. There is a natural isomorphism between the Hermitian forms on V and $W_{\mathbb{R}}^{1,1}$ given by

$$h \mapsto \omega = -\operatorname{Im} h.$$

and

$$\omega \mapsto h : V \times V \to \mathbb{C}, (u, v) \ \mapsto \ \omega(u, J(v)) - i\omega(u, v).$$

Proof. Write
$$h = \operatorname{Re} h + i \operatorname{Im} h,$$
then for all u, v in V
$$\operatorname{Re} h(u,v) + i \operatorname{Im} h(u,v) = h(u,v) = \overline{h(v,u)} = \operatorname{Re} h(v,u) - i \operatorname{Im} h(v,u),$$
so $\omega = -\operatorname{Im} h$ is an alternating real form. In order to see that it is of type $(1,1)$, note first that by construction $W^{1,1} = (V^{1,1})^*$ where
$$\bigwedge^2 V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2},$$
so $\omega \in W^{1,1}$ if and only if it vanishes on every couple of vectors (u,v) of type $V^{1,0}$ or $V^{0,1}$. Since $\omega = \overline{\omega}$ and $V^{1,0} = \overline{V^{0,1}}$, it is sufficient to check the first case. A generating family of $V^{1,0}$ is given by $v - iJ(v)$ for $v \in V$. For such vectors we have
$$\omega(u - iJ(u), v - iJ(v)) = \omega(u,v) - \omega(J(u), J(v)) - i\left(\omega(u, J(v)) + \omega(J(u), v)\right).$$
Since
$$h(J(u), J(v)) = ih(u, J(v)) = -i^2 h(u,v) = h(u,v)$$
and $h(u, J(v)) = -h(J(u), v)$, we have
$$\omega(u,v) = \omega(J(u), J(v)) \text{ and } \omega(u, J(v)) = -\omega(J(u), v)$$
This shows one inclusion, we leave the inverse construction as an exercise to the reader. □

It is crucial for our purposes that the isomorphism does not depend on the choice of a basis of the complex vector space V. Nevertheless it is enlightening to see the correspondence for a basis $z_1, \ldots, z_n$ of V. Then any Hermitian form on V can be written as
$$h = \sum_{1 \leqslant j,k \leqslant n} h_{j,k} z_j^* \otimes \overline{z_k^*}.$$
The corresponding (1,1)-form is then
$$\omega = \frac{i}{2} \sum_{1 \leqslant j,k \leqslant n} h_{j,k} z_j^* \wedge \overline{z_k^*}.$$

3.2. **Definition.** We say that a real form of type $(1,1)$ is positive if the corresponding Hermitian form is positive definite.

3.3. **Definition.** Let X be a complex manifold. A Hermitian metric on X is a Hermitian metric on the tangent bundle T_X, and we denote by ω the corresponding differentiable $(1,1)$-form. We say that ω is a Kähler form if it is closed, that is
$$d\omega = 0.$$
We say that a complex manifold is Kähler if it admits a Kähler form.

3.4. **Example.** The real $(1,1)$-form

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z_j}$$

induced by the standard Hermitian product on $\mathbb{C}^n$ certainly defines a Kähler metric.

In the same spirit, let $\Lambda \subset \mathbb{C}^n$ be a lattice. Then the complex torus $\mathbb{C}^n/\Lambda$ admits a Kähler form: let

$$\omega' = \frac{i}{2} \sum_{1 \leqslant j,k \leqslant n} h_{j,k} dz_j \wedge d\overline{z_k},$$

be any Kähler form on $\mathbb{C}^n$ with *constant* coefficients $h_{j,k} \in \mathbb{C}$. Then ω' is invariant by the translations $\lambda : \mathbb{C}^n \to \mathbb{C}^n, z \mapsto z + \lambda$ for all $\lambda \in \Lambda$, so ω' induces a Kähler form ω on $\mathbb{C}^n/\Lambda$ such that $\omega' = \pi^*\omega$ where $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda$ is the quotient map.

3.5. **Exercise.** Let X be a complex manifold. Let ω be a $(1,1)$-form corresponding to a Hermitian metric on X. Prove that we have an equivalence

$$d\omega = 0 \ \Leftrightarrow\ \partial\omega = 0 \ \Leftrightarrow\ \overline{\partial}\omega = 0.$$

□

3.6. **Exercise.** Let X be a complex curve. Show that X is Kähler. □

3.7. **Exercise.** Let X be a complex manifold endowed with a Hermitian metric h. Let ω be the corresponding $(1,1)$-form. Then $\frac{\omega^n}{n!}$ is a volume form on X. □

3.8. **Exercise.** Let X be a complex manifold, and let L be a positive holomorphic line bundle over X. Show that the first Chern class $[\frac{i}{2\pi}\Theta_L]$ is represented by a Kähler form. □

3.9. **Example.** Exercise 2.25 shows that the curvature form associated to the Fubini-Study metric on the line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$ is a Kähler form. Thus the projective space is Kähler.

3.10. **Exercise.** Let X be a Kähler manifold/ Show that the blow-up of X in a point (cf. Exercise 2.49) is a Kähler manifold. □

3.11. **Exercise.** Let X be a compact Kähler manifold, and let $\pi : \mathbb{P}(E) \to X$ be a projectivised bundle over X (cf. Exercise 2.55). Show that $\mathbb{P}(E)$ is a compact Kähler manifold. □

Let (X,ω) be a complex manifold of dimension n endowed with a Hermitian form. We have seen in Exercise 1.73 that the underlying real differentiable manifold is endowed with a canonical orientation. Furthermore if ω is the $(1,1)$-form associated to a Hermitian metric on T_X, then $\frac{\omega^n}{n!}$ is the volume form (Exercise 3.7). In particular we see that if (X,ω) is a compact Kähler manifold, then

$$\int_X \frac{\omega^n}{n!} > 0.$$

Since $d\omega^k = 0$ for all $k \in \{1,\ldots,n\}$, we can consider the corresponding de Rham cohomology class.

3.12. **Proposition.** Let X be a compact complex manifold of dimension n, and let ω be a Kähler form on X. Then for all $k \in \{1, \dots, n\}$, the cohomology class

$$[\omega^k] \in H^{2k}(X, \mathbb{R})$$

is non-zero.

Proof. If $\omega^k = d\eta$, we have $\omega^n = d(\eta \wedge \omega^{n-k})$, so by Stokes's theorem

$$\int_X \omega^n = \int_X d(\eta \wedge \omega^{n-k}) = \int_{\partial X} \eta \wedge \omega^{n-k} = 0,$$

a contradiction. □

3.13. **Exercise.** Show that a Hopf variety (cf. Exercise 1.20) is not a Kähler manifold. (Hint: Künneth formula) □

3.14. **Exercise.** Let X be a compact complex manifold of dimension n, and let ω be a Kähler form on X. Show that for all $k \in \{1, \dots, n\}$, the cohomology class

$$[\omega^k] \in H^{k,k}(X)$$

is non-zero. □

The following property of Kähler manifolds will turn out to be very useful in the proof of the Kähler identities (Proposition 3.30).

Let (X, ω) be a Kähler manifold, and fix a point $x \in X$. Let $z_1, \dots, z_n$ be local holomorphic coordinates around x, then we have

$$\omega = \frac{i}{2} \sum_{1 \leqslant j,k \leqslant n} h_{j,k} dz_j \wedge d\overline{z_k},$$

where the $h_{j,k}$ are differentiable functions. In these coordinates, the condition $\partial\omega = 0$ is equivalent to

$$\frac{\partial h_{j,k}}{\partial z_l} = \frac{\partial h_{l,k}}{\partial z_j} \qquad \forall\ 1 \leqslant j, k, l \leqslant n. \tag{3.12}$$

3.15. **Theorem.** Let (X, ω) be a Kähler manifold. Then locally we can choose holomorphic coordinates $\zeta_1, \dots \zeta_n$ such that $h_{j,k} = \delta_{j,k} + O(|\zeta|^2)$. We call $\zeta_1, \dots \zeta_n$ normal coordinates for the Kähler form ω.

Proof. Starting with any choice of local coordinates $z_1, \dots, z_n$ around x, it is clear that we can make a linear change of coordinates such that $dz_1, \dots, dz_n$ induce a basis of $\Omega_{X,x}$ that is orthonormal with respect to ω. Thus we can write

$$\omega = i \sum_{1 \leqslant j,k \leqslant n} h_{j,k} dz_j \wedge d\overline{z_k},$$

where $h_{j,k} = \delta_{j,k} + O(|z|)$. The Taylor expansion to the first order is then

$$h_{j,k} = \delta_{j,k} + \sum_{1 \leqslant l \leqslant n} (a_{jkl} z_l + a'_{jkl} \overline{z_l}) + O(|z|^2). \tag{3.13}$$

Since $\omega = \overline{\omega}$, we have $\overline{h_{k,j}} = h_{j,k}$. In particular

$$(*) \qquad \overline{a_{kjl}} = a'_{jkl}.$$

Furthermore we have by Formula (3.12),

$$(**) \qquad a_{jkl} = a_{lkj}.$$

Set now

$$\zeta_k = z_k + \frac{1}{2} \sum_{1 \leqslant j,l \leqslant n} a_{jkl} z_j z_l \qquad \forall\ k = 1, \ldots, n.$$

By the theorem of invertible functions ζ_k defines a local holomorphic system of coordinates and

$$d\zeta_k = dz_k + \frac{1}{2} \sum_{1 \leqslant j,l \leqslant n} a_{jkl}(z_j dz_l + z_l dz_j) = dz_k + \frac{1}{2} \sum_{1 \leqslant j,l \leqslant n} (a_{jkl} + a_{lkj}) z_l dz_j,$$

which by $(**)$ equals

$$d\zeta_k = dz_k + \sum_{1 \leqslant j,l \leqslant n} a_{jkl} z_l dz_j.$$

Therefore

$$i \sum_{1 \leqslant k \leqslant n} d\zeta_k \wedge d\overline{\zeta_k} = i \sum_{1 \leqslant k \leqslant n} dz_k \wedge d\overline{z_k} + i \sum_{1 \leqslant j,k,l \leqslant n} \overline{a_{jkl}}\ \overline{z_l} dz_k \wedge d\overline{z_j} + a_{jkl} z_l dz_j \wedge d\overline{z_k} + O(|z|^2).$$

Now $(*)$ implies

$$\sum_{1 \leqslant j,k,l \leqslant n} \overline{a_{jkl}}\ \overline{z_l} dz_k \wedge d\overline{z_j} = \sum_{1 \leqslant j,k,l \leqslant n} \overline{a_{kjl}}\ \overline{z_l} dz_j \wedge d\overline{z_k} = \sum_{1 \leqslant j,k,l \leqslant n} a'_{jkl} \overline{z_l} dz_j \wedge d\overline{z_k},$$

thus

$$i \sum_{1 \leqslant k \leqslant n} d\zeta_k \wedge d\overline{\zeta_k} = i \sum_{1 \leqslant j,k \leqslant n} \left(\delta_{j,k} + \sum_{1 \leqslant l \leqslant n} a_{jkl} z_l + a'_{jkl} \overline{z_l} \right) dz_j \wedge d\overline{z_k} + O(|z|^2).$$

Comparing with Formula (3.13), we see that $i \sum_{1 \leqslant k \leqslant n} d\zeta_k \wedge d\overline{\zeta_k} = \omega + O(|z|^2)$. $\square$

3.16. **Exercise.** Let $i : Y \hookrightarrow X$ be a submanifold of a Kähler manifold. Show that Y is a Kähler manifold. More precisely show that if ω is a Kähler form on X, then $i^*\omega$ is a Kähler form on Y. In particular

$$\int_Y i^* \omega^{\dim Y} > 0.$$

Deduce that a projective manifold is Kähler. $\square$

3.17. **Exercise.** (hard) A difficult and important question is to figure out how the Kähler property behaves under holomorphic maps. Show that if $f : X \to Y$ is a holomorphic submersion between complex manifolds such that (X, ω) is Kähler, then any fibre X_y and Y is Kähler. (Hint: in order to show that Y is Kähler, one should look at the direct image of $\omega^{\dim X - \dim Y + 1}$ as a current, cf. [Dem96, III].)

Give an example of a holomorphic submersion between complex manifolds $f : X \to Y$ such that Y is a Kähler manifold, the fibre X_y is a Kähler manifold for every $y \in Y$, but X is not Kähler. $\square$

3.18. **Exercise.** Let X be a compact Kähler manifold.

a) Let $D \subset X$ be a smooth hypersurface. Show that

$$[D] \in H^{1,1}(X)$$

and this class is zero if and only if $D = \emptyset$ (cf. Theorem 2.41 for the definition of the Poincaré dual $[D]$).

b) Let L be a holomorphic line bundle over X. Suppose that $c_1(L) = 0 \in H^2(X, \mathbb{R})$ and that $\Gamma(X, L) \neq 0$. Show that L is trivial[11]. $\square$

3.19. **Exercise.** Let (X, ω) be a compact Kähler manifold, and let L be a holomorphic line bundle over X. We say that L is nef if for every $\varepsilon > 0$ there exists a Hermitian metric h_ε on L such that

$$\Theta_{h_\epsilon}(L) \geqslant -\varepsilon\omega,$$

i.e. $\Theta_{h_\epsilon}(L) + \varepsilon\omega$ is positive definite.

Show that if $Y \subset X$ is a submanifold of dimension d and L is nef, we have

$$\int_Y c_1(L)^d \geqslant 0.$$

Show that if M is a positive line bundle over X, then $L^{\otimes n} \otimes M$ is positive for every $n \in \mathbb{N}$. $\square$

3.B. **Differential operators.** We recall some of the basic definitions from Appendix A. Let X be a differentiable oriented manifold of dimension n. Let (E, h) be a euclidean vector bundle over X, and let

$$C^\infty(X, E) \times C^\infty(X, E) \to C^\infty(X), (\sigma, \tau) \mapsto \{\sigma, \tau\}$$

be the bilinear mapping 2.7.

Suppose now that X is endowed with an euclidean metric, and denote by *vol* the associated volume form. We define the L^2 scalar product and the corresponding L^2-norm on $C_c^\infty(X, E)$ by

$$(\alpha, \beta)_E := (\alpha, \beta)_{L^2} := \int_X \{\alpha, \beta\} vol, \qquad ||\alpha||^2 = \int_X \{\alpha, \alpha\} vol. \tag{3.14}$$

If E and F are euclidean bundles over X and $P : C_c^\infty(X, E) \to C_c^\infty(X, F)$ is a linear operator, the formal adjoint $P^* : C_c^\infty(X, F) \to C_c^\infty(X, E)$ is defined by

$$(P\alpha, \beta)_F = (\alpha, P^*\beta)_E \qquad \forall\ \alpha \in C_c^\infty(X, E), \beta \in C_c^\infty(X, F).$$

[11] In order to answer this question in full generality, you'll have to admit/prove the statement of Theorem 2.41 for singular hypersurfaces.

The case in which we will be most interested is when E is some exterior power of the cotangent bundle Ω_X^p. In this case[12] we have the Hodge $*$-operator: if $\beta \in C^\infty(X, \Omega_X^p)$ is a p-form, then $*\beta \in C^\infty(X, \Omega_X^{\dim X - p})$ is such that

$$\alpha \wedge *\beta = \{\alpha, \beta\} vol$$

for every p-form α (cf. Appendix A for an explicit description in terms of isometric frames).

For every $k \in \mathbb{N}$, the exterior differential gives a linear operator

$$d : C_c^\infty(X, \Omega_X^k) \to C_c^\infty(X, \Omega_X^{k+1})$$

and we denote by $d^* : C_c^\infty(X, \Omega_X^{k+1}) \to C_c^\infty(X, \Omega_X^k)$ the adjoint operator. By Lemma A.6 we have

$$d^* = (-1)^{nk+1} * d * .$$

Another linear operator on $C_c^\infty(X, \Omega^k)$ is given by the contraction of a vector field: let $\theta \in C^\infty(X, T_X)$ be a vector field and let $u \in C^\infty(X, \Omega_X^k)$ be a k-form. The contraction $(\theta \lrcorner u) \in C^\infty(X, \Omega_X^{k-1})$ is the $(k-1)$-form defined by

$$(\theta \lrcorner u)(\eta_1, \ldots, \eta_{k-1}) := u(\theta, \eta_1, \ldots, \eta_{k-1}) \qquad \forall\ \eta_1, \ldots, \eta_{k-1} \in T_{X,x}.$$

If $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ is a local frame for T_X and $dx_1, \ldots, dx_n$ the dual local frame for Ω_X, we have

$$\frac{\partial}{\partial x_m} \lrcorner (dx_{j_1} \wedge \ldots \wedge dx_{j_k}) = \begin{cases} 0 & \text{if } \ m \notin \{j_1, \ldots, j_k\} \\ (-1)^{l-1} dx_{j_1} \wedge \ldots \widehat{dx_{j_l}} \ldots \wedge dx_{j_k} & \text{if } \ m = j_l. \end{cases}$$

Using this formula it is not hard to see that if u, v are differentiable forms, then

$$\theta \lrcorner (u \wedge v) = (\theta \lrcorner u) \wedge v + (-1)^{\deg u} u \wedge (\theta \lrcorner v). \tag{3.15}$$

3.20. **Exercise.** Let $U \subset \mathbb{R}^n$ be an open set and endow T_U with the standard flat metric. Let $u = \sum_{|J|=k} u_J dx_J$ be a k-form. Use Lemma A.6 to show that

$$d^* u = -\sum_{l=1}^n \sum_{|J|=k} \frac{du_J}{dx_l} \frac{\partial}{\partial x_l} \lrcorner dx_J.$$

□

[12] Note that the choice of a metric h on T_X induces metrics on Ω_X and its exterior powers Ω_X^p: if $e_1, \ldots, e_n$ is an isometric frame for T_X on some open set $U \subset X$, let $e_1^*, \ldots, e_n^*$ be the corresponding dual frame for Ω_X. On U the metric h^* is then defined by imposing that $e_1^*, \ldots, e_n^*$ is an isometric frame, i.e. for arbitrary sections $\sigma, \eta \in C^\infty(X, \Omega_X)$

$$h^*(\sigma, \eta)(x) = \sum_{j=1}^n \sigma_j(x) \eta_j(x)$$

where $\sigma = \sum_{j=1}^n \sigma_j e_j^*$ and $\eta = \sum_{j=1}^n \eta_j e_j$. It is not hard to see that these local definitions glue to a global metric on X. In a similar way we define the induced metric on Ω_X^p by imposing that the induced frame $e_{j_1}^* \wedge \ldots \wedge e_{j_p}^*$ is isometric. Throughout the whole chapter we will always consider these induced metrics on Ω_X^p.

Suppose now that X is a complex manifold of dimension n endowed with a Hermitian metric h. Denote by vol the volume form associated to h, then we extend the definition of the Hodge $*$-operator: if $\beta \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$ is a k-form, then $*\beta \in C^\infty(X, \Omega^{2n-k}_{X,\mathbb{C}})$ is such that

$$\alpha \wedge *\overline{\beta} = \{\alpha, \beta\} vol$$

for every k-form $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$.

With this definition the L^2-scalar product on $C^\infty_c(X, \Omega^k_{X,\mathbb{C}})$ is given by

$$(\alpha, \beta)_{L^2} := \int_X \alpha \wedge *\overline{\beta} = \int_X \{\alpha, \beta\} vol.$$

Let us now see how the Hodge operator acts on complex vector bundles $\Omega^{p,q}_X$: fix a point $x \in X$ and choose holomorphic coordinates around x such that $dz_1, \ldots, dz_n$ is a local frame that is isometric in the point x[13]. Let

$$u = \sum_{|J|=p, |K|=q} u_{J,K} dz_J \wedge d\overline{z_K}$$

and

$$v = \sum_{|J|=p, |K|=q} v_{J,K} dz_J \wedge d\overline{z_K}$$

be forms of type (p,q), then $\{u, v\}$ is given at the point x by

$$\{u, v\}_x = \sum_{|J|=p, |K|=q} u_{J,K}(x) \overline{v_{J,K}}(x).$$

Since by definition $u \wedge *\overline{v} = \{u, v\} vol$, this shows that the Hodge star operator gives a $\mathbb{C}$-linear isometry

$$* : \Omega^{p,q}_X \to \Omega^{n-q,n-p}_X,$$

since $*\overline{v}$ is of type $(n-p, n-q)$. This implies that the decomposition

$$C^\infty(X, \Omega^k_{X,\mathbb{C}}) = \bigoplus_{p+q=k} C^\infty(X, \Omega^{p,q}_X)$$

is orthogonal with respect to the L^2-product. Indeed if v is of type (p', q') with $p' + q' = p + q$ then

$$u \wedge *\overline{v}$$

is of type $(n - p' + p, n - q' + q)$, so it is zero unless $p = p', q = q'$. In order to get an explicit formula for $*\overline{v}$ one has to take into account that we are working

[13]This can always be achieved by a linear change of coordinates, cf. the proof of Theorem 3.15. The tricky point is to understand the behaviour of the metric in a small neighbourhood. If you don't like the idea of arguing pointwise, just fix a not necessarily holomorphic isometric frame $\xi_1, \ldots, \xi_n$ for the holomorphic tangent bundle T_X in a neighbourhood U of x. Then a form of type (p,q) can be written as

$$u = \sum_{|J|=p, |K|=q} u_{J,K} \xi^*_J \wedge \overline{\xi}^*_K$$

and we get the same computations as before, but on the open set U.

with alternate forms. An elementary but somewhat tedious computation [Wel80, Ch.V,Prop.1.1] shows that

$$*\overline{v}(x) = \sum_{|J|=p,|K|=q} \varepsilon_{J,K} \overline{v_{J,K}} dz_{CJ} \wedge d\overline{z_{CK}}$$

where $\varepsilon_{J,K} := (-1)^{q(n-p)} \mathrm{sign}(J, CJ)\mathrm{sign}(K, CK)$ and $\mathrm{sign}(\bullet, \bullet)$ is the sign of the permutation.

As in the real case, we can consider the exterior differential d and its formal adjoint d^*. Since the real dimension of X is even we have by Lemma A.6 that $d^* = - * d *$. Consider now the differential operators

$$\partial : C_c^\infty(X, \Omega_X^{p,q}) \to C_c^\infty(X, \Omega_X^{p+1,q})$$

and

$$\overline{\partial} : C_c^\infty(X, \Omega_X^{p,q}) \to C_c^\infty(X, \Omega_X^{p,q+1}).$$

Then we define the operators

$$\partial^* : C_c^\infty(X, \Omega^{p+1,q}) \to C_c^\infty(X, \Omega^{p,q})$$

and

$$\overline{\partial}^* : C_c^\infty(X, \Omega^{p,q}) \to C_c^\infty(X, \Omega^{p,q-1})$$

in analogy with the preceding formula by

$$\partial^* := - * \overline{\partial} * \tag{3.16}$$

$$\overline{\partial}^* := - * \partial * . \tag{3.17}$$

The following lemma shows that the notation ∂^* and $\overline{\partial}^*$ is justified, i.e. the operators are the formal adjoints of ∂ and $\overline{\partial}$.

3.21. **Lemma.** Let X be a complex manifold of dimension n. For all $\alpha \in C_c^\infty(X, \Omega^{p,q})$ and $\beta \in C_c^\infty(X, \Omega^{p,q+1})$, we have

$$(\overline{\partial}\alpha, \beta)_{L^2} = (\alpha, \overline{\partial}^* \beta)_{L^2}.$$

For all $\alpha \in C_c^\infty(X, \Omega^{p,q})$ and $\beta \in C_c^\infty(X, \Omega^{p+1,q})$, we have

$$(\partial\alpha, \beta)_{L^2} = (\alpha, \partial^* \beta)_{L^2}.$$

Proof. We show the first statement, the proof of the second is analogous. By definition

$$(\overline{\partial}\alpha, \beta)_{L^2} = \int_X \overline{\partial}\alpha \wedge *\overline{\beta}.$$

Since $\alpha \wedge *\overline{\beta} \in C_c^\infty(X, \Omega^{n,n-1})$, we have

$$d(\alpha \wedge *\overline{\beta}) = \overline{\partial}(\alpha \wedge *\overline{\beta}) = \overline{\partial}\alpha \wedge *\overline{\beta} + (-1)^{p+q} \alpha \wedge \overline{\partial} * \overline{\beta}.$$

Hence by Stokes' theorem

$$\int_X \overline{\partial}\alpha \wedge *\overline{\beta} = - \int_X (-1)^{p+q} \alpha \wedge \overline{\partial} * \overline{\beta}.$$

Since $*$ is a real operator, we have

$$\overline{\partial} * \overline{\beta} = \overline{\partial * \beta} = * \overline{*^{-1} \partial * \beta}.$$

By Exercise A.2 we have $*^{-1}\gamma = (-1)^{(2n-k)k} * \gamma$ for any k-form γ, so $*^{-1}\partial * \beta = (-1)^{p+q} * \partial * \beta$. Therefore

$$-\int_X (-1)^{p+q}\alpha \wedge \overline{\partial} * \overline{\beta} = -\int_X \alpha \wedge * \overline{* \partial * \beta} = (\alpha, - * \partial * \beta)_{L^2}.$$

This implies the claim. □

As in Exercise 3.20, one can use this description to give a local expression of the formal adjoint in terms of contraction by vector fields. Let $U \subset \mathbb{C}^n$ be an open set and endow T_U with the standard hermitian metric $\sum_{j=1}^n 2dz_j \otimes d\overline{z_j}$ where $z_1, \ldots, z_n$ are the linear coordinates on $\mathbb{C}^n$. Thus $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ is an isometric holomorphic frame for T_U. Let u be a form with compact support of type (p,q) given in these local coordinates by $u = \sum_{|J|=p,|K|=q} u_{J,K} dz_J \wedge d\overline{z_K}$, then

$$\partial^* u = -\sum_{l=1}^n \sum_{|J|=p,|K|=q} \frac{\partial u_{J,K}}{\partial z_l} \frac{\partial}{\partial \overline{z_l}} \lrcorner dz_J \wedge d\overline{z_K} = -\sum_{l=1}^n \frac{\partial}{\partial \overline{z_l}} \lrcorner \frac{\partial u}{\partial z_l}, \tag{3.18}$$

where

$$\frac{\partial u}{\partial z_l} := \sum_{|J|=p,|K|=q} \frac{\partial u_{J,K}}{\partial z_l} dz_J \wedge d\overline{z_K}$$

and the contraction by a vector field is defined analogously to the real case.

Now that we have defined the adjoint operators, we define the corresponding Laplacians by

$$\begin{aligned} \Delta_d &:= dd^* + d^*d \\ \Delta_\partial &:= \partial\partial^* + \partial^*\partial \\ \Delta_{\overline{\partial}} &:= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}. \end{aligned}$$

3.22. **Lemma.** Let X be a compact complex manifold, then we have

$$\begin{aligned} (\alpha, \Delta_d \beta)_{L^2} &= (d\alpha, d\beta)_{L^2} + (d^*\alpha, d^*\beta)_{L^2} \\ (\alpha, \Delta_\partial \beta)_{L^2} &= (\partial\alpha, \partial\beta)_{L^2} + (\partial^*\alpha, \partial^*\beta)_{L^2} \\ (\alpha, \Delta_{\overline{\partial}} \beta)_{L^2} &= (\overline{\partial}\alpha, \overline{\partial}\beta)_{L^2} + (\overline{\partial}^*\alpha, \overline{\partial}^*\beta)_{L^2}. \end{aligned}$$

Proof. Immediate from the formal adjoint property, i.e. Lemma 3.21. □

3.23. **Definition.** Let X be a complex manifold. We say that a form α is harmonic (resp. Δ_∂-harmonic, resp. $\Delta_{\overline{\partial}}$-harmonic) if $\Delta_d \alpha = 0$ (resp. $\Delta_\partial \alpha = 0$, resp. $\Delta_{\overline{\partial}} \alpha = 0$).

3.24. **Lemma.** Let X be a complex compact manifold. A form α is harmonic (resp. Δ_∂-harmonic, resp. $\Delta_{\overline{\partial}}$-harmonic) if and only if it is d- *and* d^*-closed (resp. ∂- *and* ∂^*-closed, resp. $\overline{\partial}$- *and* $\overline{\partial}^*$-closed).

Proof. The proof is the same as the one for Lemma A.19, just apply Lemma 3.22. □

3.25. Lemma. The symbol of the differential operator Δ_∂ and $\Delta_{\overline{\partial}}$ is

$$\xi \mapsto \frac{-1}{2}||\xi||^2 Id.$$

In particular the operators Δ_∂ and $\Delta_{\overline{\partial}}$ are elliptic operators.

Proof. The proof is exactly the same as for Δ_d (cf. Examples A.36 or [Voi02, Lemma 5.18]). □

This lemma shows that we can apply the theory of elliptic operators to $\Delta_{\overline{\partial}}$.

3.26. Theorem. Let X be a compact complex manifold endowed with a Hermitian metric h. Let $\mathscr{H}^{p,q}(X)$ be the space of $\Delta_{\overline{\partial}}$-harmonic forms of type (p, q). Then:

(1) $\mathscr{H}^{p,q}(X)$ is finite dimensional; and
(2) we have a decomposition

$$C^\infty(X, \Omega^{p,q}) = \mathscr{H}^{p,q}(X) \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega^{p,q})),$$

which is orthogonal for the L^2 scalar product.

Proof. Apply Theorem A.42. □

As in the case of the de Rham cohomology, we obtain immediately a series of corollaries.

3.27. Corollary. Let X be a compact complex manifold endowed with a Hermitian metric h. Then we have an orthogonal decomposition

$$C^\infty(X, \Omega^{p,q}) = \mathscr{H}^{p,q}(X) \oplus \overline{\partial}(C^\infty(X, \Omega^{p,q-1})) \oplus \overline{\partial}^*(C^\infty(X, \Omega^{p,q+1}))$$

and $\operatorname{im} \overline{\partial}^* \cap \ker \overline{\partial} = \{0\} = \operatorname{im} \overline{\partial} \cap \ker \overline{\partial}^*$. In particular

$$\ker \overline{\partial} = \mathscr{H}^{p,q}(X) \oplus \overline{\partial}(C^\infty(X, \Omega^{p,q-1})) \tag{3.19}$$

$$\ker \overline{\partial}^* = \mathscr{H}^{p,q}(X) \oplus \overline{\partial}^*(C^\infty(X, \Omega^{p,q+1})). \tag{3.20}$$

Proof. By Theorem 3.26, we have

$$C^\infty(X, \Omega^{p,q-1}) = \mathscr{H}^{p,q-1}(X) \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega^{p,q-1})),$$

so Lemma 3.24 and $\overline{\partial}^2 = 0$ imply that

$$\overline{\partial}(C^\infty(X, \Omega^{p,q-1})) = \overline{\partial}\overline{\partial}^*(C^\infty(X, \Omega^{p,q})).$$

Analogously we get

$$\overline{\partial}^*(C^\infty(X, \Omega^{p,q+1})) = \overline{\partial}^*\overline{\partial}(C^\infty(X, \Omega^{p,q})).$$

So Theorem 3.26 and $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ imply that

$$C^\infty(X, \Omega^{p,q}) = \mathscr{H}^{p,q}(X) \oplus \overline{\partial}(C^\infty(X, \Omega^{p,q-1})) + \overline{\partial}^*(C^\infty(X, \Omega^{p,q+1})),$$

and

$$(\overline{\partial}^*\beta, \overline{\partial}\alpha)_{L^2} = (\beta, \overline{\partial}\overline{\partial}\alpha)_{L^2} = 0$$

shows that the last two spaces are orthogonal.

The second statement is now immediate: if $\alpha \in \ker \overline{\partial} \cap \operatorname{im} \overline{\partial}^*$, then $\overline{\partial}\alpha = 0$ and $\overline{\partial}^*\alpha = 0$, so α is $\Delta_{\overline{\partial}}$-harmonic by Lemma 3.24. Yet the intersection $\mathscr{H}^{p,q}(X) \cap \overline{\partial}^*(C^\infty(X, \Omega^{p,q+1}))$ is zero by the first statement. The proof of the third statement is analogous. □

Since a $\Delta_{\overline{\partial}}$-harmonic form is $\overline{\partial}$-closed by Lemma 3.24, we have a linear map $\mathscr{H}^{p,q}(X) \to H^{p,q}(X)$. Formula (3.19) shows that this map is fact an isomorphism, i.e. every Dolbeault cohomology class is represented by a unique $\Delta_{\overline{\partial}}$-harmonic form:

3.28**. Corollary.** Let X be a compact complex manifold endowed with a Hermitian metric h. Then we have

$$H^{p,q}(X) \simeq \mathscr{H}^{p,q}(X).$$

In particular the Dolbeault cohomology groups have finite dimension.

3.C. **Differential operators on Kähler manifolds.** Let now (X, ω) be a Kähler manifold, and denote by $vol_\omega = \frac{\omega^n}{n!}$ the volume form (cf. Exercise 3.7) of the corresponding Hermitian metric. The exterior product with the Kähler form defines a differential operator

$$L : C_c^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C_c^\infty(X, \Omega^{k+2}_{X,\mathbb{C}}), \alpha \mapsto \omega \wedge \alpha$$

of degree zero. We denote by

$$\Lambda : C_c^\infty(X, \Omega^{k+2}_{X,\mathbb{C}}) \to C_c^\infty(X, \Omega^k_{X,\mathbb{C}})$$

its formal adjoint. We claim that

$$\Lambda\beta = (-1)^k (*L*)\beta$$

for every k-form β.

Proof. Note that since ω is a real differential form, it is sufficient to show the claim for $\beta \in C^\infty(X, \Omega^{k+2}_{X,\mathbb{R}})$. Then we have

$$\begin{aligned} \{L\alpha, \beta\} vol_\omega &= L\alpha \wedge *\beta = (\omega \wedge \alpha) \wedge *\beta \\ &= \alpha \wedge \omega \wedge *\beta = \alpha \wedge * \big((-1)^k * (\omega \wedge *\beta)\big) \\ &= \{\alpha, \big((-1)^k * L*\big)\, \beta\} vol_\omega, \end{aligned}$$

where we used Exercise A.2. □

If A, B are two differential operators of degree a and b respectively, their Lie bracket is a differential operator of degree $a + b$ defined by

$$[A, B] := AB - (-1)^{ab} BA.$$

Note that if C is another differential operator of degree c, then for purely formal reasons, the Jacobi identity

$$(-1)^{ac}[A, [B, C]] + (-1)^{ab}[B, [C, A]] + (-1)^{bc}[C, [A, B]] = 0 \tag{3.21}$$

holds.

3.29. **Lemma.** Let $U \subset \mathbb{C}^n$ be an open set endowed with the constant Kähler metric

$$\omega = i \sum_{1 \leqslant j \leqslant n} dz_j \wedge d\overline{z_j}.$$

Then we have

$$[\overline{\partial}^*, L] = i\partial.$$

Proof. Let $u \in C_c^\infty(U, \Omega^l_{U,\mathbb{C}})$ be a l-form, then by Formula (3.18)

$$\overline{\partial}^* u = - \sum_{1 \leqslant k \leqslant n} \frac{\partial}{\partial \overline{z_k}} \lrcorner \frac{\partial u}{\partial z_k}.$$

Therefore we have

$$[\overline{\partial}^*, L]u = - \sum_{1 \leqslant k \leqslant n} \frac{\partial}{\partial \overline{z_k}} \lrcorner \frac{\partial}{\partial z_k} (\omega \wedge u) + \omega \wedge \sum_{1 \leqslant k \leqslant n} \frac{\partial}{\partial \overline{z_k}} \lrcorner \frac{\partial u}{\partial z_k}.$$

Since the coefficients of ω are constant, we have $\frac{\partial}{\partial z_k}(\omega \wedge u) = \omega \wedge \frac{\partial u}{\partial z_k}$. Furthermore by Formula (3.15)

$$\frac{\partial}{\partial \overline{z_k}} \lrcorner (\omega \wedge \frac{\partial u}{\partial z_k}) = \left(\frac{\partial}{\partial \overline{z_k}} \lrcorner \omega \right) \wedge \frac{\partial u}{\partial z_k} + \omega \wedge \left(\frac{\partial}{\partial \overline{z_k}} \lrcorner \frac{\partial u}{\partial z_k} \right)$$

thus we get

$$[\overline{\partial}^*, L]u = - \sum_{1 \leqslant k \leqslant n} \left(\frac{\partial}{\partial \overline{z_k}} \lrcorner \omega \right) \wedge \frac{\partial u}{\partial z_k}.$$

Yet it is clear that $\frac{\partial}{\partial \overline{z_k}} \lrcorner (\omega) = -i dz_k$, so

$$[\overline{\partial}^*, L]u = i \sum_{1 \leqslant k \leqslant n} dz_k \wedge \frac{\partial u}{\partial z_k} = i\partial u.$$

□

The preceding lemma is a 'local' version of the following Kähler identities that will be the corner stones of the proof of the Hodge decomposition Theorem 3.36.

3.30. **Proposition.** Let (X, ω) be a Kähler manifold. Then we have

$$[\overline{\partial}^*, L] = i\partial \tag{3.22}$$

$$[\partial^*, L] = -i\overline{\partial}. \tag{3.23}$$

$$[\Lambda, \overline{\partial}] = -i\partial^* \tag{3.24}$$

$$[\Lambda, \partial] = i\overline{\partial}^*. \tag{3.25}$$

Proof. Since $\overline{L} = L$ and $\overline{\Lambda} = \Lambda$, it is clear that the first (resp. third) identity implies the second (resp. fourth) one by conjugation. Furthermore the third relation follows from the first by the formal adjoint property: let u, v be (p, q) forms, then

$$([\Lambda, \overline{\partial}]u, v)_{L^2} = (u, [\overline{\partial}^*, L]v)_{L^2} = (u, i\partial v)_{L^2} = (-i\partial^* u, v)_{L^2}.$$

Thus we are left to show the first relation. Note now that the local expressions of the differential operators only use the coefficients of the metric up to first order: indeed the operator L uses the metric only up to order zero and $\overline{\partial}^* = - * \partial *$

(cf. Formula 3.16) shows that we only use the metric and its first derivatives. By Theorem 3.15, we can choose holomorphic coordinates such that

$$h = Id + O(|z|^2),$$

thus it is sufficient to consider the situation of an open set in $\mathbb{C}^n$ endowed with the standard Kähler metric. Conclude with Lemma 3.29. $\square$

3.31**. Theorem.** Let (X, ω) be a Kähler manifold, and let Δ_d, Δ_∂ and $\Delta_{\overline{\partial}}$ be the Laplacians associated to the operators d, ∂ and $\overline{\partial}$. Then we have

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.$$

In particular a k-form is harmonic if and only if it is Δ_∂-harmonic if and only if it is $\Delta_{\overline{\partial}}$-harmonic.

Proof. We will show the first equality, the proof of the second one is analogous. We have $d = \partial + \overline{\partial}$, so

$$\Delta_d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}).$$

Since $\overline{\partial}^* = -i[\Lambda, \partial] = -i\Lambda\partial + i\partial\Lambda$ by Formula (3.25)and $\partial^2 = 0$, we have

$$(\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) = \partial\partial^* - i\partial\Lambda\partial + \overline{\partial}\partial^* - i\overline{\partial}\Lambda\partial + i\overline{\partial}\partial\Lambda$$

and

$$(\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) = \partial^*\partial + i\partial\Lambda\partial + \partial^*\overline{\partial} + i\partial\Lambda\overline{\partial} - i\Lambda\partial\overline{\partial}.$$

By Formula (3.24) we have $\partial^* = i[\Lambda, \overline{\partial}] = i\Lambda\overline{\partial} - i\overline{\partial}\Lambda$, which implies

$$\partial^*\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial} = -\overline{\partial}\partial^*. \tag{3.26}$$

Thus we get

$$\Delta_d = \partial\partial^* - i\overline{\partial}\Lambda\partial + i\overline{\partial}\partial\Lambda + \partial^*\partial + i\partial\Lambda\overline{\partial} - i\Lambda\partial\overline{\partial} = \Delta_\partial - i\Lambda\partial\overline{\partial} - i\overline{\partial}\Lambda\partial + i\partial\Lambda\overline{\partial} + i\overline{\partial}\partial\Lambda.$$

Since $\partial\overline{\partial} = -\overline{\partial}\partial$, we have

$$-i\Lambda\partial\overline{\partial} - i\overline{\partial}\Lambda\partial + i\partial\Lambda\overline{\partial} + i\overline{\partial}\partial\Lambda = i(\Lambda\overline{\partial} - \overline{\partial}\Lambda)\partial + i\partial(\Lambda\overline{\partial} - \overline{\partial}\Lambda),$$

so another application of the Kähler identity (3.24) gives

$$i(\Lambda\overline{\partial} - \overline{\partial}\Lambda)\partial + i\partial(\Lambda\overline{\partial} - \overline{\partial}\Lambda) = \Delta_\partial.$$

$\square$

3.32**. Exercise.** Show that

$$[\Delta_d, L] = 0.$$

and

$$[L, \Lambda]\alpha = (k - n)\alpha$$

for every $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$. $\square$

The comparison Theorem 3.31 has numerous important consequences.

3.33**. Corollary.** Let (X, ω) be a Kähler manifold, and let α be a form of type (p, q). Then $\Delta_d\alpha$ has type (p, q).

Proof. Obvious, since $\Delta_d \alpha = 2\Delta_\partial \alpha$ has type (p,q). □

3.34. **Theorem.** Let (X,ω) be a Kähler manifold, and let $\alpha = \sum_{k=p+q} \alpha^{p,q}$ be the decomposition of a k-form in its components of type (p,q). Then α is harmonic if and only if $\alpha^{p,q}$ is harmonic for all p,q. In particular we have

$$\mathscr{H}^k(X) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}(X)$$

where $\mathscr{H}^{p,q}(X)$ is the space of harmonic forms of type (p,q). Furthermore we have

$$\mathscr{H}^{p,q}(X) = \overline{\mathscr{H}^{q,p}(X)} \qquad \forall\ p,q \in \mathbb{N}.$$

3.35. **Remark.** Note that by the comparison theorem we are entitled to speak simply of harmonic forms without specifying the Laplacian.

Proof. By the preceding corollary,

$$\Delta_d \alpha = \sum_{k=p+q} \Delta_d \alpha^{p,q}$$

is the decomposition of $\Delta_d \alpha$ in forms of type (p,q). It is is zero if and only if all components are zero.

Let β be a harmonic form of type (p,q), then $\overline{\beta}$ has type (q,p). By hypothesis $\Delta_\partial \beta = 0$, so by the comparison theorem

$$\overline{\Delta_\partial \overline{\beta}} = \overline{\Delta_\partial} \beta = \Delta_{\overline{\partial}} \beta = \Delta_\partial \beta = 0,$$

hence $\Delta_\partial \overline{\beta} = 0$. □

If X is a compact Kähler manifold, we can combine Theorem 3.34 with the Corollaries A.23 and 3.28 on the representability of cohomology classes by harmonic forms. So we get the famous

3.36. **Theorem.** (Hodge decomposition theorem) Let X be a compact Kähler manifold. Then we have the Hodge decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{k=p+q} H^{p,q}(X)$$

and the Hodge duality

$$H^{q,p}(X) = \overline{H^{p,q}(X)}.$$

Note that the isomorphisms $H^{p,q}(X) \simeq \mathscr{H}^{p,q}(X)$ depend on the choice of the Kähler metric, so a priori we only have an isomorphism $H^k(X,\mathbb{C}) \simeq \bigoplus_{k=p+q} H^{p,q}(X)$. We will see in Subsection 3.D that the isomorphism is canonical which justifies the stronger statement $H^k(X,\mathbb{C}) = \bigoplus_{k=p+q} H^{p,q}(X)$.

3.37. **Corollary.** Let X be a compact Kähler manifold. Then we have

(1) $b_k = \sum_{k=p+q} h^{p,q} \qquad \forall\ k \in \mathbb{N}$.
(2) $h^{p,q} = h^{q,p} \qquad \forall\ p,q \in \mathbb{N}$.
(3) $H^{k,k}(X) \neq 0 \qquad \forall\ k \in 1,\ldots,n$.
(4) b_k is even if k is odd.

Proof. The first and second statement are trivial by the Hodge decomposition and Hodge duality. The third follows from the Hodge decomposition and Proposition 3.12. The fourth is a consequence of the first and the second one. □

3.38. **Corollary.** The Dolbeault cohomology groups of $\mathbb{P}^n$ are

$$H^{p,p}(\mathbb{P}^n) \simeq \mathbb{C} \qquad \forall\, 0 \leqslant p \leqslant n$$

and

$$H^{p,q}(\mathbb{P}^n) = 0 \qquad \forall\, p \neq q.$$

Proof. It is a well-known exercise in differential geometry (use induction and the Mayer-Vietoris sequence !) that

$$H^{2p}(X, \mathbb{C}) \simeq \mathbb{C} \qquad \forall\, 0 \leqslant p \leqslant n.$$

Since $\mathbb{P}^n$ is Kähler, we have $H^{p,p}(\mathbb{P}^n) \neq 0$ by Corollary 3.37. We conclude with the Hodge decomposition theorem. □

3.39. **Exercise.** Consider the Kähler manifold $(\mathbb{C}, \omega)$ where ω is the standard Kähler form on $\mathbb{C}$. Show that

$$\mathscr{H}^1(\mathbb{C}, \mathbb{C}) = \mathscr{H}^{1,0}(\mathbb{C}) \oplus \mathscr{H}^{0,1}(\mathbb{C}),$$

but

$$H^1(\mathbb{C}, \mathbb{C}) \neq H^{1,0}(\mathbb{C}) \oplus H^{0,1}(\mathbb{C}).$$

□

3.40. **Exercise.** Let X be a compact Kähler variety, and let $\omega \in H^0(X, \Omega^p)$ be a a holomorphic p-form. Then $d\omega = 0$. □

3.41. **Exercise.** (Iwasawa manifold) Let $G \subset GL_3(\mathbb{C})$ be the subgroup of matrix

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{C}$. Let $\Gamma \subset G$ be the subgroup of matrices such that $x, y, z \in \mathbb{Z}[i]$. Show that G/Γ is a compact complex manifold.

Show that $dx, dy, dz - xdy \in H^0(G, \Omega_G)$ induce holomorphic 1-forms on G/Γ. Deduce that G/Γ is not Kähler. □

3.42. **Exercise.** Let X be a compact complex surface, that is a compact complex variety of dimension two.

a) Let ω be a global 2-form of type $(2, 0)$. Show that

$$\int_X \omega \wedge \overline{\omega} \geqslant 0.$$

When do we have equality ?

b) Show that if $\omega \in \Gamma(X, \Omega^k)$ is a holomorphic k-form, then $d\omega = 0$ (we do *not* assume that X is Kähler).

c) Let $f : X \to \mathbb{C}$ be a differentiable function such that $\partial\overline{\partial} f = 0$. Show that f is constant.

Hint: show that in a coordinate neighborhood, the function f is pluriharmonic.

Deduce that if $\omega \in H^{1,0}(X)$ such that there exists a differentiable function f such that $\omega = \partial f$, then $\omega = 0$. Show that we have an inclusion

$$H^{1,0}(X) \hookrightarrow H^{0,1}(X), \omega \mapsto [\overline{\omega}].$$

d) Show that if $\omega \in H^{2,0}(X)$ such that there exists $\eta \in C^\infty(X, \Omega_X^{1,0})$ such that $\omega = \partial \eta$, then $\omega = 0$. Deduce that we have an inclusion

$$H^{2,0}(X) \hookrightarrow H^{0,2}(X), \omega \mapsto [\overline{\omega}]$$

e) Show that we have inclusions

$$H^{1,0}(X) \hookrightarrow H^1(X, \mathbb{C}), \omega \mapsto [\omega]$$

and

$$H^{2,0}(X) \hookrightarrow H^2(X, \mathbb{C}), \omega \mapsto [\omega].$$

For $k = 1, 2$, we can thus identify $H^{k,0}(X)$ to a subspace of $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$. Show that

$$H^{k,0}(X) \cap \overline{H^{k,0}(X)} = 0.$$

Deduce that for every compact surface

$$2h^{1,0} \leqslant b_1 \leqslant 2h^{0,1}.$$

Give an example where the inequalities are strict.

Hint: for the second inequality, you can consider the exact sequence

$$0 \to \mathbb{C} \to \mathscr{O}_X \xrightarrow{d} Z^1 \to 0,$$

where Z^1 is the sheaf of holomorphic 1-forms that are d-closed. $\square$

3.43. **Exercise.** Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Show that $T_X \simeq \mathscr{O}_X^{\oplus n}$. Compute the Hodge numbers $h^{p,q}$ for every $p, q \in \{0, \dots, n\}$. $\square$

3.D. **Bott-Chern cohomology and $\partial\overline{\partial}$-Lemma.** Since the definition of the Laplacian operations $\Delta, \Delta_\partial, \Delta_{\overline{\partial}}$ depends on the Hodge $*$-operator and thus on the Hermitian metric on T_X, it is a priori not clear if the isomorphism in the Hodge decomposition Theorem 3.36 depends on the choice of a Kähler metric. In this paragraph, we show that this is not the case.

3.44. **Definition.** Let X be a complex manifold. We define the Bott-Chern cohomology groups of X to be

$$H^{p,q}_{BC}(X) = \frac{\{\alpha \in C^\infty(X, \Omega_X^{p,q}) \mid d\alpha = 0\}}{\partial\overline{\partial} C^\infty(X, \Omega_X^{p-1,q-1})}.$$

Since $\partial\overline{\partial}\beta = d\overline{\partial}\beta$ for any $\beta \in C^\infty(X, \Omega_X^{p-1,q-1})$, the natural morphism

$$\{\alpha \in C^\infty(X, \Omega^{p,q}) \mid d\alpha = 0\} \to H^{p+q}(X, \mathbb{C})$$

passes to the quotient, so we have a *canonical* map

$$H^{p,q}_{BC}(X) \to H^{p+q}(X, \mathbb{C}).$$

Since a (p,q)-form that is d-closed is also $\overline{\partial}$-closed, we have a natural morphism

$$\{\alpha \in C^\infty(X, \Omega^{p,q}) \mid d\alpha = 0\} \to H^{p,q}(X).$$

Since $\partial\overline{\partial}\beta = \overline{\partial}(-\partial\beta)$ for any $\beta \in C^\infty(X, \Omega^{p-1,q-1})$, the morphism passes to the quotient, so we have again a *canonical* map

$$H^{p,q}_{BC}(X) \to H^{p,q}(X).$$

The following, $\partial\overline{\partial}$-Lemma allows to compare the Bott-Chern cohomology with the Dolbeault and the de Rham cohomology.

3.45**. Lemma.** Let X be a compact Kähler manifold and let ω be a (p,q)-form that is d-closed. If ω is ∂- or $\overline{\partial}$-exact, there exists a $(p-1,q-1)$-form φ such that $\omega = \partial\overline{\partial}\varphi$.

Proof. We will prove the case where ω is $\overline{\partial}$-exact, the other case is analogous.

Since ω has type (p,q) and is d-closed, it is ∂- and $\overline{\partial}$-closed. By hypothesis $\omega = \overline{\partial}\eta$ for some $\eta \in C^\infty(X, \Omega_X^{p,q-1})$ and the analogue of Corollary 3.27 for ∂ shows that we have a decomposition

$$\eta = \alpha + \partial\beta + \partial^*\gamma$$

with $\alpha \in \mathscr{H}^{p,q}_{\partial}$, $\beta \in C^\infty(X, \Omega_X^{p-1,q-1})$ and $\gamma \in C^\infty(X, \Omega_X^{p+1,q-1})$. Since X is Kähler $\mathscr{H}^{p,q}_{\partial} = \mathscr{H}^{p,q}_{\overline{\partial}}$, so we get

$$\omega = \overline{\partial}\partial\beta + \overline{\partial}\partial^*\gamma.$$

Since $\overline{\partial}\partial^* = -\partial^*\overline{\partial}$ by Formula (3.26), this implies

$$\omega = -\partial\overline{\partial}\beta - \partial^*\overline{\partial}\gamma.$$

Since $\partial\omega = 0$ and $\partial\partial\overline{\partial}\beta = 0$, we have $\partial\partial^*\overline{\partial}\gamma = 0$. Thus we have $\partial^*\overline{\partial}\gamma \in (\mathrm{Im}\partial^* \cap \ker\partial)$, so Corollary 3.27 shows that $\partial^*\overline{\partial}\gamma = 0$. $\square$

3.46**. Corollary.** Let X be a compact Kähler manifold. Then the canonical morphisms

$$H^{p,q}_{BC}(X) \to H^{p,q}(X)$$

and

$$\bigoplus_{p+q=k} H^{p,q}_{BC}(X) \to H^k(X, \mathbb{C})$$

are isomorphisms.

In particular the isomorphisms in the Hodge decomposition Theorem 3.36 are canonical.

Proof. We have $H^{p,q}(X) \simeq \mathscr{H}^{p,q}(X)$ by Corollary 3.28, so every Dolbeault class can be represented by a unique $\Delta_{\overline{\partial}}$-harmonic form. By the comparison Theorem 3.31 a form is $\Delta_{\overline{\partial}}$-harmonic if and only if it is Δ-harmonic, in particular it is d-closed. This shows that the morphism $H^{p,q}_{BC}(X) \to H^{p,q}(X)$ is surjective.

Let now $\gamma \in H^{p,q}_{BC}(X)$ such that its image in $H^{p,q}(X)$ is zero. Let $\alpha \in C^\infty(X, \Omega^{p,q})$ be d-closed such that $[\alpha] = \gamma$ in $H^{p,q}_{BC}(X)$. Furthermore $[\alpha] = 0$ in $H^{p,q}(X)$, so α is $\overline{\partial}$-exact. Therefore the $\partial\overline{\partial}$-lemma 3.45 applies and we see that $[\alpha] = 0$ in $H^{p,q}_{BC}(X)$. This shows that the morphism $H^{p,q}_{BC}(X) \to H^{p,q}(X)$ is injective.

The proof of the second statement is left to the reader, it follows from the first statement and

$$H^k(X, \mathbb{C}) \simeq \mathscr{H}^k(X) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}(X).$$

□

3.E. **Applications of Hodge theory.**

Picard and Albanese variety

Let X be a compact Kähler manifold. We consider the long exact cohomology sequence associated to the exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_X \stackrel{\exp(2\pi i\bullet)}{\to} \mathscr{O}_X^* \to 0.$$

Since $\mathbb{C} \simeq H^0(X, \mathscr{O}_X) \stackrel{\exp}{\to} H^0(X, \mathscr{O}_X^*) \simeq \mathbb{C}^*$ is surjective, we have an injection

$$i : H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathscr{O}_X),$$

so $H^1(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of finite type and we want to understand its image[14]. The inclusion $\mathbb{Z} \subset \mathbb{R}$ gives an inclusion $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$ and by the universal coefficient theorem [Hat02, Cor.3.A.6] we see that $H^1(X, \mathbb{Z}) \otimes \mathbb{R} = H^1(X, \mathbb{R})$, so it is a lattice of rank b_1 in $H^1(X, \mathbb{R})$. The inclusions $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$ thus gives an inclusion

$$H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

such that $H^1(X, \mathbb{Z})$ is invariant under conjugation. Since $H^{1,0}(X) = \overline{H^{0,1}(X)}$ by the Hodge theorem 3.36, we see that $H^1(X, \mathbb{Z}) \cap H^{1,0}(X) = 0$. Thus the projection on $H^{0,1}(X)$ maps $H^1(X, \mathbb{Z})$ isomorphically onto a lattice of rank $b_1 = \dim_{\mathbb{R}} H^{0,1}(X)$. Using the de Rham and the Dolbeault complex, one sees that this map identifies to i, so we have shown

3.47**. Lemma.** Let X be a compact Kähler manifold. The quotient

$$H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z})$$

is a complex torus called the Picard variety $\mathrm{Pic}^0(X)$.

[14]Throughout the whole paragraph we will denote by $H^k(X, \mathbb{Z})$ the singular cohomology of X with values in $\mathbb{Z}$. If you prefer, you can replace this by the Čech cohomology groups $\check{\mathrm{H}}^k(X, \mathbb{Z})$ of the sheaf of locally constant functions with values in $\mathbb{Z}$.

We claim that the Picard variety parametrises the holomorphic line bundles L on X such that $c_1(L) = 0 \in H^2(X, \mathbb{Z})$. Indeed we have seen in Exercise 1.91 that $H^1(X, \mathscr{O}_X^*)$ parametrises the holomorphic line bundles on X. Moreover the cohomology sequence associated to the exponential sequence yields

$$0 \to \mathrm{Pic}^0(X) = H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z}) \to \mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^*) \to H^2(X, \mathbb{Z})$$

and the edge operator $\delta : H^1(X, \mathscr{O}_X^*) \to H^2(X, \mathbb{Z})$ composed with the natural morphism $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ is given by the first Chern class. So $\mathrm{Pic}^0(X)$ is the kernel of this map. Denote by

$$NS(X) := \mathrm{im}\, \delta$$

the Neron-Severi group[15]. Then we get an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to NS(X) \to 0$$

and by the Lefschetz theorem on $(1,1)$-classes we have

$$\mathrm{im}\left(NS(X) \to H^2(X, \mathbb{C})\right) = \mathrm{im}\left(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})\right) \cap H^{1,1}(X).$$

Thus the Neron-Severi group is (modulo torsion) the intersection of the lattice $H^2(X, \mathbb{Z})$ with the subspace $H^{1,1}(X)$. The rank of this intersection, the Picard number $\rho(X)$ depends on the complex structure of X and can vary between 0 and $h^{1,1}$.

3.48. **Example.** If X is a compact complex curve, then

$$\check{\mathrm{H}}^2(X, \mathbb{Z}) \simeq \mathbb{Z}$$

and $h^{2,0} = h^{0,2} = 0$, so we get $NS(X) \simeq \mathbb{Z}$. Since $\check{\mathrm{H}}^2(X, \mathbb{Z})$ is torsion-free, we have an inclusion $\check{\mathrm{H}}^2(X, \mathbb{Z}) \subset \check{\mathrm{H}}^2(X, \mathbb{R}) \simeq H^2(X, \mathbb{R})$. Since the top de Rham cohomology group is canonically isomorphic to $\mathbb{R}$ via the integration morphism, we get an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathbb{Z} \to 0,$$

where the last arrow is given by $[L] \mapsto \int_X c_1(L)$.

3.49. **Example.** Let X be the projective space $\mathbb{P}^n$, then $H^2(\mathbb{P}^n, \mathbb{Z}) \simeq Z$. Moreover by Corollary 3.38 we have $H^1(X, \mathscr{O}_X) = 0$, so the Picard torus is trivial. Hence

$$\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$$

and it is not very hard to see that the tautological bundle $\mathscr{O}_{\mathbb{P}^n}(-1)$ is a generator.

3.50. **Exercise.** a) Show that the Néron-Severi group of a complex torus $X = \mathbb{C}^n/\Lambda$ identifies to the set of Hermitian forms $H = (h_{j,k})_{j,k=1,\dots,n}$ on $\mathbb{C}^n$ such that

$$\mathrm{Im}\, H(\gamma, \gamma') \in \mathbb{Z} \qquad \forall\, \gamma, \gamma' \in \Lambda.$$

Hint : [BL04, Sect.2].

[15]Note that in the literature, the Neron-Severi group is also often defined as the image of the composed morphism $H^1(X, \mathscr{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$. This amounts to killing the torsion part, so that one is left with a free abelian group of finite type.

b) Let $a, b, c, d \in \mathbb{R} \setminus \mathbb{Q}$ that are linearly independent over $\mathbb{Q}$ and such that $ad - bc \in \mathbb{R} \setminus \mathbb{Q}$. Let $\Lambda \subset \mathbb{C}^2$ be the lattice generated by the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} ia \\ ib \end{pmatrix}, \begin{pmatrix} ic \\ id \end{pmatrix},$$

and let X be the torus $\mathbb{C}^2/\Lambda$. Show that $\mathrm{Pic}(X) = \mathrm{Pic}^0(X)$, i.e. the Néron-Severi group of X is zero. Deduce that X is not projective.

c) Show that a complex torus $X = \mathbb{C}^n/\Lambda$ admits a positive line bundle if and only if there exists a definite positive Hermitian form $H = (h_{j,k})_{j,k=1,\ldots,n}$ on $\mathbb{C}^n$ such that

$$\operatorname{Im} H(\gamma, \gamma') \in \mathbb{Z} \qquad \forall\ \gamma, \gamma' \in \Lambda.$$

Such a torus is called an abelian variety.

Hint: let $d\mu$ be a mesure of volume 1 on X that is invariant under the group action. Let ω be a Kähler form on X, then ω is cohomologous to a Kähler form with constant coefficients

$$\tilde{\omega}(z) = \int_{a \in X} \tau_a^* \omega(z) d\mu(a) = \int_{a \in X} \omega(z + a) d\mu(a)$$

where $\tau_a : X \to X, z \mapsto z + a$ is the translation map by $a \in X$, $\square$

A second torus attached to a compact Kähler manifold X is the Albanese variety: the inclusions $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$ induce maps

$$H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}(X, \mathbb{R}) \to H^{2n-1}(X, \mathbb{C}) = H^{n,n-1}(X) \oplus H^{n-1,n}(X)$$

and we see as before that

$$Alb(X) := H^{n-1,n}(X)/\mathrm{Im} H^{2n-1}(X, \mathbb{Z})$$

is a complex torus. Since by Poincaré duality the first homology group $H_1(X, \mathbb{Z})$ is naturally isomorphic to $H^{2n-1}(X, \mathbb{Z})$, and by Serre duality 4.3 and the comparison theorem $H^{n-1,n}(X) = H^{1,0}(X)^* = H^0(X, \Omega_X)^*$, we have

$$AlbX = H^0(X, \Omega_X)^*/\mathrm{Im} H_1(X, \mathbb{Z})$$

where the map $H_1(X, \mathbb{Z}) \to H^0(X, \Omega_X)$ is defined as follows: let $[\gamma] \in H_1(X, \mathbb{Z})$ be a class represented by 1-cycle γ on X then we can associate a linear map

$$I_\gamma : H^0(X, \Omega_X) \to \mathbb{C}, \omega \ \mapsto \ \int_\gamma \omega.$$

3.51. **Exercise.** Show that I_γ depends only on the homology class $[\gamma]$. (This is not true on a compact complex manifold that is not Kähler!) $\square$

Actually the difference between the Picard and the Albanese variety is not that big: they are dual tori (cf. [BL04, Ch.2.4]). The advantage of the Albanese variety is that we can define a holomorphic map $\alpha : X \to Alb(X)$ as follows: fix a point $x_0 \in X$. Then we set

$$\alpha : X \to Alb(X), x \mapsto (\omega \mapsto \int_{x_0}^x \omega),$$

where $\int_{x_0}^{x} \omega$ is the integration over some path from x_0 to x. Let us show that this expression actually makes sense: let $\gamma : [0,1] \to X$ be a path connecting x_0 to x. The linear map given by integration over the path

$$I_\gamma : H^0(X, \Omega_X) \to \mathbb{C}, \omega \mapsto \int_\gamma \omega.$$

is of course not independent of the choice of the path, but if γ' is a second path connecting x_0 to x, then the composition of the paths $\gamma^{-1}\gamma'$ defines an element of $H_1(X, \mathbb{Z})$, so $I_{\gamma^{-1}\gamma'} \in \mathrm{Im} H_1(X, \mathbb{Z})$. Thus I_γ gives a point in $Alb(X)$ that does not depend on the choice of the path.

3.52. **Exercise.** Show that α is holomorphic. □

The Albanese torus has the following universal property:

3.53. **Proposition.** Let X be a compact Kähler manifold. Let $\varphi : X \to T$ be a holomorphic map to a compact complex torus T. Then there exists a unique holomorphic map $\psi : Alb(X) \to T$ such that $\psi \circ \alpha = \varphi$.

Proof. Omitted. Follows from Exercise 3.56 and functorial properties of the cohomology groups. □

3.54. **Exercise.** Let X be a compact Kähler manifold such that $b_1 = 0$. Then any holomorphic map to a torus is constant. □

3.55. **Exercise.** Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Show that the Albanese map $\alpha : X \to Alb(X)$ is an isomorphism. □

3.56. **Exercise.** (Hodge structures of weight one) A Hodge structure of weight one is a free $\mathbb{Z}$-module V of finite type such that the complexification $V_\mathbb{C} = V \otimes_\mathbb{Z} \mathbb{C}$ admits a decomposition

$$V_\mathbb{C} = V^{1,0} \oplus V^{0,1}$$

such that $V^{1,0} = \overline{V^{0,1}}$. A morphism of abstract Hodge structures is a $\mathbb{Z}$-linear map $\phi : V \to V'$ such that the complexified morphism $\phi_\mathbb{C}$ respects the Hodge decomposition.

Show that we have an equivalence of categories between Hodge structures of weight one and compact complex tori. □

The Lefschetz decomposition

Let (X, ω) be a Kähler manifold of dimension n. The Kähler form ω is real[16], so it defines a linear map

$$L : \Omega^k_{X,\mathbb{R}} \to \Omega^{k+2}_{X,\mathbb{R}}, \alpha \mapsto \omega \wedge \alpha.$$

We claim that for $k \leqslant n$, the map

$$L^{n-k} : \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}}.$$

[16]Note that the whole theory developed in this paragraph also works for the de Rham cohomology with complex coefficients.

is an isomorphism. Since the two vector bundles have the same rank, it is sufficient to show that for every $U \subset X$, the morphism

$$L^{n-k} : C^\infty(U, \Omega^k_{X,\mathbb{R}}) \to C^\infty(U, \Omega^{2n-k}_{X,\mathbb{R}}), \alpha \mapsto \omega \wedge \alpha$$

is injective: by Exercise 3.32, we have

$$[L, \Lambda]\alpha = (k-n)\alpha$$

for every $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$. By definition

$$[L^r, \Lambda] = L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1},$$

so we get inductively for every k-form α

$$(*) \qquad [L^r, \Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha.$$

We will prove by induction over $k \in \{0, \ldots, n\}$ and $r \in \{0, \ldots, n-k\}$ that L^r is injective. For $k=0$ and $r=0$ this is clear, thus let α be a k-form such that $L^r\alpha = 0$. By $(*)$ this implies that

$$L^{r-1}(L\Lambda\alpha - (r(k-n) + r(r-1))\alpha) = 0,$$

so by the induction hypothesis on r

$$L\Lambda\alpha - (r(k-n) + r(r-1))\alpha = 0.$$

Therefore we get $(r(k-n) + r(r-1))\alpha = L\beta$ with $\beta = \Lambda\alpha$ of degree $k-2$. Furthermore $L^{r+1}\beta = 0$, so by the induction hypothesis on k, we have $\beta = 0$.

3.57**. Definition.** Let (X, ω) be a Kähler manifold of dimension n. We say that $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$, $k \leqslant n$ is primitive if $L^{n-k+1}\alpha = 0$.

3.58**. Exercise.**

a) Show that $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$, $k \leqslant n$ is primitive if and only if $\Lambda\alpha = 0$.

b) Show that every element $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$, admits a unique decomposition of the form

$$\alpha = \sum_r L^r \alpha_r$$

such that α_r is primitive of degree $k - 2r \leqslant \inf(2n-k, k)$. We call this the Lefschetz decomposition of α. $\square$

Note now that since ω is d-closed, the linear map L induces a linear map on the de Rham cohomology groups

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R}), \ [\alpha] \mapsto [\omega \wedge \alpha].$$

3.59**. Theorem.** (Hard Lefschetz theorem) Let (X, ω) be a compact Kähler manifold of dimension n. Then for every $k \leqslant n$, the map

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism.

Proof. Since $[\Delta_d, L] = 0$ by Exercise 3.32, the image $L(\alpha)$ of a harmonic form α is harmonic. Since X is compact, Corollary A.23 shows that it is equivalent to show that the induced morphism

$$L^{n-k} : \mathscr{H}^k(X,\mathbb{R}) \to \mathscr{H}^{2n-k}(X,\mathbb{R})$$

is an isomorphism. By Poincaré duality (Corollary A.24) both vector spaces have the same dimension furthermore by the claim above, the morphism

$$L^{n-k} : C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{2n-k}_{X,\mathbb{R}})$$

is injective. In particular it is injective on the harmonic forms, so the statement follows. □

3.60. **Definition.** Let (X,ω) be a compact Kähler manifold of dimension n. We say that $[\alpha] \in H^k(X,\mathbb{R})$, $k \leqslant n$ is primitive if $L^{n-k+1}[\alpha] = 0$. We denote by $H^k(X,\mathbb{R})_{\text{prim}} \subset H^k(X,\mathbb{R})$ the subspace of primitive classes.

As a consequence of Exercise 3.58, we obtain

3.61. **Corollary.** . Let (X,ω) be a compact Kähler manifold. Then every element $[\alpha] \in H^k(X,\mathbb{R})$ admits a unique decomposition of the form

$$[\alpha] = \sum_r L^r[\alpha_r]$$

such that $\alpha_r \in H^{k-2r}(X,\mathbb{R})_{\text{prim}}$ with $k - 2r \leqslant \inf(2n-k, k)$. In particular we have

$$H^k(X,\mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\text{prim}}.$$

This Lefschetz decomposition plays an important role in the Hodge index theorem (cf. Theorem 3.67 below).

The Hard Lefschetz theorem also holds for the de Rham cohomology with complex coefficients. Since the Kähler form ω is $\overline{\partial}$-closed, the Hodge decomposition shows immediately that for all $p+q \leqslant n$, we have an isomorphism

$$L^{n-p-q} : H^{p,q}(X) \to H^{n-q,n-p}(X).$$

3.62. **Exercise.** Let X be a compact Kähler manifold of dimension n.

$$h^{p-1,q-1} \leqslant h^{p,q},\ b_k \leqslant b_{k+2} \qquad \forall\ k = p+q \leqslant n$$

and

$$h^{p,q} \geqslant h^{p+1,q+1},\ b_k \geqslant b_{k+2} \qquad \forall\ k = p+q \geqslant n.$$

Hint: note that if α is a harmonic k-form, the Lefschetz decomposition commutes with the decomposition into forms of type (p,q). □

The Hodge Index Theorem

We will now consider the Hodge index theorem for the intersection form on $H^2(X, \mathbb{C})$. For the sake of simplicity of notation, we will restrict ourselves to the case of a compact Kähler surface (cf. [Voi02, Ch.6.3] for a full account).

We start with a technical lemma.

3.63. **Lemma.** Let $U \subset \mathbb{C}^2$ be an open subset and endow T_U with the standard metric $h = 2(dz_1 \otimes d\overline{z_1} + dz_2 \otimes d\overline{z_2})$. Let $\alpha \in C^\infty(U, \Omega_U^{p,q})$ be a primitive two-form. Then we have

$$*\alpha = (-1)^q \alpha.$$

Proof. We will prove the claim where α is of type $(1,1)$, the other cases are analogous. Let $\omega = i(dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2})$ be the Kähler form, and set

$$\alpha = \alpha_{1,2} dz_1 \wedge d\overline{z_2} + \alpha_{2,1} dz_2 \wedge d\overline{z_1} + \alpha_{1,1} dz_1 \wedge d\overline{z_1} + \alpha_{2,2} dz_2 \wedge d\overline{z_2}$$

where the $\alpha_{j,k}$ are differentiable functions. The volume form is

$$\frac{\omega^2}{2!} = i^2 dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2},$$

so $\bullet \wedge *\overline{\alpha} = \{\bullet, \alpha\} vol$ implies

$$*\alpha = -\alpha_{1,2} dz_1 \wedge d\overline{z_2} - \alpha_{2,1} dz_2 \wedge d\overline{z_1} + \alpha_{1,1} dz_2 \wedge d\overline{z_2} + \alpha_{2,2} dz_1 \wedge d\overline{z_1}.$$

Furthermore

$$L\alpha = \omega \wedge \alpha = i(\alpha_{1,1} + \alpha_{2,2}) dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2}$$

equals zero if and only if $\alpha_{1,1} = -\alpha_{2,2}$. This implies the claim. □

Arguing as in the proof of Proposition 3.30, we deduce:

3.64. **Lemma.** Let (X, ω) be a Kähler manifold of dimension two, and let $\alpha \in C^\infty(X, \Omega_X^{p,q})$ be a primitive 2-form. Then we have

$$*\alpha = (-1)^q \alpha.$$

Let now X be a compact Kähler variety of dimension two. Then the Poincaré duality Theorem A.24 shows that we have a non-degenerate symmetric bilinear pairing

$$Q : H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to \mathbb{R}, \ ([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta.$$

Therefore

$$H(\alpha, \beta) := Q(\alpha, \overline{\beta})$$

defines a non-degenerate hermitian form on $H^2(X, \mathbb{C})$.

3.65. **Lemma.** Let (X, ω) be a compact Kähler variety of dimension two. The Lefschetz decomposition

$$H^2(X, \mathbb{C}) = H^2(X, \mathbb{C})_{\text{prim}} \oplus LH^0(X, \mathbb{C})_{\text{prim}} = H^2(X, \mathbb{C})_{\text{prim}} \oplus \mathbb{C}[\omega]$$

is orthogonal.

Proof. As before we reduce the statement on cohomology to a statement on harmonic forms. By definition a two-form α is primitive if $\omega \wedge \alpha = 0$. Therefore

$$H(\alpha, \omega) = \int_X \alpha \wedge \overline{\omega} = \int_X \alpha \wedge \omega = 0.$$

□

3.66. **Proposition.** The subspaces $H^{p,q} \subset H^2(X, \mathbb{C})$ are orthogonal with respect to H. Furthermore $(-1)^q H$ is definite positive on the subspace

$$H^{p,q}_{\text{prim}} := H^2(X, \mathbb{C})_{\text{prim}} \cap H^{p,q}(X).$$

Proof. The orthogonality is obvious for reasons of type (cf. page 73). Note that for reasons of type $H^{2,0}_{\text{prim}} = H^{2,0}(X)$ and $H^{0,2}_{\text{prim}} = H^{0,2}(X)$. Let $\gamma \in H^{p,q}_{\text{prim}}$ be a non-zero class. Note also that $\mathscr{H}^{p,q}(X) \simeq H^{p,q}(X)$ and the Lefschetz operator commutes with $\Delta_{\overline{\partial}}$. Thus if α is the harmonic representative of γ, then the form α (hence $\overline{\alpha}$) is primitive, i.e. $\omega \wedge \alpha$ is zero. Then we have by Lemma 3.64

$$*\overline{\alpha} = (-1)^q \overline{\alpha},$$

so

$$(-1)^q H(\alpha, \alpha) = (-1)^q \int_X \alpha \wedge \overline{\alpha} = (-1)^{2q} \int_X \alpha \wedge *\overline{\alpha} = ||\alpha||_{L^2} > 0.$$

□

As a corollary, we obtain

3.67. **Theorem.** (Hodge Index Theorem) Let X be a compact Kähler surface. Then the signature of the intersection form

$$Q([\alpha], [\beta]) = \int_X \alpha \wedge \beta$$

on $H^2(X, \mathbb{R}) \cap H^{1,1}(X)$ is $(1, h^{1,1} - 1)$.

Proof. By Lemma 3.65 we have an orthogonal decomposition

$$H^2(X, \mathbb{C}) = H^2(X, \mathbb{C})_{\text{prim}} \oplus \mathbb{C}[\omega].$$

Since ω is a real form, this implies

$$H^2(X, \mathbb{R}) \cap H^{1,1}(X) = H^2(X, \mathbb{R}) \cap H^{1,1}(X)_{\text{prim}} \oplus \mathbb{C}[\omega].$$

By the preceeding proposition, the intersection form Q is negative definite on $H^{1,1}(X)_{\text{prim}}$. But we have already seen (cf. Proposition 3.12) that

$$\int_X \omega \wedge \overline{\omega} = \int_X \omega \wedge \omega > 0.$$

□

3.68. **Exercise.** Let (X, ω) be a compact Kähler surface, and let γ be a real 2-form such that $[\gamma] \in H^{1,1}(X)$. Show that

$$Q([\gamma], [\gamma]) \cdot Q([\omega], [\omega]) \leqslant Q([\gamma], [\omega])^2.$$

□

3.69**. Exercise.** (not so easy)

a) Let X be a projective surface, and let H be a positive line bundle on X. Fix $d \in \mathbb{Z}$. Show that the set of classes $\gamma \in NS(X)$ such that

$$Q(c_1(H), \gamma) = d$$

and such that there exists a curve $C \subset X$ such that $\gamma = [C]$ is finite.

b) Let C be a compact curve of genus $g \geqslant 2$. Show that the group of biholomorphic automorphisms of C is finite as follows: given an automorphism σ, let $\Gamma \subset C \times C$ be its graph, and let Δ be the diagonal.

Show first that $\mathscr{O}_{C \times C}(\Delta)|_{\Delta} \simeq K_C^*$. Deduce that $Q([\Delta], [\Delta]) < 0$.

Show that if $[\Gamma] = [\Delta]$, then $\Gamma = \Delta$, i.e. σ is the identity map.

Conclude with a). $\square$

4. Kodaira's projectivity criterion

We have seen in Section 3 that every projective manifold is a Kähler manifold, but in general the converse is not true: a complex torus $\mathbb{C}^n/\Gamma$ is always a Kähler manifold, but if we choose the lattice Γ general enough, it is not projective (Exercise 3.50). The aim of this section is to give an outline of the proof of Kodaira's famous projectivity criterion for compact Kähler manifolds (Theorem 4.15). A main step in the proof of this theorem is the equally famous Kodaira vanishing Theorem 4.8.

4.A. **Serre duality and Kodaira's vanishing theorem.** Let X be a compact complex manifold, and let E be a holomorphic vector bundle over X. We endow E with a Hermitian metric h and define the L^2 scalar product $C^\infty(X,E)$ by Formula (3.14). We define a $\mathbb{C}$-antilinear Hodge operator

$$\overline{*}_E : \Omega_X^{p,q} \otimes E \to \Omega_X^{n-p,n-q} \otimes E^*$$

as follows: the Hermitian metric induces a $\mathbb{C}$-antilinear isomorphism of complex vector bundles

$$\tau : E \to E^*, \ s \in E_x \ \mapsto h_x(s_x, \bullet).$$

For any open set $U \subset X$ and any decomposable $\varphi \otimes e \in C^\infty(U, \Omega_X^{p,q} \otimes E)$, we set

$$\overline{*}_E(\varphi \otimes e) := *\overline{\varphi} \otimes \tau(e),$$

and extend the definition by linearity to all elements of $C^\infty(U, \Omega_X^{p,q} \otimes E)$. It is then clear that

$$\alpha \wedge \overline{*}_E \beta = \{\alpha, \beta\} vol.$$

Let D_E be the Chern connection of E. Recall that the $(0,1)$-part of the Chern connection equals $\overline{\partial}_E$ and we set $\partial_E := D_E^{1,0}$. Since the Chern curvature tensor Θ_E is of type $(1,1)$, the equality

$$\Theta_E = D_E^2 = \partial_E^2 + \partial_E \overline{\partial}_E + \overline{\partial}_E \partial_E + \overline{\partial}_E^2$$

implies $\partial_E^2 = 0$ and $\Theta_E = \partial_E \overline{\partial}_E + \overline{\partial}_E \partial_E = [\partial_E, \overline{\partial}_E]$.

We define ∂_E^* (resp. $\overline{\partial}_E^*$) to be the formal adjoints of ∂_E (resp. $\overline{\partial}_E$) and the corresponding Laplacians by

$$\Delta_{\partial_E} := \partial_E \partial_E^* + \partial_E^* \partial_E$$

and

$$\Delta_{\overline{\partial}_E} := \overline{\partial}_E \overline{\partial}_E^* + \overline{\partial}_E^* \overline{\partial}_E.$$

As in the case where E is the trivial line bundle $\mathscr{O}_X$, one shows that these Laplacians are elliptic operators. Thus we can apply Theorem A.42. and argue as in the proof of Corollary 3.27 to show that we have an orthogonal decomposition

$$C^\infty(X, \Omega^{p,q} \otimes E) = \mathscr{H}^{p,q}(X,E) \oplus \overline{\partial}_E(C^\infty(X, \Omega^{p,q-1} \otimes E)) \oplus \overline{\partial}_E^*(C^\infty(X, \Omega^{p,q+1} \otimes E))$$

and

$$\ker \overline{\partial}_E = \mathscr{H}^{p,q}(X,E) \oplus \overline{\partial}_E(C^\infty(X, \Omega^{p,q-1} \otimes E))$$

In particular we have the

4.1. **Corollary.** Let X be a compact complex manifold, and let (E, h) be a hermitian holomorphic vector bundle over X. Then we have

$$H^{p,q}(X, E) \simeq \mathscr{H}^{p,q}(X, E).$$

In particular the Dolbeault cohomology groups have finite dimension.

4.2. **Exercise.** In analogy to Formulas 3.16 and 3.17, establish formulas for the adjoint operators ∂_E^* and $\overline{\partial}_E^*$. □

We come to one of the fundamental statements in complex geometry.

4.3. **Theorem.** (Serre duality) Let X be a compact complex manifold, and let E be a holomorphic vector bundle over X. Then the bilinear pairing

$$H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^*) \to \mathbb{C}, \ ([s], [t]) \mapsto \int_X s \wedge t$$

is a nondegenerate bilinear pairing.

4.4. **Remark.** The symbol $s \wedge t$ should be understood as follows: let $e_1, \ldots, e_r$ be a local holomorphic frame on $U \subset X$, then

$$s|_U = \sum_{j=1}^r s_j e_j, \qquad t|_U = \sum_{j=1}^r t_j e_j^*,$$

where $s_j \in C^\infty(U, \Omega_X^{p,q})$ and $t_j \in C^\infty(U, \Omega_X^{n-p,n-q})$. Then

$$(s \wedge t)|_U = \sum_{j=1}^r s_j \wedge t_j$$

and one checks easily that these local definitions glue to a global (n, n)-form.

Sketch of the proof. We note first that the map

$$C^\infty(X, \Omega_X^{p,q} \otimes E) \times C^\infty(X, \Omega_X^{n-p,n-q} \otimes E^*) \to \mathbb{C}, \ (s, t) \mapsto \int_X s \wedge t$$

induces indeed a map on the cohomology groups: if for example $s = \overline{\partial}_E \eta$, then

$$\overline{\partial}_E \eta \wedge t = \overline{\partial}(\eta \wedge t) = d(\eta \wedge t),$$

since $\eta \wedge t$ is of type $(n, n-1)$. Thus Stokes' theorem shows that $\int_X s \wedge t = 0$.

Then one shows that (cf. [Wel80, Ch. V.,2] for the details)

$$\overline{*}_E \Delta_{\overline{\partial}_E} = \Delta_{\overline{\partial}_{E^*}} \overline{*}_E,$$

so we get a commutative diagram

$$\begin{tikzcd}
C^\infty(X, \Omega_X^{p,q} \otimes E) \arrow[r, "\overline{*}_E"] & C^\infty(X, \Omega_X^{n-p,n-q} \otimes E^*) \\
\mathscr{H}^{p,q}(X, E) \arrow[r, "\overline{*}_E"] & \mathscr{H}^{n-p,n-q}(X, E^*).
\end{tikzcd}$$

Since $\overline{*}_E$ is an isomorphism of complex vector bundles, we get an isomorphism between the spaces of harmonic forms

$$\mathscr{H}^{p,q}(X,E) \overset{\overline{*}_E}{\to} \mathscr{H}^{n-p,n-q}(X,E^*).$$

By Corollary 4.1, this implies that the vector spaces $H^{p,q}(X,E)$ and $H^{n-p,n-q}(X,E^*)$ have the same dimension. Moreover the bilinear pairing is non-degenerate since for every $0 \neq u \in \mathscr{H}^{p,q}(X,E)$, we have $\overline{*}_E u \in \mathscr{H}^{n-p,n-q}(X,E^*)$ and

$$\int_X u \wedge \overline{*}_E u = ||u||^2_{L^2} > 0.$$

□

Let (X,ω) be a Kähler variety, and let (E,h) be a holomorphic Hermitian vector bundle over X. Set

$$L : C^\infty_c(X, \Omega^{p,q} \otimes E) \to C^\infty_c(X, \Omega^{p+1,q+1} \otimes E), u \mapsto \omega \wedge u$$

for the Lefschetz operator with values in E, and denote by $\Lambda = L^*$ its formal adjoint. The following lemma adapts the Kähler identities (Proposition 3.30) to this setting, we omit the proof which is analogous to the one given in Subsection 3.C.

4.5**. Lemma.** Let (X,ω) be a Kähler manifold, and let (E,h) be a holomorphic Hermitian vector bundle over X. Then we have

$$[\overline{\partial}^*_E, L] = i\partial_E \tag{4.27}$$

$$[\partial^*_E, L] = -i\overline{\partial}_E. \tag{4.28}$$

$$[\Lambda, \overline{\partial}_E] = -i\partial^*_E \tag{4.29}$$

$$[\Lambda, \partial_E] = i\overline{\partial}^*_E \tag{4.30}$$

A consequence of these commutation relations is the next theorem that should be seen as a generalisation of the second statement in the comparison Theorem 3.31.

4.6**. Theorem.** (Bochner-Kodaira-Nakano identity) Let (X,ω) be a Kähler manifold, and let (E,h) be a holomorphic Hermitian vector bundle over X. Then we have

$$\Delta_{\overline{\partial}_E} = [i\Theta_E, \Lambda] + \Delta_{\partial_E}.$$

Proof. By definition $\Delta_{\overline{\partial}_E} = \overline{\partial}_E\overline{\partial}^*_E + \overline{\partial}^*_E\overline{\partial}_E$, so by Formula (4.30)

$$\Delta_{\overline{\partial}_E} = [\overline{\partial}_E, \overline{\partial}^*_E] = -i[\overline{\partial}_E, [\Lambda, \partial_E]].$$

By the Jacobi identity (3.21)

$$-i[\overline{\partial}_E, [\Lambda, \partial_E]] = -i[\Lambda, [\partial_E, \overline{\partial}_E]] - i[\partial_E, [\overline{\partial}_E, \Lambda]],$$

so Formula (4.29) and $[\partial_E, \overline{\partial}_E] = \Theta_E$ imply that

$$\Delta_{\overline{\partial}_E} = -i[\Lambda, \Theta_E] + -i[\partial_E, i\partial^*_E] = [i\Theta_E, \Lambda] + \Delta_{\partial_E}.$$

□

4.7. **Corollary.** (Bochner-Kodaira-Nakano inequality) Let (X, ω) be a compact Kähler manifold, and let (E, h) be a holomorphic Hermitian vector bundle over X. Let $u \in C^\infty(X, \Omega_X^{p,q} \otimes E)$ be a (p,q)-form with values in E. Then we have

$$\int_X \{[i\Theta_E, \Lambda]u, u\} vol \leqslant (\Delta_{\overline{\partial}_E} u, u)_{L^2}.$$

In particular if u is $\Delta_{\overline{\partial}_E}$-harmonic, we have

$$\int_X \{[i\Theta_E, \Lambda]u, u\} vol \leqslant 0.$$

Proof. By the definition of the Laplacian and the adjoint property, we immediately get

$$(\Delta_{\partial_E} u, u)_{L^2} = ||\partial_E u||^2 + ||\partial_E^* u||^2 \geqslant 0.$$

Therefore by the Bochner-Kodaira-Nakano identity

$$(\Delta_{\overline{\partial}_E} u, u)_{L^2} = (\Delta_{\partial_E} u, u)_{L^2} + \int_X \{[i\Theta_E, \Lambda]u, u\} vol \geqslant \int_X \{[i\Theta_E, \Lambda]u, u\} vol.$$

□

4.8. **Theorem.** (Kodaira-Akizuki-Nakano vanishing theorem) Let (X, ω) be a compact Kähler manifold of dimension n, and let (L, h) be a positive line bundle on X. Then

$$H^{p,q}(X, L) = 0 \qquad \forall\ p + q \geqslant n + 1.$$

4.9. **Remark.** The most important case is when $p = n$. Then the statement simplifies to Kodaira's original statement

$$H^q(X, K_X \otimes L) = 0 \qquad \forall\ q > 0.$$

Proof. By Corollary 4.1 we have

$$H^{p,q}(X, L) \simeq \mathscr{H}^{p,q}(X, L),$$

where $\mathscr{H}^{p,q}(X, L)$ are the $\Delta_{\overline{\partial}_L}$-harmonic forms. So it sufficient that the $\Delta_{\overline{\partial}_L}$-harmonic forms are identically to zero. Let now $u \in C^\infty(X, \Omega_X^{p,q} \otimes L)$ be a (p,q)-form with values in L that is $\Delta_{\overline{\partial}_L}$-harmonic. By Corollary 4.7 we have

$$0 \geqslant \int_X \{[i\Theta_L, \Lambda]u, u\} vol,$$

so we are done if we show

$$\int_X \{[i\Theta_L, \Lambda]u, u\} vol \geqslant (p + q - n)||u||^2.$$

Technical remark: This inequality will be an immediate consequence of the *a-priori estimate* (4.31) which we will prove now. Note that the proof of Formula (4.31) does not use the hypothesis on the positivity of L and can also be adapted to the case of vector bundles.

It is certainly sufficient to establish this inequality pointwise, so fix a point $x \in X$. By Theorem 3.15 we can choose local holomorphic coordinates such that

$$\omega = i \sum_{1 \leqslant j \leqslant n} dz_j \wedge d\overline{z_j} + O(|z|^2).$$

As in the proof of Proposition 3.30 we see that in order to compute Λu in the point x, we can assume that

$$\omega = i \sum_{1 \leqslant j \leqslant n} dz_j \wedge d\overline{z_j},$$

hence

$$\Lambda u = i \sum_{1 \leqslant j \leqslant n} \frac{\partial}{\partial z_j} \rfloor \frac{\partial}{\partial \overline{z_j}} \rfloor u.$$

On the other hand we can simultaneously diagonalise Θ_L in x, so we get

$$i\Theta_L(x) = i \sum_{1 \leqslant k \leqslant n} \lambda_k dz_k \wedge d\overline{z_k}.$$

Then $\lambda_1(x), \ldots, \lambda_n(x)$ are the eigenvalues of the curvature tensor $i\Theta_L$ in x and without loss of generality we can suppose $\lambda_1(x) \leqslant \ldots \leqslant \lambda_n(x)$.

We clearly have

$$i\Theta_L \Lambda u = - \sum_{1 \leqslant j,k \leqslant n} \lambda_k dz_k \wedge d\overline{z_k} \wedge \frac{\partial}{\partial z_j} \rfloor \frac{\partial}{\partial \overline{z_j}} \rfloor u.$$

Furthermore by Formula (3.15)

$$\begin{aligned}
\Lambda(i\Theta_L u) &= \sum_{1 \leqslant j,k \leqslant n} i \frac{\partial}{\partial z_j} \rfloor \frac{\partial}{\partial \overline{z_j}} \rfloor (i\lambda_k dz_k \wedge d\overline{z_k} \wedge u) \\
&= - \sum_{1 \leqslant j,k \leqslant n} \lambda_k \frac{\partial}{\partial z_j} \rfloor \left[\frac{\partial}{\partial \overline{z_j}} \rfloor (dz_k \wedge d\overline{z_k}) \wedge u + dz_k \wedge d\overline{z_k} \wedge \frac{\partial}{\partial \overline{z_j}} \rfloor u \right] \\
&= - \sum_{1 \leqslant j,k \leqslant n} \lambda_k \frac{\partial}{\partial z_j} \rfloor \left[-\delta_{j,k} dz_k \wedge u + dz_k \wedge d\overline{z_k} \wedge \frac{\partial}{\partial \overline{z_j}} \rfloor u \right] \\
&= \sum_{1 \leqslant j \leqslant n} \lambda_j u - \sum_{1 \leqslant j \leqslant n} \lambda_j dz_j \wedge \frac{\partial}{\partial z_j} \rfloor u - \sum_{1 \leqslant j \leqslant n} \lambda_j d\overline{z_j} \wedge \frac{\partial}{\partial \overline{z_j}} \rfloor u \\
&\quad + \sum_{1 \leqslant j,k \leqslant n} \lambda_k dz_k \wedge d\overline{z_k} \wedge \frac{\partial}{\partial z_j} \rfloor \frac{\partial}{\partial \overline{z_j}} \rfloor u.
\end{aligned}$$

Thus we obtain

$$[i\Theta_L, \Lambda] u(x) = - \sum_{1 \leqslant j \leqslant n} \lambda_j u + \sum_{1 \leqslant j \leqslant n} \lambda_j dz_j \wedge \frac{\partial}{\partial z_j} \rfloor u + \sum_{1 \leqslant k \leqslant n} \lambda_k d\overline{z_k} \wedge \frac{\partial}{\partial \overline{z_k}} \rfloor u.$$

Let now e_1 be a local orthonormal frame for L, then locally

$$u = \sum_{|J|=p, |K|=q} u_{J,K} dz_J \wedge d\overline{z_K} \otimes e_1,$$

and we claim that

$$\{[i\Theta_L, \Lambda]u, u\}_x = \sum_{|J|=p, |K|=q} (\lambda_K - \lambda_{CJ})|u_{J,K}|^2$$

where $\lambda_K := \sum_{k \in K} \lambda_k$ and $\lambda_{CJ} := \sum_{j \notin J} \lambda_j$. Indeed

$$\sum_{1 \leqslant j \leqslant n} \lambda_j dz_j \wedge \frac{\partial}{\partial z_j} \rfloor u = \sum_{|J|=p, |K|=q} \sum_{j \in J} \lambda_j u_{J,K} dz_J \wedge d\overline{z_K} \otimes e_1$$

and

$$\sum_{1 \leqslant k \leqslant n} \lambda_k d\overline{z_k} \wedge \frac{\partial}{\partial \overline{z_k}} \rfloor u = \sum_{|J|=p, |K|=q} \sum_{k \in K} \lambda_k u_{J,K} dz_J \wedge d\overline{z_K} \otimes e_1,$$

thus we get finally that

$$\{[i\Theta_L, \Lambda]u, u\}_x = \sum_{|J|=p, |K|=q} \left(- \sum_{1 \leqslant j \leqslant n} \lambda_j + \sum_{j \in J} \lambda_j + \sum_{k \in K} \lambda_k \right) |u_{J,K}|^2,$$

which implies the claim. Since $\lambda_1 \leqslant \ldots \leqslant \lambda_n$, we have

$$\lambda_K - \lambda_{CJ} \geqslant \sum_{1 \leqslant k \leqslant q} \lambda_k - \sum_{1 \leqslant j \leqslant n-p} \lambda_{p+j}.$$

So we see that

$$\{[i\Theta_L, \Lambda]u, u\}_x \geqslant (\sum_{1 \leqslant k \leqslant q} \lambda_k - \sum_{1 \leqslant j \leqslant n-p} \lambda_{p+j})|u|_x^2. \tag{4.31}$$

Since L is a positive line bundle, the curvature tensor $i\Theta_L$ gives a Kähler form on X. Thus we could have supposed from the beginning that $\omega = i\Theta$ and $\lambda_1 = \ldots = \lambda_n = 1$. Therefore Formula (4.31) simplifies to

$$\{[i\Theta_L, \Lambda]u, u\}_x \geqslant (p + q - n)|u|_x^2.$$

The statement is now an immediate consequence. □

4.10**. Exercise.** We want to compute the cohomology of the line bundles $\mathscr{O}_{\mathbb{P}^n}(k)$ on $\mathbb{P}^n$ for arbitary $k \in \mathbb{Z}$.

a) Show that for $k \geqslant -n$,

$$H^q(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) = 0 \qquad \forall\ q \geqslant 1.$$

b) Use the Serre duality Theorem 4.3 to discuss the case $k \leqslant -(n+1)$. □

4.11**. Exercise.** A Fano manifold is a compact Kähler manifold such that the anticanonical bundle $K_X^* := \det T_X$ is a positive line bundle. Show that if X is a Fano manifold

$$H^i(X, \mathscr{O}_X) = 0 \qquad \forall\ i > 0.$$

□

4.12. **Exercise.** Let X be a complex projective variety, $L \to X$ a holomorphic positive line bundle, and let $\sigma \in \Gamma(X, E)$ be a global section. We suppose that the zero set $Y = \{x \in X \mid \sigma(x) = 0\}$ is smooth.

a) Show that if $\dim X \geqslant 2$, then Y is connected.

b) Give an example where X is a curve and Y is not connected. □

The Kodaira vanishing theorem generalises to Nakano positive vector bundles:

4.13. **Theorem.** (Nakano vanishing theorem) Let (X, ω) be a compact Kähler manifold of dimension n, and let E be a Nakano positive vector bundle on X. Then

$$H^q(X, K_X \otimes E) = 0 \qquad \forall\, q \geqslant 1.$$

4.14. **Example.** The Kodaira vanishing theorem does not generalise to Griffiths positive vector bundles: by Exercise 2.54 and the Euler sequence 1.1, the tangent bundle $T_{\mathbb{P}^n}$ of the projective space is Griffiths positive. Yet the long exact sequence associated to the Euler sequence tensored with $K_{\mathbb{P}^n}$ and Exercise 4.10 show that

$$H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \neq 0.$$

Since $\mathscr{O}_{\mathbb{P}^n}(1)^{n+1}$ is Nakano positive, this shows that the quotient of a Nakano positive vector bundle is not necessarily Nakano positive.

4.B. **Kodaira's embedding theorem.** We come to the last main result of these lectures.

4.15. **Theorem.** (Kodaira's embedding theorem) Let X be a compact complex manifold. The following statements are equivalent:

(1) X is projective, that is there exists a holomorphic embedding $\varphi : X \to \mathbb{P}^N$.
(2) There exists a Kähler form ω on X that is an integer class, i.e. is in the image of the morphism $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$.
(3) There exists a holomorphic line bundle $L \to X$ that is positive.

Sketch of the proof of Theorem 4.15, Part I. 1) ⇒ 2) is clear since the restriction of the Fubini-Study form ω_{FS} (Exercise 2.25) is an integer Kähler form.

2) ⇒ 3) is an immediate consequence of the Lefschetz Theorem on $(1,1)$-classes 2.30.

3) ⇒ 1) is the difficult second part, see page 100. □

Let us give an immediate application.

4.16. **Corollary.** Let X be a compact Kähler manifold such that $H^2(X, \mathscr{O}_X) = 0$. Then X is projective.

4.17. **Exercise.** Let (X, ω) be a compact Kähler manifold. Show that ω is harmonic. □

Proof. By hypothesis and Hodge duality $H^{2,0}(X) = H^{0,2}(X) = 0$, so by the Hodge decomposition theorem $H^{1,1}(X) = H^2(X,\mathbb{C}) \simeq H^2(X,\mathbb{Q}) \otimes \mathbb{C}$. Since $H^2(X,\mathbb{C}) \simeq \mathscr{H}^2(X)$, we can choose a basis $\alpha_1, \ldots, \alpha_m$ of $H^2(X,\mathbb{Q})$ such that the α_i are harmonic and of type $(1,1)$. Since the Kähler form ω is harmonic and real, we have

$$\omega = \sum_{j=1}^{m} \lambda_j \alpha_j$$

with $\lambda_j \in \mathbb{R}$. If we take $\mu_j \in \mathbb{Q}$ sufficiently close to λ_j, the Hermitian form corresponding to $\sum_{j=1}^m \mu_j \alpha_j$ is still positive definite, so $\sum_{j=1}^m \mu_j \alpha_j$ is a Kähler form. Up to multiplying with N sufficiently big and divisible, we have $\sum_{j=1}^m \mu_j \alpha_j \in H^2(X,\mathbb{Z})$. Conclude with Kodaira's Theorem 4.15. □

4.18**. Exercise.** A ruled surface is a compact Kähler manifold S that admits a submersion $f : S \to C$ onto a smooth curve such that all the fibres $S_c := f^{-1}(c)$ are isomorphic to $\mathbb{P}^1$. Show that a ruled surface is projective.

Hint: show that we have an exact sequence

$$0 \to T_{S_c} \to T_S|_{S_c} \to \mathscr{O}_{\mathbb{P}^1} \to 0.$$

Thus we have $K_S|_{S_c} \simeq \mathscr{O}_{\mathbb{P}^1}(-2)$. □

4.19**. Exercise.** (Push-forward of sheaves) Let X and Y be complex manifolds, and let $f : X \to Y$ be a holomorphic map. Let $\mathscr{F}$ be a sheaf of abelian groups on X, then we set

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$$

for every open subset $U \subset Y$. Show that $f_*\mathscr{F}$ is a sheaf of abelian groups on Y. □

Sketch of the proof of Theorem 4.15, Part II. Let L be a positive line bundle on X. The goal of the proof is to show the following three claims.

Claim 1. For sufficiently high $N \in \mathbb{N}$, the line bundle $L^{\otimes N}$ is globally generated, i.e. for every point $x \in X$ there exists a global section $s \in \Gamma(X, L^{\otimes N})$ such that $s(x) \neq 0$.

Claim 2. For sufficiently high $N \in \mathbb{N}$, the line bundle $L^{\otimes N}$ separates point, i.e. for every couple of points $x, y \in X$ there exists a global section $s \in \Gamma(X, L^{\otimes N})$ such that $s(x) \neq 0$ and $s(y) = 0$.

Claim 3. For sufficiently high $N \in \mathbb{N}$, the line bundle $L^{\otimes N}$ separates tangent vectors, i.e. for every point $x \in X$ and $u \in T_{X,x}$ there exists a global section $s \in \Gamma(X, L^{\otimes N})$ such that $s(x) = 0$ and $ds(u) \neq 0$.

Assuming these claims for the time being, let us show show how they imply the theorem. Choose $N \in \mathbb{N}$ such that all three claims hold. By the first claim, for $x \in X$ there exists $s \in \Gamma(X, L^{\otimes N})$ such that $s(x) \neq 0$. Therefore

$$H_x = \{s \in \Gamma(X, L^{\otimes N}) \mid s(x) = 0\}$$

is a hyperplane in $\Gamma(X, L^{\otimes N})$. Thus we can define a morphism

$$\varphi : X \to \mathbb{P}(\Gamma(X, L^{\otimes N})^*),\ x \mapsto H_x.$$

We can write this morphism locally around a point $x_0 \in X$ more explicitly if we choose a basis $s_0, \ldots, s_d$ of $\Gamma(X, L^{\otimes N})$. Up to renumerating we can suppose that $s_0(x_0) \neq 0$, so by continuity there exists an open neighbourhood U of x_0 such that $s_0(x) \neq 0$ for every $x \in U$. Thus

$$\frac{s_1}{s_0}, \ldots, \frac{s_d}{s_0} \in \Gamma(U, \mathscr{O}_U)$$

and one checks that

$$\varphi|_U : U \to \mathbb{P}(\Gamma(X, L^{\otimes N})^*),\ x \mapsto (1 : \frac{s_1}{s_0}(x) : \ldots : \frac{s_d}{s_0}(x)).$$

We leave it as an exercise to the reader to show that Claim 2 implies that φ is injective, and Claim 3 implies that φ is an immersion (cf. also [Har77, II, Prop.7.3]). Since X is compact, it is then clear that φ defines an embedding of X in $\mathbb{P}^n$.

We come to the proof of the claims. We will show Claim 1, and leave the other claims as an exercise to the reader. Fix a point $x \in X$ and denote by $\mathscr{I}_x$ its ideal sheaf. Consider the exact sequence

$$0 \to \mathscr{I}_x \otimes L^{\otimes N} \to L^{\otimes N} \to L^{\otimes N}|_x \to 0.$$

Taking the long exact sequence in cohomology we get

$$0 \to \Gamma(X, \mathscr{I}_x \otimes L^{\otimes N}) \to \Gamma(X, L^{\otimes N}) \to \mathbb{C} \to \check{\mathrm{H}}^1(X, \mathscr{I}_x \otimes L^{\otimes N}) \to \ldots$$

where we used that the restriction of the line bundle $L^{\otimes N}$ to the point x is the trivial bundle. Thus in order to see that $L^{\otimes N}$ is generated by global sections in x, it would be sufficient to show that for N sufficient high

$$(*) \qquad \check{\mathrm{H}}^1(X, \mathscr{I}_x \otimes L^{\otimes N}) = 0,$$

Since this implies that $L^{\otimes N}$ is generated by global sections in an open neighbourhood of x, the compactness of X implies that we can choose $N \in \mathbb{N}$ that works for every $x \in X$.

The vanishing property $(*)$ is indeed true, it is a special case of the Serre vanishing theorem and implicitly this it what the proof of Lemma 4.20 will show. Nevertheless we feel that it is a bit easier to show only that the inclusion

$$\Gamma(X, \mathscr{I}_x \otimes L^{\otimes N}) \hookrightarrow \Gamma(X, L^{\otimes N})$$

is strict. The argument goes as follows: let $\mu : X' \to X$ be the blow-up of X in the point x, then X' is a compact Kähler manifold (Exercise 2.49). Let E be the exceptional divisor, then $\mathscr{O}_{X'}(-E) \simeq \mathscr{I}_E$ and it is not hard to see that its push-forward (cf. Exercise 4.19) is the ideal sheaf of x. Therefore we get

$$\mu_*(\mathscr{O}_{X'}(-E) \otimes \mu^* L^{\otimes N}) \simeq \mathscr{I}_x \otimes L^{\otimes N}.$$

Analogously one sees that

$$\Gamma(X', \mathscr{O}_{X'}(-E) \otimes \mu^* L^{\otimes N}) = \Gamma(X, \mathscr{I}_x \otimes L^{\otimes N}),\ \Gamma(X', \mu^* L^{\otimes N}) = \Gamma(X, L^{\otimes N}).$$

Therefore it is sufficient to show that the inclusion

$$\Gamma(X', \mathscr{O}_{X'}(-E) \otimes \mu^* L^{\otimes N}) \hookrightarrow \Gamma(X', \mu^* L^{\otimes N})$$

is strict. Consider the exact sequence

$$0 \to \mathscr{I}_E \otimes \mu^* L^{\otimes N} \to \mu^* L^{\otimes N} \to \mu^* L^{\otimes N}|_E \to 0.$$

The restriction of the pull-back $\mu^* L^{\otimes N}$ to the fibre $E = \mu^{-1}(x)$ is trival, so we get

$$0 \to \Gamma(X, \mathscr{I}_E \otimes \mu^* L^{\otimes N}) \to \Gamma(X, \mu^* L^{\otimes N}) \to \mathbb{C} \to \check{\mathrm{H}}^1(X, \mathscr{I}_E \otimes L^{\otimes N}) \to \dots$$

By Lemma 4.20 below, we have $\check{\mathrm{H}}^1(X, \mathscr{I}_E \otimes L^{\otimes N}) = 0$ for N sufficiently high. ☐

4.20. **Lemma.** Let $\mu : X' \to X$ be the blow-up of a compact Kähler manifold of dimension n in a point $x \in X$, and denote by E the exceptional divisor. Let L be a positive line bundle on X, then

$$\check{\mathrm{H}}^1(X, \mathscr{I}_E \otimes \mu^* L^{\otimes N}) = 0 \tag{4.32}$$

for N sufficiently high.

Proof. We want to apply the Kodaira vanishing Theorem 4.8, so let us change slightly the formulation of the problem. We have $\mathscr{I}_E \simeq \mathscr{O}_{X'}(-E)$ and $K_{X'} = \mu^* K_X + (n-1)E$ by Exercise 2.49, so

$$\mathscr{I}_E \otimes \mu^* L^{\otimes N} \simeq K_{X'} \otimes \mathscr{O}_{X'}(-nE) \otimes \mu^*(K_X^* \otimes L^{\otimes N})$$

and we will show that for N sufficiently high, the line bundle

$$\mathscr{O}_{X'}(-nE) \otimes \mu^*(K_X^* \otimes L^{\otimes N})$$

is positive. Fix a metric h on L such that (L, h) is positive, and let h_X be any metric on K_X^*. We know by Exercise 2.27 that for N_1 sufficiently high, the curvature form

$$i(\Theta_{K_X^*, h_X} + \Theta_{L^{\otimes N_1}, h^{\otimes N_1}}) = i(\Theta_{K_X^*, h_X} + N_1 \Theta_{L,h})$$

is positive definite. In particular the pull-back $\mu^*(K_X \otimes L^{\otimes N_1})$ admits a metric with semi-positive curvature.

Consider now the restriction of the line bundle $\mathscr{O}_{X'}(-E)$ to E. By Exercise 2.49

$$\mathscr{O}_{X'}(-E)|_E \simeq \mathscr{O}_{\mathbb{P}^{n-1}}(1),$$

so we can endow $\mathscr{O}_{X'}(-E)|_E$ with the Fubini-Study metric. Fix now any metric h_E on $\mathscr{O}_{X'}(-E)$ that extends the Fubini-Study metric. For any $N_2 \in \mathbb{N}$, the curvature form Θ associated to the metric $h_E^{\otimes n} \otimes \mu^* h^{\otimes N_2}$ on $\mathscr{O}_{X'}(-nE) \otimes \mu^* L^{\otimes N_2}$ satisfies

$$\Theta(t,t) = n\Theta_{\mathscr{O}_{X'}(-E), h_E}(t,t) + N_2 \Theta_{L,h}(T_\mu(t), T_\mu(t)) \qquad \forall\ t \in T_{X',x}.$$

Fix now a point $x \in X$. Since $h_E|_E$ is the Fubini-Study metric, it is clear that

$$\Theta(t,t) = n\Theta_{\mathscr{O}_{X'}(-E), h_E}(t,t) > 0 \qquad \forall\ t \in T_{E,x}.$$

If $t \notin T_{E,x}$, we have $T_\mu(t) \neq 0$, so $\Theta_{L,h}(T_\mu(t), T_\mu(t)) > 0$. Hence for sufficiently high N_2, we get $\Theta(t,t) > 0$. This shows that Θ is positive definite on $T_{X',x}$, thus by continuity on $T_{X',x'}$ for x' in a small neighbourhood. Conclude with the compactness of X. ☐

4.21**. Remark.** Note that if X is a curve, then the blow-up in a point x is just the curve X itself. So Lemma 4.20 shows that if X compact complex curve, $x \in X$ a point, and L a positive line bundle on X, then

$$H^1(X, \mathscr{O}_X(x)^* \otimes L^N) = 0$$

for N sufficiently high.

4.C. **Sketch of proofs of results in Section 2.D.**

Proof of Theorem 2.39. Let A be a positive holomorphic line bundle on X. As in the proof of Kodaira's embedding theorem we see that for $N \gg 0$ we have

$$h^0(X, A^{\otimes N}) \neq 0,\ h^0(X, L \otimes A^{\otimes N}) \neq 0.$$

Thus there exists an effective divisor D_1 such that $A^{\otimes N} \simeq \mathscr{O}_X(D_1)$ and an effective divisor D_2 such that $L \otimes A^{\otimes N} \simeq \mathscr{O}_X(D_2)$. Thus we have $L \simeq \mathscr{O}_X(D_1 - D_2)$. □

Let us recall the Riemann-Roch theorem on curves.

4.22**. Theorem.** Let C be a compact curves, and let L be a holomorphic line bundle on C. Then we have

$$\chi(C, L) = 1 - g(C) + \int_C c_1(L).$$

Proof of Theorem 2.51. By Theorem 2.39 and Remark 2.40 we can write $L \simeq \mathscr{O}_X(C - E)$ with C and E smooth curves in X. We consider the exact sequences

$$0 \to \mathscr{O}_X(-E) \to \mathscr{O}_X \to \mathscr{O}_E \to 0$$

and

$$0 \to \mathscr{O}_X(-C) \to \mathscr{O}_X \to \mathscr{O}_C \to 0.$$

Tensoring both sequences with $\mathscr{O}_X(C)$ we obtain

$$0 \to \mathscr{O}_X(C) \otimes \mathscr{O}_X(-E) \simeq L \to \mathscr{O}_X(C) \to \mathscr{O}_X(C)|_E \to 0$$

and

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(C) \to \mathscr{O}_X(C)|_C \to 0.$$

By the additivity of the Euler characteristic for exact sequences this implies

$$\chi(X, L) + \chi(E, \mathscr{O}_E(C)) = \chi(X, \mathscr{O}_X) + \chi(C, \mathscr{O}_C(C)).$$

Yet by the Riemann-Roch theorem for curves 4.22 we have

$$\chi(E, \mathscr{O}_E(C)) = 1 - g(E) + \int_E c_1(\mathscr{O}_E(C))$$

and

$$\chi(C, \mathscr{O}_C(C)) = 1 - g(C) + \int_C c_1(\mathscr{O}_C(C)).$$

By Theorem 2.41 we have

$$\int_E c_1(\mathscr{O}_E(C)) = \int_X c_1(\mathscr{O}_X(E)) \wedge c_1(\mathscr{O}_X(C)) = C \cdot E$$

and analogously $\int_C c_1(\mathscr{O}_C(C)) = C^2$. Using the adjunction formula we see that

$$g(C) = \frac{1}{2}(K_X + C) \cdot C + 1$$

and

$$g(E) = \frac{1}{2}(K_X + E) \cdot E + 1.$$

Conclude with an elementary computation. $\square$

4.23**. Exercise.** Let X be a complex torus of dimension two. Let A be a positive line bundle on X. Show that $H^0(X, A) \neq 0$. $\square$

4.24**. Exercise.** Using the arguments from the Kodaira embedding theorem prove that a holomorphic line bundle L on a compact curve C is positive if and only if

$$\int_C c_1(L) > 0.$$

$\square$

Proof of Theorem 2.50. By Theorem 2.39 and Remark 2.40 we can write $L \simeq \mathscr{O}_X(C - E)$ with C and E smooth curves in X. We consider the exact sequence

$$0 \to \mathscr{O}_X(-C) \to \mathscr{O}_X \to \mathscr{O}_C \to 0.$$

Tensoring with $L^{\otimes m}$ we obtain an exact sequence

$$(*) \qquad 0 \to L^{\otimes m} \otimes \mathscr{O}_X(-C) \simeq L^{\otimes m-1} \otimes \mathscr{O}_X(-E) \to L^{\otimes m} \to L|_C^{\otimes m} \to 0.$$

We consider the exact sequence

$$0 \to \mathscr{O}_X(-E) \to \mathscr{O}_X \to \mathscr{O}_E \to 0.$$

Tensoring with $L^{\otimes m-1}$ we obtain an exact sequence

$$(**) \qquad 0 \to L^{\otimes m-1} \otimes \mathscr{O}_X(-E) \to L^{\otimes m-1} \to L|_E^{\otimes m-1} \to 0.$$

By hypothesis we have $L \cdot C > 0$ and $L \cdot E > 0$ so by Exercise 4.24 the line bundles $L|_E$ and $L|_C$ are positive. Arguing as in the proof of Lemma 4.20 we obtain that

$$H^1(C, L|_C^{\otimes m}) = 0, \qquad H^1(E, L|_E^{\otimes m-1}) = 0$$

for all $m \gg 0$. By the long exact sequences in cohomology associated to the exact sequences $(*)$ and $(**)$ we obtain that

$$H^2(X, L^{\otimes m}) \simeq H^2(X, L^{\otimes m-1} \otimes \mathscr{O}_X(-E)) \simeq H^2(X, L^{\otimes m-1})$$

for all $m \gg 0$. In particular we see that $h^2(X, L^{\otimes m})$ is a constant c for $m \gg 0$. In particular we have

$$\chi(X, L^{\otimes m}) = h^0(X, L^{\otimes m}) - h^1(X, L^{\otimes m}) + c.$$

By the Riemann-Roch theorem 2.51 we have

$$\chi(X, L^{\otimes m}) = \frac{m^2}{2} L^2 - \frac{m}{2} K_X \cdot L + \chi(X, \mathscr{O}_X),$$

so the hypothesis $L^2 > 0$ implies that $\chi(X, L^{\otimes m})$ goes to infinity for $m \to \infty$. Thus we see that for some $m \gg 0$ we have

$$h^0(X, L^{\otimes m}) \neq 0.$$

In order to simplify the notation in the rest of the proof we will suppose (without loss of generality) that $H^0(X, L) \neq 0$ and there exists a global section σ such that $d\sigma(x) \neq 0$ for every $x \in X$ such that $\sigma(x) = 0$. In particular

$$D := \{x \in X \mid \sigma(x) = 0\}$$

is a smooth curve. We have an exact sequence

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0.$$

Tensoring with $L^{\otimes m}$ we obtain an exact sequence

$$(***) \qquad 0 \to L^{\otimes m-1} \to L^{\otimes m} \to L|_D^{\otimes m} \to 0.$$

Since $L \cdot D > 0$ by hypothesis we see that $L|_D$ is positive, so

$$H^1(D, L|_D^{\otimes m}) = 0$$

for all $m \gg 0$. By the long exact sequences in cohomology associated to the exact sequence $(***)$ we obtain that the map

$$H^1(X, L^{\otimes m-1}) \to H^1(X, L^{\otimes m})$$

is surjective for all $m \gg 0$. In particular we see that $h^1(X, L^{\otimes m})$ is a constant d for $m \gg 0$. Yet this implies that $H^1(X, L^{\otimes m-1}) \to H^1(X, L^{\otimes m})$ is actually an isomorphism, so the restriction arrow

$$H^0(X, L^{\otimes m}) \to H^0(D, L|_D^{\otimes m})$$

is surjective for $m \gg 0$. Since $L|_D$ is positive, the line bundle $L|_D^{\otimes m}$ is globally generated for $m \gg 0$. Thus we obtain that $L^{\otimes m}$ is globally generated for $m \gg 0$. In particular we have a holomorphic map

$$\varphi : X \to \mathbb{P}^N$$

such that $L^{\otimes m} \simeq \varphi^* \mathscr{O}_{\mathbb{P}^N}(1)$. We claim that φ is finite onto its image: arguing by contradiction we suppose that there exists a curve $A \subset X$ such that $\varphi(A)$ is a point. Then we have

$$L^{\otimes m} \cdot A = \varphi^* \mathscr{O}_{\mathbb{P}^N}(1) \cdot A = 0,$$

a contradiction to our hypothesis. Yet once we know that φ is finite we can use the general machinery of algebraic geometry to show that for $m' \geqslant m \gg 0$ the line bundle $L^{\otimes m'}$ defines an embedding of X into the projective space. The isomorphism $L^{\otimes m'} \simeq \varphi^* \mathscr{O}_{\mathbb{P}^N}(1)$ then implies that L is a positive line bundle. $\square$

Appendix A. Hodge theory

by Olivier Biquard

This chapter is an introduction to Hodge theory, and more generally to the analysis on elliptic operators on compact manifolds. Hodge theory represents De Rham cohomology classes (that is topological objects) on a compact manifold by harmonic forms (solutions of partial differential equations depending on a Riemannian metric on the manifold). It is a powerful tool to understand the topology from the geometric point of view.

In this chapter we mostly follow reference [Dem96], which contains a complete concise proof of Hodge theory, as well as applications in Kähler geometry.

A.A. **The Hodge operator.** Let V be a n-dimensional oriented euclidean vector space (it will be later the tangent space of an oriented Riemannian n-manifold). Therefore there is a canonical volume element $\mathrm{vol} \in \Omega^n V$. The exterior product $\Omega^p V \wedge \Omega^{n-p} V \to \Omega^n V$ is a non degenerate pairing. Therefore, for a form $\beta \in \Omega^p V$, one can define $*\beta \in \Omega^{n-p} V$ by its wedge product with p-forms:

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \,\mathrm{vol} \tag{1.33}$$

for all $\alpha \in \Omega^p V$. The operator $* : \Omega^p \to \Omega^{n-p}$ is called the *Hodge $*$ operator*.

In more concrete terms, if $(e_i)_{i=1\dots n}$ is a direct orthonormal basis of V, then $(e^I)_{I \subset \{1,\dots,n\}}$ is an orthonormal basis of ΩV. One checks easily that

$$\begin{aligned} *1 &= \mathrm{vol}, \quad & *e^1 &= e^2 \wedge e^3 \wedge \cdots \wedge e^n, \\ *\,\mathrm{vol} &= 1, \quad & *e^i &= (-1)^{i-1} e^1 \wedge \cdots \wedge \widehat{e^i} \cdots e^n. \end{aligned}$$

More generally,

$$*e^I = \epsilon(I, \complement I) e^{\complement I}, \tag{1.34}$$

where $\epsilon(I, \complement I)$ is the signature of the permutation $(1, \dots, n) \to (I, \complement I)$.

A.1. **Exercise.** Suppose that in the basis (e_i) the quadratic form is given by the matrix $g = (g_{ij})$, and write the inverse matrix $g^{-1} = (g^{ij})$. Prove that for a 1-form $\alpha = \alpha_i e^i$ one has

$$*\alpha = (-1)^{i-1} g^{ij} \alpha_j e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n. \tag{1.35}$$

□

A.2. **Exercise.** Prove that $*^2 = (-1)^{p(n-p)}$ on Ω^p. □

If n is even, then $* : \Omega^{n/2} \to \Omega^{n/2}$ satisfies $*^2 = (-1)^{n/2}$. Therefore, if $n/2$ is even, the eigenvalues of $*$ on $\Omega^{n/2}$ are ± 1, and $\Omega^{n/2}$ decomposes accordingly as

$$\Omega^{n/2} = \Omega_+ \oplus \Omega_-. \tag{1.36}$$

The elements of Ω_+ are called *selfdual forms*, and the elements of Ω_- *antiselfdual forms*. For example, if $n = 4$, then $\Omega_\pm$ is generated by the forms

$$e^1 \wedge e^2 \pm e^3 \wedge e^4, \quad e^1 \wedge e^3 \mp e^2 \wedge e^4, \quad e^1 \wedge e^4 \pm e^2 \wedge e^3. \tag{1.37}$$

A.3. **Exercise.** If $n/2$ is even, prove that the decomposition (1.36) is orthogonal for the quadratic form $\Omega^{n/2} \wedge \Omega^{n/2} \to \Omega^n \simeq \mathbb{R}$, and

$$\alpha \wedge \alpha = \pm|\alpha|^2 \operatorname{vol} \qquad \text{if } \alpha \in \Omega_\pm. \tag{1.38}$$

□

A.4. **Exercise.** If u is an orientation-preserving isometry of V, that is $u \in SO(V)$, prove that u preserves the Hodge operator. This means the following: u induces an isometry of $V^* = \Omega^1$, and an isometry $\Omega^p u$ of $\Omega^p V$ defined by $(\Omega^p u)(x^1 \wedge \cdots \wedge x^p) = u(x^1) \wedge \cdots \wedge u(x^p)$. Then for any p-form $\alpha \in \Omega^p V$ one has

$$*(\Omega^p u)\alpha = (\Omega^{n-p} u) * \alpha.$$

This illustrates the fact that an orientation-preserving isometry preserves every object canonically attached to a metric and an orientation. □

A.B. **Adjoint operator.** Suppose (M^n, g) is an oriented Riemannian manifold, and $E \to M$ a unitary bundle. Then on sections of E with compact support, one can define the L^2 scalar product and the L^2 norm:

$$(s,t) = \int_M \langle s,t\rangle_E \operatorname{vol}^g, \qquad \|s\|^2 = \int_M \langle s,s\rangle_E \operatorname{vol}^g. \tag{1.39}$$

If E and F are unitary bundles and $P : \Gamma(E) \to \Gamma(F)$ is a linear operator, then a *formal adjoint of* P is an operator $P^* : \Gamma(F) \to \Gamma(E)$ satisfying

$$(Ps,t)_E = (s, P^*t)_F \tag{1.40}$$

for all sections $s \in C_c^\infty(E)$ and $t \in C_c^\infty(F)$.

A.5. **Example.** Consider the differential of functions,

$$d : C^\infty(M) \to C^\infty(\Omega^1).$$

Choose local coordinates (x^i) in an open set $U \subset M$ and suppose that the function f and the 1-form $\alpha = \alpha_i dx^i$ have compact support in U; write $\operatorname{vol}^g = \gamma(x) dx^1 \wedge \cdots \wedge dx^n$, then by integration by parts:

$$\begin{aligned} \int_M \langle df, \alpha\rangle \operatorname{vol}^g &= \int g^{ij} \partial_i f \alpha_j \gamma dx^1 \cdots dx^n \\ &= -\int f \partial_i (g^{ij} \alpha_j \gamma) dx^1 \cdots dx^n \\ &= -\int f \gamma^{-1} \partial_i (g^{ij} \alpha_j \gamma) \operatorname{vol}^g. \end{aligned}$$

It follows that

$$d^*\alpha = -\gamma^{-1} \partial_i (\gamma g^{ij} \alpha_j). \tag{1.41}$$

More generally, one has the following formula.

A.6. **Lemma.** The formal adjoint of the exterior derivative $d : \Gamma(\Omega^p M) \to \Gamma(\Omega^{p+1} M)$ is

$$d^* = (-1)^{np+1} * d * .$$

Proof. For $\alpha \in C_c^\infty(\Omega^p)$ and $\beta \in C_c^\infty(\Omega^{p+1})$ one has the equalities:

$$\begin{aligned}\int_M \langle d\alpha, \beta\rangle \operatorname{vol}^g &= \int_M du \wedge *v \\ &= \int_M d(u \wedge *v) - (-1)^p u \wedge d * v\end{aligned}$$

by Stokes theorem, and using exercice A.2:

$$\begin{aligned}&= (-1)^{p+1+p(n-p)} \int_M u \wedge * * d * v \\ &= (-1)^{pn+1} \int_M \langle u, *d * v\rangle \operatorname{vol}^g .\end{aligned}$$

□

A.7. **Remarks.** 1) If n is even then the formula simplifies to $d^* = - * d*$.

2) The same formula gives an adjoint for the exterior derivative $d^\nabla : \Gamma(\Omega^p \otimes E) \to \Gamma(\Omega^{p+1} \otimes E)$ associated to a unitary connection ∇ on a bundle E.

3) As a consequence, for a 1-form α with compact support one has

$$\int_M (d^*\alpha) \operatorname{vol}^g = 0 \tag{1.42}$$

since this equals $(\alpha, d(1)) = 0$.

A.8. **Exercise.** Suppose that (M^n, g) is a manifold with boundary. Note $\vec{n}$ is the normal vector to the boundary. Prove that (1.42) becomes:

$$\int_M (d^*\alpha) \operatorname{vol} = -\int_{\partial M} *\alpha = -\int_{\partial M} \alpha_{\vec{n}} \operatorname{vol}^{\partial M} . \tag{1.43}$$

□

For 1-forms we have the following alternative formula for d^*.

A.9. **Lemma.** Let E be a vector bundle with unitary connection ∇, then the formal adjoint of $\nabla : \Gamma(M, E) \to \Gamma(M, \Omega^1 \otimes E)$ is

$$\nabla^*\alpha = -\operatorname{Tr}^g(\nabla u) = -\sum_1^n (\nabla_{e_i}\alpha)(e_i).$$

Proof. Take a local orthonormal basis (e_i) of TM, and consider an E-valued 1-form $\alpha = \alpha_i e^i$. We have $*\alpha = (-1)^{i-1}\alpha_i e^1 \wedge \cdots \wedge \widehat{e^i} \wedge \cdots \wedge e^n$. One can suppose that just at the point p one has $\nabla e_i(p) = 0$, therefore $de^i(p) = 0$ and, still at the point p,

$$d^\nabla * \alpha = \sum_1^n (\nabla_i \alpha_i) e^1 \wedge \cdots \wedge e^n.$$

Finally $\nabla^*\alpha(p) = -\sum_1^n (\nabla_i \alpha_i)(p)$. □

A.10. **Remark.** Actually the same formula is also valid for p-forms. Indeed, $d^\nabla : \Gamma(M, \Omega^p) \to \Gamma(M, \Omega^{p+1})$ can be deduced from the covariant derivative $\nabla : \Gamma(M, \Omega^p) \to \Gamma(M, \Omega^1 \otimes \Omega^p)$ by the formula[17]

$$d^\nabla = (p+1)\mathbf{a} \circ \nabla,$$

where $\mathbf{a}$ is the antisymmetrization of a $(p+1)$-tensor. Also observe that if $\alpha \in \Omega^p \subset \otimes^p \Omega^1$, its norm as a p-form differs from its norm as a p-tensor by

$$|\alpha|^2_{\Omega^p} = p!|\alpha|^2_{\otimes^p \Omega^1}.$$

Putting together this two facts, one can calculate that d^* is the restriction of ∇^* to antisymmetric tensors in $\Omega^1 \otimes \Omega^p$. We get the formula

$$d^*\alpha = -\sum_1^n e_i \lrcorner \nabla_i \alpha. \tag{1.44}$$

Of course the formula remains valid for E-valued p-forms, if E has a unitary connection ∇.

A.11. **Exercise.** Consider the symmetric part of the covariant derivative,

$$\delta^* : \Gamma(\Omega^1) \to \Gamma(S^2\Omega^1).$$

Prove that its formal adjoint is the divergence δ, defined for a symmetric 2-tensor h by

$$(\delta h)_X = -\sum_1^n (\nabla_{e_i} h)(e_i, X).$$

□

A.C. **Hodge-de Rham Laplacian.**

A.12. **Definition.** Let (M^n, g) be an oriented Riemannian manifold. The *Hodge-De Rham Laplacian on p-forms* is defined by

$$\Delta\alpha = (dd^* + d^*d)\alpha.$$

Clearly, Δ is a formally selfadjoint operator. The definition is also valid for E-valued p-forms, using the exterior derivative d^∇, where E has a metric connection ∇.

A.13. **Example.** On functions $\Delta = d^*d$; using (1.41), we obtain the formula in local coordinates:

$$\Delta f = -\frac{1}{\sqrt{\det(g_{ij})}} \partial_i \big(g^{ij} \sqrt{\det(g_{ij})} \partial_j f\big). \tag{1.45}$$

In particular, for the flat metric $g = \sum_1^n (dx^i)^2$ of $\mathbb{R}^n$, one has

$$\Delta f = -\sum_1^n \partial_i^2 f.$$

[17] This formula is true as soon as ∇ is a torsion free connection on M.

In polar coordinates on $\mathbb{R}^2$, one has $g = dr^2 + r^2 d\theta^2$ and therefore

$$\Delta f = -\frac{1}{r}\partial_r(r\partial_r f) - \frac{1}{r^2}\partial_\theta^2 f.$$

More generally on $\mathbb{R}^n$ with polar coordinates $g = dr^2 + r^2 g_{S^{n-1}}$, one has

$$\Delta f = -\frac{1}{r^{n-1}}\partial_r(r^{n-1}\partial_r f) + \frac{1}{r^2}\Delta_{S^{n-1}} f.$$

Similarly, on the real hyperbolic space H^n with geodesic coordinates, $g = dr^2 + \sinh^2(r) g_{S^{n-1}}$ and the formula reads

$$\Delta f = -\frac{1}{\sinh(r)^{n-1}}\partial_r(\sinh(r)^{n-1}\partial_r f) + \frac{1}{r^2}\Delta_{S^{n-1}} f.$$

A.14. **Exercise.** On p-forms in $\mathbb{R}^n$ prove that $\Delta(\alpha_I dx^I) = (\Delta\alpha_I)dx^I$. $\square$

A.15. **Exercise.** Prove that $*$ commutes with Δ. $\square$

A.16. **Exercise.** If (M^n, g) has a boundary, prove that for two functions f and g one has

$$\int_M (\Delta f) g \operatorname{vol} = \int_M \langle df, dg\rangle \operatorname{vol} - \int_{\partial M} \frac{\partial f}{\partial \vec{n}} g \operatorname{vol}^{\partial M}.$$

Deduce

$$\int_M (\Delta f) g \operatorname{vol} = \int_M f\Delta g \operatorname{vol} + \int_{\partial M} \big(f\frac{\partial g}{\partial \vec{n}} - \frac{\partial f}{\partial \vec{n}} g\big) \operatorname{vol}^{\partial M}.$$

$\square$

A.17. **Exercise.** Prove that the radial function defined on $\mathbb{R}^n$ by (V_n being the volume of the sphere S^n)

$$G(r) = \begin{cases} \frac{1}{(n-2)V_{n-1}r^{n-2}} & \text{if } n > 2 \\ \frac{1}{2\pi}\log r & \text{if } n = 2 \end{cases}$$

satisfies $\Delta G = \delta_0$ (Dirac function at 0). Deduce the explicit solution of $\Delta f = g$ for $g \in C_c^\infty(\mathbb{R}^n)$ given by the integral formula

$$f(x) = \int_{\mathbb{R}^n} G(|x-y|)g(y)|dy|^n.$$

The function G is called Green's function.

Similarly, find the Green's function for the real hyperbolic space. $\square$

A.D. **Statement of Hodge theory.** Let (M^n, g) be a closed Riemannian oriented manifold. Consider the De Rham complex

$$0 \to \Gamma(\Omega^0) \xrightarrow{d} \Gamma(\Omega^1) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(\Omega^n) \to 0.$$

Remind that the De Rham cohomology in degree p is defined by $H^p = \{\alpha \in C^\infty(M, \Omega^p), d\alpha = 0\}/dC^\infty(M, \Omega^{p-1})$.

Other situation: (E, ∇) is a flat bundle, we have the associated complex

$$0 \to \Gamma(\Omega^0 \otimes E) \xrightarrow{d^\nabla} \Gamma(\Omega^1 \otimes E) \xrightarrow{d^\nabla} \cdots \xrightarrow{d^\nabla} \Gamma(\Omega^n \otimes E) \to 0$$

and we can define the De Rham cohomology with values in E in the same way.

In both cases, we have the Hodge-De Rham Laplacian $\Delta = dd^* + d^*d$.

A.18. **Definition.** A *harmonic form* is a C^∞ form such that $\Delta\alpha = 0$.

A.19. **Lemma.** If $\alpha \in C^\infty_c(M, \Omega^p)$, then α is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$. In particular, on a compact connected manifold, any harmonic function is constant.

Proof. It is clear that if $d\alpha = 0$ and $d^*\alpha = 0$, then $\Delta\alpha = d^*d\alpha + dd^*\alpha = 0$. Conversely, if $\Delta\alpha = 0$, because

$$(\Delta\alpha, \alpha) = (d^*d\alpha, \alpha) + (dd^*\alpha, \alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2,$$

we deduce that $d\alpha = 0$ and $d^*\alpha = 0$. □

A.20. **Remark.** The lemma remains valid on complete manifolds, for L^2 forms α such that $d\alpha$ and $d^*\alpha$ are also L^2. This is proved by taking cut-off functions χ_j, such that $\chi_j^{-1}(1)$ are compact domains which exhaust M, and $|d\chi_j|$ remains bounded by a fixed constant C. Then

$$\begin{aligned}\int_M \langle \Delta\alpha, \chi_j\alpha\rangle \operatorname{vol} &= \int_M \big(\langle d\alpha, d(\chi_j\alpha)\rangle + \langle d^*\alpha, d^*(\chi_j\alpha)\rangle\big) \operatorname{vol} \\ &= \int_M \big(\chi_j(|d\alpha|^2 + |d^*\alpha|^2) + \langle d\alpha, d\chi_j \wedge \alpha\rangle - \langle d^*\alpha, \nabla\chi_j \lrcorner \alpha\rangle\big) \operatorname{vol}\end{aligned}$$

Using $|d\chi_j| \leqslant C$ and taking j to infinity, one obtains $(\Delta\alpha, \alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2$.

Note $\mathbf{H}^p$ the space of harmonic p-forms on M. The main theorem of this section is:

A.21. **Theorem.** Let (M^n, g) be a compact closed oriented Riemannian manifold. Then:

(1) $\mathbf{H}^p$ is finite dimensional;
(2) one has a decomposition $C^\infty(M, \Omega^p) = \mathbf{H}^p \oplus \Delta(C^\infty(M, \Omega^p))$, which is orthogonal for the L^2 scalar product.

This is the main theorem of Hodge theory, and we will prove it later, as a consequence of theorem A.42. Just remark now that it is obvious that $\ker\Delta \perp \operatorname{im}\Delta$, because Δ is formally selfadjoint. Also, general theory of unbounded operators gives almost immediately that $L^2(M, \Omega^p) = \mathbf{H}^p \oplus \overline{\operatorname{im}\Delta}$. What is non trivial is: finite dimensionality of $\mathbf{H}^p$, closedness of $\operatorname{im}\Delta$, and the fact that smooth forms in the L^2 image of Δ are images of smooth forms.

Now we will derive some immediate consequences.

A.22. **Corollary.** Same hypothesis. One has the orthogonal decomposition

$$C^\infty(M, \Omega^p) = \mathbf{H}^p \oplus d\big(C^\infty(M, \Omega^{p-1})\big) \oplus d^*\big(C^\infty(M, \Omega^{p+1})\big),$$

where

$$\ker d = \mathbf{H}^p \oplus d\big(C^\infty(M, \Omega^{p-1})\big), \tag{1.46}$$

$$\ker d^* = \mathbf{H}^p \oplus d^*\big(C^\infty(M, \Omega^{p+1})\big). \tag{1.47}$$

Note that since harmonic forms are closed, there is a natural map $\mathbf{H}^p \to H^p$. The equality (1.46) implies immediately:

A.23. **Corollary.** Same hypothesis. The map $\mathbf{H}^p \to H^p$ is an isomorphism.

Using exercice A.15, we obtain:

A.24. **Corollary.**[Poincaré duality] Same hypothesis. The Hodge $*$ operator induces an isomorphism $* : \mathbf{H}^p \to \mathbf{H}^{n-p}$. In particular the corresponding Betti numbers are equal, $b_p = b_{n-p}$.

A.25. **Remark.** As an immediate consequence, if M is connected then $H^n = \mathbb{R}$ since $H^0 = \mathbb{R}$. Since $*1 = \mathrm{vol}^g$ and $\int_M \mathrm{vol}^g > 0$, an identification with $\mathbb{R}$ is just given by integration of n-forms on M.

A.26. **Remark.** In Kähler geometry there is a decomposition of harmonic forms using the (p, q) type of forms, $\mathbf{H}^k \otimes \mathbb{C} = \oplus_0^k \mathbf{H}^{p,k-p}$, and corollary A.24 can then be refined as an isomorphism $* : \mathbf{H}^{p,q} \to \mathbf{H}^{m-q,m-p}$, where $n = 2m$.

A.27. **Remark.** Suppose that n is a multiple of 4. Then by exercises A.3 and A.15, one has an orthogonal decomposition

$$\mathbf{H}^{n/2} = \mathbf{H}_+ \oplus \mathbf{H}_-. \tag{1.48}$$

Under the wedge product, the decomposition is orthogonal, $\mathbf{H}_+$ is positive and $\mathbf{H}_-$ is negative, therefore the signature of the manifold is (p, q) with $p = \dim \mathbf{H}_+$ and $q = \dim \mathbf{H}_-$.

A.28. **Exercise.** Suppose again that n is a multiple of 4. Note $d_\pm : \Gamma(\Omega^{n/2-1}) \to \Gamma(\Omega_\pm)$ the projection of d on selfdual or antiselfdual forms. Prove that on $(n/2-1)$-forms, one has $d_+^* d_+ = d_-^* d_-$. Deduce that the cohomology of the complex

$$0 \to \Gamma(\Omega^0) \xrightarrow{d} \Gamma(\Omega^1) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(\Omega^{n/2-1}) \xrightarrow{d_+} \Gamma(\Omega_+) \to 0 \tag{1.49}$$

is $\mathbf{H}^0$, $\mathbf{H}^1$, ..., $\mathbf{H}^{n/2-1}$, $\mathbf{H}_+$. □

A.29. **Exercise.** Using exercise A.14, calculate the harmonic forms and the cohomology of a flat torus $\mathbb{R}^n/\mathbb{Z}^n$. □

A.30. **Exercise.** Let (M, g) be a compact oriented Riemannian manifold.

1) If γ is an orientation-preserving isometry of (M, g) and α a harmonic form, prove that $\gamma^*\alpha$ is harmonic.

2) (requires some knowledge of Lie groups) Prove that if a *connected* Lie group G acts on M, then the action of G on $H^\bullet(M, \mathbb{R})$ given by $\alpha \to \gamma^*\alpha$ is trivial[18].

3) Deduce that harmonic forms are invariant under $\mathrm{Isom}(M, g)^o$, the connected component of the identity in the isometry group of M. Apply this observation to

[18] If ξ belongs to the Lie algebra of G and X_ξ is the associated vector field on M given by the infinitesimal action of G (that is defined by $X_\xi(x) = \frac{d}{dt} e^{t\xi} x|_{t=0}$), then one has $\frac{d}{dt}(e^{t\xi})^*\alpha|_{t=0} = \mathscr{L}_{X_\xi}\alpha = i_{X_\xi} d\alpha + d i_{X_\xi}\alpha$. Deduce that if α is closed, then the infinitesimal action of G on $H^\bullet(M, \mathbb{R})$ is trivial.

give a proof that the cohomology of the n-sphere vanishes in degrees $k = 1, \dots, n-1$ (prove that there is no $SO(n+1)$-invariant k-form on S^n using the fact that the representation of $SO(n)$ on $\Omega^k\mathbb{R}^n$ is irreducible and therefore has no fixed nonzero vector). $\square$

A.E. **Bochner technique.** Let (E, ∇) be a bundle equipped with a unitary connection over an oriented Riemannian manifold (M^n, g). Then $\nabla : \Gamma(E) \to \Gamma(\Omega^1 \otimes E)$ and we can define the *rough Laplacian* $\nabla^*\nabla$ acting on sections of E. Using a local orthonormal basis (e_i) of TM, from lemma A.9 it follows that

$$\nabla^*\nabla s = \sum_1^n -\nabla_{e_i}\nabla_{e_i} s + \nabla_{\nabla_{e_i} e_i} s. \tag{1.50}$$

If we calculate just at a point p and we choose a basis (e_i) which is parallel at p, then the second term vanishes.

In particular, using the Levi-Civita connection, we get a Laplacian $\nabla^*\nabla$ acting on p-forms. It is not equal to the Hodge-De Rham Laplacian, as follows from:

A.31. **Lemma.**[Bochner formula] Let (M^n, g) be an oriented Riemannian manifold. Then for any 1-form α on M one has

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha).$$

A.32. **Remark.** There is a similar formula (Weitzenböck formula) on p-forms: the difference $\Delta\alpha - \nabla^*\nabla\alpha$ is a zero-th order term involving the curvature of M.

Proof of the lemma. We have $d\alpha_{X,Y} = (\nabla_X\alpha)_Y - (\nabla_Y\alpha)_X$, therefore

$$d^*d\alpha_X = -\sum_1^n (\nabla_{e_i} d\alpha)_{e_i,X} = \sum_1^n -(\nabla_{e_i}\nabla_{e_i}\alpha)_X + (\nabla_{e_i}\nabla_X\alpha)_{e_i},$$

where in the last equality we calculate only at a point p, and we have chosen the vector fields (e_i) and X parallel at p.

Similarly, $d^*\alpha = -\sum_1^n (\nabla_{e_i}\alpha)_{e_i}$, therefore

$$dd^*\alpha_X = -\sum_1^n \nabla_X((\nabla_{e_i}\alpha)_{e_i}) = -\sum_1^n (\nabla_X\nabla_{e_i}\alpha)_{e_i}.$$

Therefore, still at the point p, comparing with (1.50),

$$(\Delta\alpha)_X = (\nabla^*\nabla\alpha)_X + \sum_1^n (R_{e_i,X}\alpha)_{e_i} = (\nabla^*\nabla\alpha)_X + \mathrm{Ric}(\alpha)_X. \tag{1.51}$$

$\square$

A.33. **Remark.** There is a similar formula if the exterior derivative is coupled with a bundle E equipped with a connection ∇. The formula for the Laplacian $\Delta = (d^\nabla)^* d^\nabla + d^\nabla (d^\nabla)^*$ becomes

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha) + \mathscr{R}^\nabla(\alpha), \tag{1.52}$$

where the additional last term involves the curvature of ∇,

$$\mathscr{R}^{\nabla}(\alpha)_X = \sum_1^n R^{\nabla}_{e_i,X}\alpha(e_i). \tag{1.53}$$

The proof is exactly the same as above, a difference arises just in the last equality of (1.51), when one analyses the curvature term: the curvature acting on α is that of $\Omega^1 \otimes E$, so equals $R \otimes 1 + 1 \otimes R^{\nabla}$, from which:

$$\sum_1^n (R_{e_i,X}\alpha)_{e_i} = \mathrm{Ric}(\alpha)_X + \sum_1^n R^{\nabla}_{e_i,X}\alpha(e_i).$$

Now let us see an application of the Bochner formula. Suppose M is compact. By Hodge theory, an element of $H^1(M)$ is represented by a harmonic 1-form α. By the Bochner formula, we deduce $\nabla^*\nabla\alpha + \mathrm{Ric}(\alpha) = 0$. Taking the scalar product with α, one obtains

$$\|\nabla\alpha\|^2 + (\mathrm{Ric}(\alpha), \alpha) = 0. \tag{1.54}$$

If $\mathrm{Ric} \geqslant 0$, this equality implies $\nabla\alpha = 0$ and $\mathrm{Ric}(\alpha) = 0$. If $\mathrm{Ric} > 0$, then $\alpha = 0$; if $\mathrm{Ric} \geqslant 0$ we get only that α is parallel, therefore the cohomology is represented by parallel forms. Suppose that M is connected, then a parallel form is determined by its values at one point p, so we get an injection

$$\mathbf{H}^1 \hookrightarrow \Omega^1_p.$$

Therefore $\dim \mathbf{H}^1 \leqslant n$, with equality if and only if M has a basis of parallel 1-forms. This implies that M is flat, and by Bieberbach's theorem that M is a torus. Therefore we deduce:

A.34. **Corollary.** If (M^n, g) is a compact connected oriented Riemannian manifold, then:

- if $\mathrm{Ric} > 0$, then $b_1(M) = 0$;
- if $\mathrm{Ric} \geqslant 0$, then $b_1(M) \leqslant n$, with equality if and only if (M, g) is a flat torus.

This corollary is a typical example of application of Hodge theory to prove vanishing theorems for the cohomology: one uses Hodge theory to represent cohomology classes by harmonic forms, and then a Weitzenböck formula to prove that the harmonic forms must vanish or be special under some curvature assumption. For examples in Kähler geometry see [Dem96].

A.F. **Differential operators.** A linear operator $P : \Gamma(M, E) \to \Gamma(M, F)$ between sections of two bundles E and F is a *differential operator of order* d if, in any local trivialisation of E and F over a coordinate chart (x^i), one has

$$Pu(x) = \sum_{|\alpha| \leqslant d} a^{\alpha}(x)\partial_{\alpha}u(x),$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multiindex with each $\alpha_i \in \{1 \ldots n\}$, $|\alpha| = k$, $\partial_\alpha = \partial_{\alpha_1} \ldots \partial_{\alpha_k}$, and $a^{\alpha}(x)$ is a matrix representing an element of $\mathrm{Hom}(E_x, F_x)$.

The *principal symbol* of P is defined for $x \in M$ and $\xi \in T_x^*M$ by taking only the terms of order d in P:

$$\sigma_P(x,\xi) = \sum_{|\alpha|=d} a^\alpha(x)\xi_\alpha,$$

where $\xi_\alpha = \xi_{\alpha_1}\cdots\xi_{\alpha_d}$ if $\xi = \xi_i dx^i$. It is a degree d homogeneous polynomial in the variable ξ with values in $\mathrm{Hom}(E_x, F_x)$.

A priori, it is not clear from the formula in local coordinates that the principal symbol is intrinsically defined. But it is easy to check that one has the following more intrinsic definition: suppose $f \in C^\infty(M)$, $t \in \mathbb{R}$ and $u \in \Gamma(M,E)$, then

$$e^{-tf(x)}P\big(e^{tf(x)}u(x)\big)$$

is a polynomial of degree d in the variable t, whose monomial of degree d is a homogeneous polynomial of degree d in $df(x)$. It is actually

$$t^d\sigma_P(x, df(x))u(x).$$

The following property of the principal symbol is obvious.

A.35. **Lemma.** $\sigma_{P\circ Q} = \sigma_P \circ \sigma_Q$.

A.36. **Examples.** 1) If one has a connection $\nabla : \Gamma(E) \to \Gamma(\Omega^1 \otimes E)$, then $e^{-tf}\nabla(e^{tf}u) = tdf \otimes u + \nabla u$. Therefore

$$\sigma_\nabla(x,\xi) = \xi\otimes : \Gamma(E_x) \to \Gamma(\Omega^1_x \otimes E_x).$$

2) The principal symbol of the exterior derivative $d^\nabla : \Gamma(\Omega^p \otimes E) \to \Gamma(\Omega^{p+1} \otimes E)$ is $\sigma_d(x,\xi) = \xi\wedge$.

3) The principal symbol of $d^* : \Gamma(\Omega^{p+1} \otimes E) \to \Gamma(\Omega^p \otimes E)$ is $\sigma_{d^*}(x,\xi) = -\xi\lrcorner$.

4) The principal symbol of the composite $\nabla^*\nabla$ is the composite $-(\xi\lrcorner) \circ (\xi\otimes) = -|\xi|^2$.

A.37. **Exercise.** Prove that the principal symbol of the Hodge-De Rham Laplacian is also $\sigma_\Delta(x,\xi) = -|\xi|^2$. □

A.38. **Lemma.** Any differential operator $P : \Gamma(E) \to \Gamma(F)$ of order d has a formal adjoint P^*, whose principal symbol is

$$\sigma_{P^*}(x,\xi) = (-1)^d\sigma_P(x,\xi)^*.$$

A.39. **Exercise.** Prove the lemma in the following way. In local coordinates, write $\mathrm{vol}^g = v(x)dx^1 \wedge \cdots \wedge dx^n$. Choose orthonormal trivialisations of E and F, and write $P = \sum a^\alpha(x)\partial_\alpha$. Then prove that

$$P^*t = \sum_{|\alpha|\leqslant d} (-1)^{|\alpha|}\frac{1}{v(x)}\partial_\alpha\big(v(x)a^\alpha(x)^*t\big).$$

The proof is similar to that in example A.5. □

A.40. **Remark.** In analysis, the principal symbol is often defined slightly differently: ξ_j corresponds to $D_j = \frac{1}{i}\frac{\partial}{\partial x^j}$. The advantage is that D_j is formally selfadjoint, so with this definition the principal symbol of P^* is always $\sigma_P(x,\xi)^*$ and the principal symbol of the Laplacian becomes positive.

A.41. **Definition.** A differential operator $P : \Gamma(E) \to \Gamma(F)$ is *elliptic* if for any $x \in M$ and $\xi \neq 0$ in T_xM, the principal symbol $\sigma_P(x,\xi) : E_x \to F_x$ is injective.

Here is our main theorem on elliptic operators. It will be proved in section A.H.

A.42. **Theorem.** Suppose (M^n, g) is a compact oriented Riemannian manifold, and $P : \Gamma(E) \to \Gamma(F)$ is an elliptic operator, with $\operatorname{rank} E = \operatorname{rank} F$. Then

(1) $\ker(P)$ is finite dimensional;
(2) there is a L^2 orthogonal sum
$$C^\infty(M,F) = \ker(P^*) \oplus P\big(C^\infty(M,E)\big).$$

The Hodge theorem A.21 follows immediately, by applying to the Hodge-De Rham Laplacian Δ.

Remark that $\ker(P^*)$ is also finite dimensional, since P^* is elliptic if P is elliptic. The difference $\dim\ker P - \dim\ker P^*$ is the *index* of P, defined by
$$\operatorname{ind}(P) = \dim\ker P - \dim\operatorname{coker} P.$$
Operators with finite dimensional kernel and cokernel are called *Fredholm operators*, and the index is invariant under continuous deformation among Fredholm operators. Since ellipticity depends only on the principal symbol, it follows immediately that the index of P depends only on σ_P. The fundamental index theorem of Atiyah-Singer gives a topological formula for the index, see the book [BGV04].

A useful special case is that of a formally selfadjoint elliptic operator. Its index is of course zero. The invariance of the index then implies that any elliptic operator with the same symbol (or whose symbol is a deformation through elliptic symbols) has also index zero.

A.G. **Basic elliptic theory.** In this section we explain the basic results enabling to prove theorem A.42.

Sobolev spaces. The first step is to introduce the Sobolev space $H^s(\mathbb{R}^n)$ of tempered distributions f on $\mathbb{R}^n$ such that the Fourier transform satisfies

$$\|f\|_s^2 := \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s |d\xi|^n < +\infty. \tag{1.55}$$

Equivalently, $H^s(\mathbb{R}^n)$ is the space of functions $f \in L^2(\mathbb{R}^n)$ which admit s derivatives in distribution sense[19] in L^2, and

$$\|f\|_s^2 \sim \sum_{|\alpha|\leqslant s} \|\partial_\alpha s\|_{L^2}^2. \tag{1.56}$$

[19] Weak derivative: $g = D_\alpha f$ if for any $\phi \in C_c^\infty(\mathbb{R}^n)$ one has $\int_{\mathbb{R}^n}(D_\alpha\phi)f = \int_{\mathbb{R}^n}\phi g$.

(But observe that the definition (1.55) is valid also for any real s).

If M is a compact manifold and E a vector bundle over M, then one can define the space $C^k(M,E)$ of sections of E whose coefficients are of class C^k in any trivialisation of E, and $H^s(M,E)$ the space of sections of E whose coefficients in any trivialisation and any coordinate chart are functions of class H^s in $\mathbb{R}^n$. If M is covered by a finite number of charts (U_j) with trivialisations of $E|_{U_j}$ by a basis of sections $(e_{j,a})_{a=1,\dots,r}$, choose a partition of unity (χ_j) subordinate to (U_j), then a section u of E can be written $u=\sum \chi_j u_{j,\alpha}e_{j,\alpha}$ with $\chi_j u_{j,\alpha}$ a function with compact support in $U_j\subset\mathbb{R}^n$, therefore

$$\|u\|_{C^k}=\sup_{j,\alpha}\|\chi_j u_{j,\alpha}\|_{C^k(\mathbb{R}^n)},\quad \|u\|_s^2=\sum\|\chi_j u_{j,\alpha}\|^2_{H^s(\mathbb{R}^n)}. \tag{1.57}$$

Up to equivalence of norms, the result is independent of the choice of coordinate charts and trivialisations of E.

There is another approach to define C^k and H^s norms for sections of E. Suppose that M^n has a Riemannian metric, and E is equipped with a unitary connection ∇. Then one can define

$$\|u\|_{C^k}=\sup_{j\leqslant k}\sup_M|\nabla^j u|,\quad \|u\|_s=\sum_0^k\int_M|\nabla^j u|^2\operatorname{vol}^g. \tag{1.58}$$

A.43. **Remark.** On a noncompact manifold, the definition (1.57) does not give a well defined class of equivalent norms when one changes the trivialisations. On the contrary, definition (1.58), valid only for integral s, can be useful if (M,g) is non compact; the norms depend on the geometry at infinity of g and ∇.

A.44. **Example.** If M is a torus T^n, then the regularity can be seen on the Fourier series: $f\in H^s(T^n)$ if and only if

$$\|f\|_s^2=\sum_{\xi\in\mathbb{Z}^n}(1+|\xi|^2)^s|\hat f(\xi)|^2<+\infty.$$

From the inverse formula $f(x)=\sum_\xi\hat f(\xi)\exp(i\langle\xi,x\rangle)$, by the Cauchy-Schwartz inequality,

$$|f(x)|\leqslant\sum_{\xi\in\mathbb{Z}^n}|\hat f(\xi)|\leqslant\|f\|_s\big(\sum_\xi(1+|\xi|^2)^{-s}\big)^{1/2}<+\infty\quad\text{if } s>\frac{n}{2}.$$

It follows that there is a continuous inclusion $H^s\subset C^0$ is $s>\frac{n}{2}$. Similarly it follows that $H^s\subset C^k$ if $s>k+\frac{n}{2}$.

Of course the same results are true on $\mathbb{R}^n$ using Fourier transform, and one obtains the following lemma.

A.45. **Lemma.**[Sobolev] If M^n is compact, $k\in\mathbb{N}$ and $s>k+\frac{n}{2}$, then there is a continuous and compact injection $H^s\subset C^k$.

The fact that the inclusion is compact follows from the following lemma (which is obvious on a torus, and the general case follows):

A.46. **Lemma.**[Rellich] If M^n is compact and $s > t$, then the inclusion $H^s \subset H^t$ is compact.

Action of differential operators. If $P : \Gamma(M, E) \to \Gamma(M, F)$ is a differential operator of order d, then looking at P in local coordinates it is clear that P induces continuous operators $P : H^{s+d}(M, E) \to H^s(M, F)$.

In general a *weak solution* of the equation $Pu = v$ is a L^2 section u of E such that for any $\phi \in C_c^\infty(M, F)$ one has

$$(u, P^*\phi) = (v, \phi).$$

We can now state the main technical result in this section.

A.47. **Lemma.**[Local elliptic estimate] Let $P : \Gamma(M, E) \to \Gamma(M, F)$ be an elliptic operator. Fix a ball B in a chart with local coordinates (x^i) and the smaller ball $B_{1/2}$. Suppose that $u \in L^2(B, E)$ and $Pu \in H^s(B, F)$, then $u \in H^{s+d}(B_{1/2}, E)$ and

$$\|u\|_{H^{s+d}(B_{1/2})} \leqslant C\big(\|Pu\|_{H^s(B)} + \|u\|_{L^2(B)}\big). \tag{1.59}$$

A.48. **Remark.** An important addition to the lemma is the fact that for a family of elliptic operators with bounded coefficients and bounded inverse of the principal symbol, one can take the constant C to be uniform.

A.49. **Remark.** Elliptic regularity is not true in C^k spaces, that is $Pu \in C^k$ does not imply $u \in C^{k+d}$ in general.

We will not prove lemma A.47, which is a difficult result. There are basically two ways to prove it. The first way is to locally approximate the operator on small balls by an operator with constant coefficients on $\mathbb{R}^n$ or T^n, where an explicit inverse is available using Fourier transform: one then glues together these inverses to get an approximate inverse for P which will give what is needed on u. See [War83] for this method. The second way is more modern and uses microlocal analysis: one inverts the operator "microlocally", that is fiber by fiber on each cotangent space—this is made possible by the theory of pseudodifferential operators. See a nice and concise introduction in [Dem96].

This implies immediately the following global result:

A.50. **Corollary.**[Global elliptic estimate] Let $P : \Gamma(M, E) \to \Gamma(M, F)$ be an elliptic operator. If $u \in L^2(M, E)$ and $Pu \in H^s(M, F)$, then $u \in H^{s+d}(M, E)$ and

$$\|u\|_{s+d} \leqslant C\big(\|Pu\|_s + \|u\|_{L^2}\big). \tag{1.60}$$

From the elliptic estimate and the fact that $\cap_s H^s = C^\infty$, we obtain:

A.51. **Corollary.** If P is elliptic and $Pu = 0$, then u is smooth. More generally, if Pu is C^∞ then u is C^∞.

A.52. **Exercise.** Prove (1.60) for operators with constant coefficients on the torus.
□

A.H. **Proof of the main theorem.** We can now prove theorem A.42.

First let us prove the first statement: the kernel of P is finite dimensional. By the elliptic estimate (1.60), for $u \in \ker(P)$ one has

$$\|u\|_{s+d} \leqslant C\big\|u\|_{L^2}.$$

Therefore the first identity map in the following diagram is continuous:

$$(\ker P, L^2) \longrightarrow (\ker P, H^{s+d}) \longrightarrow (\ker P, L^2).$$

The second inclusion is compact by lemma A.46. The composite map is the identity of $\ker P$ equipped with the L^2 scalar product, it is therefore a compact map. This implies that the closed unit ball of $\ker(P)$ is compact, therefore $\ker(P)$ is a finite dimensional vector space.

Now let us prove the theorem in Sobolev spaces. We consider P as an operator

$$P : H^{s+d}(M, E) \longrightarrow H^s(M, F), \tag{1.61}$$

and in these spaces we want to prove

$$H^s(M, F) = \ker(P^*) \oplus \operatorname{im}(P). \tag{1.62}$$

We claim that for any $\epsilon > 0$, there exists an L^2 orthonormal family $(v_1, \ldots, v_N)$ in H^{s+d}, such that

$$\|u\|_{L^2} \leqslant \epsilon\|u\|_{s+d} + \big(\sum_1^N |(v_j, u)|^2\big)^{1/2}. \tag{1.63}$$

Suppose for the moment that the claim is true. Then combining with the elliptic estimate (1.60), we deduce

$$(1 - C\epsilon)\|u\|_{s+d} \leqslant C\|Pu\|_s + C\big(\sum_1^N |(v_j, u)|^2\big)^{1/2}.$$

Choose $\epsilon = \frac{1}{2C}$, and let T be the subspace of sections in $H^{s+d}(M, E)$ which are L^2 orthogonal to the $(v_i)_{i=1\ldots N}$. Then we obtain

$$2\|u\|_{s+d} \leqslant C\|Pu\|_s \quad \text{for } u \in T.$$

It follows that $P(T)$ is closed in $H^s(M, F)$. But $\operatorname{im}(P)$ is the sum of $P(T)$ and the image of the finite dimensional space generated by the $(v_i)_{i=1\ldots N}$, so $\operatorname{im}(P)$ is closed as well in $H^s(M, F)$.

Finally the statement (1.61) in the Sobolev spaces H^s implies the statement for the space C^∞, which finishes the proof of the theorem. Indeed, suppose that $v \in C^\infty(M, F)$ is L^2 orthogonal to $\ker(P^*)$. Fix any $s \geqslant 0$ and apply (1.62) in H^s: therefore there exists $u \in H^{s+d}(M, E)$ such that $Pu = v$. Then u is C^∞ by corollary A.51.

It remains to prove the claim (1.63). Choose a Hilbertian basis (v_j) of L^2, and suppose that the claim is not true. Then there exists a sequence of $(u_N) \in H^{s+d}(M, E)$ such that

(1) $\|u_N\|_{L^2} = 1$,

(2) $\epsilon \|u_N\|_{s+d} + \left(\sum_1^N |(v_j, u_N)|^2\right)^{1/2} < 1.$

From the second condition we deduce that (u_N) is bounded in $H^{s+d}(E)$, therefore there is a weakly convergent subsequence in $H^{s+d}(E)$, and the limit satisfies

$$\epsilon \|u\|_{s+d} + \|u\|_0 \leqslant 1.$$

By the compact inclusion $H^{s+d} \subset L^2$ this subsequence is strongly convergent in $L^2(E)$ so by the first condition, the limit u satisfies

$$\|u\|_0 = 1,$$

which is a contradiction.

A.53. **Remark.** The same proof applies for an elliptic operator $P : \Gamma(E) \to \Gamma(F)$ where the ranks of E and F are not the same. The results are

(1) $\ker P$ is finite dimensional (this can be also obtained by identifying $\ker P$ with $\ker P^*P$, and by noting that P^*P is elliptic if P is elliptic);
(2) the image of the operator $P : H^{s+d}(M,E) \to H^s(M,F)$ is closed, and there is a L^2 orthogonal decomposition $H^s(M,F) = \ker P^* \oplus \operatorname{im} P$; note that here $\ker P^*$ depend on s as P^* is not elliptic if $\operatorname{rank} F > \operatorname{rank} E$.

References

[BGV04] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004.

[Biq08] Olivier Biquard. *Introduction to differential geometry*. Lecture Notes UPMC, http://people.math.jussieu.fr/ biquard/poly.html, 2008.

[BL04] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2004.

[Dem96] Jean-Pierre Demailly. *Complex Analytic and Differential Geometry*. http://www-fourier.ujf-grenoble.fr/ demailly/books.html, 1996.

[Eke06] Torsten Ekedahl. *One semester of elliptic curves*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2006.

[Fis76] Gerd Fischer. *Complex analytic geometry*. Lecture Notes in Mathematics, Vol. 538. Springer-Verlag, Berlin, 1976.

[For91] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[Gun90] Robert C. Gunning. *Introduction to holomorphic functions of several variables*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990. Function theory.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[KK83] Ludger Kaup and Burchard Kaup. *Holomorphic functions of several variables*, volume 3 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1983.

[LT97] Christine Laurent-Thiébaut. *Théorie des fonctions holomorphes de plusieurs variables*. Savoirs Actuels. InterEditions, Paris, 1997. Mathématiques.

[Voi02] Claire Voisin. *Théorie de Hodge et géométrie algébrique complexe*, volume 10 of *Cours Spécialisés*. Société Mathématique de France, Paris, 2002.

[War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983.

[Wel80] R. O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1980.

UPMC Université Paris 06, UMR 7586, Institut de Mathématiques de Jussieu

E-mail address: `hoering@math.jussieu.fr`