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INTRODUCTION TO SHEAF COHOMOLOGY

LORING W. TU

CONTENTS

Lecture 1. Presheaves and Sheaves	4
1. Presheaves	4
2. The Stalk of a Presheaf	5
3. Sheaves	6
4. The Sheaf Associated to a Presheaf	7
5. Sheaf Morphisms	9
6. Exact Sequences of Sheaves	9
7. Resolutions	11
Lecture 2. Sheaf Cohomology	11
8. De Rham Cohomology	11
9. The Godement Canonical Functors	12
10. Sheaf Cohomology	13
11. Flasque Sheaves	13
12. Fine Sheaves	14
Lecture 3. Hypercohomology and Spectral Sequences	16
13. Cohomology of a Double Complex	16
14. Cohomology Sheaves	17
15. Filtrations	17
16. Spectral Sequences	18
17. Hypercohomology of a Complex of Sheaves	19
Lecture 4. Applications	20
18. The de Rham Theorem	20
19. The de Rham Theorem with Complex Coefficients	22
20. The Analytic de Rham Theorem	23
21. Acyclic Resolutions	24

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References

This is a more or less verbatim account of my lectures at the CIMPA Summer School on Algebraic Geometry and Number Theory in Istanbul, Turkey, in June 2014. My goal was to introduce to the uninitiated, in just four lectures, the wonderful techniques of sheaf cohomology, hypercohomology, and spectral sequences. Because of the diversity of the background of the audience, I decided to start at the beginning and to assume no knowledge of sheaves. Indeed, since there exists a multitude of approaches to sheaves, it may be desirable to pick one out that seems to me the best because of its simplicity.

As prerequisites, I assume that the students have a good knowledge of manifolds, including the exterior calculus of differential forms, as in [10]. Our main reference for sheaf cohomology is Part I of the article "From sheaf cohomology to the algebraic de Rham theorem" [2] that Fouad El Zein and I wrote. However, because I had only four lectures at the CIMPA School, I tried to move at a fast clip, omitting some of the proofs. Most of the omitted proofs can be found in [2].

Introduced in 1946 by Jean Leray and further developed in subsequent years by Henri Cartan ([6] and [7]), sheaves are a powerful tool for relating local and global phenomena on a space. Smooth differential forms of a given degree define a sheaf on a manifold. On a complex manifold, in addition to smooth differential forms, there are holomorphic differential forms that also define sheaves on the manifold.

Sheaf cohomology may be viewed as a generalization of singular cohomology from constant coefficients to variable coefficients. Sheaf cohomology has become absolutely essential in modern algebraic geometry as well as certain areas of topology and complex analysis. For example, cohomology with coefficients in the sheaf Ω^p of holomorphic *p*-forms on a complex manifold gives invariants of the complex structure.

Hypercohomology is a generalization of sheaf cohomology from one sheaf to a complex of sheaves. It is a functor from the category of complexes of sheaves to the category of abelian groups. There are two spectral sequences that converge to hypercohomology. As we will see, the isomorphism of the limits of the two spectral sequences often yields important isomorphism theorems in complex and algebraic geometry.

On a smooth manifold M, a smooth k-form ω can be integrated over a continuous k-chain to yield a real number. Thus, $\int_{(\cdot)} \omega$ is a k-cochain on M. Let $\mathcal{A}^k(M)$ be the group of C^{∞} k-forms on M and $S^k(M,\mathbb{R})$ the group of continuous real singular k-cochains on M. By Stokes' theorem, the map

$$arphi \colon \mathcal{A}^k(M) o S^k(M,\mathbb{R}),$$
 $oldsymbol{\omega} \mapsto \int_{(\cdot)} oldsymbol{\omega},$

satisfies

$$\varphi(d\omega) = \int_{(\)} d\omega = \int_{\partial(\)} \omega = \delta \int_{(\)} \omega = \delta(\varphi\omega),$$

where δ is the coboundary operator on $S^k(M,\mathbb{R})$. Thus, φ is a cochain map and induces a linear map

$$\varphi^* \colon H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{sing}}(M,\mathbb{R})$$

from de Rham cohomology $H^k_{dR}(M)$ to singular cohomology $H_{sing}(M,\mathbb{R})$. The de Rham theorem states that the induced map φ^* in cohomology is an isomorphism. We will show how to use sheaf cohomology, hypercohomology, and spectral sequences to prove the de Rham theorem. An immediate consequence of the de Rham theorem is that although de Rham cohomology is defined in terms of smooth forms, it is actually a topological invariant, since singular cohomology is defined in the continuous category.

A differential form ω on a manifold M is *smooth* if locally it can be written as $\omega = \sum a_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where (U, x^1, \dots, x^n) is a chart and the a_I are C^{∞} functions on U. On a complex manifold, one can define similarly *holomorphic forms*: a differential form ω on a complex manifold is *holomorphic* if locally it can be written as $\omega = \sum a_I dz^{i_1} \wedge \cdots \wedge dz^{i_k}$, where (U, z^1, \dots, z^n) is a complex chart and the a_I are holomorphic functions on U.

Let $\Omega^k(M)$ be the group of holomorphic *k*-forms on *M*. The holomorphic forms on *M* constitute a *differential complex*, called the *holomorphic de Rham complex*,

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \to \Omega^n(M) \to 0,$$

in the sense that $d \circ d = 0$. One might ask if the singular cohomology of a complex manifold with complex coefficients can be computed using only holomorphic forms. According to the analytic de Rham theorem, the answer is yes, but it is not as simple as taking the cohomology of the holomorphic de Rham complex. Here the hypercohomology of the complex Ω^{\bullet} of sheaves of holomorphic forms on *M* makes its appearance:

$$H^*_{\mathrm{sing}}(M,\mathbb{C})\simeq \mathbb{H}^*(M,\Omega^{\bullet})$$

In case *M* is a Stein manifold (a complex manifold that is a closed subset of \mathbb{C}^N), then a simpler form of analytic de Rham theorem holds:

 $H^*_{\text{sing}}(M,\mathbb{C}) \simeq h^*(\Omega^{\bullet}(M)) := \text{cohomology of the holomorphic de Rham complex.}$

In the algebraic category, a differential *k*-form ω on a smooth algebraic variety is *algebraic* if locally it can be written as $\omega = \sum f_I dg^{i_1} \wedge \cdots \wedge dg^{i_k}$, where the f_I and g^j are regular functions. If *X* is a smooth complex algebraic variety with the Zariski topology, let $X(\mathbb{C})$ be the set of complex points with the complex topology. Grothendieck's algebraic de Rham theorem states that the singular cohomology $H^*_{sing}(X(\mathbb{C}),\mathbb{C})$ with complex coefficients can be computed from the algebraic differential forms on *X*:

$$H^*_{\operatorname{sing}}(X(\mathbb{C}),\mathbb{C})\simeq \mathbb{H}^*(X,\Omega^{\bullet}_{\operatorname{alg}}),$$

where $\Omega_{\text{alg}}^{\bullet}$ is the complex of sheaves of algebraic forms on *X*. For a smooth affine variety *X*, the singular cohomology $H_{\text{sing}}^*(M, \mathbb{C})$ is the cohomology of the algebraic de Rham complex:

$$H^*_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{C})\simeq h^*(\Omega^{ullet}_{\mathrm{alg}}(X)).$$

In this short article we will not be able to prove the algebraic de Rham theorem; for a proof, see [2]. We will give an introduction to sheaf cohomology, hypercohomology, and spectral sequences, enough to prove the de Rham theorem for smooth manifolds and the analytic de Rham theorem for complex manifolds.

Lecture 1. Presheaves and Sheaves

1. PRESHEAVES

The functor \mathcal{A}^* that assigns to every open set U on a manifold the vector space of C^{∞} forms on U is an example of a presheaf. By definition a **presheaf** \mathcal{F} of abelian groups on a topological space X is a function that assigns to every open set U in X an abelian group $\mathcal{F}(U)$ and to every inclusion of open sets $i_U^V \colon V \to U$ a group homomorphism $\mathcal{F}(i_U^V) \coloneqq \rho_V^U$, called the *restriction* from U to V,

$$\rho_V^U \colon \mathfrak{F}(U) \to \mathfrak{F}(V),$$

such that the system of restrictions ρ_V^U satisfies the following properties:

- (i) (identity) $\rho_U^U = \mathbb{1}_{F(U)}$, the identity map on $\mathcal{F}(U)$; (ii) (transitivity) if $W \subset V \subset U$, then $\rho_W^V \circ \rho_V^U = \rho_W^U$.

We refer to elements of $\mathcal{F}(U)$ as sections of \mathcal{F} over U. The group $\mathcal{F}(U)$ is also written $\Gamma(U, \mathcal{F})$. Elements of $\Gamma(X, \mathcal{F})$ are called *global sections* of \mathcal{F} .

If \mathcal{F} and \mathcal{G} are presheaves on X, a *morphism* $f: \mathcal{F} \to \mathcal{G}$ of presheaves is a collection of group homomorphisms $f_U: \mathfrak{F}(U) \to \mathfrak{G}(U)$, indexed by open sets U in X, that commute with the restrictions, i.e., such that each diagram

$$\begin{array}{c|c} \mathcal{F}(U) \xrightarrow{f_U} & \mathcal{G}(U) \\ \rho_V^U & & \bigvee \rho_V^U \\ \mathcal{F}(V) \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$
(1.1)

is commutative. Although we write both vertical maps as ρ_V^U , they are in fact not the same map; the first one is $\mathcal{F}(i_U^V)$ and the second is $\mathcal{G}(i_U^V)$. If we write $\omega|_U$ for $\rho_V^U(\omega)$, then the diagram (1.1) is equivalent to $f_V(\boldsymbol{\omega}|_V) = f_U(\boldsymbol{\omega})|_V$ for all $\boldsymbol{\omega} \in \mathcal{F}(U)$. In practice, we often omit the subscripts in f_U and f_V and write them simply as f.

For any topological space X, let Open(X) be the category in which the objects are open subsets of X and for any two open subsets U, V of X, the set of morphisms from V to U is

$$\operatorname{Hom}(V,U) := \begin{cases} \{\operatorname{inclusion} i_U^V \colon V \to U\} & \text{if } V \subset U, \\ \operatorname{the empty set} \varnothing & \text{otherwise} \end{cases}$$

In functorial language, a presheaf of abelian groups is simply a contravariant functor from the category Open(X) to the category of abelian groups, and a morphism of presheaves is a natural transformation from the functor \mathcal{F} to the functor \mathcal{G} . What we have defined are presheaves of abelian groups; it is possible to define similarly presheaves of vector spaces, of algebras, and indeed of objects in any category, but all the presheaves that we consider will be presheaves of abelian groups.

Example 1.1. The *zero presheaf* \mathcal{F} on a topological space X associates to every open set U the zero group $\mathfrak{F}(U) = 0$ and to every inclusion $V \subset U$ the zero map $\mathfrak{F}(U) \to \mathfrak{F}(V)$.

Example. If G is an abelian group, we define the *presheaf of locally constant* G-valued functions on X (constant on connected components) to be the presheaf \underline{G} that associates to every open set U in X the group

$$\underline{G}(U) := \{ \text{locally constant functions } f : U \to G \}$$

and to every inclusion of open sets $V \subset U$ the restriction $\rho_V^U \colon \underline{G}(U) \to \underline{G}(V)$ of locally constant functions.

2. THE STALK OF A PRESHEAF

On a smooth manifold M, the function that assigns to every open set $U \subset M$ the group $C^{\infty}(U)$ of C^{∞} real-valued functions on U is a presheaf. As we know from manifold theory, the behavior of C^{∞} functions at a point is encoded in the the *germs* of the functions at the point. The corresponding notion for a presheaf is the *stalk* of the presheaf at a point. To define the stalk, we recall an algebraic construction called the *direct limit* of a direct system of groups.

A *directed set* is a set *I* with a binary relation \leq satisfying

- (i) (reflexivity) for all $a \in I$, $a \le a$,
- (ii) (transitivity) for all $a, b, c \in I$, if $a \leq b$ and $b \leq c$, then $a \leq c$,
- (iii) (upper bound) for all $a, b \in I$, there is an element $c \in I$, called an *upper bound* of a and b, such that $a \leq c$ and $b \leq c$.

We often write $b \ge a$ if $a \le b$.

A *direct system of groups* is a collection of groups $\{G_i\}_{i \in I}$ indexed by a directed set I and a collection of group homomorphisms $f_b^a \colon G_a \to G_b$ indexed by pairs $a \leq b$ in I such that

(i) $f_a^a = \mathbb{1}_{G_a}$, the identity map on G_a ,

(ii)
$$f_c^a = f_c^b \circ f_b^a$$
 for $a \le b \le c$ in *I*.

On the disjoint union $\coprod_i G_i$ we introduce an equivalence relation \sim by decreeing two elements g_a in G_a and g_b in G_b to be equivalent if there exists an upper bound c of a and b such that $f_c^a(g_a) = f_c^b(g_b)$ in G_c . The **direct limit** of the direct system, denoted by $\varinjlim_{i \in I} G_i$, is the quotient of the disjoint union $\coprod_i G_i$ by the equivalence relation \sim ; in other words, two elements of $\coprod_i G_i$ represent the same element in the direct limit if they are "eventually equal." We make the direct limit $\varinjlim_i G_i$ into a group by defining $[g_a] + [g_b] = [f_c^a(g_a) + f_c^b(g_b)]$, where c is an upper bound of a and b and $[g_a]$ is the equivalence class of g_a . It is easy to check that the addition + is well defined and that with this operation the direct limit $\varinjlim_i G_i$ becomes a group; moreover, if all the groups G_i are abelian, then so is their direct limit. Instead of groups, one can obviously also consider direct systems of modules, rings, algebras, and so on.

Example. Fix a point p in a manifold M and let I be the directed set consisting of all neighborhoods of p in M, with \leq being reverse inclusion: $U \leq V$ if and only if $V \subset U$. Let $C^{\infty}(U)$ be the ring of C^{∞} functions on U. Then $\{C^{\infty}(U)\}_{U \ni p}$ is a direct system of rings and its direct limit $C_p^{\infty} := \varinjlim_{U \ni p} C^{\infty}(U)$ is precisely the ring of germs of C^{∞} functions at p.

If \mathcal{F} is a presheaf of abelian groups on a topological space *X* and *p* is a point in *X*, then $\{\mathcal{F}(U)\}_{U\ni p}$, where *U* ranges over all open neighborhoods of *p*, is a direct system of abelian groups. The direct limit $\mathcal{F}_p := \varinjlim_{U\ni p} \mathcal{F}(U)$ is called the *stalk* of \mathcal{F} at *p*. An element of the stalk \mathcal{F}_p is called a *germ* of sections at *p*. For example, the ring C_p^{∞} is the stalk at *p* of the presheaf $C^{\infty}()$ of C^{∞} functions on the manifold *M*.

A morphism of presheaves $\varphi: \mathcal{F} \to \mathcal{G}$ over a topological space *X* induces a morphism of stalks $\varphi_p: \mathcal{F}_p \to \mathcal{G}_p$ at each $p \in X$ by sending the germ at *p* of a section $s \in \mathcal{F}(U)$ to the germ at *p* of the section $\varphi(s) \in \mathcal{G}(U)$. The morphism $\varphi_p: \mathcal{F}_p \to \mathcal{G}_p$ of stalks is also called the *stalk map* at *p*.

3. Sheaves

The stalk of a presheaf at a point embodies in it the local character of the presheaf about the point. However, in general there is no relation between the global sections and the stalks of a presheaf.

Example 3.1. If *G* is an abelian group and *S* is the presheaf on a topological space *X* defined by S(X) = G and S(U) = 0 for all $U \neq X$, then all the stalks S_p vanish, but *S* is not the zero presheaf.

A *sheaf* is a presheaf with two additional properties that link the global and local sections of the presheaf. In practice, most of the presheaves one encounters are sheaves. Unlike the example in the preceding paragraph, a sheaf all of whose stalks vanish has no nonzero global sections (Example 5.2).

Definition 3.2. A *sheaf* \mathcal{F} of abelian groups on a topological space X is a presheaf satisfying two additional conditions for any open set $U \subset X$ and any open cover $\{U_i\}$ of U:

- (i) (uniqueness axiom) if $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all *i*, then s = t;
- (ii) (gluing axiom) if $\{s_i \in \mathcal{F}(U_i)\}$ is a collection of sections such that

$$s_i|_{U_i \cap U_i} = s_j|_{U_i \cap U_i}$$
 for all i, j ,

then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each *i*.

Suppose there is an ordering on the index set *I* of the open cover $\{U_i\}_{i \in I}$, and consider the sequence of maps

$$0 \to \mathcal{F}(U) \xrightarrow{r} \prod_{i} \mathcal{F}(U_{i}) \xrightarrow{\delta} \prod_{i < j} \mathcal{F}(U_{i} \cap U_{j}),$$
(3.1)

where *r* is the restriction $r(\omega)_i = \omega|_{U_i}$ and δ is the Čech coboundary operator

$$(\boldsymbol{\delta}\boldsymbol{\omega})_{ij} := \boldsymbol{\omega}_j|_{U_{ij}} - \boldsymbol{\omega}_i|_{U_{ij}}.$$

Then the two sheaf axioms (i) and (ii) are equivalent to the exactness of the sequence (3.1) at $\mathcal{F}(U)$ and at $\prod_i \mathcal{F}(U_i)$, respectively; i.e., the map *r* is injective and ker $\delta = \operatorname{im} r$.

Example. For any open subset U of a topological space X, let $\mathcal{F}(U)$ be the abelian group of constant real-valued functions on U. If $V \subset U$, let $\rho_V^U \colon \mathcal{F}(U) \to \mathcal{F}(V)$ be the restriction of functions. Then \mathcal{F} is a presheaf on X. Suppose X has nonempty disjoint subsets (for

example, $X = \mathbb{R}^n$ with the standard topology). Then the presheaf \mathcal{F} satisfies the uniqueness axiom but not the gluing axiom of a sheaf: if U_1 and U_2 are disjoint open sets in X, and $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ have different values, then there is no constant function s on $U_1 \cup U_2$ that restricts to s_1 on U_1 and to s_2 on U_2 .

Example. Let \mathbb{R} be the presheaf on a topological space *X* that associates to every open set $U \subset X$ the abelian group $\mathbb{R}(U)$ consisting of all *locally* constant real-valued functions on *U*. Then \mathbb{R} is a sheaf. More generally, if *G* is an abelian group, then the presheaf <u>*G*</u> of locally constant functions with values in *G* is a sheaf, called the *constant sheaf* with values in *G*.

Example. The zero presheaf 0 on a topological space in Example 1.1 is a sheaf.

Example. The presheaf \mathcal{A}^k on a manifold that assigns to each open set U the abelian group of C^{∞} k-forms on U is a sheaf.

Example. The presheaf \mathbb{Z}^k on a manifold that associates to each open set U the abelian group of closed C^{∞} k-forms on U is a sheaf.

4. THE SHEAF ASSOCIATED TO A PRESHEAF

Associated to a presheaf \mathcal{F} on a topological space X is another topological space $E_{\mathcal{F}}$, called the *étalé space* of \mathcal{F} . As a set, the étalé space $E_{\mathcal{F}}$ is the disjoint union $\coprod_{p \in X} \mathcal{F}_p$ of all the stalks of \mathcal{F} . There is a natural projection map $\pi \colon E_{\mathcal{F}} \to X$ that maps \mathcal{F}_p to p. A *section* of the étalé space $\pi \colon E_{\mathcal{F}} \to X$ over $U \subset X$ is a map $s \colon U \to E_{\mathcal{F}}$ such that $\pi \circ s = \mathbb{1}_U$, the identity map on U. For any open set $U \subset X$, element $s \in \mathcal{F}(U)$, and point $p \in U$, let $s_p \in \mathcal{F}_p$ be the germ of s at p. Then the element $s \in \mathcal{F}(U)$ defines a section \tilde{s} of the étalé space over U,

$$\tilde{s} \colon U \to E_{\mathcal{F}},$$
$$p \mapsto s_p \in \mathcal{F}_p.$$

The collection

$$\{\tilde{s}(U) \mid U \text{ open in } X, s \in \mathfrak{F}(U)\}$$

of subsets of $E_{\mathcal{F}}$ satisfies the conditions to be a basis for a topology on $E_{\mathcal{F}}$. With this topology, the étalé space $E_{\mathcal{F}}$ becomes a topological space. By construction, the topological space $E_{\mathcal{F}}$ is locally homeomorphic to X. For any element $s \in \mathcal{F}(U)$, the function $\tilde{s} \colon U \to E_{\mathcal{F}}$ is a continuous section of $E_{\mathcal{F}}$. A section t of the étalé space $E_{\mathcal{F}}$ is continuous if and only if every point $p \in X$ has a neighborhood U such that $t = \tilde{s}$ on U for some $s \in \mathcal{F}(U)$.

Let \mathcal{F}^+ be the presheaf that associates to each open subset $U \subset X$ the abelian group

 $\mathcal{F}^+(U) := \{ \text{continuous sections } t \colon U \to E_{\mathcal{F}} \}.$

Under pointwise addition the presheaf \mathcal{F}^+ is easily seen to be a sheaf, called the *sheafification* or the *associated sheaf* of the presheaf \mathcal{F} . There is an obvious presheaf morphism $\theta: \mathcal{F} \to \mathcal{F}^+$ that sends a section $s \in \mathcal{F}(U)$ to the section $\tilde{s} \in \mathcal{F}^+(U)$.

Example. For an open set U in a topological space X, let $\mathcal{F}(U)$ be the group of all *constant* real-valued functions on U. At each point $p \in X$, the stalk \mathcal{F}_p is \mathbb{R} . The étalé space $E_{\mathcal{F}}$ is $X \times \mathbb{R}$, but not with its usual topology. A set in $E_{\mathcal{F}}$ is open if and only if it is of the form $U \times \{r\}$ for an open set $U \subset X$ and a number $r \in \mathbb{R}$. Thus, the topology on $E_{\mathcal{F}} = X \times \mathbb{R}$ is the product of the given topology on X and the discrete topology on \mathbb{R} . The presheaf \mathcal{F} is not a sheaf. The sheafification \mathcal{F}^+ is the constant sheaf $\underline{\mathbb{R}}$.

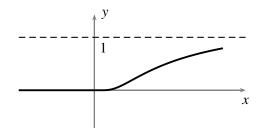


FIGURE 4.1. A C^{∞} function all of whose derivatives vanish at 0

Remark 4.1. The topology of the étalé space of a sheaf can be quite weird. For example, if \mathcal{F} is the sheaf of C^{∞} real-valued functions on \mathbb{R} , then the étalé space $E_{\mathcal{F}}$ is not Hausdroff: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

Then *f* is C^{∞} on \mathbb{R} (see [10, Example 1.3, p. 5]). The germ of the function *f* at 0 and the germ of the zero function at 0 cannot be separated by open sets. Indeed, if *U* is any neighborhood of the germ f_0 and *V* is any neighborhood of the germ 0_0 , then there exists an $\varepsilon < 0$ such that $f_{\varepsilon} = 0_{\varepsilon} \in U$ and $0_{\varepsilon} \in V$.

EXERCISE 4.2 Prove that if \mathcal{F} is a sheaf, then $\mathcal{F} = \mathcal{F}^+$. (*Hint*: Show that every continuous section $t: U \to E_{\mathcal{F}}$ is \tilde{s} for some $s \in \mathcal{F}(U)$.)

Proposition 4.3. For every sheaf \mathfrak{G} and every presheaf morphism $\varphi: \mathfrak{F} \to \mathfrak{G}$, there is a unique sheaf morphism $\varphi^+: \mathfrak{F}^+ \to \mathfrak{G}$ such that the diagram



commutes.

PROOF. The proof is straightforward and is left as an exercise.

5. Sheaf Morphisms

Recall that all our sheaves are sheaves of abelian groups. A *morphism* $\varphi \colon \mathcal{F} \to \mathcal{G}$ of sheaves, also called a *sheaf map*, is by definition a morphism of presheaves. If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the *presheaf kernel*

$$U \mapsto \ker (\varphi_U \colon \mathfrak{F}(U) \to \mathfrak{G}(U))$$

is a sheaf, called the *kernel* of φ and written ker φ . The *presheaf image*

$$U \mapsto \operatorname{im} (\varphi_U \colon \mathfrak{F}(U) \to \mathfrak{G}(U)),$$

however, is not always a sheaf. The *image* of φ , denoted im φ , is defined to be the sheaf associated to the presheaf image of φ .

A sheaf \mathcal{F} over a space X is a *subsheaf* of a sheaf \mathcal{G} if for every open set U in X the group $\mathcal{F}(U)$ is a subgroup of $\mathcal{G}(U)$, and the inclusion map $i: \mathcal{F} \to \mathcal{G}$ is a presheaf morphism. If \mathcal{F} is a subsheaf of \mathcal{G} , the *quotient sheaf* is defined to be the sheaf associated to the presheaf $U \mapsto \mathcal{G}(U)/\mathcal{F}(U)$.

A morphism of sheaves $\varphi \colon \mathcal{F} \to \mathcal{G}$ is said to be *injective* if ker $\varphi = 0$, and *surjective* if im $\varphi = \mathcal{G}$.

Proposition 5.1.

- (i) A morphism of sheaves $\varphi \colon \mathcal{F} \to \mathcal{G}$ is injective if and only if the stalk map $\varphi_p \colon \mathcal{F}_p \to \mathcal{G}_p$ is injective for every p.
- (ii) A morphism of sheaves $\varphi \colon \mathfrak{F} \to \mathfrak{G}$ is surjective if and only if the stalk map $\varphi_p \colon \mathfrak{F}_p \to \mathfrak{G}_p$ is surjective for every p.

PROOF. Exercise (see [5, Exercise 1.2 (a),(b), p. 66]).

In this proposition, neither (i) nor (ii) are true for morphisms of presheaves. Let G be an abelian group and S the presheaf on a topological space X such that S(X) = G and S(U) = 0 for all open sets $U \neq X$ (This is the presheaf of Example 3.1). Then the stalks S_p are all zero. A counterexample of Proposition 5.1(i) for presheaves is $S \rightarrow 0$; a counterexample to Proposition 5.1(ii) for presheaves is $0 \rightarrow S$. It is the truth of this proposition for sheaves that makes sheaves so much more useful than general presheaves.

Example 5.2. If the stalk \mathcal{F}_p of a sheaf \mathcal{F} vanishes for every $p \in X$, then by Proposition 5.1, the sheaf map $\mathcal{F} \to 0$ is both injective and surjective, since the stalk maps $\mathcal{F}_p \to 0_p = 0$ are injective and surjective for all $p \in X$. Hence, \mathcal{F} is isomorphic to the zero sheaf and has no nonzero global sections.

6. EXACT SEQUENCES OF SHEAVES

A sequence of sheaves and (pre)sheaf morphisms

$$\cdots \longrightarrow \mathcal{F}^1 \xrightarrow{d_1} \mathcal{F}^2 \xrightarrow{d_2} \mathcal{F}^3 \xrightarrow{d_3} \cdots$$

on a topological space X is said to be *exact* at \mathcal{F}^k if $\operatorname{im} d_{k-1} = \ker d_k$; the sequence is said to be *exact* if it is exact at every \mathcal{F}^k . By Proposition 5.1, the exactness of a sequence of

sheaves on X is equivalent to the exactness of the sequence of stalk maps at every point $p \in X$. An exact sequence of sheaves of the form

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \tag{6.1}$$

is said to be a *short exact sequence*. The exactness of a sequence of groups is defined in the same way.

It is not too difficult to show that the exactness of the sheaf sequence (6.1) over a topological space X implies the exactness of the sequence of sections

$$0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \tag{6.2}$$

for every open set $U \subset X$, but the last map $\mathcal{F}(U) \to \mathcal{G}(U)$ need not be surjective. In fact, as we see in [2, Theorem 2.8], the cohomology $H^1(U, \mathcal{E})$ is a measure of the nonsurjectivity of the map of global sections $\mathcal{F}(U) \to \mathcal{G}(U)$.

Fix an open subset U of a topological space X. To every sheaf \mathcal{F} on X, we can associate the abelian group $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$ of sections over U and to every sheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$, the group homomorphism $\varphi_U \colon \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$. This makes $\Gamma(U,)$ a functor from sheaves to abelian groups, henceforth called the *section functor*.

Example. Let \mathcal{O} be the sheaf of holomorphic functions on the complex plane \mathbb{C} and \mathcal{O}^* the sheaf of nowhere-vanishing holomorphic functions on \mathbb{C} . For any open set $U \subset \mathbb{C}$, if $f \in \mathcal{O}(U)$, then $\exp 2\pi i f \in \mathcal{O}^*(U)$. The kernel of the sheaf map $\exp 2\pi i ()$ on any open set U consists of the holomorphic (hence locally constant) integer-valued functions on U. Hence, there is an exact sequence of sheaves, called the *exponential sequence*,

$$0 \to \underline{\mathbb{Z}} \to \mathbb{O} \to \mathbb{O}^* \to 0$$

The surjectivity of $\mathcal{O} \to \mathcal{O}^*$ follows from the fact that simply connected neighborhoods form a basis at each point and if U is simply connected and $f \in \mathcal{O}^*(U)$, then f(U) is a simply connected set in \mathbb{C} not containing the origin, and hence log f is defined on U.

If U is the punctured plane $\mathbb{C} - \{0\}$, then the exponential map

$$\exp 2\pi i(): \mathcal{O}(U) \to \mathcal{O}^*(U)$$

is not surjective, since it is not possible to define its inverse, $(1/2\pi i)\log$, on $\mathbb{C} - \{0\}$: for example, $\log z = \log(re^{i\theta})$ must be defined as $(\log r) + i\theta$, but the angle θ cannot be defined as a continuous function around a puncture at the origin.

A functor F from the category of sheaves on X to the category of abelian groups is said to be *exact* if it maps a short exact sequence of sheaves

$$0 \to \mathfrak{E} \to \mathfrak{F} \to \mathfrak{G} \to 0$$

to a short exact sequence of abelian groups

$$0 \to F(\mathcal{E}) \to F(\mathcal{F}) \to F(\mathcal{G}) \to 0.$$

If instead one has only the exactness of

$$0 \to F(\mathcal{E}) \to F(\mathcal{F}) \to F(\mathcal{G})$$

then F is said to be a *left-exact functor*. Thus, the section functor $\Gamma(U,)$ is left-exact but not exact.

7. RESOLUTIONS

Recall that $\underline{\mathbb{R}}$ is the sheaf of locally constant functions with values in \mathbb{R} and \mathcal{A}^k is the sheaf of C^{∞} *k*-forms on a manifold *M*. The exterior derivative $d: \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)$, defined for every open set *U* in *M*, induces a morphism of sheaves $d: \mathcal{A}^k \to \mathcal{A}^{k+1}$.

Proposition 7.1. On any manifold M of dimension n, the sequence of sheaves

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^n \to 0 \tag{7.1}$$

is exact.

PROOF. Exactness at \mathcal{A}^0 is equivalent to the exactness of the sequence of stalk maps $\underline{\mathbb{R}}_p \to \mathcal{A}_p^0 \xrightarrow{d} \mathcal{A}_p^1$ for all $p \in M$. Fix a point $p \in M$. Suppose $[f] \in \mathcal{A}_p^0$ is the germ of a C^{∞} function $f: U \to \mathbb{R}$, where U is a neighborhood of p, such that d[f] = [0] in \mathcal{A}_p^1 . Then there is a neighborhood $V \subset U$ of p on which $df \equiv 0$. Hence, f is locally constant on V and $[f] \in \underline{\mathbb{R}}_p$. Conversely, if $[f] \in \underline{\mathbb{R}}_p$, then d[f] = 0. This proves the exactness of the sequence (7.1) at \mathcal{A}^0 .

Next, suppose $[\omega] \in \mathcal{A}_p^k$ is the germ of a C^{∞} k-form on some neighborhood of p such that $d[\omega] = 0 \in \mathcal{A}_p^{k+1}$. This means there is a neighborhood V of p on which $d\omega \equiv 0$. By making V smaller, we may assume that V is contractible. By the Poincaré lemma, ω is exact on V, say $\omega = d\tau$ for some $\tau \in \mathcal{A}^{k-1}(V)$. Hence, $[\omega] = d[\tau]$.

In general, an exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$$

on a topological space X is called a *resolution* of the sheaf \mathcal{A} . On a complex manifold M of complex dimension n, the analogue of the Poincaré lemma is the $\bar{\partial}$ -Poincaré lemma [4, p. 25], from which it follows that for each fixed integer $p \ge 0$, the sheaves $\mathcal{A}^{p,q}$ of $C^{\infty}(p,q)$ -forms on M give rise to a resolution of the sheaf Ω^p of holomorphic p-forms on M:

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\partial} \mathcal{A}^{p,1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{A}^{p,n} \to 0.$$
(7.2)

The cohomology of the Dolbeault complex

$$0 \to \mathcal{A}^{p,0}(M) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{p,n}(M) \to 0$$

is by definition the **Dolbeault cohomology** of the complex manifold M. (For (p,q)-forms on a complex manifold, see [4].)

Lecture 2. Sheaf Cohomology

8. DE RHAM COHOMOLOGY

To define sheaf cohomology, one might try to imitate the definition of de Rham cohomology. Recall that the de Rham cohomology of a manifold M may be defined as follows:

i) First take a resolution of the sheaf $\underline{\mathbb{R}}$ of locally constant functions on *M* by the sheaves of C^{∞} forms:

$$0 \to \mathbb{R} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \xrightarrow{d} \cdots$$

ii) Apply the global section functor:

$$0 \to \Gamma(M,\underline{\mathbb{R}}) \to \mathcal{A}^0(M) \xrightarrow{d} \mathcal{A}^0(M) \xrightarrow{d}$$

iii) Omitting the initial term $\Gamma(M, \mathbb{R})$, take the cohomology of the resulting complex $\mathcal{A}^{\bullet}(M)$:

$$H^{k}_{\mathrm{dR}}(M) = \frac{\ker d_{k} \colon \mathcal{A}^{k}(M) \to \mathcal{A}^{k+1}(M)}{\operatorname{im} d_{k-1} \colon \mathcal{A}^{k-1}(M) \to \mathcal{A}^{k}(M)}.$$

The trouble with this approach is that there could be many different resolutions of a given sheaf \mathcal{F} , and the resulting cohomology groups might not be isomorphic. Fortunately, every sheaf has a unique, canonical resolution called the *Godement canonical resolution*. Using the Godement canonical resolution, sheaf cohomology $H^*(X, \mathcal{F})$ will be well defined.

9. THE GODEMENT CANONICAL FUNCTORS

If \mathcal{F} is a sheaf on a topological space *X* and $\mathcal{E}_{\mathcal{F}}$ its étalé space, then

 $\mathcal{F}(U) = \mathcal{F}^+(U) = \{ \text{continuous sections of } \mathcal{E}_{\mathcal{F}} \to X \}.$

Let $\mathcal{C}^0\mathcal{F}(U) = \{ \text{all (continuous or discontinuous) sections of } \mathcal{E}_{\mathcal{F}} \to X \}$. Thus, \mathcal{F} is a subsheaf of $\mathcal{C}^0\mathcal{F}$. If \mathcal{Q}^1 is the quotient sheaf, then there is an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{C}^0 \mathcal{F} \to \mathcal{Q}^1 \to 0. \tag{9.1}$$

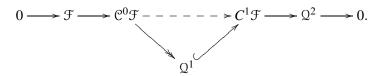
Repeat the construction to Q^1 , call $C^0Q^1 = C^1 \mathcal{F}$ the *first Godement functor of* \mathcal{F} , and let Q^2 be the quotient of C^0Q^1 by Q^1 :

$$0 \longrightarrow \mathcal{Q}^{1} \longrightarrow \mathcal{C}^{0}\mathcal{Q}^{1} \longrightarrow \mathcal{Q}^{2} \longrightarrow 0.$$

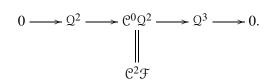
$$(9.2)$$

$$e^{1}\mathcal{F}$$

We can splice together the two short exact sequences (9.1) and (9.2) to obtain a four-term exact sequence:



Repeat the construction on Ω^2 and so on:



By recursion, we obtain a long exact sequence

$$0 \to \mathcal{F} \to \mathcal{C}^0 \mathcal{F} \to \mathcal{C}^1 \mathcal{F} \to \mathcal{C}^2 \mathcal{F} \to \cdots$$

of sheaves. This is the *Godement canonical resolution* of \mathcal{F} .

12

The Godement functors are not easy to picture except possibly $\mathcal{C}^0\mathcal{F}$. An element of $\mathcal{C}^0\mathcal{F}$ is a continuous or discontinuous section of the étalé space $\mathcal{E}_{\mathcal{F}} \to X$ and hence associates to each $x \in X$ an element of the stalk \mathcal{F}_x . Therefore, $\mathcal{C}^0\mathcal{F}$ is also the direct product $\prod_{x \in X} \mathcal{F}_x$.

10. Sheaf Cohomology

Let \mathcal{F} be a sheaf of abelian groups on a topological space *X*. To define the sheaf cohomology $H^*(X, \mathcal{F})$,

(i) Take the Godement canonical resolution of \mathcal{F} :

$$0 \to \mathcal{F} \to \mathcal{C}^0 \mathcal{F} \xrightarrow{\delta} \mathcal{C}^1 \mathcal{F} \xrightarrow{\delta} \mathcal{C}^2 \mathcal{F} \to \cdots$$

(ii) Apply the global section functor $\Gamma(X, \cdot)$ to the Godement resolution:

$$0 \to \mathcal{F}(X) \to \mathcal{C}^0 \mathcal{F}(X) \xrightarrow{\delta} \mathcal{C}^1 \mathcal{F}(X) \xrightarrow{\delta} \mathcal{C}^2 \mathcal{F}(X) \xrightarrow{\delta} \dots$$

(Recall that $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$.)

(iii) Take the cohomology of the resulting complex $C^{\bullet} \mathcal{F}(X)$ of global sections (omitting the initial term $\mathcal{F}(X)$):

$$H^{k}(X,\mathcal{F}) = h^{k}(\mathcal{C}^{\bullet}\mathcal{F}(X)) = \frac{\ker \delta \colon \mathcal{C}^{k}\mathcal{F}(X) \to \mathcal{C}^{k+1}\mathcal{F}(X)}{\operatorname{im} \delta \colon \mathcal{C}^{k-1}\mathcal{F}(X) \to \mathcal{C}^{k}\mathcal{F}(X)}.$$

Proposition 10.1. Let \mathcal{F} be a sheaf of abelian groups on a topological space X. Then the zeroth cohomology $H^0(X, \mathcal{F})$ is the group of global sections:

$$H^0(X,\mathcal{F}) = \mathcal{F}(X).$$

PROOF. Because the global section functor is left exact, the exactness of the sequence

$$0 \to \mathcal{F} \to \mathcal{C}^0 \mathcal{F} \to \mathcal{C}^1 \mathcal{F}$$

implies the exactness of

Hence,

$$0 \to \mathcal{F}(X) \to \mathcal{C}^0 \mathcal{F}(X) \xrightarrow{\delta} \mathcal{C}^1 \mathcal{F}(X).$$
$$H^0(X, \mathcal{F}) = \ker \delta = \mathcal{F}(X).$$

11. FLASQUE SHEAVES

A sheaf \mathcal{F} on a topological space X is said to be *acyclic* if $H^k(X,\mathcal{F}) = 0$ for all k > 0. Acyclic sheaves are sheaves with the simplest possible cohomology groups. They play a special role in sheaf theory. We will now introduce two types of acyclic sheaves—flasque sheaves and fine sheaves.

Definition 11.1. A sheaf \mathcal{F} on a topological space *X* is said to be *flasque* if for every open set $U \subset X$, the restriction map $\rho_U^X \colon \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

Example 11.2. A continuous function such as sec: $(-\pi/2, \pi/2) \to \mathbb{R}$ blows up at the endpoints $-\pi/2$ and $\pi/2$ and so cannot be extended to a continuous function on \mathbb{R} . This example shows that the sheaf \mathcal{C} of continuous functions on \mathbb{R} is not flasque, since $\mathcal{C}(\mathbb{R}) \to \mathbb{R}$

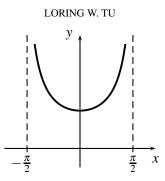


FIGURE 11.1. The graph of $y = \sec x$

 $C((-\pi/2, pi/2))$ is not surjective. For the same reason, none of the sheaves \mathcal{A}^k of smooth *k*-forms on a manifold is flasque.

Example 11.3. Recall that for any sheaf \mathcal{F} on a topological space X, we defined $\mathcal{C}^0\mathcal{F}$ to be the sheaf of all sections of the étalé space of \mathcal{F} . If U is an open set in X, then

$$\mathcal{C}^{0}\mathcal{F}(U) = \prod_{p \in U} \mathcal{F}_{p}$$
 and $\mathcal{C}^{0}\mathcal{F}(X) = \prod_{p \in X} \mathcal{F}_{p}.$

Clearly, the restriction map $\rho \colon C^0 \mathcal{F}(X) \to C^0 \mathcal{F}(U)$ is surjective. Thus, for any sheaf \mathcal{F} , the sheaf $C^0 \mathcal{F}$ is flasque.

Since the *k*th Godement sheaf $\mathcal{C}^k \mathcal{F}$ is $\mathcal{C}^0 \mathcal{Q}^{k-1}$ for some sheaf \mathcal{Q}^{k-1} , the Godement sheaf $\mathcal{C}^k \mathcal{F}$ is flasque for any $k \ge 0$.

We summarize several important properties of flasque sheaves and the Godement sheaves $\mathcal{C}^k \mathcal{F}$ in the proposition below.

Proposition 11.4. Let \mathcal{F} be a sheaf on a topological space X.

- (i) For any $k \ge 0$, the Godement sheaf $\mathcal{C}^k \mathcal{F}$ is flasque.
- (ii) The Godement functor $\mathcal{C}^k()$ is an exact functor from sheaves to sheaves [2, Prop. 2.2.1].
- (iii) The Godement section functor $\Gamma(X, \mathbb{C}^k())$ is an exact functor from sheaves to abelian groups [2, Cor. 2.2.7].
- (iv) A flasque sheaf is acyclic [2, Cor. 2.2.5].

12. FINE SHEAVES

A morphism $f: \mathcal{F} \to \mathcal{G}$ of sheaves over a topological space X induces at each point $x \in X$ a stalk map $f_x: \mathcal{F}_x \to \mathcal{G}_x$. Unlike in the case of continuous real-valued functions, the zero set of stalk maps

$$Z = \{x \in X | f_x = 0\}$$

is an open subset of X. This is because if two stalk maps agree at a point p, then being germs, they agree in a neighborhood of the point. Since Z is where the stalk map f_x agrees with the zero map 0_x , we see that Z is open in X. Thus,

$$\operatorname{supp} f = \{x \in X | f_x \neq 0\}$$

is a closed subset of X. (Unlike the support of a real-valued function, the support of a sheaf morphism is automatically closed without having to take closure.)

Definition 12.1. Let \mathcal{F} be a sheaf of abelian groups on a topological space *X* and $\{U_{\alpha}\}$ a locally finite open cover of *X*. A *partition of unity* for \mathcal{F} subordinate to $\{U_{\alpha}\}$ is a collection $\{\eta_{\alpha} : \mathcal{F} \to \mathcal{F}\}$ of sheaf morphisms such that

- (i) supp $\eta_{\alpha} \subset U_{\alpha}$,
- (ii) at each point $x \in X$, the stalk maps $\eta_{\alpha,x}$ sum up to the identity map on the stalk \mathcal{F}_x : $\sum \eta_{\alpha,x} = \mathbb{1}_{\mathcal{F}_x}$.

The local finiteness condition guarantees that the sum $\sum_{\alpha} \eta_{\alpha,x}$ is a finite sum, since x has a neighborhood that meets only finitely many of the U_{α} and supp $\rho_{\alpha} \subset U_{\alpha}$.

Definition 12.2. A sheaf \mathcal{F} on a topological space *X* is *fine* if for every locally finite open cover $\{U_{\alpha}\}$ of *X*, the sheaf \mathcal{F} admits a partition of unity subordinate to $\{U_{\alpha}\}$.

Proposition 12.3. For each integer $k \ge 0$, the sheaf \mathcal{A}^k of smooth k-forms on a manifold M is a fine sheaf on M.

PROOF. Let $\{U_{\alpha}\}$ be a locally finite open cover of M. A partition of unity on M subordinate to $\{U_{\alpha}\}$ is a collection of functions $\{\rho_{\alpha} : M \to \mathbb{R}\}$ such that supp $\rho_{\alpha} \subset U_{\alpha}$ and $\sum_{\alpha} \rho_{\alpha} = 1$. By [10, Th. 13.7, p. 147], there is a partition of unity $\{\rho_{\alpha}\}$ on M subordinate to $\{U_{\alpha}\}$. We will show that a C^{∞} partition of unity on the manifold M gives rise to a partition of unity for the sheaf \mathcal{A}^k . Let $\mathcal{F} = \mathcal{A}^k$. Define a collection of morphisms $\{\eta_{\alpha} : \mathcal{F} \to \mathcal{F}\}$ for \mathcal{F} by

$$\eta_{\alpha,U} \colon \mathcal{F}(U) \to \mathcal{F}(U),$$

$$\eta_{\alpha,U}(s) = \rho_{\alpha}s \text{ for } s \in \mathcal{F}(U).$$

Then

$$\operatorname{supp} \eta_{\alpha} \subset \operatorname{supp} \rho_{\alpha} \subset U_{\alpha}$$

and

$$\sum_{\alpha} \eta_{\alpha,x}(v) = \sum_{\alpha} \rho_{\alpha}v = v.$$

Hence, $\{\eta_{\alpha}\}$ is a partition of unity for \mathcal{F} .

A topological space X is *paracompact* if every open cover of X has a locally finite open refinement. Continuous partitions of unity exist on a paracompact Hausdorff space [8, Th. 41.7]. Thus, if S^k is the sheaf of continuous k-cochains on a paracompact Hausdorff space, then the same proof as for Proposition 12.3 with $\mathcal{F} = S^k$ proves that S^k is a fine sheaf.

Similarly, the sheaves of continuous forms on a manifold are fine. However, in the holomorphic or algebraic category, partitions of unity generally do not exist, and the sheaves Ω^k of holomorphic forms on a complex manifold and the sheaves Ω^k_{alg} of algebraic forms on a smooth variety are generally not fine.

The most important property of a fine sheaf is that it is acyclic on sufficiently nice spaces.

Theorem 12.4. Let \mathcal{F} be a fine sheaf on a topological space X in which every open subset is paracompact (a manifold has this property). Then $H^k(X, \mathcal{F}) = 0$ for all $k \ge 1$.

Lecture 3. Hypercohomology and Spectral Sequences

13. COHOMOLOGY OF A DOUBLE COMPLEX

A *double complex* is a collection of bigraded abelian groups $K^{p,q}$ with two commuting differentials

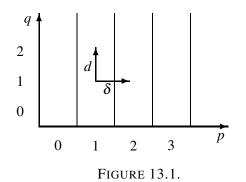
 $d: K^{p,q} \to K^{p,q+1}$ in the vertical direction, and

 $\delta: K^{p,q} \to K^{p+1,q}$ in the horizontal direction,

i.e., $d^2 = 0$, $\delta^2 = 0$, and $d\delta = \delta d$. We usually denote a double complex by

$$K^{\bullet,\bullet} = \bigoplus K^{p,q}$$

or graphically,



Every row and every column of a double complex $(K^{\bullet,\bullet}, d, \delta)$ is a differential complex. The cohomology of the columns is denoted by H_d and the cohomology of the rows by H_{δ} . The groups H_d and H_{δ} each again has a bigrading; for example,

$$H_d^{p,q} = \frac{\ker d \colon K^{p,q} \to K^{p,q+1}}{\operatorname{im} d \colon K^{p,q-1} \to K^{p,q}}$$

Because $d\delta = \delta d$, the map δ carries *d*-cocycles to *d*-cocycles, and *d*-coboundaries to *d*-coboundaries, and induces a differential, also denoted by δ , on H_d . Thus, $H_{\delta}H_d$ is defined.

Associated to a double complex $K^{\bullet,\bullet}$ is a single complex $K^{\bullet} = \bigoplus K^k$, where $K^k = \bigoplus_{p+q=k} K^{p,q}$, obtained by summing along the antidiagonals of K. Because d and δ commute,

$$(d+\delta)(d+\delta) = d\delta + \delta d \neq 0,$$

so that $\delta + d$ is not a differential. However, if we alternate the sign of the vertical differential, with $D := \delta + (-1)^p d$, then $D^2 = 0$ on the single complex K^{\bullet} . The cohomology $H_D(K^{\bullet})$ of the associated single complex K^{\bullet} is called the *total cohomology* of the double complex $K^{\bullet,\bullet}$. Note that the cohomology of the columns is unchanged by switching *d* for -d every other column.

14. COHOMOLOGY SHEAVES

We have already introduced on several occasions the cohomology of a complex of abelian groups. In fact, in any category in which the notions of kernel and cokernel are defined, the cohomology of a complex makes sense. A *complex* in such a category is a sequence of objects and morphisms

$$C^{\bullet}: \cdots \to C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \to \cdots$$

such that the composition of two successive morphisms is zero:

$$d_k \circ d_{k-1} = 0.$$

The *k*th cohomology object of the complex C^{\bullet} is

$$h^k(C^{\bullet}) = \frac{\ker d_k}{\operatorname{im} d_{k-1}}.$$

In particular, if \mathcal{C}^{\bullet} is a complex of sheaves of abelian groups on a topological space X, then we can define the *k*th *cohomology sheaf* of \mathcal{C}^{\bullet} to be

$$\mathfrak{H}^k := h^k(\mathfrak{C}^{\bullet}) = \frac{\ker d_k \colon \mathfrak{C}^k \to \mathfrak{C}^{k+1}}{\operatorname{im} d_{k-1} \colon \mathfrak{C}^{k-1} \to \mathfrak{C}^k}.$$

Note that the *cohomology sheaf* \mathcal{H}^k is a sheaf, not to be confused with *sheaf cohomology* $H^k(X, \mathcal{F})$, which is an abelian group.

Example 14.1. The de Rham complex of sheaves. If \mathcal{A}^{\bullet} is the complex

$$0 \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \xrightarrow{d} \cdots$$

of sheaves of C^{∞} forms on a manifold M, then by the Poincaré lemma the sequence

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \xrightarrow{d} \cdots$$

is exact. Therefore, the cohomology sheaves of \mathcal{A}^{\bullet} are

$$\mathcal{H}^{k} = \begin{cases} \underline{\mathbb{R}} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

15. FILTRATIONS

A *filtration* on an abelian group A is a sequence of nested subgroups

$$A = F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_n = 0.$$

An abelian group with a filtration is called a *filtered group*. Given a filtered group A, its *associated graded group* is

$$\operatorname{Gr}(A) = \bigoplus_{p=0}^{n-1} \frac{F_p}{F_{p+1}}.$$

A double complex $\bigoplus K^{p,q}$ has a natural filtration by p as in Figure 15.1. Each F_p is actually a *subcomplex* of (K^{\bullet}, D) . Therefore, the filtration F_{\bullet} by p on K induces a filtration on $H_D(K)$, which we denote by $F_{\bullet}(H_D(K))$. It is defined by

$$F_p(H_D(K))$$
: image of $(H_D(F_p) \to H_D(K))$.

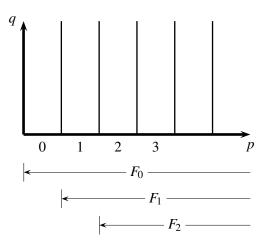


FIGURE 15.1. The filtration by p

16. SPECTRAL SEQUENCES

A spectral sequence is a sequence $\{(E_r, d_r)\}_{r=0}^{\infty}$ of differential complexes such that each E_r is the cohomology of its predecessor (E_{r-1}, d_{r-1}) for all $r \ge 1$. We will consider only spectral sequences in which each E_r has a bigrading so that $E_r = \bigoplus_{p,q=0}^{\infty} E_r^{p,q}$.

Fix (p,q) and consider $E_r^{p,q}$ as $r \to \infty$. If the sequence $E_r^{p,q}$ becomes stationary, i.e., there is an r_0 such that

$$E_{r_0}^{p,q} = E_{r_0+1}^{p,q} = E_{r_0+2}^{p,q} = \cdots,$$

then we define $E_{\infty}^{p,q}$ to be the stationary value of $E_r^{p,q}$. If for every (p,q), the sequence $E_r^{p,q}$ eventually becomes stationary as r goes to infinity, then E_{∞} is defined. It is important to note that there may not exist an r_0 such that $E_{\infty}^{p,q} = E_{r_0}^{p,q}$ for all p,q.

One of the main tools for computing the total cohomology of a double complex is Leray's theorem on the spectral sequence of a double complex [1, Th. 14.14].

Theorem 16.1 (Leray). *Given a double complex* $(K^{\bullet,\bullet}, d, \delta)$ *filtered by p, there is a spectral sequence*

$$\{(E_r, d_r: E_r^{p,q} \to E_r^{p+r,q-r+1})\}$$

with $(E_0, d_0) = (K^{\bullet, \bullet}, d)$, $(E_1, d_1) = (H_d, \delta)$, and $E_2 = H_{\delta}H_d$ such that E_{∞} is defined and isomorphic to the associated graded group of $H_D(K^{\bullet})$ with the induced filtration by p. (In this case, it is customary to say that the spectral sequence **converges** to $H_D(K^{\bullet})$.)

Instead of filtering the double complex $K^{\bullet,\bullet}$ by p, one can also filter it by q. In this case, by interchanging p and q in Leray's theorem, one obtains a second spectral sequence with $(E_0, d_0) = (K^{\bullet,\bullet}, \delta), (E_1, d_1) = (H_{\delta}, d), \text{ and } E_2 = H_d H_{\delta}$ that converges to the associated graded group $H_D(K^{\bullet})$ with the induced filtration by q.

The existence of these two spectral sequences, both converging to $H_D(K^{\bullet})$, is a particularly poweful tool for proving isomorphism theorems.

17. HYPERCOHOMOLOGY OF A COMPLEX OF SHEAVES

Let \mathcal{F} be a sheaf of abelian groups on a topological space *X*. Recall from Section 10 that the *sheaf cohomology* $H^*(X, \mathcal{F})$ is defined in three steps:

(1) Take the Godement canonical resolution of \mathcal{F} :

$$\mathcal{C}^{\bullet}\mathcal{F}\colon 0\to \mathcal{F}\to \mathcal{C}^0\mathcal{F}\to \mathcal{C}^1\mathcal{F}\to\cdots.$$

(2) Apply the global section functor $\Gamma(X, \cdot)$ to $\mathcal{C}^{\bullet} \mathcal{F}$:

$$0 \to \mathcal{C}^0 \mathcal{F}(X) \to \mathcal{C}^1 \mathcal{F}(X) \to \cdots$$

(Note that the initial term $\mathcal{F}(X)$ has been dropped.)

(3) Take the cohomology of $\mathcal{C}^{\bullet}\mathcal{F}(X)$:

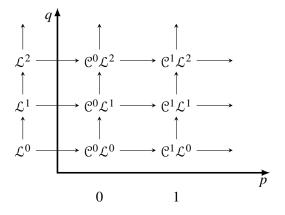
$$H^k(X,\mathcal{F}) = h^k(\mathcal{C}^{\bullet}\mathcal{F}(X)).$$

Let

$$\mathcal{L}^{\bullet} \colon \mathcal{L}^{0} \to \mathcal{L}^{1} \to \mathcal{L}^{2} \to \cdots$$

be a complex of sheaves of abelian groups over a topological space X. The *hypercohomology* of the complex \mathcal{L}^{\bullet} generalizes the sheaf cohomology of a single sheaf. The hypercohomology $\mathbb{H}^{k}(X, \mathcal{L}^{\bullet})$ is also defined in three steps:

(1) Take the Godement canonical resolution of every sheaf \mathcal{L}^q :



The vertical maps $\mathbb{C}^p \mathcal{L}^q \to \mathbb{C}^p \mathcal{L}^{q+1}$ are induced from $\mathcal{L}^q \to \mathcal{L}^{q+1}$ by the Godement functors $\mathbb{C}^p()$. This gives rise to a double complex $\bigoplus_{p,q=0}^{\infty} \mathbb{C}^p \mathcal{L}^q$ of sheaves.

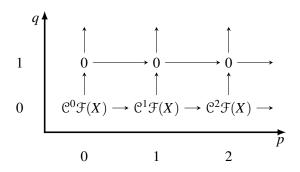
- (2) Apply the global section functor to $\bigoplus \mathbb{C}^p \mathcal{L}^q$ to obtain a double complex $\bigoplus \Gamma(X, \mathbb{C}^p \mathcal{L}^q)$ of abelian groups.
- (3) Take the total cohomology of the double complex $\bigoplus \Gamma(X, \mathcal{C}^p \mathcal{L}^q)$:

$$\mathbb{H}^{k}(X,\mathcal{L}^{\bullet}) = H_{D}\{\bigoplus \Gamma(X,\mathbb{C}^{p}\mathcal{L}^{q})\}.$$

Example 17.1. If the complex \mathcal{L}^{\bullet} consists of a single sheaf

$$0 \to \mathcal{F} \to 0 \to 0 \to \cdots,$$

then the double complex $\bigoplus \Gamma(X, \mathbb{C}^p \mathcal{L}^q)$ has only one possibly nonzero row, which occurs in degree q = 0:



In this case the total cohomology of the double complex $\bigoplus \Gamma(X, \mathbb{C}^p \mathcal{L}^q)$ is the cohomology of the zeroth row. Hence,

$$\mathbb{H}^*(X,\mathcal{L}^{\bullet}) = h^*\big(\mathcal{C}^{\bullet}\mathcal{F}(X)\big) = H^*(X,\mathcal{F}).$$

It is in this sense that hypercohomology generalizes sheaf cohomology.

Lecture 4. Applications

In the computation of hypercohomology using spectral sequences, there are two facts about exact functors that we will repeatedly use:

- (i) The Godement section functors $\Gamma(X, \mathbb{C}^p())$ are exact functors from sheaves on X to abelian groups [2, Cor. 2.2.7].
- (ii) Exact functors commute with cohomology [2, Prop. 2.2.10]: if T is an exact functor from sheaves to abelian groups and \mathcal{L}^{\bullet} is a complex of sheaves, then

$$T(\mathcal{H}(\mathcal{L}^{\bullet})) = H(T(\mathcal{L}^{\bullet})).$$

18. The de Rham Theorem

Let \mathcal{A}^k be the sheaf of C^{∞} k-forms of dimension n. By the Poincaré lemma,

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \to \dots \to \mathcal{A}^n \to 0$$

is an exact sequence. In particular, the cohomology sheaves of the complex \mathcal{A}^{\bullet} are

$$\mathcal{H}^{q} = \begin{cases} \underline{\mathbb{R}} & \text{for } q = 0, \\ 0 & \text{for } q \ge 1 \end{cases}$$

We will now compute the hypercohomology $\mathbb{H}^*(M, \mathcal{A}^{\bullet})$ using the two spectral sequences of the double complex

$$E_0 = \bigoplus K^{p,q} = \bigoplus \Gamma(M, \mathcal{C}^p \mathcal{A}^q).$$

Since $\Gamma(M, \mathbb{C}^p())$ is an exact functor and exact functors commute with cohomology,

$$E_1 = H_d(E_0) = \bigoplus \Gamma(M, \mathbb{C}^p \mathcal{H}^q) = \begin{cases} \bigoplus \Gamma(M, \mathbb{C}^p \underline{\mathbb{R}}), & \text{for } q = 0; \\ 0 & \text{for } q > 0. \end{cases}$$

$$E_{1} = \left[\begin{array}{c|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline C^{0}\underline{\mathbb{R}}(M) & C^{1}\underline{\mathbb{R}}(M) & C^{2}\underline{\mathbb{R}}(M) \\ \hline 0 & 1 & 2 \end{array}\right]$$

Thus,

$$E_{2} = H_{\delta}H_{d} = \underbrace{\left[\begin{array}{ccc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ H^{0}(M,\underline{\mathbb{R}}) & H^{1}(M,\underline{\mathbb{R}}) & H^{2}(M,\underline{\mathbb{R}}) \\ 0 & 1 & 2 \end{array}\right]}_{0}$$

Since d_r moves down r-1 rows, all differentials d_r , $r \ge 2$, are zero. It follow that the spectral sequence degenerates at the E_2 term and

$$\mathbb{H}^{k}(M,\mathcal{A}^{\bullet}) \simeq H^{k}(M,\underline{\mathbb{R}}).$$
(18.1)

There is no extension problem because the associated graded group of $H^k(M, \mathbb{R})$ has only one nonzero term, so that $H^k(M, \mathbb{R}) = \text{Gr}(H^k(M, \mathbb{R}))$.

On the other hand, the second spectral sequence of the double complex $E_0 = \bigoplus \Gamma(M, \mathbb{C}^p \mathcal{A}^q)$ starts with

$$E_1^{p,q} = H^{p,q}_{\delta} = H^p(M, \mathcal{A}^q) = \begin{cases} \mathcal{A}^q(M) & \text{for } p = 0, \\ 0 & \text{for } p > 0, \end{cases}$$

since \mathcal{A}^q is a fine sheaf and hence acyclic. Thus,

$$E_{1} = H_{\delta} = \left[\begin{array}{c|c} \mathcal{A}^{2}(M) & 0 & 0 \\ \mathcal{A}^{1}(M) & 0 & 0 \\ \mathcal{A}^{0}(M) & 0 & 0 \\ 0 & 1 & 2 & P \end{array} \right]$$

and

$$E_{2} = H_{d}H_{\delta} = \underbrace{\left| \begin{array}{ccc} H_{dR}^{2}(M) & 0 & 0 \\ H_{dR}^{1}(M) & 0 & 0 \\ H_{dR}^{0}(M) & 0 & 0 \\ 0 & 1 & 2 & p \end{array} \right|}_{0}$$

This spectral sequence also degenerates at the E_2 term and

$$\mathbb{H}^{k}(M,\mathcal{A}^{\bullet}) \simeq H^{k}_{\mathrm{dR}}(M).$$
(18.2)

Combining (18.1) and (18.2) gives an isomorphism $H^*_{dR}(M) \simeq H^k(M, \underline{\mathbb{R}})$ between de Rham cohomology and sheaf cohomology.

Theorem 18.1. On a manifold M, there is a canonical isomorphism between de Rham cohomology $H^*_{d\mathbb{R}}(M)$ and the sheaf cohomology $H^*(M,\underline{\mathbb{R}})$ of the sheaf $\underline{\mathbb{R}}$ of locally constant functions with values in \mathbb{R} .

Instead of the complex \mathcal{A}^{\bullet} of sheaves of C^{∞} forms on a manifold M, one may also consider on a topological space X the sheaf S^q of continuous q-cochains with values in \mathbb{R} . There is an exact sequence

$$0 \to \mathbb{R} \to \mathbb{S}^0 \xrightarrow{\delta} \mathbb{S}^1 \xrightarrow{\delta} \mathbb{S}^2 \xrightarrow{\delta} \cdots$$

of sheaves. By the same computation as for $\mathbb{H}^*(M, \mathcal{A}^{\bullet})$ above, one can show that if all the open subsets of *X* are paracompact, then

$$\mathbb{H}^*(X, \mathbb{S}^{\bullet}) \simeq H^*(X, \underline{\mathbb{R}})$$

and

$$\mathbb{H}^*(X, \mathbb{S}^{\bullet}) \simeq H^*_{\operatorname{sing}}(X, \mathbb{R}).$$

Theorem 18.2. On a topological space X in which all the open subsets are paracompact, there is a canonical isomorphism between singular cohomology with real coefficients and the sheaf cohomology of the sheaf $\underline{\mathbb{R}}$:

$$H^k_{\operatorname{sing}}(X,\mathbb{R})\simeq H^k(X,\underline{\mathbb{R}}).$$

Theorems 18.1 and 18.2 together prove the de Rham theorem: on a manifold M, there is a canonical isomorphism

$$H_{\mathrm{dR}}(M) \simeq H^k_{\mathrm{sing}}(M,\mathbb{R})$$

between de Rham cohomology and real singular cohomology.

19. THE DE RHAM THEOREM WITH COMPLEX COEFFICIENTS

On a complex manifold, instead of differential forms with real coefficients, we can consider forms with complex coefficient. Let $\mathcal{A}^k_{\mathbb{C}}$ be the sheaf of smooth *k*-forms with complex coefficients on *M*. By the Poincaré lemma with complex coefficients, the sequence

$$0 \to \underline{\mathbb{C}} \to \mathcal{A}^0_{\mathbb{C}} \to \mathcal{A}^1_{\mathbb{C}} \to \mathcal{A}^2_{\mathbb{C}} \to \cdots$$

is exact. The same computations as above show that the hypercohomology of $\mathcal{A}^{\bullet}_{\mathbb{C}}$ is isomorphic to sheaf cohomology or to de Rham cohomology:

$$\begin{split} & \mathbb{H}^*(M, \mathcal{A}^{\bullet}_{\mathbb{C}}) \simeq H^*(M, \underline{\mathbb{C}}), \\ & \mathbb{H}^*(M, \mathcal{A}^{\bullet}_{\mathbb{C}}) \simeq H^*_{\mathrm{dR}}(M, \mathbb{C}) \end{split}$$

Combined with Theorem 18.2, these isomorphisms yield the de Rham theorem with complex coefficients:

$$H^*_{\mathrm{dR}}(M,\mathbb{C})\simeq H^*(M,\mathbb{C})\simeq H^*_{\mathrm{sing}}(M,\mathbb{C}).$$

20. The Analytic de Rham Theorem

On a complex manifold M, let Ω^k be the sheaf of holomorphic k-forms and let $\mathcal{A}^{p,q}$ be the sheaf of $C^{\infty}(p,q)$ -forms. The Poincaré lemma has a holomorphic analogue: the sequence

$$0 \to \underline{\mathbb{C}} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots$$

is exact (for a proof, see [2, Th. 2.5.1]). It also has a $\bar{\partial}$ -analogue: on a complex manifold *M* of complex dimension *n*, the sequence

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \to \dots \to \mathcal{A}^{p,n} \to 0$$

is exact [4, p. 25].

By the holomorphic Poincaré lemma, the cohomology sheaves of the complex Ω^{\bullet} are

$$\mathfrak{H}^{q}(\Omega^{\bullet}) = \begin{cases} \underline{\mathbb{C}} & \text{for } q = 0, \\ 0 & \text{for } q > 0. \end{cases}$$

With the double complex $E_0 = \bigoplus K^{p,q} = \bigoplus \Gamma(M, \mathbb{C}^p \Omega^q)$ filtered by p,

$$E_1 = H_d(E_0) = \bigoplus \Gamma(M, \mathbb{C}^p \mathcal{H}^q) = \begin{cases} \bigoplus \Gamma(M, \mathbb{C}^p \mathbb{C}) & \text{for } q = 0\\ 0 & \text{for } q > 0, \end{cases}$$

or

$$E_{1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Gamma(M, \mathcal{C}^{0}\mathbb{C}) & \Gamma(M, \mathcal{C}^{1}\mathbb{C}) & \Gamma(M, \mathcal{C}^{2}\mathbb{C}) \\ 0 & 1 & 2 & p \end{bmatrix}}_{0}$$

Hence,

$$E_{2} = H_{\delta}H_{d} = \underbrace{\left[\begin{array}{ccc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ H^{0}(M,\underline{\mathbb{C}}) & H^{1}(M,\underline{\mathbb{C}}) & H^{2}(M,\underline{\mathbb{C}}) \\ 0 & 1 & 2 & p \end{array}\right]}_{0}$$

Therefore, the spectral sequence degenerates at the E_2 term and $E_2 = E_{\infty}$, so that there is a group isomorphism

$$H^k(M,\underline{\mathbb{C}})\simeq \mathbb{H}^k(M,\Omega^{\bullet}).$$

By an earlier result, there is an isomorphism between singular cohomology and sheaf cohomology

$$H^k_{\operatorname{sing}}(M,\mathbb{C})\simeq H^k(M,\underline{\mathbb{C}}).$$

Hence,

$$H^k_{\operatorname{sing}}(M,\underline{\mathbb{C}})\simeq \mathbb{H}^k(M,\Omega^{\bullet}).$$

This is the analytic de Rham theorem. It shows that the singular cohomology with complex coefficients of a complex manifold can be computed using only holomorphic forms.

Unlike in the smooth case, however, because the sheaves Ω^q of holomorphic forms are not acyclic, one cannot conclude that $H^k_{\text{sing}}(M, \mathbb{C})$ is the cohomology of the holomorphic de Rham complex $\Omega^{\bullet}(M)$.

21. ACYCLIC RESOLUTIONS

A resolution

$$0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \mathcal{L}^2 \to \cdots$$

of a sheaf \mathcal{F} is *acyclic* if all the sheaves \mathcal{L}^q are acyclic. Suppose $0 \to \mathcal{F} \to \mathcal{L}^{\bullet}$ is an acyclic resolution. To compute the hypercohomology $\mathbb{H}^*(X, \mathcal{L}^{\bullet})$, we start with the double complex

$$E_0 = \bigoplus E_0^{p,q} = \bigoplus \Gamma(X, \mathcal{C}^p \mathcal{L}^q),$$

filtered by p. Then

$$E_1 = H_d(E_0) = \bigoplus H_d(\Gamma(X, \mathcal{C}^p \mathcal{L}^{\bullet}))$$
$$= \bigoplus \Gamma(X, \mathcal{C}^p \mathcal{H}^q),$$

where \mathcal{H}^q is the *q*th cohomology sheaf of \mathcal{L}^{\bullet} and the last equality follows from the fact that cohomology commutes with the exact functor $\Gamma(X, \mathcal{C}^p(\cdot))$.

Because $0\to \mathcal{F}\to \mathcal{L}^\bullet$ is exact, the cohomology sheaves of \mathcal{L}^\bullet are

$$\mathcal{H}^q = \begin{cases} \mathcal{F} & \text{for } q = 0, \\ 0 & \text{for } q > 0. \end{cases}$$

Therefore, E_1 has only one nonzero row and it is row 0:

$$E_{1} = H_{d} = \underbrace{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Gamma(X, \mathcal{C}^{0}\mathcal{F}) & \Gamma(X, \mathcal{C}^{1}\mathcal{F}) & \Gamma(X, \mathcal{C}^{2}\mathcal{F}) \\ 0 & 1 & 2 & p \end{bmatrix}}_{0}$$

It follows that

Because each antidiagonal has only one nonzero box, there is no extension problem and

$$\mathbb{H}^{k}(X, \mathcal{L}^{\bullet}) \simeq H^{k}(X, \mathcal{F}).$$
(21.1)

On the other hand, we can also filter

$$E_0 = \bigoplus E_0^{p,q} = \bigoplus \Gamma(X, \mathfrak{C}^p \mathcal{L}^q),$$

by q. Note that the cohomology of the qth row is precisely the sheaf cohomology $H^*(X, \mathcal{L}^q)$:

$$E_1 = H_{\delta}(E_0) = \bigoplus H^p(X, \mathcal{L}^q).$$

Since \mathcal{L}^q is acyclic, the only nonzero column is the zeroth column

$$E_1 = H_{\delta}(E_0) = \underbrace{\left| \begin{array}{ccc} H^0(X, \mathcal{L}^2) & 0 & 0 \\ H^0(X, \mathcal{L}^1) & 0 & 0 \\ H^0(X, \mathcal{L}^0) & 0 & 0 \\ \hline 0 & 1 & 2 & p \end{array} \right|}_{0}$$

Thus,

$$E_2 = H_d(E_1) = h^* \big(\mathcal{L}^{\bullet}(X) \big).$$

This spectral sequence also degenerates at the E_2 page and

$$\mathbb{H}^{k}(X,\mathcal{L}^{\bullet}) \simeq E_{\infty}^{0,k} \simeq h^{k}(\mathcal{L}^{\bullet}(X)).$$
(21.2)

Combining (21.1) and (21.2), we have proven the following theorem.

Theorem 21.1. *Sheaf cohomology can be computed as the cohomology of the complex of global sections of an acyclic resolution: let*

$$0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \mathcal{L}^2 \to \cdots$$

be an acyclic resolution of a sheaf \mathcal{F} over a topological space X. Then

$$H^k(X, \mathcal{F}) \simeq h^k(\mathcal{L}^{\bullet}(X)).$$

Example 21.2. Since the Godement resolution $\mathcal{C}^{\bullet}\mathcal{F}$ of any sheaf \mathcal{F} is flasque and hence acyclic, applying Theorem 21.1 to the Godement resolution returns

$$H^k(X, \mathcal{F}) = h^k \big(\mathcal{C}^{\bullet} \mathcal{F}(X) \big),$$

the definition of sheaf cohomology.

Example 21.3. The complex \mathcal{A}^{\bullet} of sheaves of C^{∞} forms on a manifold M is a resolution of \mathbb{R} . Since the sheaves \mathcal{A}^q are fine sheaves and hence acyclic, by Theorem 21.1 there is an isomorphism between sheaf cohomology and de Rham cohomology:

$$H^{k}(M,\underline{\mathbb{R}}) \simeq h^{k} \left(\mathcal{A}^{\bullet}(M) \right) = H^{k}_{\mathrm{dR}}(M).$$
(21.3)

Example 21.4. The complex S^{\bullet} of sheaves of continuous cochains on a topological space X all of whose open subsets are paracompact is an acyclic resolution of \mathbb{R} . By Theorem 21.1

$$H^{k}(X,\underline{\mathbb{R}}) \simeq h^{k}\left(\mathbb{S}^{\bullet}(X)\right) = H^{k}_{\operatorname{sing}}(X,\mathbb{R}).$$
(21.4)

Combining (21.3) and (21.4) gives the de Rham theorem again.

Example 21.5. In the holomorphic category there do not generally exist partitions of unity and the sheaves Ω^k of holomorphic *k*-forms are in general not acyclic. This explains why the analytic de Rham theorem does not have the same form as the usual de Rham theorem.

Example 21.6. Let $\mathcal{A}^{p,q}$ be the sheaf of $C^{\infty}(p,q)$ -forms on a complex manifold M of complex dimension n. By the $\bar{\partial}$ -Poincaré lemma,

$$0 \to \Omega^{p} \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \cdots \to \mathcal{A}^{p,n} \to 0$$

is a resolution of Ω^p . Since the $\mathcal{A}^{p,q}$ are fine sheaves and hence acyclic, by Theorem 21.1,

$$H^q(M, \Omega^p) \simeq h^q(\mathcal{A}^{p, \bullet}(M)) \simeq H^{p, q}(M),$$

which is Dolbeault's theorem.

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DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, MEDFORD, MA 02155 *Email address*: loring.tu@tufts.edu